Multiscale and Wavelet Methods for Operator Equations

(1) Examples, motivation, key features of wavelet bases

Wolfgang Dahmen

Institut für Geometrie und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
e–mail: dahmen@igpm.rwth-aachen.de
WWW: http://www.igpm.rwth-aachen.de/∼dahmen/

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- Examples, motivation, key features of wavelet bases
- Multiresolution, construction and analysis principles
- Scope of problems - Transformation to sequence spaces
- Adaptive wavelet schemes I
- Adaptive wavelet schemes II - Applications, outlook

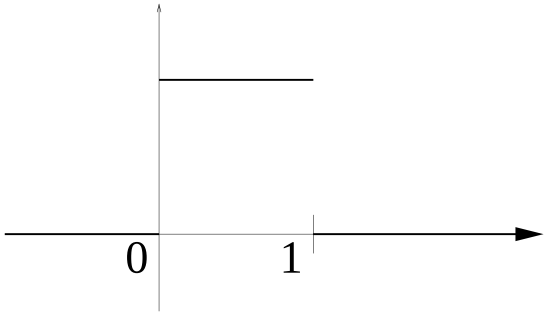
Sparse Representations of Functions, an Example

Single scale Approximations

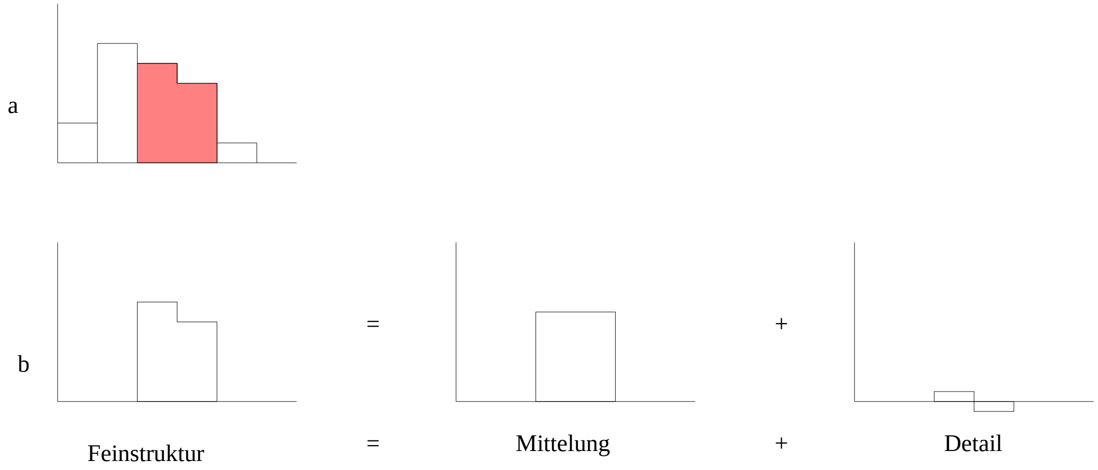
Box function $\phi(x) = \chi_{[0,1)}(x)$

$\phi_{j,k} = 2^{j/2}\phi\left(2^j \cdot -k\right)$

$P_j(f) := \sum_{k=0}^{2^j-1}\langle f, \phi_{j,k}\rangle\phi_{j,k}$

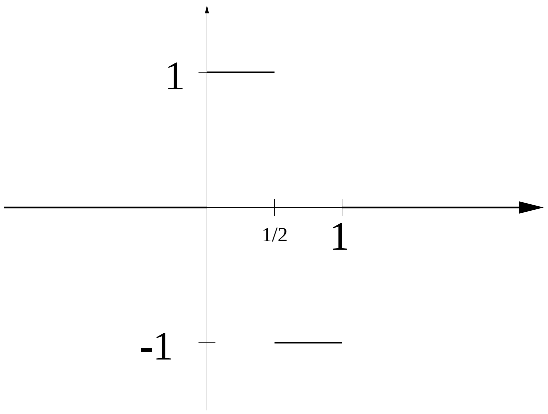


Splitting averages and details:



Multiscale Representations: functions – digits

- $\psi(x) := \phi(2x) - \phi(2x-1)$
- $\psi_{j,k} := 2^{j/2}\psi(2^j \cdot - k)$
- $(P_{j+1} - P_j)f = \sum_{k=0}^{2^j-1} d_{j,k}(f)\psi_{j,k}$
- $d_{j,k}(f) = \langle f, \psi_{j,k}\rangle$, $d_{-1,0}(f) = \langle f, \phi_{0,0}\rangle$



$$f = P_0 f + \sum_{j=1}^{\infty}(P_j - P_{j-1})f = + \sum_{j=-1}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}(f)\psi_{j,k} =: \boldsymbol{d}(f)^T\Psi$$

