Multiscale and Wavelet Methods for Operator Equations

(5) Adaptive wavelet schemes (II) - Applications, outlook

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About some Concepts used in the Proof

Thresholding and best N-term approximation in ℓ_2:

$$\mathcal{T}_{\eta}\mathbf{v} := \begin{cases} v_\lambda, & |v_\lambda| \geq \eta \\ 0, & |v_\lambda| < \eta \end{cases}$$

$$\ell_\tau^w := \left\{\mathbf{v} \in \ell_2 : N(\mathbf{v},\eta) := \#\mathrm{supp}\, \mathcal{T}_\eta \mathbf{v} \leq C_v \eta^{-\tau}\right\}, \quad \text{(for some } \tau < 2\text{)}$$

$\mathbf{v} \in \ell_\tau^w \rightsquigarrow$

$$\begin{aligned} \|\mathbf{v} - \mathcal{T}_\eta \mathbf{v}\|_{\ell_2}^2 &= \sum_{l=0}^{\infty} \sum_{2^{-l-1}\eta < |v_\lambda| \leq 2^{-l}\eta} |v_\lambda|^2 \leq C_f \sum_{l=0}^{\infty} (2^{-l}\eta)^2 (2^{-l-1}\eta)^{-\tau} \\ &= \frac{4C_f}{2^{2-\tau}-1}\eta^{2-\tau} \qquad (1) \end{aligned}$$

What about the constant C_v? - Lorentz spaces [DeV]

Decreasing rearrangement: $\mathbf{v} \;\to\; \{v_n^*\}_{n\in\mathbb{N}}\; v_{n+1}^* \le v_n^*,\; v_n^* = |v_{\lambda_n}| \rightsquigarrow$

$$v_{N(\mathbf{v},\eta)+1}^* N(\mathbf{v},\eta)^{1/\tau} \le \eta N(\mathbf{v},\eta)^{1/\tau} \le C_v^{1/\tau} \quad \rightsquigarrow$$

$$C_v^{1/\tau} = \sup_{n\in\mathbb{N}} n^{1/\tau} v_{1+n}^* =: |\mathbf{v}|_{\ell_\tau^w} \qquad \rightsquigarrow \|\mathbf{v}\|_{\ell_\tau^w} := \|\mathbf{v}\|_{\ell_2} + |\mathbf{v}|_{\ell_\tau^w} \tag{2}$$

$$n^{1/\tau} v_{n+1}^* \le (n(v_n^*)^\tau)^{1/\tau} \le \left(\sum_{j\le n} (v_j^*)^\tau\right)^{1/\tau} \le \|\mathbf{v}\|_{\ell_\tau}$$

$$\rightsquigarrow \ell_\tau \subset \ell_\tau^w \subset \ell_{\tau+\epsilon} \subset \ell_2, \quad \tau < \tau+\epsilon < 2$$

(1), (2) $\quad \rightsquigarrow$ (see [CDD1])

Lemma: Suppose that $\mathbf{v} \in \ell^w_\tau$ for $0 < \tau < 2$ and $\mathbf{w} \in \ell_2$ satisfies

$$\|\mathbf{v} - \mathbf{w}\|_{\ell_2} \le \epsilon \quad \text{for some} \quad \epsilon > 0,$$

Then one has for any $\eta > 0$

$$\|\mathbf{v} - \mathcal{T}_\eta \mathbf{w}\|_{\ell_2} \le 2\epsilon + \bar{C}\|\mathbf{v}\|^{\tau/2}_{\ell^w_\tau} \eta^{1-\tau/2}, \quad \#\mathrm{supp}\, \mathcal{T}_\eta \mathbf{w} \le \frac{4\epsilon^2}{\eta^2} + 4\bar{C}\|\mathbf{v}\|^\tau_{\ell^w_\tau} \eta^{-\tau}$$

Corollary: If $\mathbf{v} \in \ell^w_\tau$ and $\|\mathbf{v} - \mathbf{w}\|_{\ell_2} \le \eta/5$ with $\#\mathrm{supp}\, \mathbf{w} < \infty$. Then $\bar{\mathbf{w}}_\eta := \mathbf{COARSE}\,[\mathbf{w}, 4\eta/5]$ satisfies

$$\#\mathrm{supp}\, \bar{\mathbf{w}}_\eta \lesssim \eta^{-1/s}, \quad \|\mathbf{v} - \bar{\mathbf{w}}_\eta\|_{\ell_2} \le \eta$$

Characterization of Best N-Term Approximation Rates $\sim N^{-s}$

$$\sigma_{N,\ell_2}(\mathbf{v}) := \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} = \min_{\#\mathrm{supp}\,\mathbf{w} \le N} \|\mathbf{v} - \mathbf{w}\|_{\ell_2}$$

Theorem: Let $\frac{1}{\tau} = s + \frac{1}{2}$ then $\mathbf{v} \in \ell_\tau^w \iff \sigma_{N,\ell_2}(\mathbf{v}) \lesssim N^{-s}$ and

$$\|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \lesssim N^{-s} \|\mathbf{v}\|_{\ell_\tau^w}$$

Proof of $\Longrightarrow$:

$$\begin{aligned} \sigma_{N,\ell_2}(\mathbf{v})^2 &= \sum_{n>N} (v_n^*)^2 \le \left(\sum_{n \ge N} n^{-2/\tau} \right) |\mathbf{v}|_{\ell_\tau^w}^2 \le C\, N^{1-\frac{2}{\tau}} |\mathbf{v}|_{\ell_\tau^w}^2 \\ &= C\, N^{-2s} |\mathbf{v}|_{\ell_\tau^w}^2, \end{aligned}$$

Key Complexity Bounds

Suppose that $\mathbf{C} \in \mathcal{C}_{s^*}$ and let $\frac{1}{\tau} = s + \frac{1}{2}, \quad s < s^*$

Charact. of $\sigma_{N,\ell_2}(\mathbf{v}) = \mathcal{O}(N^{-s}) \quad \leadsto \quad \mathbf{w}_\eta = \mathbf{MULT}\,[\eta, \mathbf{C}, \mathbf{v}]$ satisfies

