# On the convergence of a multigrid method for linear reaction-diffusion problems

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**Abstract**

In this note we consider discrete linear reaction-diffusion problems. For the discretization a standard conforming finite element method is used. For the approximate solution of the resulting discrete problem a multigrid method with a damped Jacobi or symmetric Gauss-Seidel smoother is applied. We analyze the convergence of the multigrid V- and W-cycle in the framework of the aproximation- and smoothing property. The multigrid method is shown to be robust in the sense that the contraction number can be bounded by a constant smaller than one which does not depend on the mesh size or on the diffusion-reaction ratio.

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## 1 Introduction

In this paper we consider the linear reaction-diffusion boundary-value problem: Given $0 < \varepsilon < 1$ and functions $f$ and $d$, with $0 < d_0 \leq d(\mathbf{x}) \leq d_1$ in $\Omega$,

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find $u$ such that

$$\left\{ \begin{array}{rcll} -\varepsilon\Delta\, u + d(\mathbf{x})\, u & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where $\Omega$ is a convex polyhedral domain in $R^N$, $N = 2, 3$. For the discretization of the variational formulation of this problem a standard finite element method is applied based on a quasi-uniform family of nested triangulations of $\Omega$, with mesh size parameter denoted by $h$, and conforming finite elements. In [7, 8] a convergence analysis of this finite element method applied to the problem (1) is presented in which local and global error estimates are derived and their possible dependence on the parameter $\varepsilon$ is studied. In general the solution of (1) has exponential boundary layer behaviour and a discretization method with polynomial finite elements on a quasi-uniform family of partitions will result in large discretization errors in these boundary layers. The analyses in [7, 8], however, show that this discretization method is stable (for $\varepsilon \downarrow 0$) and that the pollution effects are not severe in this problem: Outside the boundary layer error estimates which are uniform w.r.t. $\varepsilon$ and of optimal order (as a function of the mesh size parameter) are shown to hold. Hence for the numerical solution of (1) a discretization method based on a Galerkin technique with standard finite element spaces can be useful in practice.

For the approximate solution of the resulting discrete problem we apply a multigrid method with canonical intergrid transfer operators and damped Jacobi or symmetric Gauss-Seidel smoothing. An interesting topic related to the efficiency of this multigrid solver is the dependence of its convergence rate on the parameter $\varepsilon$. In this paper we present a convergence analysis which shows that the multigrid method is *robust* in the sense that the contraction number can be bounded by a constant smaller than one which does not depend on the mesh size parameter $h$ or on $\varepsilon$. Both the multigrid W-cycle and multigrid V-cycle will be considered. The analysis will use the framework of the smoothing- and approximation property as introduced by Hackbusch (cf. [5, 6]). For the proof of the approximation property we use regularity estimates and finite element error bounds from [7, 8]. The smoothing property will be proved using a standard technique from [5]. The smoothing property and approximation property that will be proved in this paper can be combined with results from [5, 6] for the convergence of the multigrid W- or V-cycle. The analysis shows that the deterioration of the approximation property for $\varepsilon \downarrow 0$ (caused by the boundary layer) is compensated by an improved smoothing property. The combined effect is such that the multigrid method can be shown to be robust.

In the literature we did not find a theoretical analysis of the smoothing and approximation property which shows the robustness of classical multigrid applied to reaction-diffusion problems. In the literature on subspace decomposition (cf. [10, 11]) we also did not find theoretical results on the robustness of classical multigrid applied to (1). In [9] it is noted that the BPX-preconditioner [2] and the hierarchical basis multigrid method [1] are not robust for a finite element discretization of the problem (1). In [9] a hierarchical basis preconditioner is introduced which is shown to be robust for the problem (1) discretized with linear finite elements on uniform two-dimensional meshes. In [3] a multilevel method based on subspace splitting is presented which is robust for the problem (1). This method, however, is restricted to rectangular domains and discretization methods of tensor product type.

# 2 Preliminaries

Throughout the paper we use the notation $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ for the scalar product and norm in $L_2(\Omega)$. The scalar products and corresponding norms in the Sobolev spaces $\mathrm{H}^k(\Omega)$, $k = 1, 2$, are denoted by $(\cdot,\cdot)_k$ and $\|\cdot\|_k$, respectively. We also use the notation $(\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v$ for $u, v \in H^1(\Omega)$ and $|u|_1 = (\nabla u, \nabla u)^{\frac{1}{2}}$ for $u \in \mathrm{H}_0^1(\Omega)$.

