Multiscale and Wavelet Methods for Operator Equations

(3) Scope of problems - Transformation to sequence spaces

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– Typeset by FoilTEX –

Problem Setting

$\mathcal{H}$ Hilbert space, $A(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to I\!R$ bilinear form
Problem: For a given $F \in \mathcal{H}'$ find $U \in \mathcal{H}$ s.t.

$$A(V, U) = \langle V, F \rangle, \quad V \in \mathcal{H} \tag{1}$$

Well-posedness: Define $\mathcal{L} : \mathcal{H} \to \mathcal{H}'$ by

$$\langle V, \mathcal{L}U \rangle = A(V, U), \quad V \in \mathcal{H}$$

First task: verify the Mapping Property (MP):

$$c_{\mathcal{L}} \|V\|_{\mathcal{H}} \leq \|\mathcal{L}V\|_{\mathcal{H}'} \leq C_{\mathcal{L}} \|V\|_{\mathcal{H}}, \quad V \in \mathcal{H} \tag{2}$$

Thus (1) has a unique solution for every $F \in \mathcal{H}'$.

In general:

- $\mathcal{H} = H_{1,0} \times \cdots H_{m,0}$, $H_{i,0} \subseteq H_i$, e.g. $H_0^{t_i}(\Omega_i) \subseteq H_{i,0} \subseteq H^{t_i}(\Omega_i)$

- $V = (v_1, \ldots, v_m)^T$, $\langle V, W \rangle = \sum_{i=1}^m \langle v_i, w_i \rangle_{H_i}$, $\|V\|_{\mathcal{H}}^2 = \sum_{i=1}^m \|v_i\|_{H_i}^2$;

- $\|W\|_{\mathcal{H}'} = \sup_{V \in \mathcal{H}} \frac{\langle V, W \rangle}{\|V\|_{\mathcal{H}}}$

- $A(V, W) = (a_{i,l}(v_i, w_l))_{i,l=1}^m$, $\rightsquigarrow \mathcal{L} = (\mathcal{L}_{i,l})_{i,l=1}^m$

Examples . . .

Scalar 2nd Order Elliptic BVP:

$$-\mathsf{div}\big(a(x)\nabla u\big) + k(x)u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega \quad \rightsquigarrow$$

$$a(v,w) := \int_\Omega a\nabla v^T \nabla w + kvwdx, \;\; \mathcal{H} = H_0^1(\Omega), \;\; \mathcal{H}' = H^{-1}(\Omega)$$

Obstructions:

- Sparse but very large linear systems $\rightsquigarrow$ iterative solvers
- Ill-conditioning $\;\mathrm{cond}_2(a(\Phi_h, \Phi_h) \sim h^{-2}$

Global Operators - Boundary Integral Equations

$$
\begin{aligned}
-\Delta U &= 0, \text{ on } \Omega, \quad (\Omega = \Omega^- \ \text{ or } \ \Omega^+ := I\!R^3 \setminus \Omega^-) \\
U &= f \quad \text{on } \Gamma := \partial\Omega^- \quad (U(x) \to 0, \ |x| \to \infty \ \text{ when } \ \Omega = \Omega^+)
\end{aligned}
$$

Fundamental solution - Single layer potential:

$$
\mathcal{E}(x,y) = \frac{1}{4\pi|x-y|}, \quad (\mathcal{L}u)(x) = (\mathcal{V}u)(x) := \int_\Gamma \mathcal{E}(x,y)u(y)d\Gamma_y, \quad x \in \Gamma
$$

Integral equation of the first kind: Find u s.t.

$$
\mathcal{V}u = f \quad \text{on} \quad \Gamma \quad \rightsquigarrow \quad U(x) = \int_\Gamma \mathcal{E}(x,y)u(y)d\Gamma_y, \quad x \in \Omega
$$

