Multiscale and Wavelet Methods for Operator Equations

(2) Multiresolution - construction and analysis principles

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Multiscale Decompositions - Construction and Analysis Principles

Multiresolution $S_0 \subset S_1 \subset S_2 \subset \ldots L_2(\Omega), \quad \overline{\bigcup_j S_j} = L_2(\Omega)$

Single-scale bases: $S_j = \mathrm{span}\,\Phi_j =: S(\Phi_j), \quad \Phi_j = \{\phi_{j,k} : k \in \mathcal{I}_j\}$

Decomposition: $S_{j+1} = S_j \bigoplus W_j$

Complement bases: $W_j = \mathrm{span}\,\Psi_j, \quad \Psi_j = \{\psi_\lambda : \lambda \in \mathcal{J}_j\}$

Multi-scale basis: $\Psi := \bigcup_{j \in \mathbb{N}_0} \Psi_j \qquad (\Psi_{-1} := \Phi_0)$

Uniform single scale stability:

$$\|\mathbf{c}\|_{l_2(\mathcal{I}_j)} \sim \|\mathbf{c}^T \Phi_j\|_{L_2} \quad \|\boldsymbol{d}\|_{l_2(\mathcal{J}_j)} \sim \|\boldsymbol{d}^T \Psi_j\|_{L_2}$$

Stability over all levels ? $\|\boldsymbol{d}\|_{\ell_2} \sim \|\boldsymbol{d}^T \Psi\|_{L_2}$

Multiscale Transformations (cf. [CDP, D1])

$$\sum_{l=-1}^{j-1} \sum_{\lambda \in \mathcal{J}_l} d_\lambda \psi_\lambda \quad = \quad \sum_{k \in \mathcal{I}_j} c_k \phi_{j,k}$$

$$\mathbf{T}_j \; : \; \boldsymbol{d} \; \mapsto \; \mathbf{c}$$

Remark 3: Assume that the Φ_j are uniformly L_2-stable. Then

$$\|\mathbf{T}_j\|, \; \|\mathbf{T}_j^{-1}\| = \mathcal{O}(1) \quad \Longleftrightarrow \quad \Psi \; \text{is a Riesz basis in} \; L_2$$

Proof: Let $\Psi^j = \{\psi_\lambda : |\lambda| < j\}$ and $v = \boldsymbol{d}_j^T \Psi^j = \mathbf{c}_j^T \Phi_j$. Assume that Ψ is a Riesz basis $\rightsquigarrow$

$$\|\mathbf{c}_j\|_{\ell_2} \sim \|v\|_{L_2} = \|\boldsymbol{d}_j^T \Psi^j\|_{L_2} \sim \|\boldsymbol{d}_j\|_{\ell_2} = \|\mathbf{T}_j^{-1} \mathbf{c}_j\|_{\ell_2}$$

Conversely: $\|\boldsymbol{d}_j\|_{\ell_2} = \|\mathbf{T}_j^{-1} \mathbf{c}_j\|_{\ell_2} \sim \|\mathbf{c}_j\|_{\ell_2} \sim \|v\|_{L_2}$

Main Issues

- Uniform stability of the Φ_j (easy)
- (Spectral) Stability over all levels - Riesz-basis-property in L_2 (more involved)

Biorthogonality in L_2 is necessary !

Stability of single scale bases Φ_j

Key: Local dual bases $\langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}$, $\|\tilde{\phi}_{j,k}\|_{L_2} \sim 1 \leadsto$

$$v = \mathbf{c}^T \Phi_j \quad \leadsto \|\mathbf{c}\|_{\ell_2}^2 = \|\langle v, \tilde{\Phi}_j \rangle\|_{\ell_2}^2 \lesssim \sum_{k \in \mathcal{I}_j} \|v\|_{L_2(\tilde{\sigma}_{j,k})}^2 \lesssim \|v\|_{L_2}^2$$

$$\|v\|_{L_2(\square_{j,k})}^2 = \left(\sum_{\sigma_{j,m} \cap \square_{j,k} \neq \emptyset} |c_m| \|\phi_{j,m}\|_{L_2(\square_{j,k})} \right)^2 \lesssim \sum_{\sigma_{j,m} \cap \square_{j,k} \neq \emptyset} |c_m|^2$$

summation $\leadsto$

$$\|v\|_{L_2}^2 \lesssim \|\mathbf{c}\|_{\ell_2}^2$$

Construction of biorthogonal bases

- Biorthogonal wavelets on $\mathbb{R}$ or $\mathbb{R}^d$ - Fourier methods (see [CDF, Co, Dau])

