

A Demonstration of the Advantage of Asymptotic Preserving Schemes over Standard Finite Volume Schemes

Jochen Schütz

Bericht Nr. 366

Juni 2013

Key words: conservation laws, asymptotic methods,
finite volume methods, finite element methods.

AMS Subject Classifications: 35L65, 76M45, 65M08, 65M60

Institut für Geometrie und Praktische Mathematik

RWTH Aachen

Templergraben 55, D-52056 Aachen (Germany)

A DEMONSTRATION OF THE ADVANTAGE OF ASYMPTOTIC PRESERVING SCHEMES OVER STANDARD FINITE VOLUME SCHEMES.

JOCHEN SCHÜTZ

June 7, 2013

Abstract. We apply the concept of Asymptotic Preserving (AP) schemes [17] to the linearized p -system and discretize the resulting elliptic equation using standard continuous Finite Elements instead of Finite Differences. The fully discrete method is analyzed with respect to consistency, and we compare it numerically with more traditional methods such as Implicit Euler's method. Numerical results indicate that the AP method is indeed superior to more traditional methods.

1. Introduction. We consider the p -system [12] with a linear pressure function $p(v) := -\frac{1}{\varepsilon^2}v$ and a right-hand side g ,

$$v_t - u_x = 0 \quad \forall (x, t) \in \Omega \times \mathbb{R}^+ \quad (1.1)$$

$$u_t + p(v)_x = g(x, t) \quad \forall (x, t) \in \Omega \times \mathbb{R}^+ \quad (1.2)$$

on a domain $\Omega \subset \mathbb{R}$ subject to suitable initial and boundary values, where for simplicity we choose the latter to be

$$v(x) = 0 \quad \forall x \in \partial\Omega.$$

In a (simplified) physical application, u and v could denote velocity and (variations of) the specific volume of the fluid.

Obviously, the equation can be written as

$$w_t + f(w)_x = G(x, t) \quad \forall (x, t) \in \Omega \times \mathbb{R}^+ \quad (1.3)$$

for $w := (v, u)^T$, $f(w) := (-u, -\frac{1}{\varepsilon^2}v)^T$ and $G(x, t) := (0, g(x, t))^T$.

The eigenvalues of the Jacobian of the flux function f are $\pm\frac{1}{\varepsilon}$, and so a fully explicit Finite-Volume scheme will not be feasible for small values of ε , as the time-step will decrease with ε . Inspired by *Asymptotic Preserving Schemes* (AP), we develop a new solver for (1.1)-(1.2) based on a combination of Finite Volumes and Finite Elements. Its (fully discrete) consistency is investigated, and it is compared with more traditional numerical schemes with respect to error versus mesh size. We put this in the simple framework of the p -system because it was on a similar system that Jin [17] derived his famous asymptotic preserving schemes for the first time, and because it is simple (but not too simple), so that each step can be easily computed, which is not the case for more involved systems such as Euler's equations.

As already mentioned, the concept of asymptotic preserving schemes that we pursue in this publication has been introduced by Jin [17], building on joint work with Pareschi and Toscani [18]. In these publications, a scheme is called asymptotic preserving if

- it is for $\varepsilon \rightarrow 0$ a consistent scheme for the multiscale limiting equations of (1.1)-(1.2) and
- is stable with a *cfl*-number independent of ε .

This class of schemes has since been extended to various kinds of equations, such as, e.g., Euler's equation [13, 3], Shallow-water equations [2] and many more.

The current paper is a first attempt to extend the schemes, which have been presented for Finite-Volume discretizations, to Galerkin-type schemes. Based on a flux-splitting, we derive an elliptic equation whose diffusion coefficient is dependent on ε and Δt . This equation is solved by continuous Finite-Element methods, and not, as usual, by finite-difference schemes. The approach, though it can of course also be written in terms of finite differences, has the advantage that we can investigate the elliptic equation and its discretization in a rigorous setting in the context of Sobolev spaces. In a first step, we show that the elliptic equation is well-posed and uniformly well-conditioned for all values of ε and Δt . This is achieved by introducing problem-dependent spaces and norms. In a second step, we restrict ourselves to 'small' ε and 'large' Δt , i.e., $0 < \varepsilon \leq \varepsilon_0 < 1$ and $\Delta t \geq \varepsilon$, as it is only in this setting that we can use standard Finite-Element schemes [6, 15] instead of stabilized ones [5]. Also for this setting, we can derive rigorous and uniform (in ε) stability and consistency bounds.

Solutions to (1.1)-(1.2) that allow for a limit solution as $\varepsilon \rightarrow 0$ have a certain structure (see (2.6)-(2.7) in Sec. 2). Our consistency analysis for the fully discrete algorithm heavily relies on this structure, and we believe that it is only in this setting that one can derive suitable bounds on the consistency error that do not behave like $O(\varepsilon^{-1})$ or even worse. As an easy consequence, we can indeed show that the proposed scheme is AP. This is different to other authors [13, 2] who show that their scheme is asymptotic preserving by a Taylor series argument on the semi-discrete stage.

Having presented our scheme, we compare it numerically with two other schemes. The surprising outcome is that the scheme to be presented performs better by orders of magnitude in comparison to more traditional schemes.

The outline of the paper is as follows: In Sec. 2.1, we derive the multiscale limit solution of the linearized p -system for $\varepsilon \rightarrow 0$. In Sec. 2.2, we split the conservative flux f into a stiff \tilde{f} and a non-stiff \hat{f} . Based on this splitting, we derive a semi-discretization in Sec. 2.3. This yields an elliptic equation, which is investigated in Sec. 2.4. Finally, in Sec. 2.5, we formulate the fully discrete algorithm and investigate its consistency in Sec. 2.6. In Sec. 3, we show numerical results. Sec. 4 offers conclusions and outlook.

2. Asymptotic Preserving Discretization.

2.1. Multiscale limit of the equation. In this section, we follow a multiscale approach to obtain the limiting equations of (1.3). To this end, we assume that our unknown solution (v, u) admits a two-scale expansion as

$$v(x, t) = v^{(0)}(x, t) + \varepsilon v^{(1)}(x, t) + \varepsilon^2 v^{(2)}(x, t) + O(\varepsilon^3) \quad (2.1)$$

$$u(x, t) = u^{(0)}(x, t) + \varepsilon u^{(1)}(x, t) + \varepsilon^2 u^{(2)}(x, t) + O(\varepsilon^3). \quad (2.2)$$

Note that this approach does not include fast waves, i.e., contributions depending on $\frac{1}{\varepsilon}$, so one has a uniform limit as $\varepsilon \rightarrow 0$. Plugging (2.1)-(2.2) into (1.1)-(1.2) and balancing the powers of ε yields that both $v^{(0)}(x, t)$ and $v^{(1)}(x, t)$ are independent of x . Therefore, $v^{(1)}(x, t)$ can be absorbed into $v^{(0)}(x, t)$, and (2.1) reduces to

$$v(x, t) = v^{(0)}(t) + \varepsilon^2 v^{(2)}(x, t). \quad (2.3)$$

The remaining limiting equations can be easily seen to be

$$v_t^{(0)} - u_x^{(0)} = 0 \quad \forall (x, t) \in \Omega \times \mathbb{R}^+ \quad (2.4)$$

$$u_t^{(0)} - v_x^{(2)} = g(x, t) \quad \forall (x, t) \in \Omega \times \mathbb{R}^+. \quad (2.5)$$

A suitable algorithm approximating (1.1)-(1.2) for small values of ε should, in the vanishing ε -limit, be a consistent approximation to (2.4)-(2.5). In reference [17], such a consistency requirement is called *asymptotic preserving*.