Here (see e.g.[Kr])

$$a(v,w) = \langle v, \mathcal{V}w\rangle_\Gamma, \quad \mathcal{H} = H^{-1/2}(\Gamma),\ \mathcal{H}' = H^{1/2}(\Gamma)$$

Integral equation of the second kind - double layer potential:

$$(\mathcal{K}v)(x) := \int\limits_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x,y)v(y)d\Gamma_y = \int\limits_\Gamma \frac{1}{4\pi}\frac{\nu_y^T(x-y)}{|x-y|^3}u(y)\,d\Gamma_y, \quad x \in \Gamma$$

$$\mathcal{L}u := (\frac{1}{2} \pm \mathcal{K})u = f \quad \rightsquigarrow \quad U(x) = \int\limits_\Gamma \mathcal{K}(x,y)u(y)d\Gamma_y$$

Here:

$$a(v,w) = \langle v, (\frac{1}{2} \pm \mathcal{K})w\rangle_\Gamma, \quad \mathcal{H} = L_2(\Gamma) = H_2 = \mathcal{H}'$$

Similarly for Neumann boundary conditions and Hypersingular operator $\mathcal{H} = H^{1/2}(\Gamma)$

Obstructions:

- Reduction of spatial dimension, discretization of compact domain – **BUT** – **densely populated matrices**
- **growing condition numbers** if the **order** of the operator is not zero (e.g. $\mathcal{L} = \mathcal{V}$)

Saddle Point Problems [BF, GR]

Given

$$|a(v,w)| \lesssim \|v\|_X \|w\|_X, \quad |b(q,v)| \lesssim \|v\|_X \|q\|_M.$$

find $U := (u,q) \in X \times M = \mathcal{H}$ s.t.

$$A(U,V) = \begin{cases} a(u,v) \quad + \quad b(p,v) & = \quad \langle f,v\rangle_X \quad \forall\, v \in X, \\ b(q,u) & = \quad \langle q,g\rangle_M \quad \forall\, q \in M, \end{cases}$$

$$a(v,w) =: \langle v, Aw\rangle_X, \quad v \in X, \quad b(v,p) =: \langle Bv, q\rangle_M, \quad q \in M$$

$$\mathcal{L}U := \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} =: F$$

Well-posedness/mapping property: (inf - sup condition)

$$a(v,v) \sim \|v\|_X^2, \quad v \in \ker B, \qquad \inf_{q\in M} \sup_{v\in X} \frac{b(v,q)}{\|v\|_X \|q\|_M} > \beta \quad \rightsquigarrow$$

$$c_{\mathcal{L}} \left(\|v\|_X^2 + \|q\|_M^2\right)^{1/2} \leq \|\mathcal{L}\begin{pmatrix} v \\ q \end{pmatrix}\|_{X'\times M'} \leq C_{\mathcal{L}} \left(\|v\|_X^2 + \|q\|_M^2\right)^{1/2}$$

Obstructions:

- . . . as before
- **Indefinite** problem
- Finite dimensional trial spaces for X and M must be **compatible - LBB condition**

Trace Theorem: $\|v\|_{H^{1/2}(\partial\Omega)} \lesssim \|v\|_{H^1(\Omega)}$ [BF, GR]

Second Order Problem - Fictitious Domains $\Omega \subset \Box$:

Find $U = (u, p) \in \mathcal{H} := H^1(\Box) \times H^{-1/2}(\Gamma)$, $\Gamma := \partial\Omega$, such that [BaH, Br, DK2, GG]

$$
\begin{aligned}
\langle \nabla v, \mathbf{a}\nabla u\rangle + \langle v, p\rangle_\Gamma &= \langle v, f\rangle \qquad \text{for all } v \in H^1(\Box),\\
\langle q, u\rangle_\Gamma &= \langle g, q\rangle \qquad \text{for all } q \in H^{-1/2}(\Gamma)
\end{aligned}
$$