Norm Equivalence: $f \leftrightarrow \boldsymbol{d}(f)$

$$\|f\|_{L_2} = \left(\sum_{j=0}^{\infty} \|(P_j - P_{j-1})f\|_{L_2}^2 \right)^{1/2} = \|\boldsymbol{d}(f)\|_{\ell_2}$$

Small changes in $\boldsymbol{d} \Longleftrightarrow$ small changes in f

Vanishing Moments:

$$|d_{j,k}| = \inf_{c \in \mathbb{R}} |\langle f - c, \psi_{j,k} \rangle| \leq \inf_{c \in \mathbb{R}} \|f - c\|_{L_2(I_{j,k})} \leq 2^{-j} \|f'\|_{L_2(I_{j,k})},$$

$d_{j,k}$ is mall when $f|_{I_{j,k}}$ is smooth

Two-Scale Relations:

$$\phi_{j,k} = \frac{1}{\sqrt{2}}(\phi_{j+1,2k} + \phi_{j+1,2k+1}), \qquad \psi_{j,k} := \frac{1}{\sqrt{2}}(\phi_{j+1,2k} - \phi_{j+1,2k+1})$$

$$\phi_{j+1,2k} = \frac{1}{\sqrt{2}}(\phi_{j,k} + \psi_{j,k}), \qquad \phi_{j+1,2k+1} = \frac{1}{\sqrt{2}}(\phi_{j,k} - \psi_{j,k})$$

Change of Bases: $\Phi_{j+1}^T \mathbf{c}_{j+1} = \Phi_j^T \mathbf{c}_j + \Psi^T \boldsymbol{d}_j$

$$c_{j,k} = \frac{1}{\sqrt{2}}(c_{j+1,2k} + c_{j+1,2k+1}), \qquad d_{j,k} = \frac{1}{\sqrt{2}}(c_{j+1,2k} - c_{j+1,2k+1})$$

$$c_{j+1,2k} = \frac{1}{\sqrt{2}}(c_{j,k} + d_{j,k}), \qquad c_{j+1,2k+1} = \frac{1}{\sqrt{2}}(c_{j,k} - d_{j,k})$$

Fast (Orthogonal) Transform (linear time):

$\mathbf{T}_J : \boldsymbol{d}^J := (\mathbf{c}_0, \boldsymbol{d}_0, \ldots, \boldsymbol{d}_{J-1}) \to \mathbf{c}_J$

$$\begin{array}{ccccccccccc}
\mathbf{c}_0 & \to & \mathbf{c}_1 & \to & \mathbf{c}_2 & \to & \cdots & \to & \mathbf{c}_{J-1} & \to & \mathbf{c}_J \\
 & \nearrow & & \nearrow & & & & \nearrow & & \nearrow & \\
\boldsymbol{d}_0 & & \boldsymbol{d}_1 & & \boldsymbol{d}_2 & & \cdots & & \boldsymbol{d}_{J-1} & &
\end{array}$$

$\mathbf{T}^{-1} = \mathbf{T}^T : \mathbf{c}_J \to \boldsymbol{d}^J$

$$\begin{array}{ccccccccccc}
\mathbf{c}_J & \to & \mathbf{c}_{J-1} & \to & \mathbf{c}_{J-2} & \to & \cdots & \to & \mathbf{c}_1 & \to & \mathbf{c}_0 \\
 & \searrow & & \searrow & & & & \searrow & & \searrow & \\
 & & \boldsymbol{d}_{J-1} & & \boldsymbol{d}_{J-2} & & \cdots & & \boldsymbol{d}_1 & & \boldsymbol{d}_0
\end{array}$$

(Quasi-) Sparse Representation of Operators

The Hilbert Transform

$$(\mathcal{L}f)(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

Wavelet (standard) representation:

$$\mathcal{L}f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \mathcal{L}\psi_{j,k} = \sum_{j,k} \left(\sum_{l,m} \langle \mathcal{L}\psi_{j,k}, \psi_{l,m} \rangle \psi_{l,m} \right) \langle f, \psi_{j,k} \rangle = \boldsymbol{\Psi^T \mathbf{L} d}$$

where

$$\mathbf{L} := (\langle \psi_{j,k}, \mathcal{L}\psi_{l,m} \rangle)_{(j,k),(l,m)} =: \langle \Psi, \mathcal{L}\Psi \rangle, \quad \boldsymbol{d} := (\langle \psi_{j,k}, f \rangle)_{(j,k)} := \langle \Psi, f \rangle$$