- $\|\mathbf{w}_\eta\|_{\ell_\tau^w} \;\lesssim\; \|\mathbf{v}\|_{\ell_\tau^w}$
- $\#\text{flops} \sim \#\text{supp}\, \mathbf{w}_\eta \;\lesssim\; \|\mathbf{v}\|_{\ell_\tau^w}^{1/s} \eta^{-1/s}$

Optimal balance: accuracy $\quad \eta \quad \leftrightarrow \quad$ cost $\quad \eta^{-1/s}$

$$\Longrightarrow$$

Theorem: Let $\mathbf{L} \in \mathcal{C}_{s^*}$ and suppose that $\mathbf{U} \in \ell^w_\tau$ for $\frac{1}{\tau} = s + \frac{1}{2}, \quad s < s^*$. Then in all the above cases the output $\mathbf{G}_\eta$ of the right hand side scheme $\mathbf{RHS}\,[\eta, \mathbf{G}]$ satisfies

- $\|\mathbf{G}_\eta\|_{\ell^w_\tau} \;\lesssim\; \|\mathbf{G}\|_{\ell^w_\tau}$
- $\#\mathrm{flops} \sim \#\mathrm{supp}\,\mathbf{G}_\eta \;\lesssim\; \|\mathbf{G}\|^{1/s}_{\ell^w_\tau}\eta^{-1/s}$

Moreover, for $\mathbf{APPLY} \in \{\mathbf{MULT}, \mathbf{APPLY}_{ls}, \mathbf{APPLY}_{Uz}\}$ the output $\mathbf{W}_\eta$ of $\mathbf{APPLY}\,[\eta, \mathbf{M}, \mathbf{V}]$ satisfies for $s < s^*$

- $\|\mathbf{W}_\eta\|_{\ell^w_\tau} \;\lesssim\; \|\mathbf{V}\|_{\ell^w_\tau}$
- $\#\mathrm{flops} \sim \#\mathrm{supp}\,\mathbf{W}_\eta \;\lesssim\; \|\mathbf{V}\|^{1/s}_{\ell^w_\tau}\eta^{-1/s}$

$\mathbf{APPLY}\,[\epsilon, \mathbf{M}^{-1}, \mathbf{G}] \;:=\; \mathbf{SOLVE}\,[\epsilon, \mathbf{M}, \mathbf{G}]$ exibits the same work/accuracy balance as its ingredients!

Compressibility Criteria

In all examples the operator $\mathcal{L}$ is either local or of the form

$$(\mathcal{L}u)(x) = \int_\Gamma K(x,y)u(y)\,d\Gamma_y, \qquad \left|\partial_x^\alpha \partial_y^\beta K(x,y)\right| \;\lesssim\; \mathrm{dist}(x,y)^{-(d+2t+|\alpha|+|\beta|)}$$

Theorem:[DPS, PS1] Suppose that $\mathcal{L}$ has order $2t$ and satisfies for some $r > 0$

$$\|\mathcal{L}v\|_{H^{-t+a}} \;\lesssim\; \|v\|_{H^{t+a}}, \qquad v \in H^{t+a}, 0 \le |a| \le r.$$

Assume that $\mathbf{D}^{-s}\Psi$ is a Riesz-basis for H^s for $-\tilde{\gamma} < s < \gamma$ **(NE)** and has cancellation properties **(CP)** of order $\tilde{m}$. Then for any $0 < \sigma \le \min\{r, d/2 + \tilde{m} + t\}, \quad t + \sigma < \gamma, \quad t - \sigma > -\tilde{\gamma}$ one has

$$2^{-(|\lambda'|+|\lambda|)t}|\langle \psi_\lambda, \mathcal{L}\psi_{\lambda'}\rangle| \;\lesssim\; \frac{2^{-||\lambda|-|\lambda'||\sigma}}{(1 + 2^{\min(|\lambda|,|\lambda'|)}\,\mathrm{dist}(\Omega_\lambda, \Omega_{\lambda'}))^{d+2\tilde{m}+2t}}. \qquad (3)$$

Sketch of argument: Apply **(CP)** of order $\tilde{m}$ – apply to kernel

$$\int_\Gamma \int_\Gamma K(x,y)\psi_\lambda \psi_{\lambda'} dx dy = \langle K, \psi_\lambda \otimes \psi_{\lambda'} \rangle \quad \rightsquigarrow$$

Let $\Omega_\lambda := \operatorname{supp} \psi_\lambda$. If $\operatorname{dist}(\Omega_\lambda, \Omega_{\lambda'}) \gtrsim 2^{-\min(|\lambda|,|\lambda'|)}$ ([DPS, PS1, PS2]) $\rightsquigarrow$

$$|\langle \mathcal{L}\psi_{\lambda'}, \psi_\lambda \rangle| \lesssim \frac{2^{-(|\lambda|+|\lambda'|)(d/2+\tilde{m})}}{(\text{dist}(\Omega_\lambda, \Omega_{\lambda'}))^{d+2\tilde{m}+2t}}$$

If $\operatorname{dist}(\Omega_\lambda, \Omega_{\lambda'}) \lesssim 2^{-\min(|\lambda|,|\lambda'|)}$ use **continuity of $\mathcal{L}$ and (NE)** [DDHS]

$$\|\mathcal{L}v\|_{H^{-t+s}} \lesssim \|v\|_{H^{t+s}}, \quad v \in H^{t+s}, 0 \le |s| \le \tau$$

w.l.o.g. $|\lambda| > |\lambda'|$ (Schwarz inequality) $\rightsquigarrow$

$$|\langle \mathcal{L}\psi_{\lambda'}, \psi_\lambda\rangle| \le \|\mathcal{L}\psi_{\lambda'}\|_{H^{-t+\sigma}} \|\psi_\lambda\|_{H^{t-\sigma}} \lesssim \|\psi_{\lambda'}\|_{H^{t+\sigma}} \|\psi_\lambda\|_{H^{t-\sigma}}$$

Assumptions on σ $\rightsquigarrow$ $|\langle \mathcal{L}\psi_{\lambda'}, \psi_\lambda\rangle| \le 2^{t(|\lambda|+|\lambda'|)} 2^{\sigma(|\lambda'|-|\lambda|)}$ $\square$