We assume $d \in L_\infty(\Omega)$ with $0 < d_0 \le d(\mathbf{x}) \le d_1$ a.e. in $\Omega$ and $f \in L_2(\Omega)$. $\Omega$ is assumed to be a convex polyhedral domain in $R^N$, $N = 2, 3$. The variational formulation of (1) reads: Find $u \in \mathrm{U} := H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_0 \quad \text{for all} \quad v \in \mathrm{U}, \tag{2}$$

with the symmetric bilinear form

$$a(u, v) = \varepsilon(\nabla u, \nabla v) + (d\,u, v)_0 \quad \text{for } u, v \in \mathrm{U}.$$

Note that $a(\cdot,\cdot)$ is continuous and elliptic on U. Thus the problem (2) has a unique solution. Using standard regularity theory the following a priori estimates can be proved.

**Lemma 1** *Let $u$ be the solution to (2). Then $u \in \mathrm{H}^2(\Omega)$ and*

$$\begin{aligned} \|u\|_0 &\le c\|f\|_0 \ , && (3) \\ \|u\|_1 &\le \frac{c}{\sqrt{\varepsilon}}\|f\|_0 \ , && (4) \\ \|u\|_2 &\le \frac{c}{\varepsilon}\|f\|_0 \ , && (5) \end{aligned}$$

*with constants $c$ that are independent of $\varepsilon$ and $f$.*

Proof. From (2) we obtain using Young's inequality

$$\varepsilon\,|u|_1^2 + d_0\|u\|_0^2 \le \varepsilon\,|u|_1^2 + (d\,u,u)_0 = a(u,u) = (f,u)_0 \le \frac{1}{2d_0}\|f\|_0^2 + \frac{d_0}{2}\|u\|_0^2. \quad (6)$$

Now (3) follows. The result (6) in combination with the Friedrichs inequality $\|u\|_1 \le c|u|_1$ yields (4). Set $\tilde{f} = \frac{1}{\varepsilon}(f-u)$, then $u$ clearly solves the weak formulation of the Poisson problem: $(\nabla u, \nabla v) = (\tilde{f}, v)_0$ for all $v \in U$. Since $\tilde{f} \in L_2(\Omega)$ and the domain $\Omega$ is convex it follows from regularity results for the Poisson problem (e.g. Theorem 4.3.1.4 and §8.2 in [4]) that $u \in \mathrm{H}^2(\Omega)$ and

$$\|u\|_2 \le c\|\tilde{f}\|_0 \le c\frac{1}{\varepsilon}(\|f\|_0 + \|u\|_0). \quad (7)$$

Hence (5) follows from (3) and (7). □

For the discretization of (2) we introduce a quasi-uniform family of nested triangulations of $\Omega$ (triangles in 2D, tetrahedra in 3D) based on *global regular refinement*. We use conforming finite elements with piecewise polynomial functions. This results in a hierarchy of nested finite element spaces

$$\mathrm{U}_0 \subset \mathrm{U}_1 \subset \cdots \subset \mathrm{U}_k \subset \cdots \subset \mathrm{U} \ .$$

The corresponding mesh size parameter is denoted by $h_k$ and satisfies

$$c_0 2^{-k} \le h_k/h_0 \le c_1 2^{-k}$$

with positive constants $c_0$ and $c_1$ independent of $k$.

The discrete problem on level $k$ is given by: Find $u_k \in \mathrm{U}_k$ such that

$$a(u_k, v_k) = (f, v_k)_0 \quad \text{for all} \quad v_k \in \mathrm{U}_k. \quad (8)$$

The next lemma provides error bounds for the finite element solution. For $N = 2$ the result was proved in [7]. However, the arguments used in [7] are also applicable for the case $N = 3$. For completeness we present a proof here which follows the arguments in [7, 8].