- More general domains: One needs good complement spaces W_j $\rightsquigarrow$

  - Construct stable dual multiresolution sequences

    $$\mathcal{S} = \{S_j\}_{j\in\mathbb{N}_0}, \ \ S_j = \mathrm{span}(\Phi_j), \ \ \tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j\in\mathbb{N}_0}, \ \ \tilde{S}_j = \mathrm{span}(\tilde{\Phi}_j), \ \ \langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}$$

  - Construct some simple initial complement spaces $\check{W}_j$

  - Change the initial complements into better ones W_j

Stable Completions – Lifting [CDP, SW1, SW2]

Refinement equation: Stability, nestedness $\rightsquigarrow$

$$\phi_{j,k} = \sum_{l \in \mathcal{I}_{j+1}} m_{j,l,k}\phi_{j+1,l} \quad \longleftrightarrow \quad \Phi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,0}$$

Find:

$$\Psi_{j+1}^T = \Phi_{j+1}^T \mathbf{M}_{j,1}$$

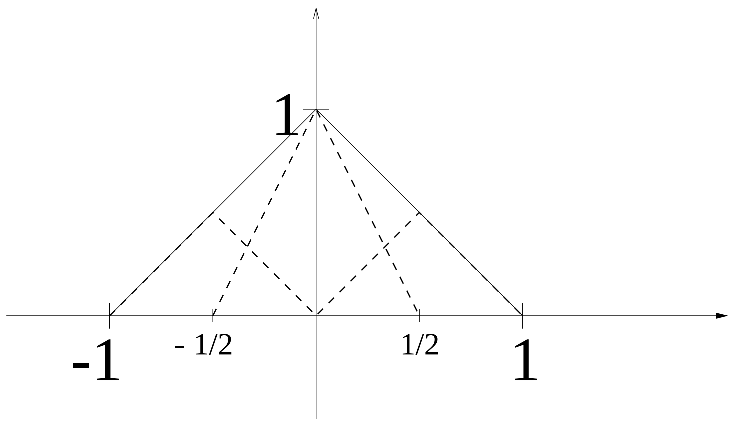
s.t. $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ is invertible, i.e, for $\mathbf{M}^{-1} =: \mathbf{G}_j = \binom{\mathbf{G}_{j,0}}{\mathbf{G}_{j,1}}$ one has

$$\Phi_{j+1}^T = \Phi_j^T \mathbf{G}_{j,0} + \Psi_{j+1}^T \mathbf{G}_{j,1} \quad \Longleftrightarrow \quad \mathbf{M}_j \mathbf{G}_j = \mathbf{G}_j \mathbf{M}_j = \mathbf{I}$$

Remark 4: Ψ_j, Φ_j uniformly **stable** $\Longleftrightarrow \|\mathbf{M}_j\|, \|\mathbf{G}_j\| = \mathcal{O}(1)$