$$\pi\left|\mathbf{L}_{(j,k),(l,m)}\right| = \left|\int\limits_{2^{-j}k}^{2^{-j}(k+1)}\left\{\int\limits_{2^{-l}m}^{2^{-l}(m+1)}\left(\frac{1}{x-y}-\frac{1}{x-2^{-l}m}\right)\psi_{l,m}(y)\,dy\right\}\psi_{j,k}(x)\,dx\right|$$

$$\text{Taylor} \rightsquigarrow \quad = \left|\int\limits_{2^{-l}m}^{2^{-l}(m+1)}\left\{\int\limits_{2^{-j}k}^{2^{-j}(k+1)}\frac{(y-2^{-l}m)}{(x-y_{l,m})^2}\psi_{j,k}(x)\,dx\right\}\psi_{l,m}(y)\,dy\right|$$

$$= \left|\int\limits_{2^{-l}m}^{2^{-l}(m+1)}\left\{\int\limits_{2^{-j}k}^{2^{-j}(k+1)}\left(\frac{(y-2^{-l}m)}{(x-y_{l,m})^2}-\frac{(y-2^{-l}m)}{(2^{-j}k-y_{l,m})^2}\right)\psi_{j,k}(x)\,dx\right\}\psi_{l,m}(y)\,dy\right|$$

$$\int\limits_{I\!R}|\psi_{j,k}(x)|\,dx \le 2^{-j/2} \quad (l \ge j) \quad \rightsquigarrow \quad \pi|\mathbf{L}_{(j,k),(m,l)}| \lesssim 2^{-(l+j)\frac{3}{2}}|2^{-j}k-2^{-l}m|^{-3} = \frac{2^{-\frac{3}{2}|j-l|}}{|k-2^{j-l}m|^3}$$

Preconditioning

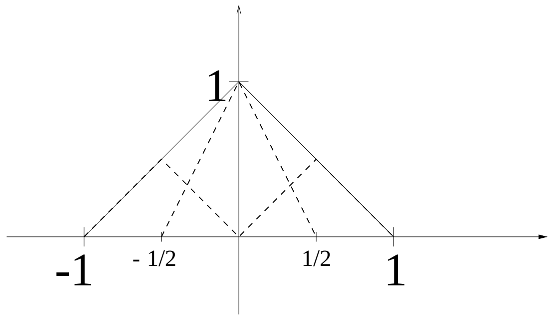
A simple example: $\quad -u'' = f$ on $[0,1], \quad u(0) = u(1) = 0$

Weak formulation: $\quad \langle u', v' \rangle = \langle f, v \rangle, \quad v \in H_0^1([0,1])$

Hat-function: $\quad \phi(x) := (1 - |x|)_+$

$\phi_{j,k} = 2^{j/2}\phi\left(2^j \cdot -k\right)$

$S_j := \mathrm{span}\,\{\phi_{j,k} : k = 1, \ldots, 2^j - 1\}$

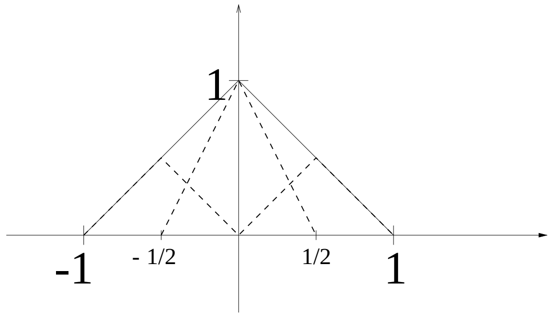


Galerkin scheme $\quad \langle u_j', v' \rangle = \langle f, v \rangle, \quad v \in S_j,$

$$u_J = \sum_{k=1}^{2^j-1} u_{J,k}\phi_{J,k} \quad \rightsquigarrow \quad \boldsymbol{A}_J \mathbf{u}_J = \mathbf{f}_J, \quad \boldsymbol{A}_J := \langle \Phi'_J, \Phi'_J \rangle, \quad \mathbf{f}_J := \langle \Phi_j, f \rangle$$

Multilevel splitting:

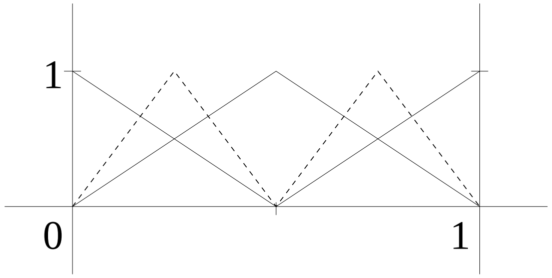
$$\begin{aligned} \phi(x) &= \tfrac{1}{2}\phi(2x+1) + \phi(2x) + \tfrac{1}{2}\phi(2x-1) \\ \phi_{j,k} &= \frac{1}{\sqrt{2}}\left(\tfrac{1}{2}\phi_{j+1,2k-1} + \phi_{j+1,2k} + \tfrac{1}{2}\phi_{j+1,2k+1}\right) \end{aligned}$$



Hierarchical (complement) bases [Y]

$$\Psi_j := \{\psi_{j,k} := \phi_{j+1,2k+1} : k = 0, \ldots, 2^j - 1\}$$

$$S_{j+1} = S_j \oplus \operatorname{span}(\Psi_j)$$



Note $\frac{d}{dx}\psi_{j,k}(x) = \frac{d}{dx}\phi_{j+1,2k+1}(x) = 2^{j+\frac{3}{2}}\psi^H_{j,k}(x) \quad \rightsquigarrow$

$$\langle \frac{d}{dx}\psi_{j,k}, \frac{d}{dx}\psi_{l,m}\rangle = 2^{2j+3}\delta_{(j,k),(l,m)} \quad \rightsquigarrow$$

Diagonal scaling produces uniformly bounded condition numbers!

Summary

Vanishing Moments – Cancellation Property (CP) $\rightsquigarrow$

- sparse representation of functions
- sparse representation of operators

Norm Equivalence (NE) $\rightsquigarrow$

- Tight relation: function $\leftrightarrow$ digits
- Well conditioned systems

Possible Leeway?

- Is orthogonality of basis necessary?
- Is diagonalization of stiffness matrix necessary?

Wavelet Bases - Functions $\leftrightarrow$ Sequences [Co, D3, D4]

Goal: Develop flexible concepts for the construction of multiscale bases for realistic domain geometries Ω (bounded Euclidean domains, surfaces, manifolds) sharing in essence the features of the above examples

General Format: $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\} \subset L_2(\Omega)$, $\|\psi_\lambda\|_{L_2} \sim 1$, $\dim \Omega = d$, $\mathcal{J} = \mathcal{J}_\phi \cup \mathcal{J}_\psi$ infinite index set where:

- $\#\mathcal{J}_\phi < \infty$, $\leftrightarrow$ "scaling functions" (spanning polynomials);
- $\mathcal{J}_\psi \leftrightarrow$ "true" wavelets spanning complements between refinement levels;
- $\lambda \leftrightarrow (j = \text{scale}, e = \text{type}, k = \text{spatial location})$, $|\lambda| := j$

Example: $\psi_\lambda(x,y) = 2^j\psi^{1,0}(2^j(x,y)-(k,l)) = 2^{j/2}\psi(2^jx-k)2^{j/2}\phi(2^jy-l)$

Notational Conventions:

- Diagonal scaling matrix: $\mathbf{D}^s := \left(\delta_{\lambda,\mu} 2^{s|\lambda|}\right)_{\lambda,\mu\in\mathcal{J}}$;

  scaled basis: $\mathbf{D}^{-s}\Psi = \{2^{-s|\lambda|}\psi_\lambda\}$;

- Arrays of wavelet coeffcients: $\mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}}$, $\boldsymbol{d}, \mathbf{u}, \ldots,$

- Wavelet expansions: $\boldsymbol{d}^T\Psi := \sum_{\lambda\in\mathcal{J}} d_\lambda\psi_\lambda$

- Gramian matrices: $c(\cdot,\cdot) : X \times Y \to I\!R$ bilinear form, $\Xi \subset X, \Theta \subset Y \rightsquigarrow$

$$c(\Xi,\Theta) := (c(\xi,\theta))_{\xi\in\Xi,\theta\in\Theta}, \quad \langle\Psi, f\rangle = (\langle f, \psi_\lambda\rangle)^T_{\lambda\in\mathcal{J}}$$

Main Features

- **Locality (L)**
- **Cancellation Properties (CP)**
- **Norm Equivalences (NE)**

Locality (L):

$$\Omega_\lambda := \operatorname{supp}\psi_\lambda, \quad \operatorname{diam}(\Omega_\lambda) \sim 2^{-|\lambda|}$$