For a general conservation law, it is nontrivial to obtain more precise results concerning $v^{(0)}$ and $u^{(0)}$, see, e.g., [19] for results in the context of Euler's equations. However, in the very simple setting of the linearized p -system, we can clarify even more the relation between v and u . Due to the boundary conditions, it can be easily shown that the only admissible functions are

$$v(x, t) = \varepsilon^2(t\xi(x) + \eta(x)) + O(\varepsilon^3) \quad (2.6)$$

$$u(x, t) = u^{(0)}(t) + \varepsilon u^{(1)} + \varepsilon^2 \int_d^x \xi(y) dy + O(\varepsilon^3) \quad (2.7)$$

with a suitable definition of $g(x, t) := u_t(x, t) - v_x(x, t)$ and arbitrary functions $\xi, \eta, u^{(0)}$ and constants $u^{(1)}, d \in \mathbb{R}$. Especially, both $v(x, t)$ and $u_x(x, t)$ are of order $O(\varepsilon^2)$, which will help performing a consistency analysis.

2.2. Flux Splitting. The way of splitting the flux into stiff and non-stiff parts has an influence on the final algorithm. We choose our splitting according to the following definition:

DEFINITION 2.1. *Let the flux function f be split into $f(w) = \widehat{f}(w) + \widetilde{f}(w)$. We consider such a splitting to be admissible if for all $0 < \varepsilon < 1$*

- *both $\widehat{f}(w)$ and $\widetilde{f}(w)$ induce a hyperbolic system, i.e., the eigenvalues of both $\widehat{f}'(w)$ and $\widetilde{f}'(w)$ are distinct and real,*
- *the eigenvalues of $\widehat{f}'(w)$ are of order one,*
- *$\widehat{f}(w)$ approaches $f(w)$ as $\varepsilon \rightarrow 1$, and*
- *$\widetilde{f}(w)$ approaches $f(w)$ for $\varepsilon \rightarrow 0$ in the sense that $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 (\widetilde{f}(w) - f(w)) = 0$.*

$\widehat{f}(w)$ is called the 'non-stiff', and $\widetilde{f}(w)$ the 'stiff' part of the flux function for obvious reasons.

To identify stiff and non-stiff parts of the flux function, we make the following ansatz:

$$f(w) = \widehat{f}(w) + \widetilde{f}(w) =: \begin{pmatrix} -\alpha(\varepsilon)u \\ -\frac{\beta(\varepsilon)}{\varepsilon^2}v \end{pmatrix} + \begin{pmatrix} -(1 - \alpha(\varepsilon))u \\ -\frac{1 - \beta(\varepsilon)}{\varepsilon^2}v \end{pmatrix}.$$

Both $\alpha(\cdot)$ and $\beta(\cdot)$ are yet unknown. One reasonable requirement is $\alpha(1) = \beta(1) = 1$, and $\alpha(0) = \beta(0) = 0$, so that one has no stiff contribution given that ε is one, and no non-stiff contribution given that ε vanishes. We make the simple ansatz of $\alpha(\varepsilon) = \varepsilon^a$, $\beta(\varepsilon) = \varepsilon^b$. An easy computation shows that for $a, b > 0$, $a + b = 2$, the eigenvalues of $\widehat{f}'(w)$ are independent of ε . A particularly simple choice is $a = b = 1$, which we will use throughout this work. In summary, for this choice of a and b , we have

$$\widehat{f}(w) = \begin{pmatrix} -\varepsilon u \\ -\frac{1}{\varepsilon}v \end{pmatrix}, \quad \widetilde{f}(w) = \begin{pmatrix} -(1 - \varepsilon)u \\ -\frac{1 - \varepsilon}{\varepsilon^2}v \end{pmatrix}$$

with corresponding eigenvalues of the Jacobians

$$\widehat{\lambda} = \pm 1, \quad \widetilde{\lambda} = \pm \frac{1 - \varepsilon}{\varepsilon}.$$

2.3. Semi-Discretization. We start the description of our algorithm with a discretization in time only. For simplicity, we assume that we work on space-time slabs of (uniform) size Δt , although uniformity is not a necessary condition. Throughout this work, we will use standard notation and set $w^n := w(t^n)$, where $t^n := n\Delta t$. Based on the flux splitting defined in Sec. 2.2, we obtain a first-order implicit / explicit semidiscretization of (1.3) in time, given by

$$\frac{w^{n+1} - w^n}{\Delta t} + \widehat{f}(w^n)_x + \widetilde{f}(w^{n+1})_x = G^n. \quad (2.8)$$

or, in terms of (v, u) ,

$$\frac{v^{n+1} - v^n}{\Delta t} = \varepsilon u_x^n + (1 - \varepsilon)u_x^{n+1} \quad (2.9)$$

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{\varepsilon}v_x^n + \frac{1 - \varepsilon}{\varepsilon^2}v_x^{n+1} + g^n. \quad (2.10)$$

One way of dealing with such a system of implicit equations that has become a standard ingredient in asymptotic preserving schemes, is to equivalently reformulate (2.9)-(2.10) in such a way that one obtains an equation for either v^{n+1} or u^{n+1} alone. We have decided to formulate an equation for v^{n+1} . To this end, we note that (2.10) is equivalent to

$$u^{n+1} = u^n + \Delta t \left(\frac{1}{\varepsilon}v_x^n + \frac{1 - \varepsilon}{\varepsilon^2}v_x^{n+1} + g^n \right), \quad (2.11)$$

and plug this into (2.9):

$$v^{n+1} = v^n + \Delta t \left(\varepsilon u_x^n + (1 - \varepsilon) \left(u^n + \Delta t \left(\frac{1}{\varepsilon}v_x^n + \frac{1 - \varepsilon}{\varepsilon^2}v_x^{n+1} + g^n \right) \right)_x \right) \quad (2.12)$$

$$= v^n + \Delta t u_x^n + \Delta t^2(1 - \varepsilon) \left(\frac{1}{\varepsilon}v_{xx}^n + \frac{1 - \varepsilon}{\varepsilon^2}v_{xx}^{n+1} + g_x^n \right) \quad (2.13)$$

$$= v^n + \Delta t u_x^n + \frac{\Delta t^2(1 - \varepsilon)}{\varepsilon}v_{xx}^n + \frac{\Delta t^2(1 - \varepsilon)^2}{\varepsilon^2}v_{xx}^{n+1} + \Delta t^2(1 - \varepsilon)g_x^n. \quad (2.14)$$

