ANALYSIS OF A STOKES INTERFACE PROBLEM

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Abstract. We consider a stationary Stokes problem with a piecewise constant viscosity coefficient. For the variational formulation of this problem we prove a well-posedness result in which the constants are uniform with respect to the jump in the viscosity coefficient. We apply a standard discretization with a pair of LBB stable finite element spaces. The main result of the paper is an infsup result for the discrete problem that is uniform with respect to the jump in the viscosity coefficient. From this we derive a robust estimate for the discretization error. We prove that the mass matrix with respect to some suitable scalar product yields a robust preconditioner for the Schur complement. Results of numerical experiments are presented that illustrate this robustness property.

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1. Introduction.

In this paper we treat the following Stokes problem on a bounded connected Lipschitz domain Ω in d-dimensional Euclidean space ($d = 2, 3$): Find a velocity $\mathbf{u}$ and a pressure p such that

$$-\operatorname{div}(\nu(\mathbf{x})\nabla\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

with a *piecewise constant viscosity*:

$$\nu = \begin{cases} 1 & \text{in } \Omega_1 \\ \varepsilon > 0 & \text{in } \Omega_2. \end{cases} \tag{1.4}$$

The subdomains $\Omega_1,\ \Omega_2$ are assumed to be Lipschitz domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. By Γ we denote the interface between the subdomains: $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$.

An important motivation for considering this type of Stokes equations comes from two-phase incompressible flows. Often such problems are modeled by Navier-Stokes equations with discontinuous density and viscosity coefficients. The effect of interface tension is taken into account by using a special localized force term at the interface. The latter approach is known as the continuum surface force (CSF) model, cf. [5]. A well-known technique for capturing the unknown interface is based on the level set method, cf. [23, 17, 13] and the references therein. If in such a setting one has highly viscous flows then the Stokes equations with discontinuous viscosity are a reasonable model problem.

For pure diffusion problems (Poisson equation) with a discontinuous diffusion coefficient one can find analyses of discretization methods [1, 2, 6, 14, 19], error estimators [18, 4] and iterative solvers [8, 9, 20, 25] in the literature. For the Stokes problem with discontinuous viscosity, however, we did not find any theoretical analysis. This

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paper presents an analysis of a finite element method and of a solver for the discretized Stokes interface problem together with some results for the weak formulation of (1.1)-(1.3).

Consider a variational formulation of the equations (1.1)-(1.3). We use the notation $\mathbf{V} := H_0^1(\Omega)^d$ for the velocity space. For the pressure space some factorization of $L^2(\Omega)$ is used. It appears that for this problem it is convenient to use:

$$M := \{\, p \in L^2(\Omega) \mid \int_\Omega \nu^{-1} p(x)\, dx = 0 \,\} \; . \tag{1.5}$$

The variational problem reads as follows: given $\mathbf{f} \in \mathbf{V}'$ find $\{\mathbf{u}, p\} \in \mathbf{V} \times M$ such that

$$\left\{ \begin{array}{lll} (\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) & = (\mathbf{f}, \mathbf{v}) & \text{for} \;\; \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad\quad (\operatorname{div}\mathbf{u}, q) & = 0 & \text{for} \;\; q \in M \; . \end{array} \right. \tag{1.6}$$

Here and in the remainder the L^2 scalar product and associated norm are denoted by $(\cdot,\cdot)$, $\|\cdot\|$, respectively. The bilinear form $(\nu\nabla\cdot, \nabla\cdot)$ defines a scalar product on $\mathbf{V}$. We use the norm induced by this scalar product:

$$\|\mathbf{u}\|_{\mathbf{V}} := (\nu\nabla\mathbf{u}, \nabla\mathbf{u})^{\frac{1}{2}} \qquad \text{for} \;\; \mathbf{u} \in \mathbf{V} \tag{1.7}$$

On M, apart from the L^2 scalar product we will also use a weighted L^2 scalar product:

$$(p, q)_M := \int_\Omega \nu^{-1}\, p\, q\, dx = (\nu^{-1} p, q) \qquad \text{for} \;\; p, q \in M \; , \tag{1.8}$$

and $\|p\|_M := (p,p)_M^{\frac{1}{2}}$. In the analysis we use the ν-dependent norm $\left(\|\cdot\|_{\mathbf{V}}^2 + \|\cdot\|_M^2\right)^{\frac{1}{2}}$ on the product space $\mathbf{V} \times M$. In section 2 we prove a continuity and an infsup result that are uniform with respect to the parameter ε. Using standard arguments this then yields uniform well-posedness of the continuous Stokes problem.

In section 3 we consider the discrete variational problem in a pair of finite element spaces ($M_h \subset M$, $\mathbf{V}_h \subset \mathbf{V}$) that are assumed to be LBB stable. As a main result of this paper we present a discrete infsup result that is uniform with respect to the parameters h (mesh size) and ε. This result is used to derive a (sharp) uniform bound for the discretization error.

In section 4 we prove that the mass-matrix with respect to the scalar product (1.5) in the pressure subspace M_h is spectrally equivalent to the Schur complement *uniform in the parameters h and ε*. In combination with known results on block-preconditioning and on multigrid this then implies optimality results for certain iterative methods. For the Uzawa method and a preconditioned MINRES method we present results of numerical experiments in section 5. In section 6 instead of M we use the more standard space $L_0^2(\Omega)$ for the pressure and we present (without proofs) infsup results and discretization error bounds for this case. These results show that for a theoretical analysis the space M is more natural than $L_0^2(\Omega)$.

2. The continuous problem. In this section we analyze the variational problem (1.6). We introduce the piecewise constant function

$$\bar{p} = \left\{ \begin{array}{rl} |\Omega_1|^{-1} & \text{on } \Omega_1 \\ -\varepsilon|\Omega_2|^{-1} & \text{on } \Omega_2. \end{array} \right. \tag{2.1}$$

and the one-dimensional subspace $M_0 := \operatorname{span}\{\bar{p}\}$ of M. We consider an $(\cdot,\cdot)_M$-orthogonal decomposition $M = M_0 \oplus M_0^\perp$. For $p \in M$ we use the notation

$$p = p_0 + p_0^\perp \; , \quad p_0 \in M_0, \;\; p_0^\perp \in M_0^\perp \tag{2.2}$$

One easily verifies that

$$M_0^\perp = \{\, p \in M \mid \int_{\Omega_1} p\,dx = \int_{\Omega_2} p\,dx = 0\,\}$$

By definition we have ellipticity and continuity of the bilinear form $(\nu\nabla\cdot,\nabla\cdot)$ in the space $(\mathbf{V},\|\cdot\|_{\mathbf{V}})$: $(\nu\nabla\mathbf{u},\nabla\mathbf{u}) = \|\mathbf{u}\|_{\mathbf{V}}^2$. Continuity of the bilinear form $(\operatorname{div}\cdot,\cdot)$ is shown in the following lemma:

LEMMA 2.1. *The inequality*

$$|(\operatorname{div}\mathbf{u},p)| \le \sqrt{d}\|\mathbf{u}\|_{\mathbf{V}}\|p\|_M$$

holds for all $\mathbf{u}\in\mathbf{V},\ p\in M$.

Proof. This result immediately follows from the Cauchy inequality and the estimate $\|\nu^{\frac12}\operatorname{div}\mathbf{u}\| \le \sqrt{d}\|\mathbf{u}\|_{\mathbf{V}}$ for $\mathbf{u}\in\mathbf{V}$. ∎

In the next theorem we prove a uniform (w.r.t. ν) infsup property corresponding to the problem (1.6). It generalizes the well-known Nečas inequality:

$$c(\Omega)\|p\| \le \|\nabla p\|_{-1} := \sup_{\mathbf{u}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{u},p)}{\|\nabla\mathbf{u}\|} \quad \forall\, p\in L_2 :\ (p,1)=0, \tag{2.3}$$

with $c(\Omega)>0$. We will need an equivalent form of (2.3): for any $p\in L_2$ such that $(p,1)=0$ their exists $\mathbf{u}\in\mathbf{V}$ such that

$$\|p\|^2 = (\operatorname{div}\mathbf{u},p) \quad \text{and} \quad c(\Omega)\|\nabla\mathbf{u}\| \le \|p\|. \tag{2.4}$$

THEOREM 2.2. *There exists a constant* $C>0$ *independent of* ν *such that*

$$\sup_{\mathbf{u}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{u},p)}{\|\mathbf{u}\|_{\mathbf{V}}} \ge C\|p\|_M \quad \textit{for all}\ \ p\in M$$

Proof. Fix an arbitrary $p\in M$. We first consider the component $p_0^\perp$ from the decomposition $p = p_0 + p_0^\perp$ in (2.2). Since $p_0^\perp|_{\Omega_k} \in L^2(\Omega_k)$ and $(p_0^\perp,1)_{\Omega_k} = 0$ for $k=1,2$, we can apply the Nečas inequality in the form (2.4) in each subdomain. Thus there exists a function $\mathbf{u}_1 \in H_0^1(\Omega_1)^d$ such that the following relations hold with a constant $c(\Omega_1)>0$:

$$\|p_0^\perp\|_{\Omega_1}^2 = (\operatorname{div}\mathbf{u}_1,p_0^\perp)_{\Omega_1} \quad \text{and} \quad c(\Omega_1)\|\nabla\mathbf{u}_1\|_{\Omega_1} \le \|p_0^\perp\|_{\Omega_1} \tag{2.5}$$

Similarly, using a scaling argument, it follows that there exists $\mathbf{u}_2 \in H_0^1(\Omega_2)^d$ such that

$$\|\varepsilon^{-\frac12}p_0^\perp\|_{\Omega_2}^2 = (\operatorname{div}\mathbf{u}_2,p_0^\perp)_{\Omega_2}, \quad c(\Omega_2)\|\varepsilon^{\frac12}\nabla\mathbf{u}_2\|_{\Omega_1} \le \|\varepsilon^{-\frac12}p_0^\perp\|_{\Omega_2}, \tag{2.6}$$

with $c(\Omega_2)>0$. Extending $\mathbf{u}_1$ and $\mathbf{u}_2$ by zero on the whole domain Ω and taking a sum of (2.5) and (2.6) we get

$$\|p_0^\perp\|_M^2 = (\operatorname{div}\tilde{\mathbf{u}},p_0^\perp) \quad \text{and} \quad c_1\|\tilde{\mathbf{u}}\|_{\mathbf{V}} \le \|p_0^\perp\|_M, \quad \tilde{\mathbf{u}} := \mathbf{u}_1+\mathbf{u}_2 \tag{2.7}$$

with $c_1 = \min\{c(\Omega_1),c(\Omega_2)\}$.