Cancellation Property (CP) of Order $\tilde{m}$:

$$|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|\left(\tilde{m}+\frac{d}{2}-\frac{d}{p}\right)} |v|_{W_p^{\tilde{m}}(\Omega_\lambda)}, \quad \lambda \in \mathcal{J}_\psi$$

$\Omega \subset I\!R^d$, **vanishing moments** of order $\tilde{m} \Longrightarrow$ (CP):

$$\begin{aligned} |\langle v, \psi_\lambda \rangle| &= \inf_{P \in \Pi_{\tilde{m}}} |\langle v - P, \psi_\lambda \rangle| \le \inf_{P \in \mathbb{P}_{\tilde{m}}} \|v - P\|_{L_p(\Omega_\lambda)} \|\psi_\lambda\|_{L_{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1 \\ &\lesssim 2^{-|\lambda|\left(\frac{d}{2} - \frac{d}{p}\right)} \inf_{P \in \mathbb{P}_{\tilde{m}}} \|v - P\|_{L_p(\Omega_\lambda)} \\ &\qquad (\|\psi_\lambda\|_{L_{p'}} \sim 2^{|\lambda|\left(\frac{d}{p'} - \frac{d}{2}\right)} \sim 2^{|\lambda|\left(\frac{d}{2} - \frac{d}{p}\right)}) \end{aligned}$$

Local polynomial approximation

$$\inf_{P \in \mathbb{P}_k} \|v - P\|_{L_p(\Omega)} \lesssim (\mathrm{diam}\, \Omega)^k |v|_{W_p^k(\Omega)}$$

Norm Equivalences (NE): For some $\gamma, \tilde{\gamma} > 0$ and $s \in (-\tilde{\gamma}, \gamma)$ $\exists$ c_s, C_s, s.t.

$$c_s \|\mathbf{v}\|_{\ell_2} \le \|\mathbf{v}^T \mathbf{D}^{-s} \Psi\|_{H^s} \le C_s \|\mathbf{v}\|_{\ell_2}$$

where for $s \ge 0$ $\quad H_0^s(\Omega) \subset H^s \subset H^s(\Omega), \quad H^s := (H^{-s})'$ when $s < 0$,
i.e. $\mathbf{D}^{-s}\Psi$ is a **Riesz-basis** for H^s.

Remark 1 - A first consequence: Let $\|v\|^2_{\mathcal{H}_\epsilon} := \epsilon \langle \nabla v, \nabla v \rangle + \langle v, v \rangle$,
$\mathbf{D} := \left((1 + \sqrt{\epsilon} 2^{|\lambda|}) \delta_{\lambda,\mu}\right)_{\lambda,\mu}$, $\rightsquigarrow$

$$\left(2(c_0^{-2} + c_1^{-1})\right)^{-1/2} \|\mathbf{v}\|_{\ell_2} \le \|\mathbf{v}^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}_\epsilon} \le \left(C_0^2 + C_1^2\right)^{1/2} \|\mathbf{v}\|_{\ell_2}$$

Proof of Remark 1: Let $v = \boldsymbol{d}^T \Psi$

$$
\begin{aligned}
\|\{(1+\sqrt{\epsilon}2^{|\lambda|})d_\lambda\}_{\lambda\in\mathcal{J}}\|_{\mathcal{J}}^2 &\leq 2\sum_{\lambda\in\mathcal{J}}\left(|d_\lambda|^2 + \epsilon 2^{2|\lambda|}|d_\lambda|^2\right)\\
&\stackrel{(NE)}{\leq} 2\left(c_0^{-2}+c_1^{-2}\right)\left\{\|v\|_{L_2}^2 + \epsilon|v|_{H^1}^2\right\}\\
&= 2\left(c_0^{-2}+c_1^{-2}\right)\|v\|_{\mathcal{H}_\epsilon}^2
\end{aligned}
$$

$$
\begin{aligned}
\|v\|_{L_2}^2 + \epsilon|v|_{H^1}^2 &\stackrel{(NE)}{\leq} C_0^2\|\boldsymbol{d}\|_{\ell_2}^2 + \epsilon C_1^2\|\mathbf{D}^1\boldsymbol{d}\|_{\ell_2}^2 \leq (C_0^2+C_1^2)\sum_{\lambda\in\mathcal{J}}(1+\epsilon 2^{2|\lambda|})|d_\lambda|^2\\
&\leq (C_0^2+C_1^2)\sum_{\lambda\in\mathcal{J}}(1+\sqrt{\epsilon}2^{|\lambda|})^2|d_\lambda|^2
\end{aligned}
$$