Rearranging terms yields an elliptic equation for v^{n+1} :

$$-\frac{\Delta t^2(1 - \varepsilon)^2}{\varepsilon^2}v_{xx}^{n+1} + v^{n+1} = v^n + \Delta t u_x^n + \frac{\Delta t^2(1 - \varepsilon)}{\varepsilon}v_{xx}^n + \Delta t^2(1 - \varepsilon)g_x^n. \quad (2.15)$$

REMARK 2.2. *Using standard theory of partial differential equations, it is easy to see that (2.15) is a well-posed equation for $\Delta t > 0$ and $0 < \varepsilon < 1$. However, in standard Sobolev norms, one can see that only for $\Delta t \gg 0$ and $\varepsilon \ll 1$, one obtains uniform bounds on the condition number of the problem. We will show in Sec. 2.4 that one can put the equation in a framework that allows for a uniform bound on the condition number of (2.15), however, in a norm that depends on the diffusion coefficient $\lambda := \frac{\Delta t^2(1 - \varepsilon)^2}{\varepsilon^2}$. (Which is, of course, not surprising.)*

The weak formulation of (2.15) can be cast in a variational framework as

$$a(v^{n+1}, \varphi) = \iota(\varphi) \quad \forall \varphi \in H_0^1(\Omega), \quad (2.16)$$

where

$$a(v^{n+1}, \varphi) := \int_{\Omega} \left(\frac{\Delta t^2 (1 - \varepsilon)^2}{\varepsilon^2} v_x^{n+1} \varphi_x + v^{n+1} \varphi \right) dx \quad \text{and} \quad (2.17)$$

$$\iota(\varphi) := \int_{\Omega} \left(v^n + \Delta t u_x^n \right) \varphi - \Delta t^2 (1 - \varepsilon) \left(\frac{v_x^n}{\varepsilon} + g^n \right) \varphi_x dx. \quad (2.18)$$

Boundedness and coercivity properties of $a(\cdot, \cdot)$ will be discussed in the next sections. What concerns ι , we can state the following lemma:

LEMMA 2.3. *Let us assume that $u^n \equiv u(t^n)$ and $v^n \equiv v(t^n)$ are functions in $H^1(\Omega)$; $g^n \equiv g(t^n)$ is a function in $L^2(\Omega)$; and $0 < \varepsilon < 1$. Then $\iota \in H_0^1(\Omega)'$.*

Proof. It is enough to show that both $v^n + \Delta t u_x^n$ and $\Delta t^2 (1 - \varepsilon) \left(\frac{v_x^n}{\varepsilon} + g^n \right)$ are functions in $L^2(\Omega)$, which is correct because of the assumptions on u^n, v^n and g^n . \square

2.4. A note on the elliptic equation. Let us now turn to the operator equation (2.16). To make it a well-defined and a uniformly well-conditioned problem for all $0 < \varepsilon \leq 1$, we put it in a variational framework with weighted Sobolev spaces as follows:

DEFINITION 2.4. *Let the coefficient of the viscous term of (2.16) be denoted by λ , i.e.,*

$$\lambda := \frac{\Delta t^2 (1 - \varepsilon)^2}{\varepsilon^2}. \quad (2.19)$$

We define a weighted norm $\|\cdot\|_{\lambda}$ as

$$\|\varphi\|_{\lambda}^2 := \|\varphi\|_{L^2}^2 + \lambda \|\varphi_x\|_{L^2}^2 \quad (2.20)$$

and a corresponding 'Sobolev-space'

$$V_{\lambda}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_{\lambda}}. \quad (2.21)$$

COROLLARY 2.5. *For $\lambda > 0$, i.e., $\varepsilon < 1$, the weighted norm $\|\cdot\|_{\lambda}$ is equivalent to the standard Sobolev norm, as can be seen from a Poincaré-Friedrichs inequality. However, the equivalence constants gets worse as ε approaches one. With this equivalence in mind, it is easy to see that*

$$V_{\lambda}(\Omega) = \begin{cases} H_0^1(\Omega), & \varepsilon < 1 \\ L^2(\Omega), & \varepsilon = 1 \end{cases}, \quad (2.22)$$

as $\lambda = 0$ for $\varepsilon = 1$ and $\lambda > 0$ for $0 < \varepsilon < 1$.

REMARK 2.6. *The weighted norm $\|\cdot\|_{\lambda}$ is the energy norm associated to (2.16), i.e.,*

$$\|\varphi\|_{\lambda}^2 = a(\varphi, \varphi). \quad (2.23)$$

Furthermore, for $\lambda = 0$, the problem (2.16) is not well-posed in $H_0^1(\Omega)$ any more, so the choice of $V_{\lambda}(\Omega)$ is actually very natural.

The following lemma computes both coercivity and boundedness constants of $a(\cdot, \cdot)$ on $V_{\lambda}(\Omega)$:

LEMMA 2.7. *The bilinear form $a(\cdot, \cdot)$ as defined in (2.17) is coercive on $V_\lambda(\Omega) \times V_\lambda(\Omega)$ with ellipticity constant one, and bounded on $V_\lambda(\Omega) \times V_\lambda(\Omega)$ with boundedness constant also one.*

Proof. It is easy to see that

$$a(\varphi, \varphi) = \lambda \|\varphi_x\|_{L^2}^2 + \|\varphi\|_{L^2}^2 = \|\varphi\|_\lambda^2, \quad (2.24)$$

so the bilinear form is elliptic with ellipticity constant one. Furthermore, using Cauchy-Schwartz inequality (this is possible because of (2.23)), one has

$$a(\varphi, \psi) \leq \|\varphi\|_\lambda \|\psi\|_\lambda. \quad (2.25)$$

□

The following theorem guarantees that (2.16) is, for the full range of $0 < \varepsilon \leq 1$, a well-conditioned problem:

THEOREM 2.8. *The equation (2.16) is well-conditioned in $V_\lambda(\Omega)$ independently of ε , i.e., for two functionals $\iota, \tilde{\iota} \in V_\lambda(\Omega)'$, and their corresponding solutions v and \tilde{v} , one has the relation*

$$\frac{\|v - \tilde{v}\|_\lambda}{\|v\|_\lambda} \leq \frac{\|\iota - \tilde{\iota}\|_{V_\lambda'}}{\|\iota\|_{V_\lambda'}}. \quad (2.26)$$