**Lemma 2** *Let $u$ be the solution of (2) and $u_k$ be the corresponding finite solution of (8). Then*

$$\|u - u_k\|_0 \le c \min\left\{1, \frac{h_k^2}{\varepsilon}\right\} \|f\|_0 \tag{9}$$

*holds with a constant $c$ independent of $f$, $\varepsilon$, $k$.*

PROOF. In the proof we use constants $c$ which are independent of $f$, $\varepsilon$, $k$. Define $e_k = u - u_k$. Noting that $a(e_k, v_k) = 0 \quad \text{for all} \quad v_k \in \mathrm{U}_k$, one obtains

$$d_0\|e_k\|_0^2 \le a(e_k, e_k) = a(u, e_k) = (f, e_k)_0 \le \|f\|_0\|e_k\|_0$$

and thus

$$\|e_k\|_0 \le d_0^{-1}\|f\|_0. \tag{10}$$

For arbitrary $v_k \in \mathrm{U}_k$ we have

$$\begin{aligned}
&\varepsilon|e_k|_1^2 + d_0\|e_k\|_0^2 \le a(e_k, e_k) = a(u - v_k, e_k) \\
&\le \varepsilon|u - v_k|_1|e_k|_1 + d_1\|u - v_k\|_0\|e_k\|_0 \\
&\le (\varepsilon|u - v_k|_1^2 + \frac{d_1^2}{d_0}\|u - v_k\|_0^2)^{\frac{1}{2}}(\varepsilon|e_k|_1^2 + d_0\|e_k\|_0^2)^{\frac{1}{2}}
\end{aligned}$$

For $v_k$ we take the $(\cdot,\cdot)_1$-projection of $u$ on $\mathrm{U}_k$ for which the standard approximation results $\|u - v_k\|_0 \le c\, h_k^2\|u\|_2$ and $|u - v_k|_1 \le c\, h_k\|u\|_2$ hold. Using this and the regularity results of Lemma 1 we get

$$\varepsilon|e_k|_1^2 + d_0\|e_k\|_0^2 \le c\frac{h_k^2}{\varepsilon}(1 + \frac{h_k^2}{\varepsilon})\|f\|_0^2. \tag{11}$$

Now we use Nitsche's duality argument. Let $w \in \mathrm{U}$ be such that $a(w, v) = (e_k, v)_0 \quad \text{for all} \quad v \in \mathrm{U}$. From Lemma 1 we have $w \in \mathrm{H}^2(\Omega)$ and $\|w\|_2 \le \frac{c}{\varepsilon}\|e_k\|_0$. Let $w_k$ be the $(\cdot,\cdot)_1$-projection of $w$ on $\mathrm{U}_k$. Then the following holds:

$$\begin{aligned}
\|e_k\|_0^2 &= a(w, e_k) = a(w - w_k, e_k) \le \varepsilon|w - w_k|_1|e_k|_1 + d_1\|w - w_k\|_0\|e_k\|_0 \\
&\le c\,(\varepsilon\, h_k\|w\|_2|e_k|_1 + d_1 h_k^2\|w\|_2\|e_k\|_0) \le c(h_k|e_k|_1 + d_1\frac{h_k^2}{\varepsilon}\|e_k\|_0)\|e_k\|_0.
\end{aligned}$$

Thus using (10) and (11), we get for $\frac{h_k^2}{\varepsilon} \leq 1$

$$
\begin{aligned}
\|e_k\|_0 &\leq c(h_k|e_k|_1 + \frac{h_k^2}{\varepsilon}\|f\|_0) \\
&\leq c\, h_k \frac{h_k}{\varepsilon}\big(1 + \frac{h_k^2}{\varepsilon}\big)^{\frac{1}{2}}\|f\|_0 + c\frac{h_k^2}{\varepsilon}\|f\|_0 \leq c\frac{h_k^2}{\varepsilon}\|f\|_0.
\end{aligned}
\tag{12}
$$

Combination of (10) and (12) proves the bound in (9). $\square$

# 3 Multigrid convergence analysis

For the approximate solution of the discrete problem we apply a multigrid method. The method and its convergence analysis will be presented in a matrix-vector form as in Hackbush [5]. To this end consider the standard nodal basis in $U_k$ denoted by $\{\phi_i\}_{1\leq i\leq n_k}$ and the isomorphism:

$$
P_k : \mathrm{X}_k := \mathbb{R}^{n_k} \to \mathrm{U}_k, \quad P_k x = \sum_{i=1}^{n_k} x_i\phi_i.
$$