Proposition: ([CDD1]) Suppose that

$$|\mathbf{C}_{\lambda,\nu}| \lesssim \frac{2^{-\sigma||\lambda|-|\nu||}}{(1+d(\lambda,\nu))^\beta}, \quad d(\lambda,\nu) := 2^{\min\{|\lambda|,|\nu|\}} \mathrm{dist}\,(\Omega_\lambda, \Omega_\nu)$$

and define

$$s^* := \min\left\{\frac{\sigma}{d} - \frac{1}{2}, \frac{\beta}{d} - 1\right\}.$$

Then $\mathbf{C} \in \mathcal{C}_{s^*}$

The proof is based on the

Schur Lemma: If for some $C < \infty$ and any positive sequence $\{\omega_\lambda\}_{\lambda\in\mathcal{J}}$

$$\sum_{\nu\in\mathcal{J}} |\mathbf{C}_{\lambda,\nu}|\omega_\nu \le C\,\omega_\lambda, \quad \sum_{\lambda\in\mathcal{J}} |\mathbf{C}_{\lambda,\nu}|\omega_\lambda \le C\,\omega_\nu, \quad \nu \in \mathcal{J}$$

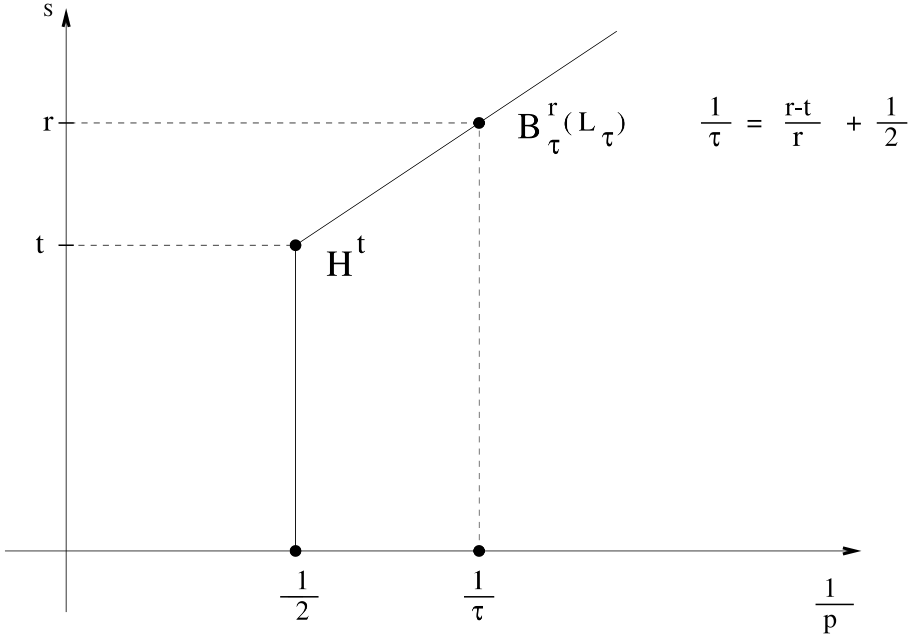
then $\|\mathbf{C}\|_{\ell_2\to\ell_2} \le C$.

Apply this to $\|\mathbf{C} - \mathbf{C}_j\|_{\ell_2}$ with $\omega_\lambda = 2^{-d|\lambda|/2}$

What about $v = \mathbf{v}^T \mathbf{D}^{-1}\Psi$ when $\mathbf{v} \in \ell^w_\tau$? [DDD, DeV]

Recall: When $\mathcal{H} = H^t$, $\mathbf{D} = \mathbf{D}^t$, $\mathbf{D}^{-t}\Psi$ Riesz basis for H^t $\rightsquigarrow$

$$\mathbf{u} \in \ell_\tau \quad \Longleftrightarrow \quad u = \sum_\lambda u_\lambda 2^{-t|\lambda|}\psi_\lambda \in B^{t+sd}_\tau(L_\tau(\Omega)) \quad \left(\frac{1}{\tau} = s + \frac{1}{2}\right)$$



Besov Regularity of Solutions - Stokes System [Dahl, DDU, DD]

$$u \in B_\tau^{1+sd}(L_\tau(\Omega)), \qquad q \in B_\tau^{sd}(L_\tau(\Omega)), \qquad \frac{1}{\tau} = s + \frac{1}{2}$$

$$\Longrightarrow$$

$$\sigma_{N,H_0^1(\Omega)}(u) \;\lesssim\; N^{-s}, \quad \sigma_{N,L_2(\Omega)}(q) \;\lesssim\; N^{-s}.$$

Theorem: *For $d = 2$ the strongest singularity solutions (u_S, q_S) belong to the above scale of Besov spaces for any $s > 0$.*
The Sobolev regularity is limited by $1.544483...$, resp. $0.544483...$ $\rightsquigarrow$