First Order Systems [BLP1, BLP2]:

$$-\mathrm{div}(\mathbf{a}\nabla u) + ku = f \text{ on } \Omega,\ u = 0 \text{ on } \partial\Omega,\ \boldsymbol{\theta} := -\mathbf{a}\nabla u \rightsquigarrow$$

$$\begin{array}{rcccll} \langle \boldsymbol{\theta}, \boldsymbol{\eta} \rangle & + & \langle \boldsymbol{\eta}, \mathbf{a}\nabla u \rangle & = & 0, & \forall\ \boldsymbol{\eta} \in \mathbf{L}_2(\Omega) \\ -\langle \boldsymbol{\theta}, \nabla v \rangle & + & \langle ku, v \rangle & = & \langle f, v \rangle, & \forall v \in H^1_{0,\Gamma_D}(\Omega) \end{array}$$

$$U = (\boldsymbol{\theta}, u)\ \in\ \mathcal{H} := \mathbf{L}_2(\Omega) \times H^1_{0,\Gamma_D}(\Omega)$$

Stokes System [BF, GR]:

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \Omega,\\
\text{div } \mathbf{u} &= 0 \qquad \text{in } \Omega,\\
\mathbf{u}|_\Gamma &= \mathbf{0},
\end{aligned}
$$

Weak Formulation:

$$
X = \mathbf{H}_0^1(\Omega) := (H_0^1(\Omega))^d, \quad M = L_{2,0}(\Omega) := \{q \in L_2(\Omega) : \int\limits_\Omega q = 0\}
$$

$$
\begin{aligned}
\nu\langle \nabla\mathbf{v}, \nabla\mathbf{u}\rangle_{\mathbf{L}_2(\Omega)} + \langle \text{ div } \mathbf{v}, p\rangle_{L_2(\Omega)} &= \langle \mathbf{f}, \mathbf{v}\rangle \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega)\\
\langle \text{ div } \mathbf{u}, q\rangle_{L_2(\Omega)} &= \mathbf{0} \qquad q \in L_{2,0}(\Omega)
\end{aligned}
$$

Energy space: $\mathcal{H} = X \times M = \mathbf{H}_0^1(\Omega) \times L_{2,0}(\Omega), \quad U = (u, p)$

Stokes System - Fictitious Domain Formulation [GuH]:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \text{div } \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u}|_\Gamma &= \mathbf{g}, \\ \int_\Omega p\,dx &= 0, && (\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,ds = 0) \end{aligned}$$

$$U = (\mathbf{u}, \boldsymbol{\lambda}, p) \in \mathcal{H} := \mathbf{H}^1(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times L_{2,0}(\Omega)$$

$$\begin{aligned} \nu\langle\nabla\mathbf{v}, \nabla\mathbf{u}\rangle_{\mathbf{L}_2(\Omega)} + \langle\mathbf{v}, \boldsymbol{\lambda}\rangle_{\mathbf{L}_2(\Gamma)} + \langle \text{ div } \mathbf{v}, p\rangle_{L_2(\Omega)} &= \langle\mathbf{f}, \mathbf{v}\rangle && \forall\ \mathbf{v} \in \mathbf{H}^1(\Omega) \\ \langle\mathbf{u}, \boldsymbol{\mu}\rangle_{\mathbf{L}_2(\Gamma)} &= \langle\mathbf{g}, \boldsymbol{\mu}\rangle && \forall\ \boldsymbol{\mu} \in \mathbf{H}^{-1/2}(\Gamma) \\ \langle \text{ div } \mathbf{u}, q\rangle_{L_2(\Omega)} &= \mathbf{0} && \forall\ q \in L_{2,0}(\Omega) \end{aligned}$$

First Order Stokes System [CLMM]:

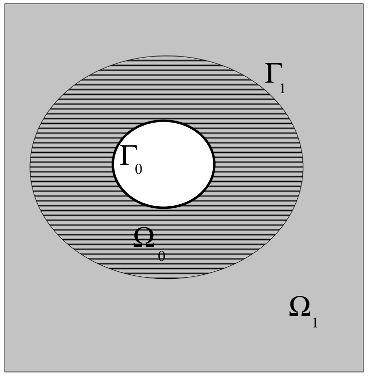
$$
\begin{aligned}
\underline{\boldsymbol{\theta}} + \nabla \mathbf{u} &= \underline{\mathbf{0}} \quad \text{in} \quad \Omega, \\
-\nu(\text{ div } \underline{\boldsymbol{\theta}})^T + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\
\text{div } \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\
\mathbf{u} &= \mathbf{g} \quad \text{on} \quad \Gamma,
\end{aligned}
$$

$$
U = (\underline{\boldsymbol{\theta}}, \mathbf{u}, p, \boldsymbol{\lambda}) \in \mathcal{H} := \underline{\mathbf{L}}_2(\Omega) \times \mathbf{H}^1(\Omega) \times L_{2,0}(\Omega) \times \mathbf{H}^{-1/2}(\Omega),
$$

$$
\begin{array}{llllcll}
\langle \underline{\boldsymbol{\theta}}, \underline{\boldsymbol{\eta}} \rangle & +\langle \underline{\boldsymbol{\eta}}, \nabla \mathbf{u} \rangle & & & = & 0, & \underline{\boldsymbol{\eta}} \in \underline{\mathbf{L}}_2(\Omega) \\
\nu \langle \underline{\boldsymbol{\theta}}, \nabla \mathbf{v} \rangle & & -\langle p, \text{ div } \mathbf{v} \rangle & -\langle \boldsymbol{\lambda}, \mathbf{v} \rangle_\Gamma & = & \langle \mathbf{f}, \mathbf{v} \rangle, & \mathbf{v} \in \mathbf{H}^1(\Omega) \\
 & \langle \text{ div } \mathbf{u}, q \rangle & & & = & 0, & q \in L_{2,0}(\Omega) \\
 & \langle \boldsymbol{\mu}, \mathbf{u} \rangle_\Gamma & & & = & \langle \boldsymbol{\mu}, \mathbf{g} \rangle_\Gamma, & \boldsymbol{\mu} \in \mathbf{H}^{-1/2}(\Gamma)
\end{array}
$$

Transmission Problem [CS]:

$$\begin{array}{rcll} -\nabla\cdot(\mathbf{a}\nabla u) & = & f & \text{in } \Omega_0 \\ -\Delta u & = & 0 & \text{in } \Omega_1, \\ u|_{\Gamma_0} & = & 0 & \\ \mathcal{H} & := & H^1_{0,\Gamma_D}(\Omega_0)\times H^{-1/2}(\Gamma_1) & \end{array}$$



Interf. cond'ns: $u^- = u^+, \quad (\partial_{\mathbf{n}})u^- = (\partial_{\mathbf{n}})u^+ \quad \rightsquigarrow$

$$\begin{array}{rcl} \langle \mathbf{a}\nabla u, \nabla v\rangle_{\Omega_0} + \langle \mathcal{W}u - (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\sigma, v\rangle_{(\Gamma_1} & = & \langle f, v\rangle_{\Omega_0} \\ v \in H^1_{0,\Gamma_D}(\Omega_0), & & \\ \langle(\tfrac{1}{2}\mathcal{I} - \mathcal{K})u, \delta\rangle_{\Gamma_1} + \langle \mathcal{V}\sigma, \delta\rangle_{\Gamma_1} & = & 0, \\ \delta \in H^{-1/2}(\Gamma_1) & & \end{array}$$

Further Obstructions:

- Occurrence and evaluation of difficult norms

  $\|\cdot\|_{H^{1/2}(\Gamma)}, \ \|\cdot\|_{H^{-1/2}(\Gamma)}, \ \|\cdot\|_{H^{-1}(\Omega)}$