Proof. It is a classical result from the theory of elliptic pde that the quotient of boundedness constant and ellipticity constant is indeed the condition number with respect to a perturbation of the functional ι . Nevertheless, for convenience, we give a sketch of the proof. From ellipticity, we can conclude

$$\|v - \tilde{v}\|_\lambda^2 = a(v - \tilde{v}, v - \tilde{v}) = \iota(v - \tilde{v}) - \tilde{\iota}(v - \tilde{v}) \leq \|\iota - \tilde{\iota}\|_{V_\lambda'} \|v - \tilde{v}\|_\lambda \quad (2.27)$$

and from boundedness

$$\|\iota\|_{V_\lambda'} = \sup_{u \in V_\lambda(\Omega), \|u\|_\lambda=1} \iota(u) = \sup_{u \in V_\lambda(\Omega), \|u\|_\lambda=1} a(v, u) \leq \|v\|_\lambda. \quad (2.28)$$

(2.27)-(2.28) yields (2.26). □

2.5. Full discretization. In this section, we introduce a fully discrete method. To this end, we assume that our spatial domain Ω is subdivided into cells Ω_k as

$$\Omega = \bigcup_{k=1}^{N_x} \Omega_k := \bigcup_{k=1}^{N_x} [x_k, x_{k+1}] \quad (2.29)$$

with midpoints \bar{x}_k . For the ease of presentation (but without loss of generality), we consider a uniform discretization, i.e., $\Delta x := x_{k+1} - x_k$ is constant. Using standard conventions, we define w_i^n to be (an approximation to) $w(\bar{x}_i, t^n)$. By now, we have all the ingredients of formulating a fully discrete algorithm for the approximation of (1.1)-(1.2) for each timestep.

The algorithm relies on the following steps:

- Compute an approximate solution to (2.16) with (linear) Finite-Elements.
- Update u according to (2.11).

Let us make this more precise: The right-hand side ι of (2.16) includes derivatives of both u and v at time t^n . These derivatives are approximated using a standard numerical flux function \widehat{f}_{num} for the non-stiff part \widehat{f} of the flux f , in the sense that

$$(-\varepsilon(u_i^n)_x, -\frac{1}{\varepsilon}(v_i^n)_x)^T \approx \frac{1}{\Delta x} \left(\widehat{f}_{num}(w_i^n, w_{i+1}^n) - \widehat{f}_{num}(w_{i-1}^n, w_i^n) \right) \quad (2.30)$$

As we have to plug functions in Ω into ι , we assume that the derivatives are piecewise constant on each cell. (A better reconstruction is left for future publication, see also Rem. 2.9.) Plugging these quantities into ι yields an approximation ι_h . Using this approximation ι_h , one computes a Finite-Element solution to the elliptic equation, i.e., one computes a solution $v_h^{n+1} \in V_h$ to

$$a(v_h^{n+1}, \varphi_h) = \iota_h(\varphi_h) \quad \forall \varphi_h \in V_h, \quad (2.31)$$

where

$$V_h := \{\varphi_h \in C^0(\Omega) \mid \varphi_h|_{\Omega_k} \text{ is linear for all } k; \varphi_h(0) = \varphi_h(1) = 0\}. \quad (2.32)$$

Subsequently, u^{n+1} is updated using formula (2.11), employing the approximations of both $(u_i^n)_x$ and $(v_i^n)_x$.

REMARK 2.9. *In this work, we use a first-order Finite-Volume scheme with a local Lax-Friedrichs flux, given by $f_{num} \equiv f_{LF}$ with*

$$f_{LF}(u_l, u_r) := \frac{1}{2} (f(u_l) + f(u_r)) - \frac{\alpha}{2} (u_r - u_l). \quad (2.33)$$

As usual, α denotes the spectral radius of the local Jacobian (which is, in this simple setting, a constant). We have decided to put our investigations in this simple framework to allow for a fair comparison, which is not influenced by the choice of limiter or time-integration scheme.

REMARK 2.10. *Note that for $\varepsilon = 1$, the elliptic equation (2.16) becomes a triviality and does not have to be solved. In this case, the scheme reduces to a standard Finite-Volume scheme with numerical flux function \widehat{f}_{num} . Therefore, we assume $\varepsilon < 1$ in the following.*

2.6. (Order of) Consistency and some stability considerations. In this section, we show that our method is consistent, and we determine its order of consistency. We have decided to put this investigation into the more classical framework of standard H_0^1 spaces and norms (instead of using V_λ), because in this setting we can use classical Finite-Element spaces and do not have to rely on stabilized Finite-Elements such as SUPG. This, however, comes at the price of restricting ε to $0 < \varepsilon \leq \varepsilon_0 < 1$ and $\Delta t \geq \varepsilon$. Nevertheless, as we are interested in the $\varepsilon \rightarrow 0$ limit for a moderate time-step Δt , this is not a severe restriction.

To prove consistency of our scheme, we have to bound the following error parts:

$$e_1 := \|v(t^{n+1}) - v^{n+1}\|_{L^2} \quad (2.34)$$

$$e_2 := \|v^{n+1} - \bar{v}^{n+1}\|_{L^2} \quad (2.35)$$

$$e_3 := \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2}. \quad (2.36)$$

The overall consistency error in v , $e := \|v(t^{n+1}) - v_h^{n+1}\|_{L^2}$, can then be bounded by the sum of the e_i . Let us remind the reader of the following definitions:

- $v(t^{n+1})$ denotes the exact solution v to (1.3) at time t^{n+1} .
- v^{n+1} denotes the exact solution to the elliptic equation, see (2.16).
- \bar{v}^{n+1} denotes the solution to the elliptic equation (2.16) with right-hand side ι_h instead of ι , see Sec. 2.5.
- v_h^{n+1} denotes the Finite-Element solution to the elliptic equation, see (2.31).

Let us make the following important assumption which is motivated by our investigations concerning the multiscale expansion, see (2.6)-(2.7):

ASSUMPTION 2.11. *We assume that v is given by*

$$v(x, t) = \varepsilon^2 v^{(2)}(x, t) + O(\varepsilon^3). \quad (2.37)$$

and that the spatial derivative of u is given by

$$u_x(x, t) = \varepsilon^2 u_x^{(2)}(x, t) + O(\varepsilon^3). \quad (2.38)$$

REMARK 2.12. *Without this assumption, it will not be possible to perform a consistency analysis for the small ε limit, because there is no limit function as $\varepsilon \rightarrow 0$. This is very similar to the observation in [19] that the initial data has to be divergence free to allow for an incompressible limit.*

We start by bounding e_1 .