For the component p_0 we define $\tilde{p}_0 := \nu^{-1} p_0$. Note that $(\tilde{p}_0, 1) = (p_0, 1)_M = 0$ and thus we can use the Nečas inequality in Ω. Hence there exists $\bar{\mathbf{u}} \in H_0^1(\Omega)^d$ such that

$$\|\tilde{p}_0\|^2 = (\operatorname{div} \bar{\mathbf{u}}, \tilde{p}_0) \quad \text{and} \quad c(\Omega)\|\nabla \bar{\mathbf{u}}\| \leq \|\tilde{p}_0\|. \tag{2.8}$$

Due to the definition of M_0 we obtain

$$\|p_0\|_M^2 = C(\varepsilon, \Omega)\|\tilde{p}_0\|^2 \quad \text{and} \quad (\operatorname{div} \bar{\mathbf{u}}, p_0) = C(\varepsilon, \Omega)(\operatorname{div} \bar{\mathbf{u}}, \tilde{p}_0)$$

with $C(\varepsilon, \Omega) = \frac{\varepsilon|\Omega_1|+|\Omega_2|}{|\Omega_1|+|\Omega_2|}$. Note that

$$C(\varepsilon, \Omega) \geq \tilde{c}(\Omega) \max\{1, \varepsilon\}, \quad \tilde{c}(\Omega) := \min\left\{\frac{|\Omega_1|}{|\Omega|}, \frac{|\Omega_2|}{|\Omega|}\right\} \tag{2.9}$$

Thus this and from (2.8) we deduce

$$\|p_0\|_M^2 = (\operatorname{div} \bar{\mathbf{u}}, p_0) \quad \text{and} \quad c_3 \max\{1, \sqrt{\varepsilon}\}\|\nabla \bar{\mathbf{u}}\| \leq \|p_0\|_M, \tag{2.10}$$

with $c_3 = c(\Omega)\,\tilde{c}(\Omega)^{\frac{1}{2}}$ a constant independent of ν. We also have:

$$\begin{aligned} &(\operatorname{div} \tilde{\mathbf{u}}, p_0) = 0, \quad \|\nu^{\frac{1}{2}} \operatorname{div} \bar{\mathbf{u}}\| \leq \sqrt{d}\|\bar{\mathbf{u}}\|_{\mathbf{V}}, \\ &\|\bar{\mathbf{u}}\|_{\mathbf{V}} \leq \max\{1, \sqrt{\varepsilon}\}\|\nabla \bar{\mathbf{u}}\| \leq c_3^{-1}\|p_0\|_M \end{aligned}$$

Using this and the results in (2.7) and (2.10) we get for arbitrary $\alpha > 0$

$$\begin{aligned} (\operatorname{div}(\alpha\tilde{\mathbf{u}} + \bar{\mathbf{u}}), p) &= \alpha\|p_0^\perp\|_M^2 + \|p_0\|_M^2 + (\operatorname{div} \bar{\mathbf{u}}, p_0^\perp) \\ &\geq \alpha\|p_0^\perp\|_M^2 + \|p_0\|_M^2 - c_3^{-1}\sqrt{d}\|p_0\|_M\|p_0^\perp\|_M \\ &\geq \frac{1}{2}\|p\|_M^2 \quad \text{if} \ \ \alpha \geq \frac{1}{2}\Big(1 + \frac{d}{c_3^2}\Big) =: \alpha_0 \end{aligned}$$

Thus if we take $\mathbf{u} = \alpha_0\tilde{\mathbf{u}} + \bar{\mathbf{u}}$ we get

$$\|p\|_M^2 \leq 2(\operatorname{div} \mathbf{u}, p) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{V}}^2 \leq 2(\alpha_0^2\|\tilde{\mathbf{u}}\|_{\mathbf{V}}^2 + \|\bar{\mathbf{u}}\|_{\mathbf{V}}^2) \leq c\|p\|_M^2,$$

with a constant c independent of ν. ◻

It follows that we have ellipticity of the bilinear form $(\nu\nabla\cdot, \nabla\cdot)$, continuity of the bilinear forms $(\nu\nabla\cdot, \nabla\cdot)$ and $(\operatorname{div}\cdot, \cdot)$, and the infsup property in the norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_M$ *with constants that are independent of* ν. Thus we have uniform (w.r.t. ν) well-posedness of the continuous variational Stokes problem (1.6) in these norms. Using standard arguments (cf. [16, 10]) it can be shown that the problem (1.6) has a unique solution and that the a priori estimate

$$(\|u\|_{\mathbf{V}}^2 + \|p\|_M^2)^{\frac{1}{2}} \leq c\|\mathbf{f}\|_{\mathbf{V}'} \tag{2.11}$$

holds with a constant c independent of $\mathbf{f}$ and of ν.

REMARK 1. The dual norm $\|\mathbf{f}\|_{\mathbf{V}'}$ in (2.11) can replaced by a more trackable norm of $\mathbf{f}$. For this we need the Poincare type inequality

$$\|\nu^{\frac{1}{2}}\mathbf{v}\| \leq C_P\|\mathbf{v}\|_{\mathbf{V}}, \quad \text{for all} \ \ \mathbf{v} \in \mathbf{V}. \tag{2.12}$$

The optimal constant C_P in (2.12) is uniformly bounded w.r.t. ν if one of the following conditions is satisfied:

$$\text{meas}(\partial\Omega_k \cap \partial\Omega) > 0 \quad \text{for} \quad k = 1, 2 \tag{2.13}$$
$$\text{meas}(\partial\Omega_1 \cap \partial\Omega) > 0 \quad \text{and} \quad \varepsilon \leq C \tag{2.14}$$

The fact that the condition (2.13) is sufficient for uniform boundedness of C_P follows from lemma 1 and (the proof of) lemma 7 in [19]. The fact that the condition (2.14) is sufficient for uniform boundedness of C_P is proved with the following argument. Due to (2.14) $\mathbf{u}$ vanishes on a part of $\partial\Omega_1$ with nonzero measure, hence

$$\|\mathbf{u}\|_{\Omega_1} \leq c\,\|\nabla\mathbf{u}\|_{\Omega_1}. \tag{2.15}$$

holds. Therefore

$$\|\mathbf{u}|_\Gamma\| = \|\mathbf{u}|_{\partial\Omega_1}\| \leq c\,\|\nabla\mathbf{u}\|_{\Omega_1}. \tag{2.16}$$

In the subdomain Ω_2 we have

$$\varepsilon\|\mathbf{u}\|_{\Omega_2}^2 \leq c\,\varepsilon(\|\nabla\mathbf{u}\|_{\Omega_2}^2 + \|\mathbf{u}|_{\partial\Omega_2}\|^2) = c\,\varepsilon(\|\nabla\mathbf{u}\|_{\Omega_2}^2 + \|\mathbf{u}|_\Gamma\|^2). \tag{2.17}$$

Inequalities (2.15), (2.16), and (2.17) give the inequality (2.12) with a constant C independent of ε and hence independent of ν.

Assume now that $\mathbf{f} \in L_2(\Omega)^d$ and that one of conditions (2.13),(2.14) holds. Then the Cauchy inequality and (2.12) immediately yield the a-priori estimate

$$(\|\mathbf{u}\|_{\mathbf{V}}^2 + \|p\|_M^2)^{\frac{1}{2}} \leq c\,C_P\|\nu^{-\frac{1}{2}}\mathbf{f}\|, \tag{2.18}$$

with $c\,C_P$ independent of $\mathbf{f}$ and of ν.

3. Finite element discretization. In this section we consider the discretization of the variational Stokes problem using a family of pairs of conforming finite element spaces. For this we assume a family of triangulations $\{\mathcal{T}_h\}$ in the sense of [11, 12]. An important assumption for our analysis is that each triangulation $\mathcal{T}_h$ is conforming w.r.t. the two subdomains Ω_1, Ω_2 in the following sense:

$$\exists\ \mathcal{T}_h^{(i)} \subset \mathcal{T}_h \ : \quad \cup\{\,T \mid T \in \mathcal{T}_h^{(i)}\,\} = \overline{\Omega}_i, \quad i = 1, 2 \tag{3.1}$$

This assumption is easily fulfilled if Ω_1 and Ω_2 are polyhedral subdomains.