- Mixed occurrence of local and global operators $\Delta, \ \mathcal{K}, \ \mathcal{V}$

An Equivalent ℓ_2-Problem

Wavelet characterization of component spaces (NE):

$$c_\Psi \|\mathbf{D}_i \mathbf{v}\|_{\ell_2(\mathcal{J}_i)} \le \|\mathbf{v}^T \Psi^i\|_{H_i} \le C_\Psi \|\mathbf{D}_i \mathbf{v}\|_{\ell_2(\mathcal{J}_i)}, \quad i = 1, \ldots, m$$

Notational conventions for systems:

$$\mathcal{J} := \mathcal{J}_1 \times \cdots \times \mathcal{J}_m, \quad \mathbf{D} := \mathrm{diag}\,(\mathbf{D}_1, \ldots, \mathbf{D}_m)\,, \quad \mathbf{V} = (\mathbf{v}^1, \ldots, \mathbf{v}^m)^T$$

$$\mathbf{V}^T \mathbf{D}^{-1} \boldsymbol{\Psi} := ((\mathbf{v}^1)^T \mathbf{D}_1^{-1} \Psi^1, \ldots, (\mathbf{v}^m)^T \mathbf{D}_m^{-1} \Psi^m)^T \quad \rightsquigarrow$$

Wavelet characterization of energy space (NE):

$$c_\Psi \|\mathbf{V}\|_{\ell_2} \le \|\mathbf{V}^T \mathbf{D}^{-1} \boldsymbol{\Psi}\|_{\mathcal{H}} \le C_\Psi \|\mathbf{V}\|_{\ell_2}$$

Standard representations of operators: $\boldsymbol{A}^{i,l} := \mathbf{D}_i^{-1} a_{i,l}(\Psi^i, \Psi^l)\mathbf{D}_l^{-1}$

$$\mathbf{L} := (\boldsymbol{A}_{i,l})_{i,l=1}^m = \mathbf{D}^{-1}\langle \boldsymbol{\Psi}, \mathcal{L}\boldsymbol{\Psi}\rangle \mathbf{D}^{-1}, \quad \mathbf{F} := \mathbf{D}^{-1}\langle \boldsymbol{\Psi}, F\rangle$$

Theorem: Let $U = \mathbf{U}^T\mathbf{D}^{-1}\Psi$. Then **(MP) + (NE)** $\Longrightarrow$

$$\mathcal{L}U = F \quad \Longleftrightarrow \quad \mathbf{LU} = \mathbf{F} \quad \text{and} \quad c_L\|\mathbf{V}\|_{\ell_2} \leq \|\mathbf{LV}\|_{\ell_2} \leq C_L\|\mathbf{V}\|_{\ell_2}$$

with $c_L := c_\Psi^2 c_{\mathcal{L}}$, $C_L := C_\Psi^2 C_{\mathcal{L}}$ [DKS].

Proof: Let $V = \mathbf{V}^T\mathbf{D}^{-1}\boldsymbol{\Psi}$

$$\begin{aligned} \|\mathbf{V}\|_{\ell_2} &\leq c_\Psi^{-1}\|V\|_{\mathcal{H}} \overset{\text{(MP)}}{\leq} c_\Psi^{-1}c_{\mathcal{L}}^{-1}\|\mathcal{L}V\|_{\mathcal{H}'} \overset{\text{(NE)}'}{\leq} c_\Psi^{-2}c_{\mathcal{L}}^{-1}\|\mathbf{D}^{-1}\langle\boldsymbol{\Psi}, \mathcal{L}V\rangle\|_{\ell_2} \\ &= c_\Psi^{-2}c_{\mathcal{L}}^{-1}\|\mathbf{D}^{-1}\langle\boldsymbol{\Psi}, \mathcal{L}\boldsymbol{\Psi}\rangle\mathbf{D}^{-1}\mathbf{V}\|_{\ell_2} = c_\Psi^{-2}c_{\mathcal{L}}^{-1}\|\mathbf{LV}\|_{\ell_2} \end{aligned}$$