Hierarchical Bases $1 - D$

$$\mathbf{M}_{j,0} = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & \dots & \\ \frac{1}{\sqrt{2}} & 0 & 0 & \dots & \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \dots & \\ 0 & \frac{1}{\sqrt{2}} & 0 & \dots & \\ 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \dots \\ 0 & 0 & \dots & & \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 0 \\ & & & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \dots & & 0 & \frac{1}{\sqrt{2}} \\ 0 & \dots & & 0 & \frac{1}{2\sqrt{2}} \end{pmatrix}, \quad \mathbf{M}_{j,1} = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & & & \vdots \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \end{pmatrix}$$



$$
\begin{aligned}
\phi_{j+1,2k} &= \sqrt{2}\,\phi_{j,k} - \frac{1}{2}\left(\phi_{j+1,2k-1} + \phi_{j+1,2k+1}\right) \\
&= \sqrt{2}\,\phi_{j,k} - \frac{1}{2}\left(\psi_{j,k-1} + \psi_{j,k}\right), \quad k = 1, \ldots, 2^j - 1, \\
\phi_{j+1,2k+1} &= \psi_{j,k}, \quad k = 0, \ldots, 2^j - 1 \\
\phi_{j+1,0} &= \sqrt{2}\,\phi_{j,0} - \frac{1}{2}\psi_{j,0}, \quad \phi_{j+1,2^{j+1}} = \sqrt{2}\,\phi_{j,2^j} - \frac{1}{2}\psi_{j,2^j-1},
\end{aligned}
$$

$$\mathbf{G}_{j,0} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & & \ldots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & & \vdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots & & \vdots \\ 0 & & & & & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \ldots & & & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\mathbf{G}_{j,1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \ldots & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & & & & -\frac{1}{2} & 0 \\ 0 & \ldots & & & & -\frac{1}{2} & 1 \end{pmatrix}$$

Multiscale Transformation

$$\Phi_j^T \mathbf{c}^j + \Psi_j^T \boldsymbol{d}^j = \Phi_{j+1}^T \left(\mathbf{M}_{j,0} \mathbf{c}^j + \mathbf{M}_{j,1} \boldsymbol{d}^j \right),$$

$$\mathbf{T}_J : \boldsymbol{d} \to \mathbf{c}$$

$$\begin{array}{ccccccccc}
\mathbf{c}^0 & \stackrel{\mathbf{M}_{0,0}}{\longrightarrow} & \mathbf{c}^1 & \stackrel{\mathbf{M}_{1,0}}{\longrightarrow} & \mathbf{c}^2 & \stackrel{\mathbf{M}_{2,0}}{\longrightarrow} & \cdots & \stackrel{\mathbf{M}_{J-1,0}}{\longrightarrow} & \mathbf{c}^J \\
 & \stackrel{\mathbf{M}_{0,1}}{\nearrow} & & \stackrel{\mathbf{M}_{1,1}}{\nearrow} & & \stackrel{\mathbf{M}_{2,1}}{\nearrow} & \cdots & \stackrel{\mathbf{M}_{J-1,1}}{\nearrow} & \\
\boldsymbol{d}^0 & & \boldsymbol{d}^1 & & \boldsymbol{d}^2 & & & \boldsymbol{d}^{J-1} &
\end{array}$$

$$\mathbf{T}_{J,j} := \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{T}_J = \mathbf{T}_{J,J-1} \cdots \mathbf{T}_{J,0}$$

$$\mathbf{T}_J^{-1} : \mathbf{c} \rightarrow \boldsymbol{d}$$

$$\Phi_{j+1}^T \mathbf{c}^{j+1} = \Phi_j^T (\mathbf{G}_{j,0} \mathbf{c}^{j+1}) + \Psi_j^T (\mathbf{G}_{j,1} \mathbf{c}^{j+1}) = \Phi_j^T \mathbf{c}^j + \Psi_j^T \boldsymbol{d}^j$$

$$\begin{array}{ccccccccc}
\mathbf{c}^J & \xrightarrow{\mathbf{G}_{J-1,0}} & \mathbf{c}^{J-1} & \xrightarrow{\mathbf{G}_{J-2,0}} & \mathbf{c}^{J-2} & \xrightarrow{\mathbf{G}_{J-3,0}} & \cdots & \xrightarrow{\mathbf{G}_{0,0}} & \mathbf{c}^0 \\
 & \overset{\mathbf{G}_{J-1,1}}{\searrow} & & \overset{\mathbf{G}_{J-2,1}}{\searrow} & & \overset{\mathbf{G}_{J-3,1}}{\searrow} & \cdots & \overset{\mathbf{G}_{0,1}}{\searrow} & \\
 & & \boldsymbol{d}^{J-1} & & \boldsymbol{d}^{J-2} & & & & \boldsymbol{d}^0,
\end{array}$$