Remark 2. In computational fluid dynamics for two-phase flow problems it is (more) realistic to assume that $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is smooth. Then the assumption (3.1) in general does not hold. However, in such applications it is common practice to approximate Γ by a polyhedral discrete interface Γ_h. In such a setting the assumption (3.1) may still make sense. As far as we know no rigorous analysis is available which for the (Navier)-Stokes equations shows the effect of approximating the smooth interface Γ by a piecewise smooth interface Γ_h. A theoretical analysis of this effect for a Poisson interface problem can be found in [14]. The results in [14], however, are not robust with respect to the jump in the diffusion coefficient.

We assume a pair of finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset L_0^2(\Omega) = \{\,p \in L^2(\Omega) \mid (p, 1) = 0\,\}$ that is LBB stable with a constant $\hat\beta$ independent of h:

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div}\mathbf{v}_h, q_h)}{\|\nabla\mathbf{v}_h\|\|q_h\|} \geq \hat\beta > 0\ . \tag{3.2}$$

Note that due to the different normalization in the space M (namely $(p,1)_M=0$) we in general have $Q_h \not\subseteq M$. To maintain conformity we use the space

$$M_h = \{\, p_h = \tilde{p}_h + \alpha 1 \mid \ \tilde{p}_h \in Q_h, \quad \alpha \in \mathbb{R} \ \text{ such that } \ (p_h,1)_M = 0 \,\}.$$

Note that $M_h \subset M$ and that functions in M_h and in Q_h only differ by a constant.

For the analysis in this section it is convenient (but not necessary) to introduce the bilinear form $a:\ (\mathbf{V}\times M)\times(\mathbf{V}\times M)\to\mathbb{R}$

$$a(\mathbf{u},p;\mathbf{v},q) := (\nu\nabla\mathbf{u},\nabla\mathbf{v}) - (\operatorname{div}\mathbf{v},p) + (\operatorname{div}\mathbf{u},q) \tag{3.3}$$

an formulate the discrete problem as follows: find $\{\mathbf{u}_h,p_h\}\in\mathbf{V}_h\times M_h$ such that

$$a(\mathbf{u}_h,p_h;\mathbf{v}_h,q_h) = (\mathbf{f},\mathbf{v}_h) \quad \text{for all } \ \{\mathbf{v}_h,q_h\}\in\mathbf{V}_h\times M_h\ . \tag{3.4}$$

In this section we will analyze continuity (theorem 3.4) and discrete stability (theorem 3.5) of the bilinear form $a(\cdot,\cdot)$. The estimates are uniform with respect to the mesh size parameter h and the diffusion coefficient ν. As a corollary we then obtain a uniform discretization error bound.

In the proof of the discrete infsup condition below we will use a decomposition which is similar, but not identical to the one from the previous section. Let $\bar{p}_h\in M_h$ be the M-orthogonal projection of $\bar{p}$ on M_h,

$$(\bar{p}-\bar{p}_h,q_h)_M = 0 \quad \text{for all } \ q_h\in M_h$$

and define the one-dimensional subspace $M_{0,h} := \operatorname{span}(\bar{p}_h)$ of M_h. This induces an $(\cdot,\cdot)_M$-orthogonal decomposition of M (and also of M_h): $M = M_{0,h}\oplus M_{0,h}^{\perp}$, and for $p\in M$ we use the notation

$$p = p_{0,h} + p_{0,h}^{\perp}, \qquad p_{0,h}\in M_{0,h},\ p_{0,h}^{\perp}\in M_{0,h}^{\perp}. \tag{3.5}$$

We will need the following elementary result

Lemma 3.1. *For all $p_h = p_{0,h}+p_{0,h}^{\perp}\in M_h$ we have*

$$(p_{0,h}^{\perp},1)_{\Omega_k} = 0 \quad \textit{for} \ \ k=1,2$$

Proof. First note that by definition $(p_{0,h}^{\perp},\bar{p}_h)_M=0$. Using $(p_{0,h}^{\perp},\bar{p}_h)_M = p_{0,h}^{\perp},\bar{p})_M$ we then get

$$0 = \frac{1}{|\Omega_1|}(p_{0,h}^{\perp},1)_{\Omega_1} - \frac{1}{|\Omega_2|}(p_{0,h}^{\perp},1)_{\Omega_2} \tag{3.6}$$

Since $p_{0,h}^{\perp}\in M$ we have $(p_{0,h}^{\perp},1)_M=0$ and thus

$$(p_{0,h}^{\perp},1)_{\Omega_1} + \varepsilon^{-1}(p_{0,h}^{\perp},1)_{\Omega_2} = 0 \tag{3.7}$$

Combination of (3.6) and (3.7) proves the lemma. ∎

In the analysis below we use the quantity

$$\mu_h := \frac{\|\bar{p}-\bar{p}_h\|_M}{\|\bar{p}\|_M} \tag{3.8}$$

which measures the error made by approximating $\bar{p}$ in the finite element pressure space. Note that $\mu_h = 0$ if M_h contains piecewise constant finite elements. In general we have $\mu_h = \mathcal{O}(\tilde{h}^{\frac{1}{2}})$, where $\tilde{h}$ is the maximal diameter of the elements in $\mathcal{T}_h$ that have a nonempty intersection with Γ. Note that $\mu_h \leq 1$ and $\mu_h \leq c\tilde{h}^{\frac{1}{2}}$ with a ν-independent constant c. For the analysis of the discrete infsup property we need the following result:

LEMMA 3.2. *For every $p_{0,h} \in M_{0,h}$ there exist $p_0 \in M_0$ such that*

$$\|p_{0,h} - p_0\|_M = \mu_h \|p_0\|_M \tag{3.9}$$

$$\|p_{0,h}\|_M = \sqrt{1-\mu_h^2}\,\|p_0\|_M \tag{3.10}$$

Proof. For $p_{0,h} \in M_{0,h}$ we have $p_{0,h} = \alpha \bar{p}_h$ with $\alpha \in \mathbb{R}$. We set $p_0 = \alpha \bar{p}$. Since $\bar{p}_h$ is the M-orthogonal projection of $\bar{p}$ on $M_{0,h}$, $p_{0,h}$ is the M-orthogonal projection of p_0 on $M_{0,h}$. This choice of p_0 implies (3.9) by definition of μ_h. The result in (3.10) follows from

$$\|p_0\|_M^2 = \|p_{0,h} - p_0\|_M^2 + \|p_{0,h}\|_M^2 = \mu_h^2 \|p_0\|_M^2 + \|p_{0,h}\|_M^2$$

□

We need an additional assumption on $\mathbf{V}_h$. Consider $\tilde{p} = \nu^{-1}\bar{p}$ with $\bar{p}$ defined in (2.1). $\tilde{p}$ has zero mean: $(\tilde{p}, 1) = 0$. We assume that $\mathbf{V}_h$ is such that there is a constant $\hat{\beta}_c > 0$ independent of h such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, \tilde{p})}{\|\nabla \mathbf{v}_h\|} \geq \hat{\beta}_c \|\tilde{p}\| \,. \tag{3.11}$$

REMARK 3. Assumption (3.11) is rather weak. We briefly discuss two cases in which this assumption is satisfied. Let h_0 be the mesh size parameter corresponding to the coarsest triangulation. Then (3.11) trivially holds for $\mathbf{V}_{h_0}$ with a constant $\hat{\beta}_c = \hat{\beta}_c(h_0)$. If the family of spaces $\{\mathbf{V}_h\}_{h \leq h_0}$ is *nested* then (3.11) with $\hat{\beta}_c = \hat{\beta}_c(h_0)$ holds for any $\mathbf{V}_h$. The second case is when $\tilde{M}_h$ contains piecewise constant elements. Then (3.11) immediately follows from (3.2).

We now prove a discrete infsup stability result, which is a main result of this paper:

THEOREM 3.3. *There exist constants $C_1 > 0, C_2 > 0$ independent of ν and h such that*

$$\textit{if} \quad \mu_h \leq C_1 \quad \textit{then} \tag{3.12}$$

$$\sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}} \geq C_2 \|p_h\|_M \quad \textit{for all} \quad p_h \in M_h \tag{3.13}$$

Proof. The proof is based on similar arguments as used in the proof of theorem 2.2. All the constants that appear in the proof are independent of ν and of h.
Take an arbitrary $p_h \in M_h$. We first consider the component $p_{0,h}^{\perp}$ from the decomposition $p_h = p_{0,h} + p_{0,h}^{\perp}$ in (3.5). From lemma 3.1 it follows that $(p_{0,h}^{\perp}, 1)_{\Omega_k} = 0$ for $k = 1, 2$ and thus we can use the LBB property (3.2) in each subdomain. Hence there

exists a function $\mathbf{u}_1 \in \mathbf{V}_h$ with $\mathbf{u}_1 = 0$ on Ω_2 such that the following relations hold with a constant $c_1 > 0$:

$$\|p_{0,h}^{\perp}\|_{\Omega_1}^2 = (\operatorname{div}\mathbf{u}_1, p_{0,h}^{\perp})_{\Omega_1} \quad \text{and} \quad c_1\|\nabla \mathbf{u}_1\|_{\Omega_1} \leq \|p_0^{\perp}\|_{\Omega_1}. \tag{3.14}$$

