Multiscale and Wavelet Methods for Operator Equations

(4) Adaptive wavelet schemes (I)

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Adaptive Wavelet Schemes [CDD1, CDD2, DDU]

Goal: Adaptive tracking of the *significant* wavelet coefficients of $U = \mathbf{U}^T \mathbf{D}^{-1} \mathbf{\Psi}$

$$\mathbf{U} \to \mathbf{U}(\epsilon) \quad \text{s.t.} \quad \|\mathbf{U} - \mathbf{U}(\epsilon)\|_{\ell_2} \leq \epsilon$$

with $\operatorname{supp} \mathbf{U}(\epsilon)$ as small as possible

Main Theme: How to Extract Significant Information?

Explicitly given data:	**Implicitly given data:**
images, signals, geometry	PDEs, integral equations
given functions	**solutions of operator eqs.**

Nonlinear / Best N-Term – Approximation

practically feasible	**idealized bench mark**
	for adaptive techniques

Adaptivity from Several Perspectives

- **Complexity Theory:** Adaptivity does **not** help! (Novak, Traub, Woczniakowski,...[N, TW])

- **Finite Elements:** Local mesh refinements – a-posteriori error estimators/indicators (Babuska/Rheinboldt, Bank et al, Verfürth, Johnson et al., Rannacher et al.,...Dörfler [Do, BR, BW, BEK, EEHJ])

- **Wavelet analysis:** Wavelet coefficients serve as error indicators (Averbuch/Beylkin/Coifman/Israeli, Bertoluzza/Maday, Canuto/Cravero, Dahlke/D./Hochmuth/Schneider,...[ABCI, Ber, BK, CM, DDHS, DHU])

  No convergence rates/proofs ! **d.o.f** $\overset{???}{\longleftarrow\!-\!\longrightarrow}$ **error**

Implicit Data – PDEs

- Hyperbolic Problems:
  Multiresolution of functionals of the solution - compression - perturbation analysis
  A. Harten, R. Abgral, F. Arrandiga, A. Cohen, W. D., R. DeVore, R. Donat, B. Lucier, O. Kaber, S. Müller, M. Postel,...

- Elliptic/Parabolic Systems: Primal multiresolution – convergence/complexity analysis
  Scope of problems:
  - elliptic boundary value problems
  - singular integral - boundary integral equations, transmission problems
  - noncoercive problems: saddle point problems (Stokes), multiple field problems

Main Issues

- **Work/Accuracy Balance:**
  A-priori estimates for quasi-uniform meshes:

$$\|u - u_h\|_{H^t} \lesssim h^\alpha \|u\|_{H^{t+\alpha}} = \epsilon \sim N^{-\alpha/d} \qquad N = \epsilon^{-d/\alpha}$$

Adaptive/nonlinear schemes: **d.o.f/work** N $\overset{???}{\longleftarrow - \longrightarrow}$ **accuracy** ϵ

$$u \in B^{t+\alpha} \quad \Longrightarrow \quad \|u - u(\epsilon)\|_{H^t} \le \epsilon, \quad N = \epsilon^{-d/\alpha}$$

- **Stabilizing effect** **No LBB condition!**

Classical Approach

- Variational formulation $\rightarrow$ discretization $\rightarrow$ finite dimensional problem $\rightarrow$ fast solvers

- Obstructions: Large systems, ill-conditioning, compatibility constraints (like LBB condition)

New Paradigm [CDD2]

(I) Variational formulation $\rightarrow$ mapping property

(II) Transformation into equivalent ∞ - dimensional well-posed ℓ_2 problem

(III) Convergent iteration for the ∞ - dimensional ℓ_2-problem

(IV) Adaptive application of operators

(III) Convergent Iteration for the ∞-Dimensional Problem

Objective:

$$\mathbf{LU} = \mathbf{F} \quad \Longleftrightarrow \mathbf{Mp} = \mathbf{G}$$

s.t. still

$$c_M \|\mathbf{q}\|_{\ell_2} \leq \|\mathbf{Mq}\|_{\ell_2} \leq C_M \|\mathbf{q}\|_{\ell_2}$$

and

$$\mathbf{p}^{n+1} = \mathbf{p}^n + \omega(\mathbf{G} - \mathbf{Mp}^n), \qquad \|\mathbf{I} - \omega \mathbf{M}\|_{\ell_2 \to \ell_2} \leq \rho < 1$$