On $X_k$ we use a scaled Euclidean scalar product: $\langle x, y\rangle_k = h_k^N \sum_{i=1}^{n_k} x_i y_i$ and corresponding norm denoted by $\|\cdot\|$. The adjoint $P_k^* : U_k \to X_k$ satisfies $(P_k x, v)_0 = \langle x, P_k^* v\rangle_k$ for all $x \in X_k,\ v \in U_k$. Note that the following norm equivalence holds

$$
C^{-1}\|x\| \leq \|P_k x\|_0 \leq C\|x\| \quad \text{for all } x \in \mathrm{X}_k, \tag{13}
$$

with a constant $C$ independent of $k$. The stiffness matrix $A_k$ on level $k$ is defined by

$$
\langle A_k x, y\rangle_k = a(P_k x, P_k y) \quad \text{for all } x, y \in \mathrm{X}_k. \tag{14}
$$

For the prolongation and restriction in the multigrid algorithm we use the canonical choice:

$$
\begin{aligned}
&p_k : \mathrm{X}_{k-1} \to \mathrm{X}_k, \quad p_k = P_k^{-1}P_{k-1} \\
&r_k : \mathrm{X}_k \to \mathrm{X}_{k-1}, \quad r_k = P_{k-1}^*(P_k^*)^{-1} = \left(\tfrac{h_k}{h_{k-1}}\right)^N p_k^T.
\end{aligned}
\tag{15}
$$

Finally, a smoother is introduced. Let $W_k : \mathrm{X}_k \to \mathrm{X}_k$ be a nonsingular matrix. We consider a smoother of the form

$$
x^{\text{new}} = x^{\text{old}} - W_k^{-1}(A_k x^{\text{old}} - b), \quad \text{for } x^{\text{old}}, b \in \mathrm{X}_k
$$

with corresponding iteration matrix denoted by

$$S_k = I - W_k^{-1} A_k. \tag{16}$$

With the components defined above a standard multigrid algorithm with $\nu_1$ pre- and $\nu_2$ post-smoothing iterations can be formulated (cf. [6]) with an iteration matrix that satisfies the recursion

$$\begin{aligned} M_0(\nu_1, \nu_2) &= 0, \\ M_k(\nu_1, \nu_2) &= S_k^{\nu_2} \left( I - p_k (I - M_{k-1}^{\gamma}) A_{k-1}^{-1} r_k A_k \right) S_k^{\nu_1}, \quad k = 1, 2, \ldots. \end{aligned}$$

The choices $\gamma = 1$ and $\gamma = 2$ correspond to the V- and W-cycle, respectively.

For the analysis of this multigrid method we use the framework of [5, 6] based on the approximation and smoothing property. Below we derive these properties for the reaction-diffusion problem. We start with a lemma in which a few inequalities are derived that will be used in the analysis of the approximation and smoothing property.

**Lemma 3** *Let $A_k$ be the stiffness matrix from (14) and $D_k := \mathrm{diag}(A_k)$. The inequalities*

$$c_1 \left( \frac{\varepsilon}{h_k^2} + 1 \right) \leq \|A_k\| \leq c_2 \left( \frac{\varepsilon}{h_k^2} + 1 \right) \tag{17}$$

$$\|D_k^{-1}\| \leq \frac{c_3}{\|A_k\|} \tag{18}$$

*hold with constants $c_i > 0$ independent of $\varepsilon$ and $k$.*

Proof. Let $e_i$ be the $i$th basis vector in $\mathbb{R}^{n_k}$. Note that

$$\begin{aligned} (A_k)_{ii} = \frac{< A_k e_i, e_i >_k}{< e_i, e_i >_k} &= h_k^{-N} a(\phi_i, \phi_i) \\ &\geq h_k^{-N} (\varepsilon |\phi_i|_1^2 + d_0 \|\phi_i\|_0^2) \geq c_1 \left( \frac{\varepsilon}{h_k^2} + 1 \right) \end{aligned} \tag{19}$$

with a constant $c_1$ independent of $\varepsilon$ and $k$. The left inequality in (17) follows from (19) and $\|A_k\| \geq (A_k)_{ii}$. Using an inverse inequality we obtain, with constants $c$ and $c_2$ independent of $\varepsilon$ and $k$,

$$\begin{aligned} < A_k x, x >_k = a(P_k x, P_k x) &\leq \varepsilon |P_k x|_1^2 + d_1 \|P_k x\|_0^2 \\ &\leq c \left( \frac{\varepsilon}{h_k^2} + 1 \right) \|P_k x\|_0^2 \leq c_2 \left( \frac{\varepsilon}{h_k^2} + 1 \right) \|x\|^2 \ , \end{aligned}$$