Similarly, using a scaling argument, it follows that there exists $\mathbf{u}_2 \in \mathbf{V}_h$ with $\mathbf{u}_2 = 0$ on Ω_1 such that

$$\|\varepsilon^{-\frac{1}{2}} p_{0,h}^{\perp}\|_{\Omega_2}^2 = (\operatorname{div}\mathbf{u}_2, p_{0,h}^{\perp})_{\Omega_2} \quad \text{and} \quad c_2\|\varepsilon^{\frac{1}{2}}\nabla \mathbf{u}_2\|_{\Omega_2} \leq \|\varepsilon^{-\frac{1}{2}} p_{0,h}^{\perp}\|_{\Omega_2} \tag{3.15}$$

with a constant $c_2 > 0$. Taking a sum of (3.14) and (3.15) we get for $\tilde{\mathbf{u}}_h := \mathbf{u}_1 + \mathbf{u}_2$:

$$\|p_{0,h}^{\perp}\|_M^2 = (\operatorname{div}\tilde{\mathbf{u}}_h, p_{0,h}^{\perp}) \quad \text{and} \quad \hat{c}\,\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}} \leq \|p_{0,h}^{\perp}\|_M, \tag{3.16}$$

with $\hat{c} := \min\{\, c_1, c_2 \,\}$. We now consider the component $p_{0,h}$. Take $p_0 \in M_0$ as in lemma 3.2. Then we have

$$\|p_{0,h} - p_0\|_M = \mu_h \|p_0\|_M, \quad \|p_{0,h}\|_M = \sqrt{1-\mu_h^2}\,\|p_0\|_M \tag{3.17}$$

For $\tilde{p}_0 := \nu^{-1} p_0$ we get $(\tilde{p}_0, 1) = (p_0, 1)_M = 0$. From assumption (3.11) it follows that there exists $\bar{\mathbf{u}}_h \in \mathbf{V}_h$ such that

$$\|\tilde{p}_0\|^2 = (\operatorname{div}\bar{\mathbf{u}}_h, \tilde{p}_0) \quad \text{and} \quad \hat{\beta}_c\,\|\nabla \bar{\mathbf{u}}_h\| \leq \|\tilde{p}_0\|$$

Using the same arguments as in the proof of theorem 2.2 one can show that

$$\|p_0\|_M^2 = (\operatorname{div}\bar{\mathbf{u}}_h, p_0) \quad \text{and} \quad \|\bar{\mathbf{u}}_h\|_{\mathbf{V}} \leq c_3^{-1}\|p_0\|_M \tag{3.18}$$

with $c_3 = \hat{\beta}_c\,\tilde{c}(\Omega)^{\frac{1}{2}}$ and $\tilde{c}(\Omega)$ the constant from (2.9). Also note that $(\operatorname{div}\tilde{\mathbf{u}}_h, p_0) = 0$. Using this and the results in (3.16), (3.18) we obtain for arbitrary $\alpha > 0$:

$$\begin{aligned}
(\operatorname{div}(\alpha\tilde{\mathbf{u}}_h + \bar{\mathbf{u}}_h), p_h) &= \alpha\big(\operatorname{div}\tilde{\mathbf{u}}_h, p_0 + (p_{0,h} - p_0) + p_{0,h}^{\perp}\big) \\
&\quad + \big(\operatorname{div}\bar{\mathbf{u}}_h, p_0 + (p_{0,h} - p_0) + p_{0,h}^{\perp}\big) \\
&= \alpha\,\|p_{0,h}^{\perp}\|_M^2 + \alpha\,(\operatorname{div}\tilde{\mathbf{u}}_h, p_{0,h} - p_0) \\
&\quad + \|p_0\|_M^2 + \big(\operatorname{div}\bar{\mathbf{u}}_h, (p_{0,h} - p_0) + p_{0,h}^{\perp}\big)
\end{aligned}$$

We assume that $\mu_h^2 \leq \frac{1}{2}$. From (3.17) we then get

$$\|p_{0,h}\|_M^2 \leq \|p_0\|_M^2 \leq 2\,\|p_{0,h}\|_M^2$$

and using $\|\nu^{\frac{1}{2}}\operatorname{div}\mathbf{v}_h\| \leq \sqrt{d}\,\|\mathbf{v}_h\|_{\mathbf{V}}$ for $\mathbf{v}_h \in \mathbf{V}_h$ we obtain

$$\begin{aligned}
(\operatorname{div}(\alpha\tilde{\mathbf{u}}_h + \bar{\mathbf{u}}_h), p_h) &\geq \alpha\,\|p_{0,h}^{\perp}\|_M^2 + \|p_{0,h}\|_M^2 - \alpha\sqrt{d}\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}}\|p_{0,h} - p_0\|_M \\
&\quad - \sqrt{d}\|\bar{\mathbf{u}}_h\|_{\mathbf{V}}\big(\|p_{0,h} - p_0\|_M + \|p_{0,h}^{\perp}\|_M\big) \\
&\geq \alpha\,\|p_{0,h}^{\perp}\|_M^2 + \|p_{0,h}\|_M^2 - \alpha\sqrt{2d}\,\hat{c}^{-1}\mu_h\|p_{0,h}^{\perp}\|_M\|p_{0,h}\|_M \\
&\quad - c_3^{-1}\sqrt{2d}\|p_{0,h}\|_M\big(\sqrt{2}\mu_h\|p_{0,h}\|_M + \|p_{0,h}^{\perp}\|_M\big) \\
&= \alpha\,\|p_{0,h}^{\perp}\|_M^2 + (1 - 2c_3^{-1}\sqrt{d}\,\mu_h)\|p_{0,h}\|_M^2 \\
&\quad - \sqrt{2d}\big(\alpha\hat{c}^{-1}\mu_h + c_3^{-1}\big)\|p_{0,h}\|_M\|p_{0,h}^{\perp}\|_M
\end{aligned}$$

Take $\mu_h \leq \frac{c_3}{4\sqrt{d}}$ then $1 - 2c_3^{-1}\sqrt{d}\mu_h \geq \frac{1}{2}$ and we obtain, using the Cauchy inequality

$$(\operatorname{div}(\alpha\tilde{\mathbf{u}}_h + \bar{\mathbf{u}}_h), p_h) \geq \big(\alpha - 2d(\alpha\hat{c}^{-1}\mu_h + c_3^{-1})^2\big)\|p_{0,h}^{\perp}\|_M^2 + \frac{1}{4}\|p_{0,h}\|_M^2$$

We now take

$$\mu_h \leq \frac{1}{\alpha_0}, \quad \alpha_0 := \frac{1}{4} + 2d(\hat{c}^{-1} + c_3^{-1})^2$$

and then for $\mathbf{u}_h := \alpha_0\tilde{\mathbf{u}}_h + \bar{\mathbf{u}}_h$ we obtain

$$(\operatorname{div}\mathbf{u}_h, p_h) \geq \frac{1}{4}\,\|p_h\|_M^2$$
$$\|\mathbf{u}_h\|_{\mathbf{V}}^2 \leq 2(\alpha_0^2\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}}^2 + \|\bar{\mathbf{u}}_h\|_{\mathbf{V}}^2) \leq C\,\|p_h\|_M^2$$

with a constant C independent of h and ν.
Hence for $C_1 = \min\left\{\frac{1}{\sqrt{2}}, \frac{c_3}{4\sqrt{d}}, \frac{1}{\alpha_0}\right\}$ we have the desired result. ∎

Note that for h sufficiently small (independent of ν) the condition $\mu_h \leq C_1$ in (3.12) is fulfilled.

We now use standard arguments to derive continuity and stability results for the bilinear form $a(\cdot,\cdot)$. For completeness we also present the proofs. We introduce the product norm

$$|||\mathbf{u}, p||| = \big(\|\mathbf{u}\|_{\mathbf{V}}^2 + \|p\|_M^2\big)^{\frac{1}{2}} \quad \{\mathbf{u}, p\} \in \mathbf{V} \times M$$

From $(\nu\nabla\mathbf{u}, \nabla\mathbf{u}) = \|\mathbf{u}\|_{\mathbf{V}}^2$ and the result in lemma 2.1 we immediately obtain the following continuity result:

THEOREM 3.4. *There exists a constant C independent of ν such that*

$$a(\mathbf{u}, p; \mathbf{v}, q) \leq C|||\mathbf{u}, p|||\;|||\mathbf{v}, q|||$$

for all $\{\mathbf{u}, p\},\ \{\mathbf{v}, q\} \in \mathbf{V} \times M$.
A discrete infsup result is presented in the next theorem:

THEOREM 3.5. *Assume that the condition* (3.12) *is satisfied. There exists a constant $c > 0$ independent of h and of ν such that*

$$\sup_{\{\mathbf{v}_h, q_h\} \in \mathbf{V}_h \times M_h} \frac{a(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h)}{|||\mathbf{v}_h, q_h|||} \geq c\,|||\mathbf{u}_h, p_h||| \quad \forall \quad \{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times M_h$$

Proof. Take $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times M_h$. From the infsup result in theorem 3.3 it follows that there exists $\mathbf{z}_h \in \mathbf{V}_h$ such that $\|\mathbf{z}_h\|_{\mathbf{V}} = \|p_h\|_M$ and $-(\operatorname{div}\mathbf{z}_h, p_h) \geq c\|p_h\|_M^2$ with $c > 0$. Now take $\mathbf{v}_h := \mathbf{u}_h + c\mathbf{z}_h, q_h := p_h$. We then get

$$a(\mathbf{u}_h, p_h; \mathbf{u}_h, p_h) = \|\mathbf{u}_h\|_{\mathbf{V}}^2,$$
$$a(\mathbf{u}_h, p_h; \mathbf{z}_h, 0) \geq \frac{c}{2}\|p_h\|_M^2 - \frac{1}{2c}\|\mathbf{u}_h\|_{\mathbf{V}}^2.$$