Parametrization of Completions [CDP]

Theorem: Given some **initial** completion $\check{\mathbf{M}}_{j,1}$ (and $\check{\mathbf{G}}_j$), then **all** other completions have the form

$$\mathbf{M}_{j,1} = \mathbf{M}_{j,0}\mathbf{L} + \check{\mathbf{M}}_{j,1}\mathbf{K}$$

and

$$\mathbf{G}_{j,0} = \check{\mathbf{G}}_{j,0} - \check{\mathbf{G}}_{j,1}(\mathbf{K}^T)^{-1}\mathbf{L}^T \quad \mathbf{G}_{j,1} = \check{\mathbf{G}}_{j,1}(\mathbf{K}^T)^{-1}$$

Proof:

$$\mathbf{I} = \mathbf{M}_j\mathbf{G}_j = \mathbf{M}_j \begin{pmatrix} \mathbf{I} & \mathbf{L} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{L}\mathbf{K}^{-1} \\ \mathbf{0} & \mathbf{K}^{-1} \end{pmatrix} \mathbf{G}_j =: \check{\mathbf{M}}_j\check{\mathbf{G}}_j$$

Idea: **Some** initial completion is often available e.g. **Hierarchical Bases**

Applications:

- Raising **moment conditions**: Choose $\mathbf{K} = \mathbf{I}$ and $\mathbf{L}$ s.t.

$$\int_{\Omega} \Psi_j^T P dx = \int_{\Omega} \Phi_{j+1}^T \mathbf{M}_{j,1} P dx = \int_{\Omega} \Phi_j^T \mathbf{L} P + \check{\Psi}_j^T \mathbf{K} P dx = 0, \quad P \in \Pi_{d^*}$$

- F.E. based wavelets – coarse grid corrections [CDP, DK, D2, VW]

- Construction of **biorthogonal** wavelets [CTU1, CTU2, CM, DS1, DS2]

Biorthogonalization

Suppose that

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,0}, \quad \tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}, \quad \langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}$$

Theorem: Let $\tilde{\mathbf{M}}_{j,0}, \check{\mathbf{M}}_{j,1}, \check{\mathbf{G}}_j$ be as above. Then ($\mathbf{K} := \mathbf{I},\ \mathbf{L} := -\tilde{\mathbf{M}}_{j,0}^T \check{\mathbf{M}}_{j,1}$)

$$\mathbf{M}_{j,1} := (I - \mathbf{M}_{j,0} \tilde{\mathbf{M}}_{j,0}^T) \check{\mathbf{M}}_{j,1}, \qquad \tilde{\mathbf{M}}_{j,1} := \check{\mathbf{G}}_{j,1}$$

satisfy $\quad \mathbf{M}_j \tilde{\mathbf{M}}_j^T = \mathbf{I} \quad$ and

$$\Psi_j^T := \Phi_{j+1}^T \mathbf{M}_{j,1}, \qquad \tilde{\Psi}_{j+1}^T := \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,1}$$

form **biorthogonal wavelet bases**. Riesz-bases ???

What could additional conditions look like ?

Suppose (NE) holds for $0 \leq t < \gamma \rightsquigarrow$

$$\|\sum_{|\lambda|<J} d_\lambda \psi_\lambda\|_{H^t} \overset{(\mathrm{NE})}{\sim} \|\{2^{t|\lambda|} d_\lambda\}_{|\lambda|<J}\|_{\ell_2} \lesssim 2^{Jt} \|\{d_\lambda\}_{|\lambda|<J}\|_{\ell_2} \overset{(\mathrm{NE})}{\sim} 2^{Jt} \|\sum_{|\lambda|<J} d_\lambda \psi_\lambda\|_{L_2} \rightsquigarrow$$