Arbitrarily high asymptotic rates by adaptive schemes

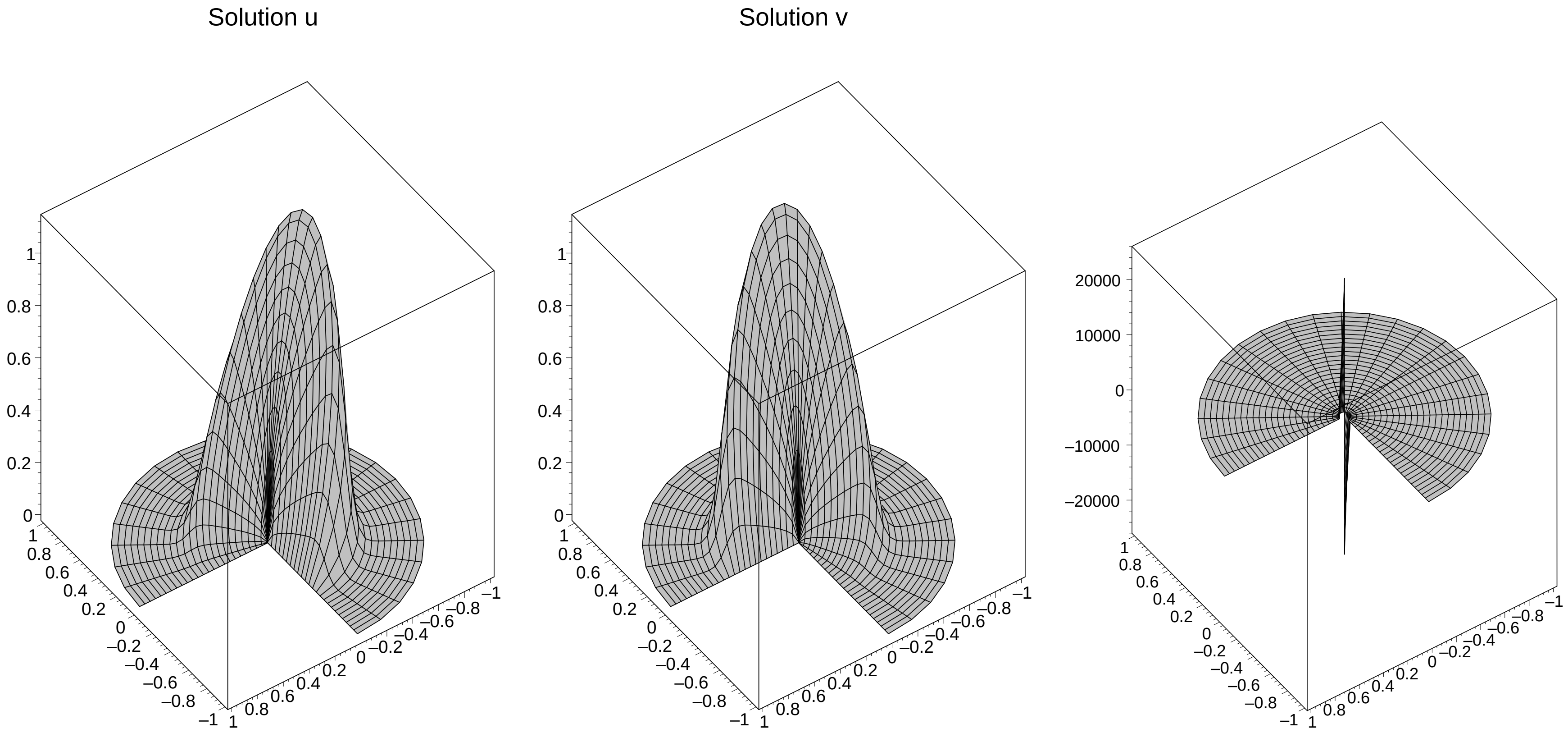


Figure 1: Exact solution for the first example. Velocity components (left and middle) and pressure (right). The pressure functions exhibits a strong singularity

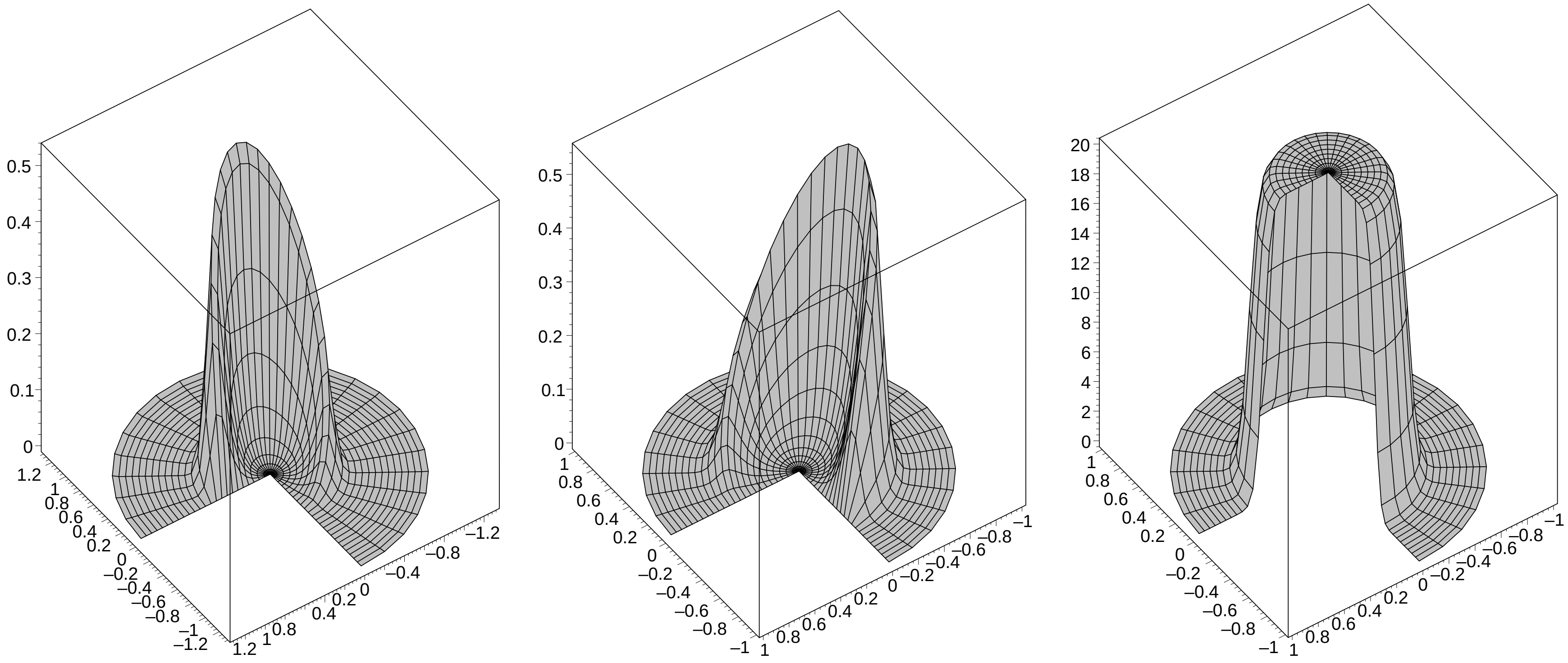


Figure 2: Exact solution for the second example. Velocity components (left and middle) and pressure (right).