We multiply the second inequality by c and add it to the first one. This gives

$$a(\mathbf{u}_h, p_h; \mathbf{u}_h + c\mathbf{z}_h, p_h) \geq \frac{1}{2}\|\mathbf{u}_h\|_{\mathbf{V}}^2 + \frac{c^2}{2}\|p_h\|_M^2 \geq c_1\,|||\mathbf{u}_h, p_h|||^2\,, \tag{3.19}$$

with $c_1 = \frac{1}{2}\min\{1, c^2\}$. Now note

$$\begin{aligned} |||\mathbf{v}_h, p_h|||^2 &\le 2(\|\mathbf{u}_h\|_{\mathbf{V}}^2 + c^2\|\mathbf{z}_h\|_{\mathbf{V}}^2) + \|p_h\|_M^2 \\ &= 2\|\mathbf{u}_h\|_{\mathbf{V}}^2 + (2c^2+1)\|p_h\|_M^2 \le 2(c^2+1)|||\mathbf{u}_h, p_h|||^2 \end{aligned} \tag{3.20}$$

Combination of (3.19) and (3.20) completes the proof. □

As for the continuous problem we get as a direct corollary that the discrete problem (3.4) has a unique solution $\{\mathbf{u}_h, p_h\}$ and the inequality

$$|||\mathbf{u}_h, p_h||| \le c^{-1}\|\mathbf{f}\|_{\mathbf{V}_h'}$$

holds, with the constant c from theorem 3.5. Moreover, if $\mathbf{f} \in L_2(\Omega)^d$, then using the Cauchy inequality and the Poincare inequality (2.12) we obtain the a-priori estimate:

$$|||\mathbf{u}_h, p_h||| \le c^{-1}\, C_P\|\nu^{-\frac{1}{2}}\mathbf{f}\|. \tag{3.21}$$

We refer to remark 1 for a discussion of the dependence of the Poincare “constant” on ν.

Using the continuity result in theorem 3.4 and the infsup result in theorem 3.5 we can prove a discretization error bound using standard arguments.

THEOREM 3.6. *Let* $\{\mathbf{u}, p\}$ *be the solution of the continuous problem* (1.6) *and* $\{\mathbf{u}_h, p_h\}$ *be the solution of the discrete problem* (3.4). *Assume that the condition* (3.12) *is satisfied. There exists a constant* C *independent of* h *and of* ν *such that the following holds:*

$$|||\mathbf{u}-\mathbf{u}_h, p-p_h||| \le C \min_{\mathbf{v}_h\in\mathbf{V}_h}\ \min_{q_h\in M_h} |||\mathbf{u}-\mathbf{v}_h, p-q_h|||. \tag{3.22}$$

Proof. For arbitrary $\mathbf{v}_h \in \mathbf{V}_h$, $q_h \in M_h$ define $\mathbf{e} := \mathbf{u}-\mathbf{v}_h$, $\mathbf{e}_h = \mathbf{u}_h - \mathbf{v}_h$, $g := p - q_h$, $g_h := p_h - q_h$. The Galerkin orthogonality property yields

$$a(\mathbf{e}_h, g_h; \mathbf{z}_h, r_h) = a(\mathbf{e}, g; \mathbf{z}_h, r_h) \quad \text{for all } \ \{\mathbf{z}_h, r_h\} \in \mathbf{V}_h \times M_h\ .$$

Using this in combination with the continuity and infsup results we obtain, for suitable $\{\mathbf{z}_h, r_h\} \in \mathbf{V}_h \times M_h$:

$$\begin{aligned} |||\mathbf{e}_h, g_h||| &\le c^{-1}\frac{a(\mathbf{e}_h, g_h; \mathbf{z}_h, r_h)}{|||\mathbf{z}_h, r_h|||} = c^{-1}\frac{a(\mathbf{e}, g; \mathbf{z}_h, r_h)}{|||\mathbf{z}_h, r_h|||} \\ &\le c^{-1}C|||\mathbf{e}, g|||. \end{aligned}$$

Now combine this with the triangle inequality $|||\mathbf{u}-\mathbf{u}_h, p-p_h||| \le |||\mathbf{e}_h, g_h||| + |||\mathbf{e}, g|||$. □

Based on the result in theorem 3.6 and using approximation properties of the finite element spaces one can derive further bounds for the discretization error. For such an analysis one needs regularity results for the continuous Stokes interface problem. As far as we know, this regularity issue is largely unsolved.

4. Preconditioner for the Schur complement. In this section we analyze convergence properties of iterative solvers for the discretized problem. For this we first introduce the matrix-vector formulation of the discrete problem.
In practice the discrete space M_h for the pressure is constructed by taking a standard finite element space, which we denote by M_h^+ (for example, continuous piecewise linear functions), and then adding an orthogonality condition:

$$M_h = \{\, p_h \in M_h^+ \mid (p_h, 1)_M = 0 \,\}$$

Note that $\dim(M_h) = \dim(M_h^+) - 1$. Let $n := \dim(\mathbf{V}_h),\ m := \dim(M_h^+)$. We assume standard (nodal) bases in $\mathbf{V}_h$ and M_h^+ and corresponding isomorphisms

$$J_V : \mathbb{R}^n \to V_h,\ ,\quad J_M : \mathbb{R}^m \to M_h^+\ .$$

Let the stiffness matrices $\mathrm{A} \in \mathbb{R}^{n\times n},\ \mathrm{B} \in \mathbb{R}^{m\times n}$ and the mass matrix $\hat{\mathrm{M}}_\nu \in \mathbb{R}^{m\times m}$ be given by

$$\begin{aligned} \langle \mathrm{Ax}, \mathrm{y}\rangle &= (\nu \nabla J_V \mathrm{x}, \nabla J_V \mathrm{y}) \quad \text{for all}\ \ \mathrm{x}, \mathrm{y} \in \mathbb{R}^n\ , \\ \langle \mathrm{Bx}, \mathrm{y}\rangle &= (\operatorname{div} J_V \mathrm{x}, J_M \mathrm{y}) \quad \text{for all}\ \ \mathrm{x} \in \mathbb{R}^n,\, \mathrm{y} \in \mathbb{R}^m\ , \\ \langle \hat{\mathrm{M}}_\nu \mathrm{x}, \mathrm{y}\rangle &= (J_M \mathrm{x}, J_M \mathrm{y})_M \quad \text{for all}\ \ \mathrm{x}, \mathrm{y} \in \mathbb{R}^m\ . \end{aligned} \tag{4.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product. We emphasize that the matrix $\hat{\mathrm{M}}_\nu$ is the mass matrix with respect to the (weighted L^2) scalar product $(\cdot,\cdot)_M$ and thus may differ very much from the usual mass matrix with respect to the L^2 scalar product $(\cdot,\cdot)$. After finite element discretization we have a linear system of the form

$$\begin{pmatrix} \mathrm{A} & \mathrm{B}^T \\ \mathrm{B} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} \mathrm{f} \\ 0 \end{pmatrix} \tag{4.2}$$

with f such that $\langle \mathrm{f}, \mathrm{y}\rangle = (\mathbf{f}, J_V \mathrm{y})$ for all $\mathrm{y} \in \mathbb{R}^n$. The Schur complement is denoted by $\mathrm{S} := \mathrm{BA}^{-1}\mathrm{B}^T$. Note that both S and the matrix in (4.2) are singular and have a one-dimensional kernel. Define the constant vector $\mathrm{e} := J_M^{-1} 1 = (1, \ldots, 1)^T \in \mathbb{R}^m$. Then we have $\ker(\mathrm{S}) = \operatorname{span}\{\mathrm{e}\}$. Note that

$$(J_M \mathrm{y}, 1)_M = 0 \ \Leftrightarrow\ (J_M \mathrm{y}, J_M \mathrm{e})_M = 0 \ \Leftrightarrow \langle \hat{\mathrm{M}}_\nu \mathrm{y}, \mathrm{e}\rangle = 0 \ \Leftrightarrow\ \langle \mathrm{y}, \hat{\mathrm{M}}_\nu \mathrm{e}\rangle = 0 \tag{4.3}$$

Hence, with

$$(\hat{\mathrm{M}}_\nu \mathrm{e})^\perp := \{\, \mathrm{y} \in \mathbb{R}^m \mid \langle \mathrm{y}, \hat{\mathrm{M}}_\nu \mathrm{e}\rangle = 0 \,\} \tag{4.4}$$

we have $M_h = \{\, J_M \mathrm{y} \mid \mathrm{y} \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp \,\}$ and we get the following matrix-vector representation of the discrete problem (3.4):

$$\text{Find}\ \ \mathrm{x} \in \mathbb{R}^n,\ \mathrm{y} \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp\ \ \text{such that (4.2) holds} \tag{4.5}$$