Choice of $\mathbf{M}$

- Wavelet least squares formulation: (A. Cohen, R. DeVore, A. Kunoth, R. Schneider, W.D., [CDD2, DKS])

$$\mathbf{M} := \mathbf{L}^T\mathbf{L}, \quad \mathbf{G} := \mathbf{L}^T\mathbf{F}, \quad \mathbf{q} = \mathbf{U}$$

$$\mathbf{M}\mathbf{U} = \mathbf{G} \quad \Longleftrightarrow \quad \|\mathcal{L}U - F\|_{\mathcal{H}'} = \min_{V\in\mathcal{H}} \|\mathcal{L}V - F\|_{\mathcal{H}'}$$

- Saddle point problems: ([DDU])

$$\mathbf{M} = \text{Schur} \;\; \text{complement}$$

...recall: in the saddle point case: $\mathcal{H} = X \times M$

$$X \leftrightarrow \Psi_X \qquad M \leftrightarrow \Psi_M$$

$$c_X \|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)} \leq \|\mathbf{v}^T \mathbf{D}_X^{-1} \Psi_X\|_X \leq C_X \|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)},$$

$$c_M \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \leq \|\mathbf{q}^T \mathbf{D}_M^{-1} \Psi_M\|_M \leq C_M \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)},$$

Stokes: $\rightsquigarrow$ $D_{X,\lambda} = 2^{|\lambda|}, \quad D_{M,\lambda} = 1$

Saddle Point Problem Continued

$$\boldsymbol{A} := \mathbf{D}_X^{-1} a(\Psi_X, \Psi_X) \mathbf{D}_X^{-1}, \quad \mathbf{B} := \mathbf{D}_M^{-1} b(\Psi_M, \Psi_X) \mathbf{D}_X^{-1}$$

$$\mathbf{f} := \mathbf{D}_X^{-1} \langle \Psi_X, f \rangle_X, \quad \mathbf{g} := \mathbf{D}_M^{-1} \langle \Psi_M, g \rangle_M, \qquad \rightsquigarrow$$

$$\mathbf{L}\mathbf{U} = \mathbf{F} \quad \Longleftrightarrow \quad \underbrace{\begin{pmatrix} \boldsymbol{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{U}} = \underbrace{\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}}_{\mathbf{F}} \quad \rightsquigarrow$$

$$c_L \|\mathbf{V}\|_{\ell_2} \le \|\mathbf{L}\mathbf{V}\|_{\ell_2} \le C_L \|\mathbf{V}\|_{\ell_2}, \quad \mathbf{V} \in \ell_2$$

$$\underbrace{\begin{pmatrix} \boldsymbol{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{U}} = \underbrace{\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}}_{\mathbf{F}} \quad \Longleftrightarrow \quad \left\{ \begin{array}{lcl} \mathbf{M}\mathbf{p} := \mathbf{B}\boldsymbol{A}^{-1}\mathbf{B}^T\mathbf{p} & = & \mathbf{B}\boldsymbol{A}^{-1}\mathbf{f} - \mathbf{g} =: \mathbf{G} \\ \boldsymbol{A}\mathbf{u} & = & \mathbf{f} - \mathbf{B}^T\mathbf{p} \end{array} \right.$$

$$\mathbf{M} := \mathbf{B}\boldsymbol{A}^{-1}\mathbf{B}^T : \ell_2(\mathcal{J}_M) \to \ell_2(\mathcal{J}_M), \quad \|\mathbf{M}\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \sim \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)}$$

$$\begin{aligned} \mathbf{p} &= \mathbf{p} + \omega\left((\mathbf{B}\boldsymbol{A}^{-1}\mathbf{f} - \mathbf{g}) - \mathbf{M}\mathbf{p}\right) \\ &= \mathbf{p} + \omega\left(\mathbf{B}\,\underbrace{\boldsymbol{A}^{-1}(\mathbf{f} - \mathbf{B}^T\mathbf{p})}_{=\mathbf{u}} - \mathbf{g}\right) = \mathbf{p} + \omega(\mathbf{B}\mathbf{u} - \mathbf{g}) \quad \rightsquigarrow \end{aligned}$$

$$\rightsquigarrow \quad \mathbf{p}^{n+1} = \mathbf{p}^n + \omega(\mathbf{B}\mathbf{u}^n - \mathbf{g}), \quad \boldsymbol{A}\mathbf{u}^n = \mathbf{f} - \mathbf{B}^T\mathbf{p}^n$$