Inverse Estimate: $\|v\|_{H^t} \lesssim 2^{jt} \|v\|_{L_2}, \quad v \in S_j,\ t < \gamma$

$Q_j v := \sum_{|\lambda|<j} \langle v, \tilde{\psi}_\lambda \rangle \psi_\lambda \rightsquigarrow$

$$\begin{aligned} \|v - Q_j v\|_{L_2} &\lesssim \left(\sum_{|\lambda| \geq j} |\langle v, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2} \leq 2^{-jt} \left(\sum_{|\lambda| \geq j} 2^{2t|\lambda|} |\langle v, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2} \\ &\overset{(\mathrm{NE})}{\lesssim} 2^{-jt} \|v\|_{H^t} \rightsquigarrow \end{aligned}$$

Direct Estimate: $\inf_{v_j \in S_j} \|v - v_j\|_{L_2} \lesssim 2^{-jt} \|v\|_{H^t}, \quad t < \gamma$

Fourier Free Criteria for (NE)

$$\sum_{|\lambda|=j} \langle v, \tilde{\psi}_\lambda \rangle \psi_\lambda = (Q_{j+1} - Q_j)v, \quad \|(Q_{j+1} - Q_j)v\|_{L_2}^2 \sim \sum_{|\lambda|=j} |\langle v, \tilde{\psi}_\lambda \rangle|^2$$

$$\leadsto \left(\sum_{j=0}^{\infty} 2^{j2s} \|(Q_{j+1} - Q_j)v\|_{L_2}^2\right)^{1/2} \sim \left(\sum_{\lambda \in \mathcal{J}} 2^{2s|\lambda|} |\langle v, \tilde{\psi}_\lambda \rangle|^2\right)^{1/2}$$

- $\mathcal{S} = \{S_j\}_{j \in \mathbb{N}_0}$: $S_j \subset H^s$ for $s < \gamma$,
  $\mathcal{Q} = \{Q_j\}_{j \in \mathbb{N}_0}$: $Q_j : L_2 \to S_j$ uniformly L_2-bounded;
- *commutator property (C):* $Q_l Q_j = Q_l, \quad l \leq j$
- *Jackson estimate (J):* $\inf_{v_j \in V_j} \|v - v_j\|_{L_2} \lesssim 2^{-m'j} \|v\|_{H^{m'}}, \quad v \in H^{m'}$
- *Bernstein estimate (B):* $\|v_j\|_{H^s} \lesssim 2^{sj} \|v_j\|_{L_2}, \quad v_j \in V_j, \; s < \gamma',$

Theorem: For $\mathcal{S}$, $\mathcal{Q}$ as above suppose that (C) and (B), (J) hold for $V_j = S_j$ with $m' = m > \gamma' = \gamma > 0$. Then

$$\|v\|_{H^s} \sim \left(\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \right)^{1/2}, \quad 0 < s < \gamma.$$

Moreover, if (J) and (B) also hold for $V_j = \tilde{S}_j := \operatorname{range} Q_j^*$ with $m' = \tilde{m} > \gamma' = \tilde{\gamma} > 0$, then the above **(NE)** also holds for $-\tilde{\gamma} < s < \gamma$ $(H^s = (H^{-s})')$.

- One sided **(NE)** are easier to realize, $s \leq 0$ is difficult
- The space H^s may have incorporated homogeneous boundary conditions
- There exist versions that do not require explicit knowledge of the projectors Q_j [DSt].

Remark: Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be *two given* multi-resolution sequences satisfying $\dim S_j = \dim \tilde{S}_j$ and

$$\inf_{v_j \in S_j} \sup_{\tilde{v}_j \in \tilde{S}_j} \frac{|\langle v_j, \tilde{v}_j \rangle|}{\|v_j\|_{L_2(\Omega)} \|\tilde{v}_j\|_{L_2(\Omega)}} \gtrsim 1 \quad \Longrightarrow \tag{1}$$

$\exists$ a sequence $\mathcal{Q}$ uniformly L_2-bounded with ranges $\mathcal{S}$ such that

$$\mathrm{range}\,(id - Q_j) = (\tilde{S}_j)^{\perp_{L_2}}, \quad \mathrm{range}\,(id - Q_j^*) = (S_j)^{\perp_{L_2}}.$$

Moreover, $\mathcal{Q}$ satisfies the commutator property (C).