In preconditioned MINRES or (inexact) Uzawa type of iterative solvers for solving this problem one needs preconditioners Q_A of A and Q_S of S. It is known that if for Q_A we take a symmetric multigrid V-cycle then we have (cf. [8, 9, 25])

$$(1 - \sigma_A)\mathrm{Q}_A \le \mathrm{A} \le \mathrm{Q}_A,$$

with a constant $\sigma_A < 1$ independent of h and of ν.
Below we show that the mass matrix $\hat{\mathrm{M}}_\nu$ is an appropriate preconditioner for S. From lemma 2.1 and theorem 3.3 we obtain

$$C_2\|p_h\|_M \le \sup_{u_h\in\mathbf{V}_h} \frac{(\operatorname{div}\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}} \le \sqrt{d}\,\|p_h\|_M \quad \text{for} \ \ p_h \in M_h \tag{4.6}$$

with $C_2 > 0$ independent of h and of ν, provided the condition $\mu_h \le C_1$ in (3.12) is fulfilled. From the definition of the Schur complement it follows that for arbitrary $\mathrm{y} \in \mathbb{R}^m$ we have

$$\langle \mathrm{Sy}, \mathrm{y}\rangle = \sup_{\mathbf{u}_h\in\mathbf{V}_h} \frac{(\operatorname{div}\mathbf{u}_h, J_M\mathrm{y})^2}{\|\mathbf{u}_h\|_{\mathbf{V}}^2} \tag{4.7}$$

As a direct consequence of (4.7) and (4.3) we get:

THEOREM 4.1. *Assume that* $\mu_h \le C_1$ *in* (3.12) *holds. For all* $\mathrm{y} \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp$ *we have*

$$C_2^2\langle \hat{\mathrm{M}}_\nu\mathrm{y}, \mathrm{y}\rangle \le \langle \mathrm{Sy}, \mathrm{y}\rangle \le d\,\langle \hat{\mathrm{M}}_\nu\mathrm{y}, \mathrm{y}\rangle$$

with constant C_2 *from* (4.6).

This theorem shows that the matrix $\hat{\mathrm{M}}_\nu^{-1}\mathrm{S}$ has a *uniformly bounded spectral condition number on the subspace* $(\hat{\mathrm{M}}_\nu\mathrm{e})^\perp$.

One further relevant issue is how to compute of $\hat{\mathrm{M}}_\nu^{-1}\mathrm{y}$ efficiently. The next lemma shows that either the matrix $\hat{\mathrm{M}}_\nu$ can be replaced by a cheap diagonal preconditioner or a good approximation of $\hat{\mathrm{M}}_\nu^{-1}\mathrm{y}$ can be obtained efficiently by applying a preconditioned CG method with a diagonal matrix as preconditioner .

LEMMA 4.2. *Define the diagonal matrix* $\bar{\mathrm{M}}_\nu$ *by* $(\bar{\mathrm{M}}_\nu)_{ii} = \sum_{j=1}^m (\hat{\mathrm{M}}_\nu)_{ij}$ *(diagonal lumping). Then for all* $\mathrm{y} \in \mathbb{R}^m$ *we have*

$$C_3\langle \bar{\mathrm{M}}_\nu\mathrm{y}, \mathrm{y}\rangle \le \langle \hat{\mathrm{M}}_\nu\mathrm{y}, \mathrm{y}\rangle \le C_4\langle \bar{\mathrm{M}}_\nu\mathrm{y}, \mathrm{y}\rangle$$

with constants $C_3 > 0$ *and* C_4 *independent of* ν *and* h.

Proof. To show the result it is sufficient to estimate the eigenvalues of $(\bar{\mathrm{M}}_\nu|_\tau)^{-1}\hat{\mathrm{M}}_\nu|_\tau$ with the local mass matrices $\bar{\mathrm{M}}_\nu|_\tau$ and $\hat{\mathrm{M}}_\nu|_\tau$ on each element τ of triangulation. On every element ν is constant and thus we can use the result from [24], which yields ν and h - independent bounds on each element. ◻

A further elementary observation is

$$\bar{\mathrm{M}}_\nu\mathrm{e} = \hat{\mathrm{M}}_\nu\mathrm{e} \tag{4.8}$$

We briefly discuss two known iterative methods for solving the linear system in (4.2). For these methods we will present numerical results in section 5.
A basic method for saddle point problems is the Uzawa method. Applying a block Gaussian elimination step to the system (4.2) yields the equivalent system

$$\begin{pmatrix} \mathrm{I} & \mathrm{A}^{-1}\mathrm{B}^T \\ 0 & \mathrm{S} \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} \mathrm{A}^{-1}\mathrm{f} \\ \mathrm{BA}^{-1}\mathrm{f} \end{pmatrix}$$

This system can be solved by block backward substitution, which yields the *Uzawa method*:

$$1.\ \textit{Solve}\ \ \mathrm{Az} = \mathrm{f} \tag{4.9}$$

$$2.\ \textit{Solve}\ \ \mathrm{Sy} = \mathrm{Bz}\ ,\quad \mathrm{y} \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp \tag{4.10}$$

$$3.\ \textit{Solve}\ \ \mathrm{Ax} = \mathrm{z} - \mathrm{B}^T\mathrm{y} \tag{4.11}$$

For the systems in steps 1. and 3. we apply a standard multigrid solver. For the system $\mathrm{Sy} = \mathrm{Bz}$ we apply a preconditioned CG method (PCG). In each matrix-vector multiplication with S we solve the linear system with A using the multigrid method. For the iterands $\mathrm{y}^1, \mathrm{y}^2, \ldots$, that are computed using the PCG method with preconditioner denoted by M and with startvector y^0, we have

$$\mathrm{y}^k - \mathrm{y}^0 \in \mathrm{span}\big\{\mathrm{M}^{-1}\mathrm{S}\mathrm{e}^0, \ldots, (\mathrm{M}^{-1}\mathrm{S})^k\mathrm{e}^0\big\}, \quad \mathrm{e}^0 := \mathrm{y} - \mathrm{y}^0\ .$$

And thus $\langle \mathrm{y}^k - \mathrm{y}^0, \mathrm{Me}\rangle = 0$ for $k \geq 1$, i.e., $\mathrm{y}^k - \mathrm{y}^0 \in (\mathrm{Me})^\perp$ for $k \geq 1$. If for the preconditioner we take $\mathrm{M} \in \{\hat{\mathrm{M}}_\nu, \bar{\mathrm{M}}_\nu\}$ then it follows, using (4.8), that

$$\mathrm{y}^k \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp \quad \text{for} \ \ k \geq 1, \quad \text{if} \ \ \mathrm{y}^0 \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp \tag{4.12}$$

This means that for both preconditioners the iterands remain in the subspace $(\hat{\mathrm{M}}_\nu \mathrm{e})^\perp$, if the starting vector y^0 is in this subspace. Since the solution y is also sought in this subspace (cf. 4.10)) this implies that the errors $\mathrm{e}^k := \mathrm{y} - \mathrm{y}^k$ remain in this subspace if $\mathrm{y}^0 \in (\hat{\mathrm{M}}_\nu \mathrm{e})^\perp$. Hence, only the spectral condition number of the preconditioned matrix on this subspace is relevant. The results in theorem 4.1 and lemma 4.2 yield that both $\hat{\mathrm{M}}_\nu^{-1}\mathrm{S}$ and $\bar{\mathrm{M}}_\nu^{-1}\mathrm{S}$ have optimal (i.e., independent of h and ν) spectral condition numbers on this subspace.

In practice the Uzawa method is not very attractive because one has to solve the A-systems accurately. In this paper we consider the Uzawa method to illustrate the robustness of the multigrid solver and of the preconditioners for the Schur complement (cf. section 5). In practical applications variants of the Uzawa method that are much more efficient are used (cf., for example, [3, 7, 15, 22, 26]). Here we consider a preconditioned MINRES method. For this we consider a symmetric positive definite preconditioner

$$\tilde{\mathrm{K}} = \begin{pmatrix} \mathrm{Q}_A & 0 \\ 0 & \mathrm{Q}_S \end{pmatrix} \quad \text{for} \ \ \mathrm{K} := \begin{pmatrix} \mathrm{A} & \mathrm{B}^T \\ \mathrm{B} & 0 \end{pmatrix}$$

Define the norm $\|\mathrm{w}\|_{\tilde{K}} := \langle \tilde{\mathrm{K}}\mathrm{w}, \mathrm{w}\rangle^{\frac{1}{2}}$ for $\mathrm{w} \in \mathbb{R}^{n+m}$. Given a starting vector w^0 with corresponding error $\mathrm{e}^0 := \mathrm{w}^* - \mathrm{w}^0$, then in the preconditioned MINRES method one computes the vector $\mathrm{w}^k \in \mathrm{w}^0 + \mathrm{span}\big\{\tilde{\mathrm{K}}^{-1}\mathrm{K}\mathrm{e}^0, \ldots, (\tilde{\mathrm{K}}^{-1}\mathrm{K})^k\mathrm{e}^0\big\}$ which minimizes the preconditioned residual $\|\tilde{\mathrm{K}}^{-1}\mathrm{K}(\mathrm{w}^* - \mathrm{w})\|_{\tilde{\mathrm{K}}}$ over this subspace. For an efficient implementation of this method we refer to the literature.
If we take $\mathrm{Q}_S \in \{\hat{\mathrm{M}}_\nu, \bar{\mathrm{M}}_\nu\}$, then again we have that the approximations (and errors) of the pressure remain in the subspace $(\hat{\mathrm{M}}_\nu \mathrm{e})^\perp$ if the starting approximation y^0 is in this subspace. From the literature (cf. [21, 22]) it is known that the convergence of the preconditioned MINRES method is fast if we have good preconditoners Q_A of A and Q_S of S (on the subspace $(\hat{\mathrm{M}}_\nu \mathrm{e})^\perp$). In the numerical experiments in section 5 we take a standard multigrid method for Q_A and $\mathrm{Q}_S = \bar{\mathrm{M}}_\nu$.