(IV) Approximate Iteration

$$\mathbf{M}\mathbf{U} = \mathbf{G} \quad \rightsquigarrow \quad \mathbf{U}^{n+1} = \mathbf{U}^n + \omega(\mathbf{G} - \mathbf{M}\mathbf{U}^n)$$

The Basic Routines: Need

RHS $[\eta, \mathbf{G}] \to \mathbf{G}_\eta$: s.t. $\|\mathbf{G} - \mathbf{G}_\eta\|_{\ell_2} \leq \eta$

APPLY $[\eta, \mathrm{M}, \mathrm{V}] \to \mathbf{W}_\eta$: s.t. $\|\mathbf{M}\mathbf{V} - \mathbf{W}_\eta\|_{\ell_2} \leq \eta$

COARSE $[\eta, \mathbf{W}] \to \bar{\mathbf{W}}_\eta$: s.t. $\|\mathbf{W} - \bar{\mathbf{W}}_\eta\|_{\ell_2} \leq \eta$

The Adaptive Algorithm

SOLVE $[\epsilon, \mathbf{M}, \mathbf{G}] \to \bar{\mathbf{U}}(\epsilon)$

(i) *Set* $\bar{\mathbf{U}}^0 = \mathbf{0}$, $\quad \epsilon_0 := c_M^{-1}\|\mathbf{G}\|_{\ell_2}$, $\quad j = 0$

(ii) *If* $\epsilon_j \le \epsilon$, *stop* $\bar{\mathbf{U}}^j \to \mathbf{U}(\epsilon)$. *Else* $\mathbf{V}^0 := \bar{\mathbf{U}}^j$.

(ii.1) **RHS** $[\rho^l \epsilon_j, \mathbf{G}] \to \mathbf{G}_l$; $\quad$ **APPLY** $[\rho^l \epsilon_j, \mathbf{M}, \mathbf{V}^l] \to \mathbf{W}^l$, $l = 0, \ldots, K-1$

$$\mathbf{V}^{l+1} := \mathbf{V}^l + \alpha(\mathbf{G}_l - \mathbf{W}^l)$$

(ii.2) **COARSE** $[\mathbf{V}^K, 2\epsilon_j/5] \to \bar{\mathbf{U}}^{j+1}$, $\quad \epsilon_{j+1} := \epsilon_j/2$, $j+1 \to j$ *go to* **(ii)**.

PROPOSITION: $\|\mathbf{U} - \bar{\mathbf{U}}^j\|_{\ell_2} \le \epsilon_j$, **Question:** $\#\mathrm{supp}\,\bar{\mathbf{U}}(\epsilon) \; \leftarrow \overset{?}{-} \rightarrow \; \epsilon$

Ideal Bench Mark - Best N-Term Approximation

$$\sigma_{N,\ell_2}(\mathbf{V}) := \|\mathbf{V} - \mathbf{V}_N\|_{\ell_2} = \min_{\#\mathrm{supp}\,\mathbf{W} \le N} \|\mathbf{V} - \mathbf{W}\|_{\ell_2}$$

$$(\mathbf{NE}) \implies \sigma_{N,\ell_2}(\mathbf{V}) \sim \inf_{\mathbf{W},\#\mathrm{supp}\,\mathbf{W} \le N} \|V - \mathbf{W}^T\mathbf{D}^{-1}\Psi\|_{\mathcal{H}}$$

Coarsening and best N-term approximation:

Let $\|\mathbf{v} - \mathbf{w}\|_{\ell_2} \le \eta/5$ with $\#\mathrm{supp}\,\mathbf{w} < \infty$, $\bar{\mathbf{w}}_\eta := \mathbf{COARSE}\,[\mathbf{w}, 4\eta/5]$.

- $\|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \lesssim N^{-s} \overset{\text{[CDD1]}}{\Longrightarrow} \#\mathrm{supp}\,\bar{\mathbf{w}}_\eta \lesssim \eta^{-1/s}, \quad \|\mathbf{v} - \bar{\mathbf{w}}_\eta\|_{\ell_2} \le \eta$