LEMMA 2.13. *The temporal discretization yields the following asymptotic error:*

$$e_1 := \|v(t^{n+1}) - v^{n+1}\|_{L^2} = O(\varepsilon^2 \Delta t^2).$$

Proof. By checking the order of consistency of (2.9), one obtains ($t^n \leq \xi_1, \xi_2 \leq t^{n+1}$):

$$\begin{aligned} & \frac{1}{\Delta t} (v(t^{n+1}) - v(t^n)) - \varepsilon u_x(t^n) - (1 - \varepsilon) u_x(t^{n+1}) \\ &= \frac{1}{\Delta t} \left(\Delta t v_t(t^n) + \frac{\Delta t^2}{2} v_{tt}(\xi_1) \right) - \varepsilon u_x(t^n) - (1 - \varepsilon) (u_x(t^n) + \Delta t u_{xt}(\xi_2)) \\ &= v_t(t^n) + \frac{\Delta t}{2} v_{tt}(\xi_1) - u_x(t^n) - (1 - \varepsilon) \Delta t u_{xt}(\xi_2) \\ &\stackrel{(1.1)}{=} \frac{\Delta t}{2} v_{tt}(\xi_1) - (1 - \varepsilon) \Delta t u_{xt}(\xi_2) \\ &\stackrel{\text{Ass.2.11}}{=} O(\varepsilon^2 \Delta t), \end{aligned}$$

which yields indeed the desired order of accuracy. \square

Let us continue by bounding e_2 . w^n denotes the (assumed smooth) exact solution $w = (v, u)^T$ at time t^n . By w_x^n , we denote the exact derivative of w at time t^n , and by \tilde{w}_x^n , we denote the approximation of the derivative by numerical flux functions. It is known that

$$\|w_x^n - \tilde{w}_x^n\|_{L^2} = O(\varepsilon^2 \Delta x), \quad (2.39)$$

which, again, is a consequence of Ass. 2.11.

Before considering the full approximation error, we have to turn to the operator equation (2.16) again in the context of classical Sobolev-spaces. Following standard convention, we define the H_0^1 -norm to be

$$\|\varphi\|_{H_0^1} := \|\varphi_x\|_{L^2}, \quad (2.40)$$

and remind the reader of Poincaré-Friedrich's inequality

$$\|\varphi\|_{L^2} \leq C_{PF} \|\varphi\|_{H_0^1}. \quad (2.41)$$

We start with the following theorem that guarantees that (2.16) is, also for small ε , 'easy' to solve.

THEOREM 2.14. *For a given $\varepsilon_0 < 1$, let $0 < \varepsilon \leq \varepsilon_0$, and $\Delta t \geq \varepsilon$. The equation (2.16) is well-conditioned in H_0^1 independently of ε , i.e., for two functionals $\iota, \tilde{\iota} \in H_0^1(\Omega)'$, and the corresponding solutions v and \tilde{v} , one has the relation*

$$\frac{\|v - \tilde{v}\|_{H_0^1}}{\|v\|_{H_0^1}} \leq \frac{M}{\gamma} \frac{\|\iota - \tilde{\iota}\|_{H_0^1'}}{\|\iota\|_{H_0^1'}}, \quad (2.42)$$

and $\frac{M}{\gamma}$ can be bounded uniformly in terms of ε .

Proof. It is easy to see that $a(\cdot, \cdot)$ fulfills, for $\varepsilon < 1$, an ellipticity condition on H_0^1 with ellipticity-constant γ , and it is a bounded bilinear form with stability constant M . Both γ and M can be explicitly given as

$$\gamma = \frac{\Delta t^2(1-\varepsilon)^2}{\varepsilon^2}, \quad M = \gamma + C_{PF}^2. \quad (2.43)$$

The rest of the proof goes along the lines of Thm. 2.8. Note that the quotient $\frac{M}{\gamma}$ is bounded for all $\varepsilon \leq \varepsilon_0 < 1$. \square

REMARK 2.15. *Thm. 2.14 is an important result that can not be taken for granted. Standard codes will suffer from instabilities when small parameters, such as ε , occur. Note that a Finite-Element approximation of (2.16) inherits the stability properties of the continuous problem, so one obtains a stability constant that is independent of the small ε limit.*

Let us return to our overall algorithm. Computing an approximate solution, we introduce two errors: One error from using a Finite-Element space instead of the whole Sobolev space, and one from considering ι_h instead of ι . We start by computing the difference between the latter two:

LEMMA 2.16. *For a given $\varepsilon_0 < 1$, let $0 < \varepsilon \leq \varepsilon_0$. Furthermore, let ι and ι_h be defined as in Sec. 2.5. Its difference can be bounded in terms of Δt and Δx as*

$$\|\iota - \iota_h\|_{H_0^1'} = O(\varepsilon^2 \Delta t \Delta x + \Delta x \Delta t^2 \varepsilon). \quad (2.44)$$

Proof. From (2.39) and Ass. 2.11, we can conclude that

$$|\iota(\varphi) - \iota_h(\varphi)| = \left| \int_{\Omega} \Delta t (u_x^n - \tilde{u}_x^n) \varphi - \frac{\Delta t^2(1-\varepsilon)}{\varepsilon} (v_x^n - \tilde{v}_x^n) \varphi_x \, dx \right| \quad (2.45)$$

$$\leq C \left(\Delta t + \frac{\Delta t^2}{\varepsilon} \right) \|w_x^n - \tilde{w}_x^n\|_{L^2} \|\varphi\|_{H_0^1} \quad (2.46)$$

$$= O(\varepsilon^2 \Delta t \Delta x + \Delta x \Delta t^2 \varepsilon) \|\varphi\|_{H_0^1} \quad (2.47)$$

for a constant $C \in \mathbb{R}$. \square

The following lemma bounds the error that occurs when using only the approximate right-hand side ι_h instead of ι :

LEMMA 2.17. *For a given $\varepsilon_0 < 1$, let $0 < \varepsilon \leq \varepsilon_0 < 1$. Furthermore, let v^{n+1} and \bar{v}^{n+1} denote the solutions to*

$$a(\bar{v}^{n+1}, \varphi) = \iota_h(\varphi) \quad \forall \varphi \in H_0^1(\Omega), \quad (2.48)$$

$$a(v^{n+1}, \varphi) = \iota(\varphi) \quad \forall \varphi \in H_0^1(\Omega). \quad (2.49)$$

One can estimate the difference as

$$e_2 = \|\bar{v}^{n+1} - v^{n+1}\|_{L^2} = O\left(\varepsilon^4 \frac{\Delta x}{\Delta t} + \varepsilon^3 \Delta x\right). \quad (2.50)$$

Proof. The difference between \bar{v}^{n+1} and v^{n+1} can be computed by

$$\gamma \|\bar{v}^{n+1} - v^{n+1}\|_{H_0^1}^2 \leq a(\bar{v}^{n+1} - v^{n+1}, \bar{v}^{n+1} - v^{n+1}) \quad (2.51)$$

$$= \iota_h(\bar{v}^{n+1} - v^{n+1}) - \iota(\bar{v}^{n+1} - v^{n+1}) \quad (2.52)$$

$$\leq \|\iota_h - \iota\|_{H_0^{1'}} \|\bar{v}^{n+1} - v^{n+1}\|_{H_0^1}, \quad (2.53)$$

and, subsequently,

$$\|\bar{v}^{n+1} - v^{n+1}\|_{L^2} \leq C_{PF} \|\bar{v}^{n+1} - v^{n+1}\|_{H_0^1} \quad (2.54)$$

$$\leq \frac{C_{PF}}{\gamma} \|\iota_h - \iota\|_{H_0^{1'}} = O\left(\varepsilon^4 \frac{\Delta x}{\Delta t} + \varepsilon^3 \Delta x\right) \quad (2.55)$$

because of La. 2.16 and $\gamma^{-1} = O(\frac{\varepsilon^2}{\Delta t^2})$ for $\varepsilon, \Delta t \rightarrow 0$. \square