5. Numerical experiments. In this section we present results of a few numerical experiments to illustrate the behaviour of the Uzawa and preconditioned MINRES method applied to the Stokes interface problem. We consider a problem as in (1.1)-(1.3) with

$$\Omega = (0,1)^3, \quad \Omega_2 = (0, \frac{1}{2})^3$$

For the discretization we start with a uniform tetrahedral grid with $h = \frac{1}{2}$ and we apply regular refinements to this starting triangulation. The resulting triangulations satisfy the conformity condition (3.1). For the finite element discretization we used the LBB stable pair of Hood-Taylor $P_2 - P_1$, i.e. continuous piecewise quadratics for the velocity and continuous piecewise linears for the pressure. We performed computations for the cases $h = 1/16$, $h = 1/32$ and with varying $\varepsilon \in (0,1]$. Note that for $h = 1/32$ we have approximately $7.5{\cdot}10^5$ velocity unknowns and $3.3{\cdot}10^4$ pressure unknowns ($n \approx 7.5{\cdot}10^5$, $m \approx 3.3{\cdot}10^4$). We consider the linear system as in (4.2) with solution $(\mathrm{x}, \mathrm{y}) = 0$. We take a fixed arbitrary starting vector $(\mathrm{x}^0, \mathrm{y}^0)$, with $\mathrm{y}^0 \in (\hat{\mathrm{M}}_\nu)^\perp$.

To test the robustness of the Schur complement preconditioning and of the multigrid solver we first consider the Uzawa method (4.9)-(4.11). The linear systems of the form $\mathrm{Ax} = \mathrm{r}$ that occur in the steps 1,2 and 3 are all solved using a standard multigrid V-cycle with one pre- and one post-smoothing iteration with a symmetric Gauss-Seidel method. The starting vector is x^0 and the iteration is stopped when for the result after k iterations, x^k, the scaled residual satisfies

$$\frac{\|\mathrm{D}^{-1}(\mathrm{Ax}^k - \mathrm{r})\|}{\|\mathrm{D}^{-1}(\mathrm{Ax}^0 - \mathrm{r})\|} \leq 10^{-10} , \quad \mathrm{D} := \mathrm{diag}(\mathrm{A}) \tag{5.1}$$

Here $\|\cdot\|$ denotes the standard Euclidean norm. The system with the Schur complement in (4.10) is solved using a PCG method with preconditioner $\hat{\mathrm{M}}_\nu$. The systems $\hat{\mathrm{M}}_\nu \mathrm{y} = \mathrm{w}$ are solved approximately using a PCG method with preconditioner $\bar{\mathrm{M}}_\nu$ and starting vector y^0 and accuracy

$$\frac{\|\bar{\mathrm{M}}_\nu^{-1}(\hat{\mathrm{M}}_\nu \mathrm{y}^k - \mathrm{w})\|}{\|\bar{\mathrm{M}}_\nu^{-1}(\hat{\mathrm{M}}_\nu \mathrm{y}^0 - \mathrm{w})\|} \leq 10^{-10} \tag{5.2}$$

The PCG method for the Schur complement system $\mathrm{Sy} = \mathrm{c}$ has starting vector y^0 and is stopped when

$$\frac{\|\bar{\mathrm{M}}_\nu^{-1}(\mathrm{Sy}^k - \mathrm{c})\|}{\|\bar{\mathrm{M}}_\nu^{-1}(\mathrm{Sy}^0 - \mathrm{c})\|} \leq 10^{-6} \tag{5.3}$$

In table 5.1 we present results for different h and ε values. Here #-MG denotes the average number of multigrid iterations needed to satisfy (5.1), #-PCG-M the average number of PCG iterations needed to satisfy (5.2) and #-PCG-S the average number of PCG iterations needed to satisfy (5.3).

These results clearly show the robustness of the multigrid solver for the velocity systems, of the preconditioner $\hat{\mathrm{M}}_\nu$ for S and of the preconditioner $\bar{\mathrm{M}}_\nu$ for $\hat{\mathrm{M}}_\nu$ with respect to variation of h and of ε. We now consider the effect of using the lumped mass matrix $\bar{\mathrm{M}}_\nu$ instead of $\tilde{\mathrm{M}}_\nu$ as a preconditioner for the Schur complement. In the PCG

TABLE 5.1
Uzawa method, preconditioner $\hat{\mathrm{M}}_\nu$

h	1/16				1/32			
ε	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
#-MG	13	13	14	14	14	14	14	14
#-PCG-M	24	25	25	26	24	25	25	25
#-PCG-S	22	29	31	34	21	29	30	34

TABLE 5.2
Uzawa method, preconditioner $\bar{\mathrm{M}}_\nu$

h	1/16				1/32			
ε	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
#-PCG-S	40	48	48	58	39	50	52	59

method we use a stopping criterion as in (5.3). Note that in this PCG method the preconditoner is now a diagonal matrix. In table 5.2 we present results for different h and ε values. As expected, the lumped mass matrix $\bar{\mathrm{M}}_\nu$ is a robust preconditioner for the Schur complement S. In the final experiment we consider the preconditioned MINRES method. For the preconditioner Q_A we take one iteration of the multigrid method described above and we take the lumped mass matrix $\bar{\mathrm{M}}_\nu$ as preconditioner for the Schur complement. In table 5.3 we show the number of iterations k (denoted by #-PMINRES), such that

$$\left\|\tilde{\mathrm{K}}^{-1}\mathrm{K}\begin{pmatrix}\mathrm{x}^k\\ \mathrm{y}^k\end{pmatrix}\right\|_{\tilde{\mathrm{K}}} \leq 10^{-6}\left\|\tilde{\mathrm{K}}^{-1}\mathrm{K}\begin{pmatrix}\mathrm{x}^0\\ \mathrm{y}^0\end{pmatrix}\right\|_{\tilde{\mathrm{K}}}$$

(recall that the right handside is 0). Note that for $h = 1/32$ one needs less iterations

TABLE 5.3
MINRES with preconditioner $\bar{\mathrm{M}}_\nu$

h	1/16				1/32			
ε	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
#-PMINRES	62	68	98	157	50	58	85	116

than for $h = 1/16$.

6. Analysis in the space $H_0^1(\Omega)^d \times L_0^2(\Omega)$. Instead of the pressure space M one may want to use the standard space

$$Q := L_0^2(\Omega) = \{\, p \in L^2(\Omega) \mid (p, 1) = 0 \,\} .$$

In this section we consider the variational formulation of the interface problem (1.1)-(1.3) in the space $\mathbf{V} \times Q$. It turns out that the analysis then requires different norms to obtain optimal estimates. These optimal estimates are ν-independent for the continuous problem. For the corresponding Galerkin discrete problem, however, opposite to the results in section 3 we now observe a ν-dependence in the estimates. A numerical experiment shows that in a certain sense the results are still sharp. The main ideas of the analysis are the same as in the sections 2 and 3. However, some further technicalities, like a mesh-dependent norm (cf. (6.5)) in the pressure space, are needed. We decided to present only the main results of this analysis here. The proofs (and some further related results) will be presented in a separate paper.

Consider the following variational problem: given $\mathbf{f} \in \mathbf{V}'$ find $\{\mathbf{u}, p\} \in \mathbf{V} \times Q$ such that

$$\begin{cases} (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) & = (\mathbf{f}, \mathbf{v}) \quad \text{for} \quad \mathbf{v} \in \mathbf{V}, \\ (\operatorname{div} \mathbf{u}, q) & = 0 \quad \text{for} \quad q \in Q\ . \end{cases} \tag{6.1}$$

We introduce a decomposition of Q. Let $\tilde{p} = \nu^{-1}\bar{p}$ with $\bar{p}$ from (2.1). Then $(\tilde{p}, 1) = 0$ and thus $\tilde{p} \in Q$. Consider the one-dimensional subspace $Q_0 := \operatorname{span}\{\tilde{p}\}$ of Q and an L^2-orthogonal decomposition $Q = Q_0 \oplus Q_0^\perp$. For $p \in Q$ we use the notation

$$p = p_0 + p_0^\perp\ , \quad p_0 \in Q_0,\ p_0^\perp \in Q_0^\perp \tag{6.2}$$

One easily checks that

$$Q_0^\perp = \{\, p \in Q \mid (p, 1)_{\Omega_1} = (p, 1)_{\Omega_2} = 0 \,\},$$

hence, $Q_0^\perp \equiv M_0^\perp$. For functions in $Q_0^\perp$ we use the M-norm from (1.8). On Q we introduce the norm

$$\|p\|_Q := \left(\|p_0\|^2 + \|p_0^\perp\|_M^2\right)^{\frac{1}{2}}.$$

On $\mathbf{V}$ we use the same norm as in (1.7). In these norms we have uniform continuity and infsup results for the bilinear form $(\operatorname{div} \cdot, \cdot)$:

Lemma 6.1. *Assume that* (2.13) *or* (2.14) *holds. There exists a constant* C *independent of* ν *such that*

$$|(\operatorname{div} \mathbf{u}, p)| \le C \|\mathbf{u}\|_{\mathbf{V}} \|p\|_Q \quad \textit{for all} \quad \mathbf{u} \in \mathbf{V},\ p \in Q$$

Theorem 6.2. *There exists a constant* $C > 0$ *independent of* ν *such that*

$$\sup_{\mathbf{u} \in \mathbf{V}} \frac{(\operatorname{div} \mathbf{u}, p)}{\|\mathbf{u}\|_{\mathbf{V}}} \ge C \|p\|_Q \quad \textit{for all} \quad p \in Q.$$

From these results it follows that we have uniform (w.r.t. ν) well-posedness of the continuous variational interface Stokes problem in the spaces $\mathbf{V}$, Q with the norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_Q$, respectively.