Remark 2 - a second consequence - duality: Let $\mathcal{H}$ be a Hilbert space, $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H}' \to I\!R$ and suppose that

$$c\|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T\Theta\|_{\mathcal{H}} \leq C\|\mathbf{v}\|_{\ell_2}$$

(Θ is Riesz-basis for $\mathcal{H}$) $\Longrightarrow$ **(NE)'**:

$$C^{-1}\|\langle\boldsymbol{\Theta}, v\rangle\|_{\ell_2} \leq \|v\|_{\mathcal{H}'} \leq c^{-1}\|\langle\boldsymbol{\Theta}, v\rangle\|_{\ell_2} \tag{1}$$

Application: $\mathcal{H} = H_0^1(\Omega)$, $\Theta := \mathbf{D}^{-1}\Psi$, $\rightsquigarrow$

$$C_1^{-1}\|\mathbf{D}^{-1}\langle\boldsymbol{\Psi}, w\rangle\|_{\ell_2} \leq \|w\|_{H^{-1}(\Omega)} \leq c_1^{-1}\|\mathbf{D}^{-1}\langle\boldsymbol{\Psi}, w\rangle\|_{\ell_2}$$

Proof of Remark 2:

1) Let $F : \ell_2 \to \mathcal{H}$, $F : \mathbf{v} \to \mathbf{v}^T\Theta$

$$c\|\mathbf{v}\|_{\ell_2} \le \|\mathbf{v}^T\Theta\|_{\mathcal{H}} \le C\|\mathbf{v}\|_{\ell_2} \quad \Longrightarrow \quad \|F\|_{\ell_2\to\mathcal{H}} = C, \quad \|F^{-1}\|_{\mathcal{H}\to\ell_2} = c^{-1}$$

2) For the adjoint $F^* : \mathcal{H}' \to \ell_2$ defined by $\langle F\mathbf{v}, w\rangle = \mathbf{v}^T F^* w$ one has

$$\|F^*w\|_{\ell_2} = \sup_{\mathbf{v}} \frac{(F^*w)^T\mathbf{v}}{\|\mathbf{v}\|_{\ell_2}} = \sup_{\mathbf{v}} \frac{\langle F\mathbf{v}, w\rangle}{\|\mathbf{v}\|_{\ell_2}} \le \sup_{\mathbf{v}} \frac{\|F\mathbf{v}\|_{\mathcal{H}}\|w\|_{\mathcal{H}'}}{\|\mathbf{v}\|_{\ell_2}} = \|F\|_{\ell_2\to\mathcal{H}}\|w\|_{\mathcal{H}'}$$

$$\|F\mathbf{v}\|_{\mathcal{H}} = \sup_{w} \frac{\langle F\mathbf{v}, w\rangle}{\|w\|_{\mathcal{H}'}} \sup_{w} \frac{(F^*w)^T\mathbf{v}}{\|w\|_{\mathcal{H}'}} \le \sup_{w} \frac{\|F^*w\|_{\ell_2}\|\mathbf{v}\|_{\ell_2}}{\|w\|_{\mathcal{H}'}} = \|F^*\|_{\mathcal{H}'\to\ell_2}\|\mathbf{v}\|_{\ell_2}$$

$$\Longrightarrow \quad \|F\|_{\ell_2\to\mathcal{H}} = \|F^*\|_{\mathcal{H}'\to\ell_2} = C, \quad \|F^{-1}\|_{\mathcal{H}\to\ell_2} = \|(F^*)^{-1}\|_{\ell_2\to\mathcal{H}'} = c^{-1}$$

3) Identify F^*w: $(F^*w)_\lambda = (F^*w)^T\mathbf{e}_\lambda = \langle F\mathbf{e}_\lambda, w\rangle = \langle \theta_\lambda, w\rangle \rightsquigarrow$

$$F^*w = \langle \Theta, w\rangle \quad \rightsquigarrow \quad \|\langle \Theta, w\rangle\|_{\ell_2} \le C\|w\|_{\mathcal{H}'}$$

Riesz-Bases - Biorthogonality

Note: $c\|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T\Theta\|_{\mathcal{H}} \leq C\|\mathbf{v}\|_{\ell_2} \leadsto$