COROLLARY 2.18. A simple consequence of the proof is that

$$\|\bar{v}^{n+1} - v^{n+1}\|_{H_0^1} = O\left(\varepsilon^4 \frac{\Delta x}{\Delta t} + \varepsilon^3 \Delta x\right). \quad (2.56)$$

It is well-known that, in order to get stable schemes, one needs to link both Δt and Δx . In our example, this can be done in two ways, based on either the non-stiff flux \tilde{f} or the total flux f . Let us therefore make the following definition:

DEFINITION 2.19. The stiff and non-stiff cfl – numbers $\widetilde{\text{cfl}}$ and $\widehat{\text{cfl}}$ are defined by

$$\widetilde{\text{cfl}} := \frac{\Delta t}{\Delta x} \lambda_{\max}, \quad \widehat{\text{cfl}} := \frac{\Delta t}{\Delta x} \widehat{\lambda}_{\max}, \quad (2.57)$$

respectively. Note that $\lambda_{\max} = \varepsilon^{-1}$ and $\widehat{\lambda}_{\max} = 1$.

In our analysis, we rely on the non-stiff cfl – number $\widehat{\text{cfl}}$, so the cfl number that is independent on ε . Let us therefore state the following assumption:

ASSUMPTION 2.20. We assume that

$$\Delta t = \widehat{\text{cfl}} \Delta x \quad (2.58)$$

for a $\widehat{\text{cfl}} \in \mathbb{R}$ (which we usually choose to be $\widehat{\text{cfl}} = 0.8$).

REMARK 2.21. This directly yields

$$e_2 = O(\varepsilon^4 + \varepsilon^3 \Delta x). \quad (2.59)$$

Having bounded e_2 , we continue by bounding e_3 .

LEMMA 2.22. Let v_h^{n+1} be the Finite-Element solution to (2.31), and let \bar{v}^{n+1} be the solution to (2.48). Then,

$$e_3 = \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2} = O\left(\frac{\varepsilon^6}{\Delta x^2} + \varepsilon^4 + \varepsilon^2 \Delta t^2\right).$$

Proof. We are using linear Finite-Elements on a symmetric problem, so one can use the Aubin-Nitsche trick (see, e.g., [6]). As it is crucial for our analysis that we get the correct dependency of the constant ε , we perform this 'trick' here explicitly. Let us define the dual solution z and its Finite-Element approximation z_h by

$$\begin{aligned} a(z, \varphi) &= \int_{\Omega} (\bar{v}^{n+1} - v_h^{n+1}) \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega), \\ a(z_h, \varphi_h) &= \int_{\Omega} (\bar{v}^{n+1} - v_h^{n+1}) \varphi_h \, dx \quad \forall \varphi_h \in V_h. \end{aligned}$$

One can conclude

$$\begin{aligned} \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2}^2 &= a(z, \bar{v}^{n+1} - v_h^{n+1}) = a(z - z_h, \bar{v}^{n+1} - v_h^{n+1}) \\ &\leq M \|z - z_h\|_{H_0^1} \|\bar{v}^{n+1} - v_h^{n+1}\|_{H_0^1} \\ &\leq M \Delta x^2 |z|_2 |\bar{v}^{n+1}|_2 \\ &\leq C \frac{M \Delta x^2}{\gamma^2} \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2} \|\iota_h\|_{H_0^1}, \\ &= \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2} \|\iota_h\|_{H_0^1} O\left(\frac{\varepsilon^4}{\Delta x^2} + \varepsilon^2\right) \\ &\leq \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2} O(\varepsilon^2 + \Delta t^2) O\left(\frac{\varepsilon^4}{\Delta x^2} + \varepsilon^2\right) \\ &\leq \|\bar{v}^{n+1} - v_h^{n+1}\|_{L^2} O\left(\frac{\varepsilon^6}{\Delta x^2} + \varepsilon^4 + \varepsilon^2 \Delta t^2\right). \end{aligned}$$

$|\bar{v}|_2$ denotes the second Sobolev semi-norm, and can be bounded by the right-hand side of the equation, if the viscosity coefficient is unity. \square

COROLLARY 2.23. In a similar way, we can deduce that

$$\|\bar{v}^{n+1} - v_h^{n+1}\|_{H_0^1} = O\left(\frac{\varepsilon^6}{\Delta x^3} + \frac{\varepsilon^4}{\Delta x} + \varepsilon^2 \Delta t\right). \quad (2.60)$$

We are now ready to state the final theorem that assures that v is approximated consistently.

THEOREM 2.24. Let v_h^{n+1} be the approximate solution according to the algorithm in Sec. 2.5 with exact initial data $w^n \equiv w(t^n)$. Under Ass. 2.11 and 2.20, we have

$$\|v_h^{n+1} - v(t^{n+1})\|_{L^2} = O\left(\varepsilon^2 \Delta t^2 + \varepsilon^4 + \varepsilon^3 \Delta x + \frac{\varepsilon^6}{\Delta x^2}\right). \quad (2.61)$$

Proof. We can just collect previous results:

$$\|v(t^{n+1}) - v_h^{n+1}\|_{L^2} \leq e_1 + e_2 + e_3 \quad (2.62)$$

$$= O(\varepsilon^2 \Delta t^2) + O(\varepsilon^4 + \varepsilon^3 \Delta x) + O\left(\frac{\varepsilon^6}{\Delta x^2} + \varepsilon^4 + \varepsilon^2 \Delta t^2\right) \quad (2.63)$$

$$= O\left(\varepsilon^2 \Delta t^2 + \varepsilon^4 + \varepsilon^3 \Delta x + \frac{\varepsilon^6}{\Delta x^2}\right) \quad (2.64)$$

\square

REMARK 2.25. Given that $\varepsilon \leq \Delta t$, one can see that v_h^{n+1} is a consistent approximation to $v(t^{n+1})$, and $\|v(t^{n+1}) - v_h^{n+1}\|_{L^2} = O(\Delta t^4)$.

By now, we have shown that v_h^{n+1} is a consistent approximation to $v(t^{n+1})$. It remains to show that also u_h^{n+1} is a consistent approximation to $u(t^{n+1})$. Thereby, u_h^{n+1} denotes the function that is obtained by evaluating (2.11) with v_h^{n+1} instead of $v(t^{n+1})$.