Theorem: If in addition to the hypotheses of the previous Remark $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfy (J) and (B) with respective parameters $m, \tilde{m} \in I\!N$ and $\gamma, \tilde{\gamma} > 0$ then one has for any $w_j \in \mathrm{range}\,(Q_j - Q_{j-1})$

$$\left\| \sum_{j=0}^{\infty} w_j \right\|_{H^s}^2 \lesssim \sum_{j=0}^{\infty} 2^{2sj} \|w_j\|_{L_2}^2, \quad s \in [-\tilde{m}, \gamma), \tag{2}$$

and

$$\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \lesssim \|v\|_{H^s}^2, \quad s \in (-\tilde{\gamma}, m]. \tag{3}$$

Thus for $s \in (-\tilde{\gamma}, \gamma)$ (2), (3) hold with ' $\lesssim$ ' replaced by ' $\sim$'.

Sketch of proof (see [D1, DSt]): Define orth. proj. $P_j : L_2 \to \tilde{S}_j$ by

$$\langle v, \tilde{v}_j \rangle = \langle P_j v, \tilde{v}_j \rangle, \quad v \in L_2,\ \tilde{v}_j \in \tilde{S}_j, \quad R_j : P_j|_{S_j} \quad \rightsquigarrow$$

$\|R_j\|_{L_2} \leq 1$, (1) $\rightsquigarrow$ $\|R_j v_j\|_{L_2} \gtrsim \|v_j\|_{L_2}$, $v_j \in S_j$. Claim: $\mathrm{range}\, R_j = \tilde{S}_j$ since otherwise $\exists\ \tilde{v}_j' \in \tilde{S}_j$, $\tilde{v}_j' \perp_{L_2} \mathrm{range}\, R_j$ contradicting (1); $\rightsquigarrow$ $R_j^{-1} : \tilde{S}_j \to S_j$ uniformly L_2-bounded; Define $Q_j := R_j^{-1} Q_j : L_2 \to S_j$ uniformly L_2 bounded, $\mathrm{range}\, Q_j = S_j$ and $\langle Q_j v, \tilde{v}_j \rangle = \langle v, \tilde{v}_j \rangle \rightsquigarrow \mathrm{range}\,(I - Q_j) \subset (\tilde{S}_j)^{\perp_{L_2}}$; conversely $v \in (\tilde{S}_j)^{\perp_{L_2}} \Longrightarrow \mathrm{range}\,(I - Q_j) \subset (\tilde{S}_j)^{\perp_{L_2}}$; $\rightsquigarrow$ analogous properties for adjoints Q_j^*; note $\tilde{S}_j \subset \tilde{S}_{j+1} \Longrightarrow$ (C) $Q_j Q_{j+1} = Q_j$. $\square$.

As for the Theorem, observe first:

$$\|w_j\|_{H^{s\pm\epsilon}} \lesssim 2^{j(s\pm\epsilon)} \|w_j\|_{L_2}, \quad \forall\ w_j \in \mathrm{range}\,(Q_j - Q_{j-1}),\ s \pm \epsilon \in [-\tilde{m}, \gamma). \qquad (4)$$

In fact, (4) follows from (B) for $s \pm \epsilon \in [0, \gamma)$.
For $t := s \pm \epsilon \in [-\tilde{m}, 0]$, $w_j \in \mathrm{range}\,(Q_j - Q_{j-1}) \rightsquigarrow$

$$\|w_j\|_{H^t} \quad = \quad \sup_{z \in H^{-t}} \frac{\langle w_j, z \rangle}{\|z\|_{H^{-t}}} = \sup_{z \in H^{-t}} \frac{\langle w_j, (Q_j^* - Q_{j-1}^*) z \rangle}{\|z\|_{H^{-t}}}$$