$$\rho_{\boldsymbol{x}} := \frac{\|\boldsymbol{x} - \boldsymbol{x}_\Lambda\|_{\ell_2}}{\|\boldsymbol{x} - \boldsymbol{x}_{\#\Lambda}\|_{\ell_2}}, \qquad r_{\boldsymbol{x}} := \frac{\|\boldsymbol{x} - \boldsymbol{x}_\Lambda\|_{\ell_2}}{\|\boldsymbol{x}\|_{\ell_2}},$$

It	$\#\Lambda_u$	$\rho_{\boldsymbol{u}}$	$r_{\boldsymbol{u}}$	$\#\Lambda_v$	$\rho_{\boldsymbol{v}}$	$r_{\boldsymbol{v}}$	$\#\Lambda_p$	$\rho_{\boldsymbol{p}}$	$r_{\boldsymbol{p}}$
1	33	1.04	0.6838	34	1.04	0.6744	768	130.35	1.0024
2	84	1.26	0.3427	83	1.24	0.3447	768	130.40	1.0028
3	193	1.32	0.1530	184	1.31	0.1541	768	15.37	0.5234
4	446	1.29	0.0821	450	1.29	0.0897	929	4.15	0.2218
5	1070	1.27	0.0434	1065	1.27	0.0456	1211	2.58	0.1034

Table 1: Results for the first example.

It	$\#\Lambda_u$	$\rho_{\boldsymbol{u}}$	$r_{\boldsymbol{u}}$	$\#\Lambda_v$	$\rho_{\boldsymbol{v}}$	$r_{\boldsymbol{v}}$	$\#\Lambda_p$	$\rho_{\boldsymbol{p}}$	$r_{\boldsymbol{p}}$
1	278	28.20	1.2936	364	60.31	2.1867	768	6.96	0.3329
2	261	8.30	0.4028	295	16.10	0.7003	768	3.76	0.1800
3	234	3.72	0.1995	274	5.63	0.2617	768	1.80	0.0863
4	180	1.25	0.0886	249	2.08	0.1056	810	1.22	0.0452
5	233	1.14	0.0615	267	1.29	0.0615	980	1.07	0.0231
6	298	1.11	0.0480	321	1.17	0.0470	1276	1.05	0.0117
7	456	1.35	0.0398	505	1.43	0.0265	1551	1.09	0.0061
8	704	1.36	0.0250	724	1.39	0.0177	1842	1.24	0.0035

Table 2: Results for the second example.

It	$\#\Lambda_u$	$\rho_{\boldsymbol{u}}$	$r_{\boldsymbol{u}}$	$\#\Lambda_v$	$\rho_{\boldsymbol{v}}$	$r_{\boldsymbol{v}}$	$\#\Lambda_p$	$\rho_{\boldsymbol{p}}$	$r_{\boldsymbol{p}}$
3	5	1.00	0.7586	5	1.00	0.7588	243	2.23810	0.1196
4	20	1.13	0.4064	24	1.45	0.3979	262	2.08107	0.0612
5	61	1.47	0.2107	77	1.79	0.2107	324	2.72102	0.0339
6	178	1.33	0.1060	198	1.52	0.1306	396	2.81079	0.0209
7	294	1.19	0.0533	286	1.46	0.0744	674	2.21371	0.0108
8	478	1.25	0.0271	531	1.46	0.0362	899	1.83271	0.0071

Table 3: Results for the second example with piecewise linear trial functions for velocity and pressure - LBB condition is violated

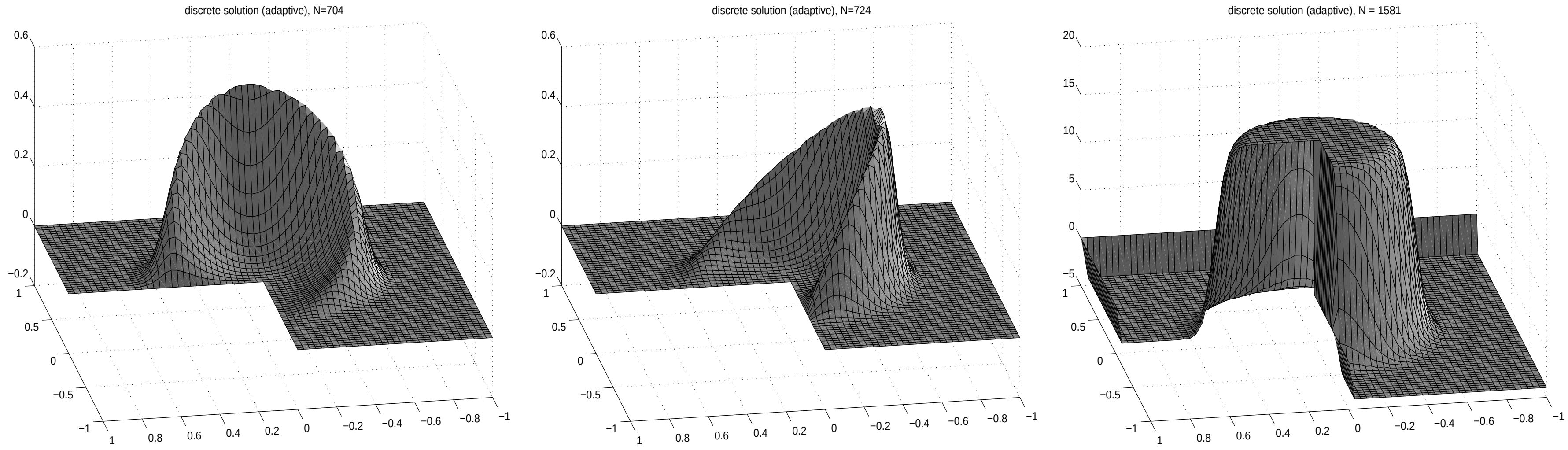


Figure 3: Approximations for the second example. First and second velocity component (left and middle column) and pressure (right column).