Compressible Matrices

Def.: $\mathbf{C} \in \mathcal{C}_{s^*}$ *if for any* $0 < s < s^*$ *and every* $j \in \mathbb{N}$ *there exists a matrix* $\mathbf{C}_j$ *obtained by replacing all but the order of* $\alpha_j 2^j$ $(\sum_j \alpha_j < \infty)$ *entries per row and column in* $\mathbf{C}$ *by zero while still*

$$\|\mathbf{C} - \mathbf{C}_j\| \leq \alpha_j 2^{-js}, \quad j \in \mathbb{N}, \quad \sum_j \alpha_j < \infty$$

$$(\textbf{Cancellation Properties}) \quad \Longrightarrow$$

Remark: The scaled wavelet representations $\mathbf{C} = \mathbf{D}^{-1} c(\Psi, \Psi) \mathbf{D}^{-1}$ of “all” differential and singular integral operators belong to $\mathcal{C}_{s^*}$ for some $s^* = s^*(\mathcal{L}, \Psi) > 0$.

Fast approximate matrix/vector multiplication [CDD1]

Define $\mathbf{v}_{[j]} := \mathbf{v}_{2^j},\ v_{[-1]} := 0$ and

$$\mathbf{w}_j := \boldsymbol{A}_j \mathbf{v}_{[0]} + \boldsymbol{A}_{j-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \boldsymbol{A}_0(\mathbf{v}_{[j]} - \mathbf{v}_{[j-1]}) \qquad (1)$$

$$\|\boldsymbol{A}\mathbf{v} - \mathbf{w}_j\|_{\ell_2} \quad \leq \quad c \underbrace{\|\mathbf{v} - \mathbf{v}_{[j]}\|_{\ell_2}}_{\sigma_{2^j,\ell_2}(\mathbf{v})} + \sum_{l=0}^{j} \alpha_l 2^{-ls} \underbrace{\|\mathbf{v}_{[j-l]} - \mathbf{v}_{[j-l-1]}\|_{\ell_2}}_{\lesssim \ \sigma_{2^{j-l-1},\ell_2}(\mathbf{v})} \quad \rightsquigarrow$$

MULT $[\eta, \boldsymbol{A}, \mathbf{v}] \to \mathbf{w}_\eta$ s.t.: $\|\boldsymbol{A}\mathbf{v} - \mathbf{w}_\eta\| \leq \eta$

Theorem: If $\boldsymbol{A} \in \mathcal{C}_{s^*}$ and $\|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \lesssim N^{-s}$, then

$$\#\mathrm{supp}\,\mathbf{w}_\eta, \quad \#\mathbf{flops} \lesssim \eta^{-1/s}, \quad (\#\mathbf{sort\ ops} \lesssim \eta^{-1/s}|\log\eta|)$$

$$\textbf{APPLY}\,[\eta, \mathbf{M}, \mathbf{V}], \qquad \textbf{RHS}\,[\eta, \mathbf{G}]$$

Composition of **COARSE** $[c\eta, \mathbf{W}]$, **MULT** $[c\eta, \boldsymbol{A}, \mathbf{V}]$ or the adaptive solution of an elliptic problem with accuracy $c\eta$.

Least squares: $\mathbf{M} = \mathbf{L}^T\mathbf{L}$, $\mathbf{G} = \mathbf{L}^T\mathbf{F}$

$$\textbf{RHS}_{ls}[\eta, \mathbf{G}] := \mathbf{MULT}\left[\tfrac{\eta}{2}, \mathbf{L}^T, \mathbf{COARSE}\,[\tfrac{\eta}{2C_L}, \mathbf{F}]\right]$$

$$\textbf{APPLY}_{ls}[\eta, \mathbf{M}, \mathbf{V}] := \mathbf{MULT}\left[\tfrac{\eta}{2}, \mathbf{L}^T, \mathbf{MULT}\,[\tfrac{\eta}{2C_L}, \mathbf{L}, \mathbf{V}]\right]$$

Application through Uzawa Iteration [DDU]

IDEAL UZAWA: *Given any* $\mathbf{p}_0 \in \ell_2(\mathcal{J}_M)$, *compute for* $i = 1, 2, \ldots$

$$\begin{aligned} \boldsymbol{A}\mathbf{u}_i &= \mathbf{f} - \mathbf{B}^T \mathbf{p}_{i-1}, \qquad (\text{set } \boldsymbol{R} := \langle \tilde{\Psi}_M, \tilde{\Psi}_M \rangle_M) \\ \mathbf{p}_i &= \mathbf{p}_{i-1} + \omega \boldsymbol{R}(\mathbf{B}\mathbf{u}_i - \mathbf{g}), \qquad \|\mathbf{I} - \omega \boldsymbol{R}\mathbf{S}\|_{\ell_2 \to \ell_2} \le \alpha < 1 \end{aligned}$$