The converse estimate works analogously in reverse order

Connection with Preconditioning

Let $\mathbf{\Psi}_\Lambda := \{\psi_\lambda : \lambda \in \Lambda \subset \mathcal{J}\}$, $S_\Lambda := \operatorname{span} \mathbf{\Psi}_\Lambda$, $\mathbf{L}_\Lambda := \mathbf{D}_\Lambda^{-1} A(\mathbf{\Psi}_\Lambda, \mathbf{\Psi}_\Lambda)\mathbf{D}^{-1}$

- If (1) is $\mathcal{H}$-elliptic $\rightsquigarrow$ $\mathbf{L}$ is symmetric positive definite (s.p.d.) $\rightsquigarrow$

$$\operatorname{cond}_2(\mathbf{L}_\Lambda) \le \frac{C_\Psi^2 C_\mathcal{L}}{c_\Psi^2 c_\mathcal{L}}$$

- If the Galerkin scheme for (1) is **stable**, i.e., $\|\mathbf{L}_\Lambda^{-1}\|_{\ell_2 \to \ell_2} = \mathcal{O}(1)$ $\rightsquigarrow$

$$\begin{aligned}
\|\mathbf{L}_\Lambda \mathbf{V}_\Lambda\|_{\ell_2} &= \|\mathbf{D}_\Lambda^{-1} A(\mathbf{\Psi}_\Lambda, \mathbf{\Psi}_\Lambda)\mathbf{D}^{-1}\mathbf{V}_\Lambda\|_{\ell_2} = \|\mathbf{D}_\Lambda^{-1}\langle \mathbf{\Psi}_\Lambda, \mathcal{L}V_\Lambda\rangle\|_{\ell_2} \\
&\le \|\mathbf{D}^{-1}\langle \mathbf{\Psi}, \mathcal{L}V_\Lambda\rangle\|_{\ell_2} \overset{(\mathrm{NE})'}{\le} c_\Psi^{-1}\|\mathcal{L}V_\Lambda\|_{\mathcal{H}'} \\
&\le c_\Psi^{-1} C_\mathcal{L} \|V_\Lambda\|_\mathcal{H} \overset{(\mathrm{NE})}{\le} c_\Psi^{-1} C_\mathcal{L} C_\Psi \|\mathbf{V}_\Lambda\|_{\ell_2}
\end{aligned}$$

Corollary: **(MP) + (NE) + stab. of Galerkin scheme** $\rightsquigarrow$

$$\mathrm{cond}_2(\mathbf{L}_\Lambda) = \mathcal{O}(1), \quad \#\Lambda \to \infty$$

Thus **(MP) + (NE)+ stab. of Galerkin scheme** imply that suitable diagonal scaling of wavelet representation of stiffness matrices ensure uniformly bounded condition numbers.

Galerkin schemes are **not** always stable! - e.g. for indefinite problems

There is always a Positive Definite Formulation -Least Squares

Theorem: Let $\mathbf{M} := \mathbf{L}^T\mathbf{L}, \quad \mathbf{G} := \mathbf{L}^T\mathbf{F}$ then

$$\mathcal{L}U = F \quad \Longleftrightarrow \quad \mathbf{MU} = \mathbf{F} \quad (\text{with } \mathbf{U} := \mathbf{U}^T\mathbf{D}^{-1}\mathbf{\Psi})$$

and

$$c_L^{-2}\|\mathbf{V}\|_{\ell_2} \leq \|\mathbf{MV}\|_{\ell_2} \leq C_L^2\|\mathbf{V}\|_{\ell_2}$$

Remark: $\mathcal{L}U = F$ iff U minimizes

$$\|\mathcal{L}V - F\|_{\mathcal{H}'} \sim \|\mathbf{MV} - \mathbf{G}\|_{\ell_2}$$