$F : \ell_2 \to \mathcal{H}$ $F : \mathbf{v} \to \mathbf{v}^T\Theta$, adjoint $F^* : \mathcal{H}' \to \ell_2$ defined by $\langle F\mathbf{v}, w\rangle = \mathbf{v}^T F^* w$

are topological isomorphisms and $F\mathbf{e}_\lambda = \theta_\lambda$ – Define: $\tilde{\theta}_\lambda := (F^*)^{-1}\mathbf{e}_\lambda \leadsto$

$$\langle \theta_\lambda, \tilde{\theta}_\mu\rangle = \langle F\mathbf{e}_\lambda, (F^*)^{-1}\mathbf{e}_\mu\rangle = \mathbf{e}_\lambda^T\mathbf{e}_\mu = \delta_{\lambda,\mu} \quad \leadsto \quad \langle \Theta, \tilde{\Theta}\rangle = \mathbf{I} \quad \leadsto$$

$$w = \langle w, \Theta\rangle\tilde{\Theta}, \quad (1) \;\leadsto\; \tilde{\Theta} \text{ is a Riesz basis for } \mathcal{H}'$$

Corollary: $\mathcal{H} = L_2(\Omega) = \mathcal{H}'$ $\leadsto$ for every Riesz basis Ψ for L_2 there exists a biorthogonal basis $\tilde{\Psi}$ which is also a Riesz basis for L_2.

Further Norm Equivalences - Besov-Spaces [BL, Co, DeV, DPJ, DP]

Let $0 < p < \infty$, $0 < q \leq \infty$. Renormalize:

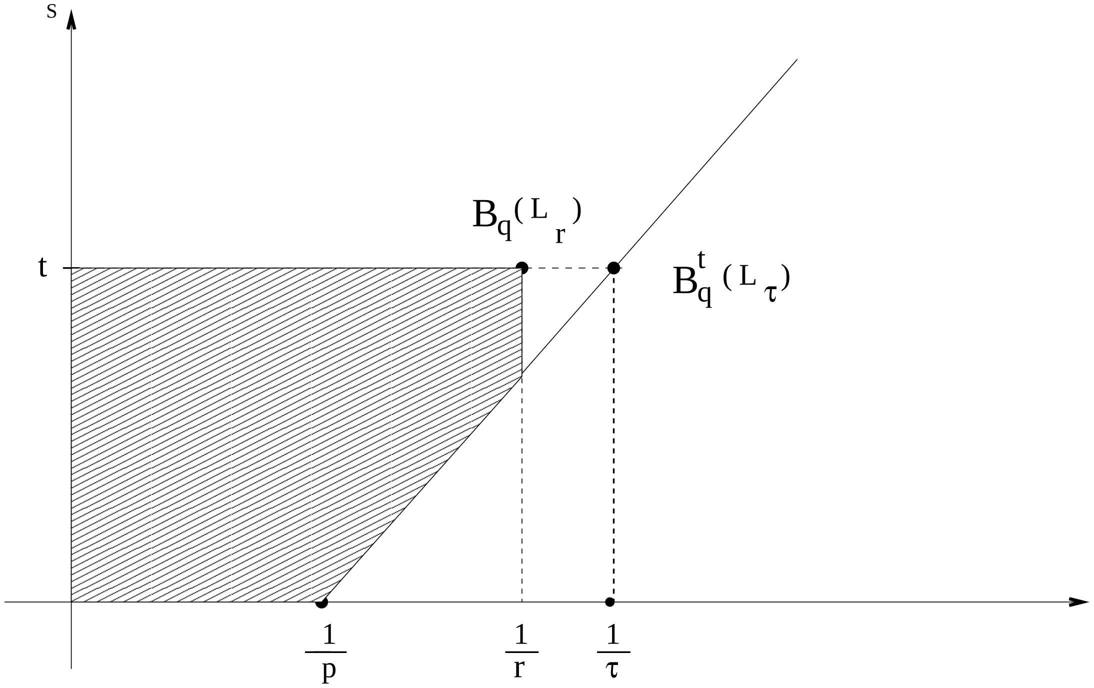
$$\psi_{\lambda,p} := 2^{d|\lambda|\left(\frac{1}{p}-\frac{1}{2}\right)}\psi_\lambda \;\rightsquigarrow\; \|\psi_{\lambda,p}\|_{L_p} \sim 1, \quad \Psi_p := \{\psi_{\lambda,p} : \lambda \in \mathcal{J}\} \;\rightsquigarrow\; \langle \Psi_p, \tilde{\Psi}_{p'} \rangle = \mathbf{I}$$

Suppose $B_q^t(L_\tau) \subset L_p$. For some $\gamma > 0$ and $0 < s < \gamma$ one has

$$\|v\|^q_{B_q^t(L_\tau)} \sim \|v\|^q_{L_\tau} + \sum_{j=0}^{\infty} \left(2^{jd\left(\frac{t}{d}+\frac{1}{p}-\frac{1}{\tau}\right)} \|\langle \tilde{\Psi}_{j,p'}, v\rangle\|_{\ell_\tau}\right)^q, \quad \tilde{\Psi}_{j,p'} := \{\tilde{\psi}_{\lambda,p'} : |\lambda| = j\}$$