Further Issues, Applications

- **Time dependent problems:**

$$\partial_t u = \mathcal{L}u \quad \rightsquigarrow \quad u(t) = e^{t\mathcal{L}}u(0)$$

$$e^{t\mathcal{L}}u_0 \quad = \quad \frac{1}{2\pi i}\int_\Gamma e^{tz}(zI - \mathcal{L})^{-1}u_0 dz \approx \sum_n \omega_n e^{tz_n}(z_n\mathbf{I} - \mathcal{L})^{-1}u_0$$

Solve the problems $(z_n\mathbf{I} - \mathcal{L})u = u_0$ in parallel by the adaptive scheme.

- **Time dependent incompressible Navier Stokes equations**

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} &= \mathbf{f}, \quad x\in\Omega,\ t>0, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} \quad \rightsquigarrow$$

$$\partial_t\mathbf{u} + (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \nabla \mathrm{P} - \nu\Delta\mathbf{u} = \mathbf{f} \quad (\mathrm{P} := \mathrm{p} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$$

$$\rightsquigarrow$$

$$(\mathbf{I} - \tau\nu\Delta)\mathbf{u} + \tau(\mathbf{w}^n\times\mathbf{u}) = \tau\mathbf{f} + \mathbf{u}^n \quad \rightsquigarrow$$

Find $\mathbf{u}^{n+1}\in V$ such that

$$a_{n,\tau}(\mathbf{u}^{n+1},\mathbf{v}) = \langle\tau\mathbf{f} + \mathbf{u}^n, \mathbf{v}\rangle, \quad \mathbf{v}\in V,$$

where

$$a_{n,\tau}(\mathbf{u},\mathbf{v}) := \langle\mathbf{u},\mathbf{v}\rangle + \tau\nu\langle\nabla\mathbf{u},\nabla\mathbf{v}\rangle + \tau\langle\mathbf{w}^n\times\mathbf{u},\mathbf{v}\rangle.$$

Note that since $\langle \mathbf{w}^n \times \mathbf{u}, \mathbf{v}\rangle = -\langle \mathbf{w}^n \times \mathbf{v}, \mathbf{u}\rangle$ one has

$$a_{n,\tau}(\mathbf{v}, \mathbf{v}) \geq |||\mathbf{v}|||_\tau,$$

where

$$|||\mathbf{v}|||_\tau^2 := \|\mathbf{v}\|_{L_2}^2 + \tau\nu|\mathbf{v}|_{H^1}^2.$$

On the other hand, one has

$$a_{n,\tau}(\mathbf{u}, \mathbf{v}) \leq \|\mathbf{u}\|_{L_2}\|\mathbf{v}\|_{L_2} + \nu\tau|\mathbf{u}|_{H^1}|\mathbf{v}|_{H^1} + \tau|\langle \mathbf{w}^n \times \mathbf{u}, \mathbf{v}\rangle|.$$

If $\|\mathbf{w}^n\|_{L_\infty} \leq C \Longrightarrow \langle \mathbf{w}^n \times \mathbf{u}, \mathbf{v}\rangle \leq C\|\mathbf{u}\|_{L_2}\|\mathbf{v}\|_{L_2} \quad \Longrightarrow \quad a_{n,\tau}(\mathbf{v}, \mathbf{v}) \sim |||\mathbf{v}|||_\tau^2$ with constants independent of τ, ν! i.e., the mapping property (MP) holds uniformly in τ, ν. $\rightsquigarrow$ with $\theta_n := \min\{1, \|\mathbf{w}^n\|_{L_\infty}^{-1}\}$

$$\langle \mathbf{u}, \mathbf{v}\rangle + \tau\nu\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle + \tau\theta_n\langle \mathbf{w}^n \times \mathbf{u}, \mathbf{v}\rangle = \langle \tau\mathbf{f} + \mathbf{u}^n, \mathbf{v}\rangle - (1-\theta_n)\tau\langle \mathbf{w}^n \times \mathbf{u}^n, \mathbf{v}\rangle, \quad v \in V$$

Nonlinear Problems [CDD3]

$$a(v,u)=\ell(v), \quad \forall \;\; v \in \mathcal{H},$$

(i) $|a(v,w)| \;\lesssim\; c(\|w\|_{\mathcal{H}})\|v\|_{\mathcal{H}}$

(ii) For each fixed w the functional $a(\cdot,w)$ is linear

(iii) $a(v,v) \geq \alpha\|v\|_{\mathcal{H}}^2$

Example:

$$a(v,w) := \langle \nabla v, \nabla w\rangle_\Omega + \langle v, u^3\rangle_\Omega, \quad \ell(v) = \langle v, f\rangle$$

$$\langle v, \mathcal{A}(u)\rangle = a(v,u), \quad v \in \mathcal{H}, \quad \rightsquigarrow \quad F(u) := \mathcal{A}(u) - f = 0$$

In wavelet coordinates

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \quad \mathbf{F}(\mathbf{u}) = \langle \Psi, F(u)\rangle$$

Solve the system e.g. by relaxation, gradient iterations or by Newton's method

$$\mathbf{u} = \mathbf{G}(u) := \mathbf{u} - \omega \mathbf{F}(\mathbf{u}) \quad \rightsquigarrow \quad \mathbf{u}^{n+1} = \mathbf{G}(\mathbf{u}^n)$$

$$\mathbf{F}'(\mathbf{u}^n)\mathbf{s} = -\mathbf{F}(\mathbf{u}^n), \quad \mathbf{u}^{n+1} = \mathbf{u}^n + \mathbf{s}$$

Use the same principles as in the linear case - perform (perturbed) iterations on the ∞-dimensional problem.

Issues:

- evaluation of nonlinear terms

- How does the nonlinearit affect the regularity and thus the compressibility of the iterates ?

Example - One can show: $u \in H^1 \implies u^3 \in H^{-1}$ for $d \leq 4$ [CDD3].

Evaluation of Nonlinear Terms [DSX, BDSX]

Objective: Given f, $u_\Lambda = \mathbf{u}_\Lambda^T \Psi_\Lambda$ compute the significant entries of $\mathbf{f} := \langle \Psi, f(u_\Lambda) \rangle$!