$$\lesssim \quad \sup_{z\in H^{-t}} \frac{\|w_j\|_{L_2} \inf_{\tilde v_{j-1}\in \tilde S_{j-1}} \|z-\tilde v_{j-1}\|_{L_2}}{\|z\|_{H^{-t}}} \overset{(J)}{\lesssim} 2^{tj}\|w_j\|_{L_2}.$$

$$\begin{aligned}\left\|\sum_j w_j\right\|_{H^s}^2 &= \left\langle \sum_j w_j, \sum_l w_l \right\rangle_{H^s} \lesssim \sum_j \sum_{l\geq j} \|w_j\|_{H^{s+\epsilon}} \|w_l\|_{H^{s-\epsilon}} \\ &\overset{(B)}{\lesssim} \sum_j \sum_{l\geq j} 2^{\epsilon j} 2^{-l\epsilon} (2^{sk}\|w_j\|_{L_2})(2^{sl}\|w_l\|_{L_2}) \lesssim \sum_j 2^{2sj}\|w_j\|_{L_2}^2\end{aligned}$$

Thus, defining $N_{s,\mathcal{Q}}(v) := \left(\sum_{j=0}^{\infty} 2^{2js}\|(Q_j - Q_{j-1})v\|_{L_2}^2\right)^{1/2}$, we have shown

$$\|v\|_{H^s} \lesssim N_{s,\mathcal{Q}}(v), \quad s\in[-\tilde m, \gamma) \tag{5}$$

which is (2). The same argument gives

$$\|v\|_{H^s} \lesssim N_{s,\mathcal{Q}^*}(v), \quad s\in[-m, \tilde\gamma). \tag{6}$$

To prove (3) note first

$$\begin{aligned} N_{s,\mathcal{Q}}(v)^2 &= \sum_j 2^{2sj}\langle (Q_j - Q_{j-1})v, (Q_j - Q_{j-1})v\rangle \\ &= \left\langle \sum_j 2^{2sj}(Q_j^* - Q_{j-1}^*)(Q_j - Q_{j-1})v, v \right\rangle \\ &\leq \underbrace{\| \sum_j 2^{2sj}(Q_j^* - Q_{j-1}^*)(Q_j - Q_{j-1})v \|}_{:=w}{}_{H^{-s}} \|v\|_{H^s} \\ &\stackrel{(6)}{\lesssim} N_{-s,\mathcal{Q}^*}(w)\|v\|_{H^s}, \quad s \in (-\tilde{\gamma}, m] \end{aligned}$$

Since by (C) $(Q_l^* - Q_{l-1}^*)(Q_j^* - Q_{j-1}^*) = \delta_{j,l}(Q_l^* - Q_{l-1}^*)$ one has

$$\|(Q_l^* - Q_{l-1}^*)w\|_{L_2} = 2^{2sl}\|(Q_l^* - Q_{l-1}^*)(Q_l - Q_{l-1})v\|_{L_2} \lesssim 2^{2sl}\|(Q_l - Q_{l-1})v\|_{L_2} \rightsquigarrow$$

$$
\begin{aligned}
N_{-s,\mathcal{Q}^*}(w) \quad &= \quad \left(\sum_l 2^{-2sl}\|(Q_l^* - Q_{l-1}^*)w\|_{L_2}^2\right)^{1/2} \\
&\lesssim \quad \left(\sum_j 2^{-2sl}2^{4sl}\|(Q_l - Q_{l-1})v\|_{L_2}^2\right)^{1/2} = N_{s,\mathcal{Q}}(v)
\end{aligned}
$$

which upon dividing by $N_{s,\mathcal{Q}}(v)$ yields (3) $\square$

Comments: (J), (B) are satisfied for all standard hierarchies of trial spaces where m is the order of polynomial exactness.

- Possible strategy for (J): Construct L_2-bounded local projectors onto S_j, use reproduction of polynomials and corresponding local polynomial inequalities;
- (B) follows from stability and rescaling arguments. Simple example: $s \in I\!N$ - estimate on reference domain - rescale and sum up.

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