and thus the right inequality in (17) holds. Using (19) and (17) it follows that

$$\|D_k^{-1}\| = (\min_i (A_k)_{ii})^{-1} \le c_1^{-1}(\frac{\varepsilon}{h_k^2} + 1)^{-1} \le \frac{c_2}{c_1}\|A_k\|^{-1}$$

holds, which proves the result in (18). $\square$

**Theorem 1** [Approximation property.] *Let $A_k$ be the stiffness matrix from (14) and $p_k, r_k$ the prolongation and restriction as in (15). Then the following approximation property holds with a constant $c$ independent of $\varepsilon$ and $k$:*

$$\|A_k^{-1} - p_k A_{k-1}^{-1} r_k\| \le c \min\left\{1, \frac{h_k^2}{\varepsilon}\right\} \le c\|A_k\|^{-1}$$

Proof. Take $y_k \in X_k$. The constants $c$ that appear in the proof do not depend on $y_k, k$ or $\varepsilon$. Let $w \in \mathrm{U}$, $w_k \in \mathrm{U}_k$, and $w_{k-1} \in \mathrm{U}_{k-1}$ be such that

$$\begin{aligned} a(w, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in \mathrm{U}, \\ a(w_k, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in \mathrm{U}_k, \\ a(w_{k-1}, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in \mathrm{U}_{k-1}. \end{aligned}$$

Putting $f = (P_k^*)^{-1} y_k \in L_2(\Omega)$ in Lemma 2, we obtain

$$\|w - w_l\|_0 \le c \min\left\{1, \frac{h_l^2}{\varepsilon}\right\} \|(P_k^*)^{-1} y_k\|_0 \quad \text{for } l \in \{k-1, k\}.$$

Due to $h_{k-1} \le c h_k$ this yields

$$\|w_k - w_{k-1}\|_0 \le c \min\left\{1, \frac{h_k^2}{\varepsilon}\right\} \|(P_k^*)^{-1} y_k\|_0.$$

From (14) and (15) it follows that $w_k = P_k A_k^{-1} y_k$ and $w_{k-1} = P_{k-1} A_{k-1}^{-1} r_k y_k$. Thus, using (13), we get

$$\begin{aligned} \|(A_k^{-1} - p_k A_{k-1}^{-1} r_k) y_k\| &\le c\|P_k A_k^{-1} y_k - P_{k-1} A_{k-1}^{-1} r_k y_k\|_0 = c\|w_k - w_{k-1}\|_0 \\ &\le c \min\left\{1, \frac{h_k^2}{\varepsilon}\right\} \|(P_k^*)^{-1} y_k\|_0 \le c \min\left\{1, \frac{h_k^2}{\varepsilon}\right\} \|y_k\| , \end{aligned}$$

which proves the first inequality. The second inequality follows from Lemma 3 and $\min\{1, \alpha\} \le 2(1 + \frac{1}{\alpha})^{-1}$ for $\alpha > 0$. $\square$

For the smoother we consider two cases, namely a damped Jacobi method and the symmetric Gauss-Seidel method. If we decompose $A_k$ as $A_k = D_k - L_k - L_k^T$ with $D_k$ diagonal and $L_k$ strictly lower triangular then these two smoothing iterations have corresponding iteration matrices as in (16) with

$$W_k = \omega^{-1} D_k, \quad \omega \in (0,1), \quad \text{and } W_k = (D_k - L_k) D_k^{-1} (D_k - L_k^T).$$

From Lemma 3 we obtain $\|D_k^{-1} A_k\| \leq \|D_k^{-1}\| \|A_k\| \leq c_3$. In the damped Jacobi method we take a fixed $\omega \leq 1$ with $0 < \omega \leq \frac{1}{c_3}$, independent of $\varepsilon$ and $k$, such that $\rho(\omega D_k^{-1} A_k) \leq 1$ holds. Note that for the symmetric Gauss-Seidel method we have

$$W_k = (D_k - L_k) D_k^{-1} (D_k - L_k^T) = A_k + L_k D_k^{-1} L_k^T \geq A_k \ .$$