We now consider the Galerkin discretization in a pair $\mathbf{V}_h \subset \mathbf{V}$, $Q_h \subset Q$ of LBB stable finite element spaces. The discrete problem is as folows: Find $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times Q_h$ such that

$$a(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = (\mathbf{f}, \mathbf{v}_h) \quad \text{for all} \quad \{\mathbf{v}_h, q_h\} \in \mathbf{V}_h \times Q_h\ . \tag{6.3}$$

($a(\cdot, \cdot)$ as in (3.3)). Let $\tilde{p}_h \in Q_h$ be the L^2-orthogonal projection of $\tilde{p}$ on Q_h,

$$(\tilde{p} - \tilde{p}_h, q_h) = 0 \quad \text{for all} \quad q_h \in Q_h$$

and define the one-dimensional subspace $Q_{0,h} := \operatorname{span}(\tilde{p}_h)$ of Q_h. This induces an L^2-orthogonal decomposition of Q: $Q = Q_{0,h} \oplus Q_{0,h}^\perp$, and for any $p \in Q$ we write

$$p = p_{0,h} + p_{0,h}^\perp, \qquad p_{0,h} \in Q_{0,h},\ p_{0,h}^\perp \in Q_{0,h}^\perp. \tag{6.4}$$

Related to this decomposition we introduce an *h-dependent* norm on Q:

$$\|p\|_{Q,h} := \left(\|p_{0,h}\|^2 + \|p_{0,h}^{\perp}\|_M^2\right)^{\frac{1}{2}} \qquad \text{for} \;\; p \in Q \tag{6.5}$$

We define

$$\tilde{\mu}_h := \frac{\|\tilde{p} - \tilde{p}_h\|}{\|\tilde{p}\|} \,,$$

which is very similar to the quantity μ_h in (3.8). In particular, $\tilde{\mu}_h = 0$ if Q_h contains piecewise constant finite elements. In general we have $\tilde{\mu}_h = \mathcal{O}(\tilde{h}^{\frac{1}{2}})$, where $\tilde{h}$ is the maximal diameter of the elements in $\mathcal{T}_h$ that have a nonempty intersection with Γ. We will assume $\tilde{\mu}_h \leq \frac{1}{2}$.

With respect to the norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_{Q,h}$ one can prove the following continuity and discrete infsup results:

LEMMA 6.3. *Assume that* (2.13) *or* (2.14) *holds. There exists a constant C independent of h and ν such that for all $\mathbf{u} \in \mathbf{V}$, $p_h \in Q_h$ the following inequalities hold*

$$|(\operatorname{div}\mathbf{u}, p_h)| \leq C\|\mathbf{u}\|_{\mathbf{V}}\left(\|p_h\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\min\{\varepsilon, 1\}}\|p_{0,h}\|^2\right)^{\frac{1}{2}} \tag{6.6}$$

$$|(\operatorname{div}\mathbf{u}, p_h)| \leq C\big(\|\mathbf{u}\|_{\mathbf{V}} + \tilde{\mu}_h\|\nabla\mathbf{u}\|\big)\|p_h\|_{Q,h}. \tag{6.7}$$

THEOREM 6.4. *There exists a constant $C > 0$ independent of h and ν such that*

$$\sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{(\operatorname{div}\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}} \geq C\|p_h\|_{Q,h} \quad \textit{for all} \;\; p_h \in Q_h$$

Based on the continuity and infsup result one can derive, using standard arguments as in section 3, a continuity and discrete infsup result for the bilinear form $a(\cdot,\cdot)$ in the product norm $(\|\cdot\|_{\mathbf{V}}^2 + \|\cdot\|_{Q,h}^2)^{\frac{1}{2}}$. This then yields the following discretization error bound.

THEOREM 6.5. *Let $\{\mathbf{u}, p\}$ be the solution of the continuous problem* (6.1) *and $\{\mathbf{u}_h, p_h\}$ the solution of the discrete problem* (6.3)*. Assume that* (2.13) *or* (2.14) *is satisfied. The following holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}}^2 + \|p - p_h\|_{Q,h}^2 \leq$$
$$\leq C\Big(\min_{\mathbf{v}_h \in \mathbf{V}_h}\left(\|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}}^2 + \tilde{\mu}_h^2\|\nabla(\mathbf{u} - \mathbf{v}_h)\|^2\right) + \min_{q_h \in Q_h}\|p - q_h\|_{Q,h}^2\Big)$$

The righthand sides in the continuity results in lemma 6.3 contain terms depending on ε and $\tilde{\mu}_h$. This dependence resulted in sub-optimal (compared to the result in theorem 3.6) error bound in theorem 6.5. It also results in a nonuniform estimate for the preconditioned Schur complement for the corresponding discrete problem. Theorem 6.4 and estimate (6.6) imply for any $p_h \in Q_h$

$$c_1\|p_h\|_{Q,h} \leq \sup_{\mathbf{u}_h \in \mathbf{V}_h}\frac{(\operatorname{div}\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}} \leq c_2\left(\|p_h\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\min\{\varepsilon, 1\}}\|p_{0,h}\|^2\right)^{\frac{1}{2}} \tag{6.8}$$

with $c_1 > 0$, c_2 independent of ν and h. Let G the mass matrix of Q_h w.r.t. the scalar product $(p_h, q_h)_{Q,h} := (p_{0,h}^{\perp}, q_{0,h}^{\perp})_M + (p_{0,h}, q_{0,h})$ and $\hat{\mathrm{M}}$ the mass matrix of Q_h w.r.t.

the L^2 scalar product $(\cdot,\cdot)$. Furthermore, the vector representation of $\tilde{p}_h$ is denoted by $\tilde{\mathrm{p}}$. i.e., $J_M\tilde{\mathrm{p}} = \tilde{p}_h$. Note that $Q_h = \{\, J_M\mathrm{y} \mid \mathrm{y} \in (\hat{\mathrm{M}}\mathrm{e})^\perp \,\}$ (recall: $\mathrm{e} = (1,\ldots,1)^T$). From (6.8) one obtains the following result:

THEOREM 6.6. *The spectral equivalences*

$$c_1^2\langle \mathrm{Gy},\mathrm{y}\rangle \le \langle \mathrm{Sy},\mathrm{y}\rangle \le c_2^2\left(1+\frac{\tilde{\mu}_h^2}{\min\{\varepsilon,1\}}\right)\langle \mathrm{Gy},\mathrm{y}\rangle \quad \textit{for all} \;\; \mathrm{y}\in(\hat{\mathrm{M}}\mathrm{e})^\perp \tag{6.9}$$

$$c_1^2\langle \mathrm{Gy},\mathrm{y}\rangle \le \langle \mathrm{Sy},\mathrm{y}\rangle \le c_2^2\langle \mathrm{Gy},\mathrm{y}\rangle \quad \textit{for all} \;\; \mathrm{y}\in(\hat{\mathrm{M}}\mathrm{e})^\perp\cap(\hat{\mathrm{M}}\tilde{\mathrm{p}})^\perp \tag{6.10}$$

hold with c_1, c_2 *as in* (6.8).

In contrast to theorem 4.1 the result in (6.9) is not robust w.r.t. ε. A simple numerical experiment shows that the upper bound in (6.9) is sharp. For this we consider a 1D Stokes problem with $\Omega = (0,1)$, $\Omega_2 = (\frac{1}{4},\frac{3}{4})$ and $P_2 isoP_1 - P_1$ finite elements on a uniform grid. In this case we have $\tilde{\mu}_h^2 \sim h$. In table 6.1 we show the values of $\langle \mathrm{Sy},\mathrm{y}\rangle/\langle \mathrm{Gy},\mathrm{y}\rangle$ for $\mathrm{y} = \tilde{\mathrm{p}}$.
This shows that the analysis is sharp and that if we consider the space $Q_h \subset L_0^2(\Omega)$ instead of $M_h \subset M$ (as in section 3) and use the norm $\|\cdot\|_{Q,h}$ in Q_h then the estimates in general can not be uniform with respect to h and ε.
Finally we note that the deterioration of the upper bound in (6.9) is not a serious problem, because it is caused by the *one*-dimensional subspace span($\tilde{\mathrm{p}}$) (cf. (6.10)). If we apply a Krylov subspace solver then already after a few iterations this one-dimensional subspace does not influence the effective spectral condition number anymore.

TABLE 6.1
Estimates for upper bound in (6.9).

	h		
ε	1/16	1/32	1/64
10^{-2}	5.0	3.5	2.7
10^{-4}	$3.1\,10^2$	$1.6\,10^2$	$8.1\,10^1$
10^{-6}	$3.1\,10^4$	$1.6\,10^4$	$7.9\,10^3$

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