Natural norm least squares, [BLP1, BLP2, DKS]

Some Useful Facts - Condition Numbers

Remark: If $a(\cdot,\cdot) \sim \|\cdot\|_{H^t}^2$ s.p.d. and Ψ is L_2-Riesz-basis then for $J := \max\{|\lambda| : \lambda \in \Lambda\}$, $\min\{|\lambda| : \lambda \in \Lambda\} \lesssim 1$, $\Longrightarrow$

$$\operatorname{cond}_2\left(a(\Psi_\Lambda, \Psi_\Lambda)\right) \sim 2^{2|t|J}, \quad J \to \infty$$

$$\max_{v \in S_\Lambda} \frac{a(v,v)}{\|v\|_{L_2}^2} \geq \frac{a(\psi_\lambda, \psi_\lambda)}{\|\psi_\lambda\|_{L_2}^2} \geq \min_{v \in S_\Lambda} \frac{a(v,v)}{\|v\|_{L_2}^2} \quad \leadsto \quad \operatorname{cond}_2(a(\Psi_\Lambda, \Psi_\Lambda)) \gtrsim \frac{\frac{a(\psi_{\lambda_1},\psi_{\lambda_1})}{\|\psi_{\lambda_1}\|_{L_2}^2}}{\frac{a(\psi_{\lambda_2},\psi_{\lambda})}{\|\psi_{\lambda_2}\|_{L_2}^2}}$$

$$a(\psi_\lambda, \psi_\lambda) \sim \|\psi_\lambda\|_{H^t}^2 \quad \textbf{(NE)} \Longrightarrow 2^{t|\lambda|} \sim \|\psi_\lambda\|_{H^t} \leadsto$$

$$|\lambda_1| = \begin{cases} \max\{|\lambda| : \lambda \in \Lambda\} & \text{if} \quad t \geq 0 \\ \min\{|\lambda| : \lambda \in \Lambda\} & \text{if} \quad t < 0 \end{cases} \qquad |\lambda_2| = \begin{cases} \min\{|\lambda| : \lambda \in \Lambda\} & \text{if} \quad t \geq 0 \\ \max\{|\lambda| : \lambda \in \Lambda\} & \text{if} \quad t < 0 \end{cases}$$

$$\leadsto \quad \operatorname{cond}_2(a(\Psi_\Lambda, \Psi_\Lambda)) \gtrsim 2^{2J|t|}$$

Upper estimate:

- $t < 0$:

$$\min_{v \in S_\Lambda} \frac{a(v,v)}{\|v\|_{L_2}^2} \sim \min_{\boldsymbol{d}_\Lambda \in \mathbb{R}^{\#\Lambda}} \frac{\|\boldsymbol{d}_\Lambda^T \Psi_\Lambda\|_{H^t}^2}{\|\boldsymbol{d}_\Lambda\|_{\ell_2}^2} \overset{\text{(NE)}}{\sim} \min_{\boldsymbol{d}_\Lambda \in \mathbb{R}^{\#\Lambda}} \frac{\|\mathbf{D}^{-t} \boldsymbol{d}_\Lambda\|_{\ell_2}^2}{\|\boldsymbol{d}_\Lambda\|_{\ell_2}^2} \gtrsim 2^{-|t|J}$$

$$\max_{v \in S_\Lambda} \frac{a(v,v)}{\|v\|_{L_2}^2} \lesssim 1$$

- $t > 0$: (B) $\rightsquigarrow$

$$\max_{v \in S_\Lambda} \frac{a(v,v)}{\|v\|_{L_2}^2} \lesssim \max_{v \in S_\Lambda} \frac{\|v\|_{H^t}^2}{\|v\|_{L_2}^2} \lesssim 2^{Jt}$$

$\rightsquigarrow \quad \mathrm{cond}_2(a(\Psi_\Lambda, \Psi_\Lambda)) \lesssim 2^{2J|t|}$

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