Hence, both for the damped Jacobi method and the symmetric Gauss-Seidel method we have

$$\sigma(W_k^{-1} A_k) \subset (0, 1]. \tag{20}$$

**Lemma 4** *Both for the damped Jacobi method and the symmetric Gauss-Seidel method the inequality*

$$\|W_k\| \leq c \|A_k\|$$

*holds with a constant $c$ independent of $\varepsilon$ and $k$.*

Proof. For the damped Jacobi method this result is a direct consequence of $\|D_k\| \leq \|A_k\|$. For the symmetric Gauss-Seidel method we note that, due to the fact that in every row of the stiffness matrix the number of nonzero entries can be bounded by a constant independent of $k$,

$$\begin{aligned} \|L_k\|^2 \leq \|L_k\|_1 \|L_k\|_\infty &= \left( \max_j \sum_{i=j+1}^{n} |(A_k)_{ij}| \right) \left( \max_i \sum_{j=1}^{i-1} |(A_k)_{ij}| \right) \\ &\leq c \max_{i,j} (A_k)_{ij}^2 \leq c \|A_k\|^2 \ , \end{aligned}$$

Hence, using Lemma 3, we obtain

$$\|W_k\| = \|A_k + L_k D_k^{-1} L_k^T\| \leq \|A_k\| + \|L_k\|^2 \|D_k^{-1}\| \leq c \|A_k\| \ . \qquad \square$$

**Corollary 1** *Theorem 1 and Lemma 4 imply*

$$\|W_k^{\frac{1}{2}}(A_k^{-1} - p_k A_{k-1}^{-1} r_k) W_k^{\frac{1}{2}}\| \le C_A \tag{21}$$

*with a constant $C_A$ independent of $\varepsilon$ and $k$.* $\square$

**Theorem 2** [Smoothing property.] *Both for the damped Jacobi and the symmetric Gauss-Seidel method the following smoothing property holds with a constant $c$ independent of $k, \varepsilon$ and $\nu$:*

$$\|A_k S_k^\nu\| \le c \frac{1}{\nu+1} \|A_k\| \ , \qquad \nu = 1, 2, \dots \ . \tag{22}$$

Proof. Denote $B := W_k^{-\frac{1}{2}} A_k W_k^{-\frac{1}{2}}$. Note that $B$ is symmetric and $\sigma(B) \subset (0,1]$. Furthermore

$$\|A_k S_k^\nu\| = \|W_k^{\frac{1}{2}} B (I-B)^\nu W_k^{\frac{1}{2}}\| \le \|W_k\| \|B(I-B)^\nu\|.$$

Note that $\|B(I-B)^\nu\| \le \max_{0 \le \lambda \le 1} \lambda(1-\lambda)^\nu \le (\nu+1)^{-1}$ (Lemma 10.6.1. in [6]) and, due to Lemma 4, $\|W_k\| \le c\|A_k\|$ with a constant $c$ independent of $k$ and $\varepsilon$. Hence (22) holds. $\square$

**Corollary 2** *For the two-grid iteration matrix with $\nu_1 = \nu$ and $\nu_2 = 0$ the smoothing and approximation property imply*

$$\|(I - p_k A_{k-1}^{-1} r_k A_k) S_k^\nu\| \le \frac{C_T}{\nu+1} \tag{23}$$

*with $C_T$ independent of $\varepsilon$ and $k$.* $\square$

For the multigrid W-cycle Theorem 10.6.25 from [6] can be applied and yields the following result.

**Theorem 3** *Take $\psi \in (0,1)$. Then there exists $\nu_0 > 0$ independent of $k$ and $\varepsilon$ such that for the contraction number of the multigrid W-cycle with damped Jacobi or symmetric Gauss-Seidel smoothing we have*

$$\|M_k(\nu, 0)\| \le \psi \quad \textit{for all } \nu \ge \nu_0. \qquad \square$$

For the analysis of the multigrid V-cycle the energy norm is used: $\|x\|_{A_k} = \langle A_k x, x\rangle_k$, $x \in X_k$. Due to Corollary 1, (20) and Theorem 10.7.15 from [6] we have the following convergence result:

**Theorem 4** *For the contraction number of the symmetric multigrid V-cycle with damped Jacobi or symmetric Gauss-Seidel smoothing the estimate*

$$\|M_k\left(\frac{\nu}{2}, \frac{\nu}{2}\right)\|_{A_k} \leq \frac{C_A}{C_A + \nu} \ , \quad \nu = 2, 4, \ldots$$

*holds with $C_A$ as in (21).*

The results in Theorem 3 and Theorem 4 prove the robustness of the multigrid method both with respect to variation in the mesh size parameter $h_k$ and with respect to variation in the parameter $\varepsilon$.

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