Embedding - the Critical Line:

$$\frac{t}{d} + \frac{1}{p} = \frac{1}{\tau} \quad \rightsquigarrow \|v\|_{B^t_\tau(L_\tau)} \sim \|v\|_{L_\tau} + \|\langle \tilde{\Psi}_{p'}, v\rangle\|_{\ell_\tau}$$



Function Spaces [A, BL, T]

Sobolev spaces: $W_p^k(\Omega) := \{f : \partial^\alpha f \in L_p(\Omega),\ |\alpha| \le k\}$

$|f|_{w_p^k(\Omega)} := \left(\sum_{|\alpha|=k} \|\partial^\alpha f\|^p_{L_p(\Omega)}\right)^{1/p}, \quad \|v\|^p_{W_p^k(\Omega)} := \sum_{m=0}^k |f|^p_{w_p^m(\Omega)}$

Fractional order spaces - An intrinsic norm: $k := \lfloor t \rfloor$

$$\|v\|_{W_p^t(\Omega)} = \left(\|v\|^p_{W_p^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x-y|^{d+tp}} dx dy \right)^{1/p}$$

Interpolation $\rightsquigarrow$ Besov-spaces: $\quad t < r$

$$|v|_{B_q^t(L_p(\Omega))} := \begin{cases} \left(\int\limits_0^\infty [s^{-t}\omega_r(f,s,\Omega)_p]^q \, \frac{ds}{s} \right)^{1/q}, & 0 < q < \infty; \\ \sup_{s>0} s^{-t}\omega_r(f,s,\Omega)_p, & q = \infty, \end{cases}$$

L_p-modulus of continuity:

$$\omega_r(f,t,\Omega)_p := \sup_{|h|\le t} \|\Delta_h^r f\|_{L_p(\Omega_{r,h})}, \quad \Delta_h f := f(\cdot + h) - f(\cdot), \ \ \Delta_h^k = \Delta_h \circ \Delta_h^{k-1}$$

where $\Omega_{r,h} := \{x : x + sh \in \Omega, \ s \in [0,1]\}$

Special cases:

- $W_p^t = B_p^t(L_p)$ for $s > 0$, $s \notin I\!N$ ($p \neq 2$)
- $H^t := W_2^t = B_2^t(L_2)$ for $s \in I\!R$ $\quad$ ($H^{-t} := (H^t)'$)

$$\|f\|_{X'} := \sup_{g \in X, g \not\equiv 0} \frac{\langle f, g\rangle}{\|g\|_X}$$

Local Polynomial Approximation, see e.g. [DS]

$$\inf_{P\in\mathbb{P}_k} \|v-P\|_{W_p^m(\Omega)} \lesssim (\operatorname{diam}\Omega)^{k-m}|v|_{W_p^k(\Omega)}, \quad (m<k)$$

$$\inf_{P\in\mathbb{P}_k} \|v-P\|_{L_p(\Omega)} \lesssim (\operatorname{diam}\Omega)^{t}|v|_{B_q^t(L_p(\Omega))}$$

$$\inf_{P\in\mathbb{P}_k} \|v-P\|_{L_p(\Omega)} \lesssim \omega_k(f,\operatorname{diam}\Omega,\Omega)_p$$

Idea of proof: By rescaling it suffices to consider reference domain with unit diameter – suppose that $\inf_{P\in\mathbb{P}_k}\|v_n-P\|_{W_p^m(\Omega)} \geq n|v_n|_{W_p^k(\Omega)}$ - rescale - $\leadsto$

$$1=\inf_{P\in\mathbb{P}_k}\|w_n-P\|_{W_p^m(\Omega)} = \|w_n\|_{W_p^m(\Omega)} \geq n|w_n|_{W_p^k(\Omega)} \quad \leadsto$$

$\{w_n\}_n$ precompact in W_p^m $\leadsto$ $\exists\ w \in W_p^m$ s.t. $|w|_{W_p^k(\Omega)} = \lim_{n\to\infty}|w_n|_{W_p^k(\Omega)} = 0 \leadsto w\in\mathbb{P}_k$. On the other hand, $\inf_{P\in\mathbb{P}_k}\|w-P\|_{W_p^m(\Omega)}=1$ a contradiction $\square$

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