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"Solving quadratic matrix equations with low rank matrix coefficients"

Abstract:
In the numerical solution of quasi-birth-and-death (QBD) Markov chains a crucial step is the efficient computation of the minimal nonnegative solution G of the quadratic matrix equation

$$X = A_{-1} + A_0 X + A_1 X^2,$$

where A_i, $i = -1, 0, 1$, are $m \times m$ nonnegative matrices such that $A_{-1} + A_0 + A_1$ is row stochastic. Recently, some attention has been addressed to the case where A_{-1} has only few non-null columns, or A_1 has only few non-null rows. These properties are in practice satisfied when the QBD has restricted transitions to higher (or lower) levels.
Here we assume that both the matrices A_{-1} and A_1 have small rank with respect to their size m. In particular, if A_{-1} and A_1 have only few non-null columns and rows, respectively, they have small rank. We show that, under this assumption, the matrix G can be computed by using the cyclic reduction algorithm, where the matrices $A_i^{(k)}$, $i = -1, 0, 1$, generated at the kth step of the algorithm, can be represented by small rank matrices. In particular, if r_{-1} is the rank of A_{-1}, and if r_1 is the rank of A_1, then each step of cyclic reduction can be performed by means of $O((r_{-1} + r_1)^3)$ arithmetic operations. This cost estimate must be compared with the cost of $O(m^3)$ arithmetic operations, needed without exploiting the structure of A_{-1} and A_1. Therefore, if r_{-1} and r_1 are much smaller than m, the advantage is evident. Moreover, the nonnegativity properties of the matrix coefficients A_i, $i = -1, 0, 1$, are preserved by the algorithm, thus maintaining the good numerical stability properties of cyclic reduction applied to the original matrix equation.
This is joint work with Dario A. Bini and Paola Favati.