THEOREM 2.26. Let u_h^{n+1} be the approximate solution that is obtained using (2.11) with v_h^{n+1} instead of $v(t^{n+1})$; and with exact initial data $w^n \equiv w(t^n)$ elsewhere. Under Ass. 2.11 and 2.20, we have

$$\|u_h^{n+1} - u(t^{n+1})\|_{L^2} = O\left(\Delta t^2 + \frac{\varepsilon^4}{\Delta x^2} + \varepsilon^2\right). \quad (2.65)$$

Proof. We can directly compute, exploiting what we have already show:

$$\|u_h^{n+1} - u(t^{n+1})\|_{L^2} \leq \|u^{n+1} - u(t^{n+1})\|_{L^2} + \|u_h^{n+1} - u^{n+1}\|_{L^2} \quad (2.66)$$

$$\leq O(\Delta t^2) + \|\Delta t \frac{1-\varepsilon}{\varepsilon^2} (v_h^{n+1} - v^{n+1})_x\|_{L^2} \quad (2.67)$$

$$\leq O(\Delta t^2) + \frac{\Delta t}{\varepsilon^2} \left(\|\bar{v}^{n+1} - v^{n+1}\|_{H_0^1} + \|v_h^{n+1} - \bar{v}^{n+1}\|_{H_0^1} \right) \quad (2.68)$$

$$\stackrel{(2.56), (2.60)}{=} O(\Delta t^2) + O(\varepsilon^2 \Delta t + \varepsilon \Delta x^2) + O\left(\frac{\varepsilon^4}{\Delta x^2} + \varepsilon^2 + \Delta t^2\right) \quad (2.69)$$

$$= O\left(\Delta t^2 + \frac{\varepsilon^4}{\Delta x^2} + \varepsilon^2\right). \quad (2.70)$$

□

There are a few remarks in order:

REMARK 2.27. The solution of the elliptic equation gets more and more difficult with decreasing time-step Δt , as the elliptic coefficient vanishes in this case. So basically, the method will only perform well as long as $\varepsilon \leq \Delta t$ (i.e., for the cfl number of the whole system there holds $\text{cfl} \leq \varepsilon$), which is a reasonable assumption. (Otherwise, one would use explicit methods instead.) However, choosing $\Delta t = O(\varepsilon^{\frac{1}{p}})$ for some $p \geq 1$, one can observe that

$$\|w_h^{n+1} - w(t^{n+1})\|_{L^2} = O(\Delta t^2).$$

This directly shows that the method works also for the $\varepsilon = 0$ case.

3. Numerical Results. We compare our scheme with an Implicit-Euler scheme, and an Implicit/Explicit scheme. Implicit-Euler scheme discretizes

$$\frac{w^{n+1} - w^n}{\Delta t} + f(w^{n+1}) = G^{n+1} \quad (3.1)$$

using a Lax-Friedrichs flux. The Implicit/Explicit scheme proceeds in two steps, discretizing

$$\frac{\hat{w}^n - w^n}{\Delta t} + \hat{f}(w^n)_x = G^n \quad (3.2)$$

explicitly, and

$$\frac{w^{n+1} - \hat{w}^n}{\Delta t} + \tilde{f}(w^{n+1})_x = 0 \quad (3.3)$$

implicitly, again both steps with Lax-Friedrichs flux.

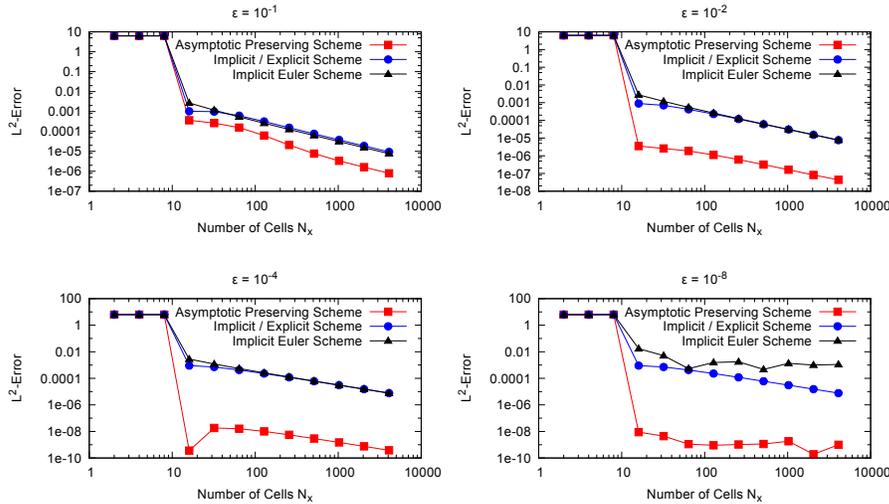


FIG. 3.1. Convergence results for the smooth test case. In dependency on ε , cfl was set to $\text{cfl} = \frac{0.8}{\varepsilon}$. Left to right, top to bottom: $\varepsilon = \{10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}\}$.

3.1. Smooth test case. As a first, simple test case, we consider a smooth solution on domain $\Omega = [0, 1]$, given by

$$v(x, t) = \varepsilon^2 t \sin(2\pi x) \quad (3.4)$$

$$u(x, t) = \sin(20\pi t) - \frac{\varepsilon^2}{2\pi} \cos(2\pi x). \quad (3.5)$$

For all methods, we use a (stiff) cfl number of $\widetilde{\text{cfl}} = \frac{0.8}{\varepsilon}$. Note that this corresponds to a cfl number of $\widehat{\text{cfl}} = 0.8$ with respect to the non-stiff flux \widehat{f} . If a method is able to cope with such a cfl number, it is called uniformly asymptotically stable. In Fig. 3.2, convergence of the l^2 -error at time $T = 0.1$ versus number of cells (N_x) is shown for all three methods under consideration. One can first observe that all three methods are stable for this unusually large cfl number, as expected. Furthermore, asymptotically (in N_x), all methods converge with order one toward the true solution (u, v), except for the Implicit Euler scheme for $\varepsilon = 10^{-8}$. We suspect that this is because the linear system of equations to be solved in each time-slab is extremely ill-conditioned. We use Matlab's in-house exact solver for linear systems of equations, which actually yields a corresponding warning. Furthermore, $\varepsilon^2 = 10^{-16}$ is close to machine zero. Note that this does not happen to the Asymptotic Preserving scheme, as its condition number is bounded for $\varepsilon \rightarrow 0$. The really surprising outcome of this research is that the AP scheme performs so much better than Implicit Euler and the mixed Implicit / Explicit scheme: Its error is up to four orders of magnitude smaller than that of the other two schemes. We can only suspect that 'traditional' Finite-Volume schemes do not take advantage of the smooth behaviour of the solution as much as the Finite-Element method does.