Idea: Find a function $g = \mathbf{g}^T \tilde{\Psi}$ such that

$$\|f(u) - g\|_{L_2} \leq \epsilon \quad \stackrel{\text{(NE)}}{\Longrightarrow} \quad \|\mathbf{f} - \mathbf{g}\|_{\ell_2} \leq c\epsilon$$

- Given u_Λ, Λ **predict** $\hat{\Lambda}$ covering the significant coefficients of $f(u)$
- Compute an approximation $g = \mathbf{g}_{\hat{\Lambda}}^T \tilde{\Psi}_{\hat{\Lambda}}$ to $f(u_\Lambda)$

$$F \;\leftrightarrow\; f(u), \quad J := \max\{|\lambda| : \lambda \in \hat{\Lambda}\}, \quad F \approx \sum_{j=-1}^{J-1} \sum_{\lambda \in \hat{\Lambda}_j} \langle F, \psi_\lambda \rangle \tilde{\psi}_\lambda$$

Sketch of the recovery scheme [BDSX]

Idea: mimic decomposition

$$F \approx \sum_{j=0}^{J} (\tilde{Q}_j - \tilde{Q}_{j-1})F$$

from top to bottom:

- Given $\hat{\Lambda}_{J-1}$, determine a possibly small safety index set $\mathcal{I}_J^0$ s.t. for a **local** multiscale decomposition applied to the array $\mathbf{c}_J := (\langle F, \phi_{J,k} \rangle : k \in \mathcal{I}_J^0)$ (completed by zeros)

$$\mathbf{c}_J \rightarrow (\mathbf{c}_{J-1}, \boldsymbol{d}_{J-1}) \quad \rightsquigarrow \quad d_{J-1,\lambda} = \langle F, \psi_\lambda \rangle, \ \lambda \in \hat{\Lambda}_{J-1}$$

  and likewise $c_{J-1,k} = \langle F, \phi_{J-1,k} \rangle, \ k \in \mathcal{I}_{J-1}(\hat{\Lambda}_{J-1}) := \{k \in \mathcal{I}_{J-1} : \lambda(k) \in \hat{\Lambda}_{J-1}$

- Compute approximations $q_{J,k}$ to $c_{J,k}$, $k \in \mathcal{I}_J^0$ by quadrature

- Decompose $\mathbf{q}_J \to (\mathbf{q}_{J-1}, \tilde{d}_{J-1})$, threshold $\tilde{d}_{J-1}$.

- Form the safety index set $\mathcal{I}_{J-1}^0$ from $\hat{\Lambda}_{J-2}$, compute (the missing) scaling coefficients $q_{J-1,k}$ by quadrature for $k \in \mathcal{I}_{J-1}^0 \setminus \mathcal{I}_{J-1}(\hat{\Lambda}_{J-1}) \leadsto \mathbf{q}_{J-1}$ with support $\hat{\mathcal{I}}_{J-1} := \mathcal{I}_{J-1}^0 \cup \mathcal{I}_{J-1}(\hat{\Lambda}_{J-1})$ continue with $\mathbf{q}_{J-1}$ as before with $\mathbf{q}_J$ $\leadsto$ approximate wavelet coefficients $\tilde{d}$

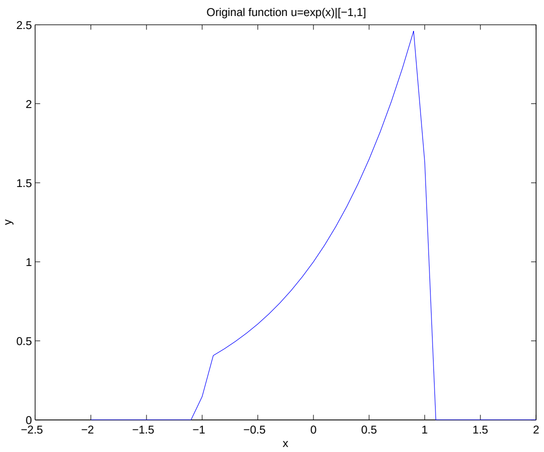
Comments:

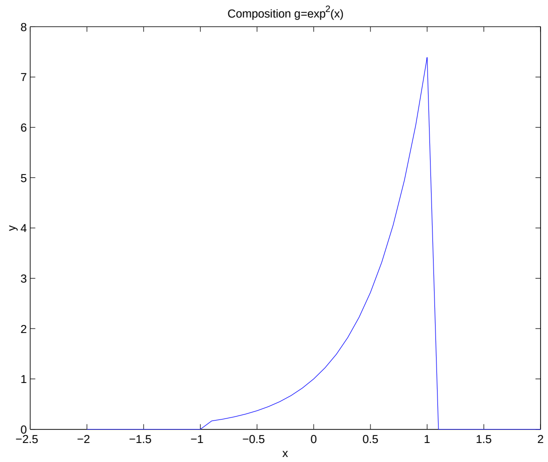
- Index sets are automatically completed to **trees** when descending to lower levels

- Quadrature on a locally highest level of resolution is sufficiently accurate when $\hat{\Lambda}$ reflects balanced local errors $\leadsto$ optimal work/accuracy rate.

Spline wavelets of order $(2,2)$,

$$u = \sum_{k \in I_{J_u}} \langle e^{(\cdot)}|_{[-1,1]}, \tilde{\phi}_{J_u,k} \rangle \phi_{J_u,k} \qquad g(x) = f \circ u, \quad g = (u) = u^2.$$





s estimated order of best n-term approximation. Refinement depth in recovery scheme: 1.

J_u = 15 ($\#\Lambda^{15}(u) = 131105$), J_g = 16 ($\#\Lambda^{16}(g) = 262179$)

ε	$\#\Lambda_\varepsilon^{15}$	$\|u_{\Lambda^{15}} - u_{\Lambda_\varepsilon^{15}}\|$(1)	s	$\#\Lambda_\varepsilon^R$	$\|g_{\Lambda^{16}} - g_{\Lambda_\varepsilon^R}\|$(2)	(2)/(1)	s
0.002	105	1.715E-4	1.9	134	4.293E-4	2.5	1.6
0.001	143	5.6125E-5	2.0	188	1.3343E-4	2.4	1.7
0.0002	318	6.7213E-6	2.1	367	2.415E-5	3.5	1.8
0.0001	688	1.1722E-6	2.1	709	5.1609E-6	4.4	1.8

$J_u = 13$ ($\#\Lambda^{13}(u) = 32797$), $J_g = 14$ ($\#\Lambda^{16}(g) = 65567$)

ε	$\#\Lambda^{13}_{\varepsilon}$	$\|u_{\Lambda^{13}} - u_{\Lambda^{13}_{\varepsilon}}\|$ (1)	s	$\#\Lambda^{R}_{\varepsilon}$	$\|g_{\Lambda^{14}} - g_{\Lambda^{R}_{\varepsilon}}\|$ (2)	(2)/(1)	s
0.002	97	1.716E-4	1.9	126	3.993E-4	2.3	1.9
0.001	135	5.6126E-5	2.1	180	1.3343E-4	2.4	1.7
0.0002	310	6.7213E-6	2.1	359	2.4150E-5	3.5	1.8
0.0001	405	3.5045E-6	2.1	573	8.2315E-6	2.3	1.8

$J_u = 10$ $(\#\Lambda^{10}(u) = 4119$), $J_g = 11$ $(\#\Lambda^{11}(g) = 8217$)

ε	$\#\Lambda_\varepsilon^{10}$	$\|u_{\Lambda^{10}} - u_{\Lambda_\varepsilon^{10}}\|$ (1)	s	$\#\Lambda_\varepsilon^{R}$	$\|g_{\Lambda^{11}} - g_{\Lambda_\varepsilon^{R}}\|$ (2)	(2)/(1)	s
0.002	85	1.716E-4	2.0	114	3.911E-4	2.28	1.7
0.001	100	1.0621E-4	2.0	135	2.48345E-4	2.33	1.7
0.0002	235	1.0803E-05	2.1	333	2.51E-05	2.3	1.8
0.0001	393	3.5026E-6	2.1	554	8.5407E-6	2.4	1.8
0.00001	1103	4.03924E-7	2.1	1074	3.17919E-6	7.8	1.8

Application to Matrix/Vector Multiplication [BDD]

In general: approximate calculation of individual entries of $\boldsymbol{A}$ is expensive - typical ingredient $2^{-|\lambda|-|\nu|}\langle a\partial_i\psi_\lambda, \partial_l\psi_\nu\rangle$ (H^1-normalized wavelets)

Observe: $\theta_{i,\nu} := 2^{-|\lambda|}\partial_i\psi_\nu$ is again a wavelet (or a difference of wavelets). Recall:

$$\mathbf{w}_j := \boldsymbol{A}_j\mathbf{v}_{[0]} + \boldsymbol{A}_{j-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \boldsymbol{A}_0(\mathbf{v}_{[j]} - \mathbf{v}_{[j-1]})$$

$$\boldsymbol{A}_{j-k}(\mathbf{v}_{[k]} - \mathbf{v}_{[k-1]}) = (\langle\theta_{i,\nu}, a\sum_{\lambda\in\sigma_k} v_\lambda\theta_{l,\lambda}\rangle : \nu\in\Gamma_{j,k})^T$$

$\sigma_k := \mathrm{supp}\,(\mathbf{v}_{[k]} - \mathbf{v}_{[k-1]})$, $\quad \Gamma_{j,k} := \bigcup\{$rm supports of columns of $\boldsymbol{A}_{j-k}$ selected by $\sigma_k\}$

Apply the recovery scheme to the function $F := a\sum_{\lambda\in\sigma_k} v_\lambda\theta_{l,\lambda}$ with prediction set $\Gamma_{j,k}$ $\quad\rightsquigarrow$ optimal work/accuracy rates.

References

[BDD] A. Barinka, S. Dahlke, W. Dahmen, Adaptive Application of Operators in Standard Representation, in preparation.

[BDSX] A. Barinka, W. Dahmen, R. Schneider, Y. Xu, An algorithm for the evaluation of nonlinear functionals of wavelet expansions, in preparation

[CDD1] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet methods for elliptic operator equations – Convergence rates, Math. Comp. 70 (2001), 27–75.

[CDD2] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet methods II - Beyond the elliptic case, IGPM Report, RWTH Aachen, Nov. 2000.

[CDD3] A. Cohen, W. Dahmen, R. DeVore, Adaptive Wavelet Schemes for Nonlinear Variational Problems, in preparation.

[Dahl] S. Dahlke: Besov regularity for elliptic boundary value problems on polygonal domains, Appl. Math. Lett., 12 (1999), 31–36.

[DD] S. Dahlke, R. DeVore, Besov regularity for elliptic boundary value problems, Comm. Partial Differential Equations, 22 (1997), 1–16.

[DDD] S. Dahlke, W. Dahmen, R. DeVore, Nonlinear approximation and adaptive techniques for solving elliptic operator equations, in: *Multiscale Wavelet Methods for PDEs*, W. Dahmen, A. Kurdila, P. Oswald (eds.), Academic Press, London, 237–283, 1997.

[DDHS] S. Dahlke, W. Dahmen, R. Hochmuth, R. Schneider, *Stable multiscale bases and local error estimation for elliptic problems*, Appl. Numer. Maths. 8, 1997, 21–47.

[DDU] S. Dahlke, W. Dahmen, K. Urban, Adaptive wavelet methods for saddle point problems – Convergence rates, IGPM Report # 204, RWTH Aachen, July 2001.

[DPS] W. Dahmen, S. Prößdorf, R. Schneider, Multiscale methods for pseudo-differential equations on smooth manifolds, in: *Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications*, C.K. Chui, L. Montefusco, L. Puccio (eds.), Academic Press, 1994, 385-424.

[DSX] W. Dahmen, R. Schneider, Y. Xu, Nonlinear functions of wavelet expansions – Adaptive reconstruction and fast evaluation, Numerische Mathematik, 86(2000), 49–101.

[DeV] R. DeVore, Nonlinear approximation, Acta Numerica, **7**, Cambridge University Press, 1998, 51-150.

[PS1] T. von Petersdorff, C. Schwab, Wavelet approximation for first kind integral equations on polygons, Numer. Math., to appear.

[PS2] T. von Petersdorff, C. Schwab, Fully discrete multiscale Galerkin BEM, in: *Multiscale Wavelet Methods for PDEs*, W. Dahmen, A. Kurdila, P. Oswald (eds.), Academic Press.