3.2. Testcase with a kink. To assess whether the good performance of the asymptotic preserving method is due to the smoothness of the solution, we perform a numerical study on a test case with a kink, more precisely, we consider again domain

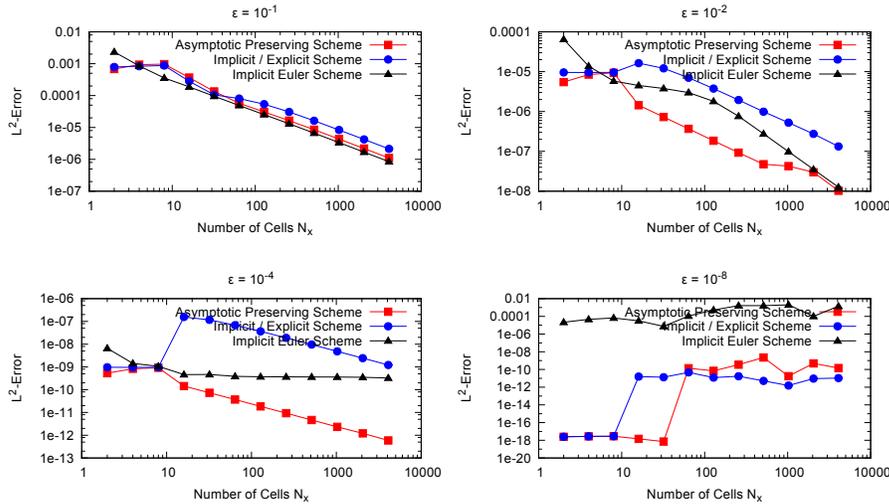


FIG. 3.2. Convergence results for the test case with a kink. In dependency on ε , cfl was set to $\text{cfl} = \frac{0.8}{\varepsilon}$. Left to right, top to bottom: $\varepsilon = \{10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}\}$.

$\Omega = [0, 1]$ and the solution

$$v(x, t) = \varepsilon^2 t \begin{cases} x & x < 0.5 \\ -x + 1 & x \geq 0.5 \end{cases} \quad (3.6)$$

$$u(x, t) = 1 + \varepsilon^2 \begin{cases} \frac{x^2}{2} & x < 0.5 \\ -\frac{x^2}{2} + x - \frac{1}{4} & x \geq 0.5 \end{cases} \quad (3.7)$$

Again, we use a (stiff) cfl number of $\widetilde{\text{cfl}} = \frac{0.8}{\varepsilon}$. In Fig. 3.2, convergence of the l^2 -norm at time $T = 0.1$ versus N_x is plotted. One can observe that the schemes converge with order one up to 10^{-10} , which is about machine zero (note that the error has to be scaled with ε^2), except for the $\varepsilon = 10^{-8}$, where Implicit Euler fails to converge for this large cfl number. For large values of ε , the schemes nearly perform equally well, while, for $\varepsilon = 10^{-4}$, the AP scheme really performs better by orders of magnitude. For $\varepsilon = 10^{-8}$, both the AP and Implicit / Explicit scheme perform about equally well. Nevertheless, as $\varepsilon^2 = 10^{-16}$ is close to machine zero, these results are not too reliable.

4. Conclusions and Outlook. We have compared the recently developed AP schemes versus more traditional Finite-Volume schemes for the p -system. It was demonstrated that the AP schemes outperform both Implicit Euler and an Implicit / Explicit scheme by orders of magnitude if there is a small parameter ε .

We are interested in the use of high-order methods, also in the context of asymptotic preserving schemes. In particular, our interest lies in the use of Discontinuous Galerkin method [10, 9, 8, 7, 11, 4, 16, 1, 14, 20]. Future work will therefore treat an asymptotic preserving discontinuous Galerkin scheme applied to (1.1)-(1.2) for various orders of consistency, and also compare performance of the AP schemes versus Diagonally-Implicit-Runge-Kutta (DIRK). It is to be expected that the high order of consistency will reduce the effect that we could observe in this publication. Neverthe-

less, the use of AP schemes has some inherent advantages, such as the occurrence of an elliptic equation, which is generally easier to solve than a hyperbolic problem. To conclude, we are positive that there will still be a benefit of using AP schemes.

Acknowledgement. I am thankful for fruitful discussions with Sebastian Noelle.

REFERENCES

- [1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of Discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2002.
- [2] K. Arun and S. Noelle. An asymptotic preserving scheme for low froude number shallow flows. *IGPM Preprint 352*, 2012.
- [3] K. Arun, S. Noelle, M. Lukacova-Medvidova, and C.-D. Munz. An asymptotic preserving all mach number scheme for the euler equations of gas dynamics. *IGPM Preprint 348*, 2012.
- [4] F. Bassi and S. Rebay. A high-order accurate discontinuous Finite-Element method for the numerical solution of the compressible Navier-Stokes equations. *Journal of Computational Physics*, 131:267–279, 1997.
- [5] A. N. Brooks and T. J. R. Hughes. Streamline upwind/ Petrov-galerkin formulations for convection-dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 32:199–259, 1982.
- [6] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, New York, Oxford, 1978.
- [7] B. Cockburn, S. Hou, and C.-W. Shu. The Runge–Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws IV: The multidimensional case. *Mathematics of Computation*, 54:545–581, 1990.
- [8] B. Cockburn and S. Y. Lin. TVB Runge-Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws III: One dimensional systems. *Journal of Computational Physics*, 84:90–113, 1989.
- [9] B. Cockburn and C.-W. Shu. TVB Runge-Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws II: General framework. *Mathematics of Computation*, 52:411–435, 1988.
- [10] B. Cockburn and C.-W. Shu. The Runge-Kutta local projection p^1 -Discontinuous Galerkin Finite Element Method for Scalar Conservation Laws. *RAIRO - Modélisation mathématique et analyse numérique*, 25:337–361, 1991.
- [11] B. Cockburn and C.-W. Shu. The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V: Multidimensional Systems. *Mathematics of Computation*, 141:199–224, 1998.
- [12] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer Berlin / Heidelberg, 2005.
- [13] P. Degond and M. Tang. All speed scheme for the low mach number limit of the isentropic euler equation. *Commun. Comput. Phys.*, 10:1–31, 2011.
- [14] H. Egger and J. Schöberl. A hybrid mixed Discontinuous Galerkin Finite Element method for convection-diffusion problems. *IMA Journal of Numerical Analysis*, 30:1206–1234, 2010.
- [15] C. Grossmann and H.-G. Roos. *Numerical Treatment of Partial Differential Equations*. Springer Berlin / Heidelberg, 2007.
- [16] R. Hartmann and P. Houston. Adaptive Discontinuous Galerkin Finite Element methods for nonlinear hyperbolic Conservation Laws. *SIAM J. Numer. Anal.*, 24:979–1004, 2002.
- [17] S. Jin. Efficient asymptotic-preserving (ap) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21:441–454, 1999.
- [18] S. Jin, L. Pareschi, and G. Toscani. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM J. Numer. Anal.*, 35:2405–2439, 1998.
- [19] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, 34:481–524, 1981.
- [20] J. Schütz and G. May. A Hybrid Mixed Method for the Compressible Navier-Stokes Equations. *Journal of Computational Physics*, 240:58–75, 2013.