REAL UZAWA: *Apply each step approximately*

Ingredients:

- $\mathbf{MULT}\,[\eta, \mathbf{C}, \mathbf{v}] \to \mathbf{w}_\eta$, s.t. $\|\mathbf{C}\mathbf{v} - \mathbf{w}_\eta\|_{\ell_2} \le \eta \quad (\mathbf{C} \in \{\boldsymbol{A}, \mathbf{B}, \mathbf{B}^T\})$
- $\mathbf{ELLSOLVE}\,[\eta, \boldsymbol{A}, \mathbf{r}] \to \bar{\mathbf{u}}(\eta)$, s.t. $\|\mathbf{u} - \bar{\mathbf{u}}(\eta)\|_{\ell_2(\mathcal{J}_X)} \le \eta$
- $\mathbf{COARSE}\,[\eta, \mathbf{v}] \to \mathbf{v}_\eta$, s.t. $\|\mathbf{v} - \mathbf{v}_\eta\|_{\ell_2} \le \eta$

$\mathbf{M}$ = Schur complement: $\rightsquigarrow$

$\mathbf{APPLY}_{Uz}[\eta, \mathbf{M}, \mathbf{q}]$ consists of finitely many (uniformly bounded) applications of approximate Uzawa iterations [DDU]

Complexity of $\mathbf{APPLY} \in \{\mathbf{MULT},\ \mathbf{APPLY}_{ls},\ \mathbf{APPLY}_{Uz}\}$?

Main Result - Convergence/Complexity [CDD2]

Theorem: *Assume that* $\mathbf{L} \in \mathcal{C}_{s^*}$*. Then if the exact solution* $U = \mathbf{U}^T \mathbf{D}^{-1} \Psi$ *satisfies for some* $s < s^*$

$$\inf_{\#\mathrm{supp}\mathbf{V} \le N} \|U - \mathbf{V}^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}} \ \lesssim \ N^{-s}$$

then, for any $\epsilon > 0$*, the approximations* $\bar{\mathbf{U}}(\epsilon)$ *produced by* **SOLVE** *satisfy*

$$\|U - \bar{\mathbf{U}}(\epsilon)^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}} \ \lesssim \ \epsilon.$$

and

$$\#\mathrm{supp}\, \bar{\mathbf{U}}(\epsilon), \ \text{comp. work} \ \lesssim \ \epsilon^{-1/s}.$$

- **No LBB condition**

References

[ABCI] A. Averbuch, G. Beylkin, R. Coifman, and M. Israeli, Multiscale inversion of elliptic operators, in: Signal and Image Representation in Combined Spaces, J. Zeevi, R. Coifman (eds.), Academic Press, 1995,

[BR] I. Babuška and W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. **15** (1978), 736–754.

[BW] R.E. Bank and A. Weiser, Some a posteriori error estimates for elliptic partial differential equations, Math. Comp., **44** (1985), 283–301.

[Ber] S. Bertoluzza, A-posteriori error estimates for wavelet Galerkin methods, Appl Math. Lett., **8** (1995), 1–6.

[BK] G. Beylkin, J. M. Keiser, An adaptive pseudo-wavelet approach for solving nonlinear partial differential equations, in: *Multiscale Wavelet Methods for PDEs*, W. Dahmen, A. J. Kurdila, P. Oswald (eds.) Academic Press, 1997, 137–197.

[BEK] F. Bornemann, B. Erdmann, and R. Kornhuber, A posteriori error estimates for elliptic problems in two and three space dimensions, SIAM J. Numer. Anal., **33** (1996), 1188–1204.

[CDD1] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet methods for elliptic operator equations – Convergence rates, Math. Comp. 70 (2001), 27–75.

[CDD2] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet methods II - Beyond the elliptic case, IGPM Report, RWTH Aachen, Nov. 2000.

[CM] A. Cohen, R. Masson, Wavelet adaptive methods for second order elliptic problems, boundary conditions and domain decomposition, Preprint 1997.

[DDHS] S. Dahlke, W. Dahmen, R. Hochmuth, R. Schneider, *Stable multiscale bases and local error estimation for elliptic problems*, Appl. Numer. Maths. 8, 1997, 21–47.

[DDU] S. Dahlke, W. Dahmen, K. Urban, Adaptive wavelet methods for saddle point problems – Convergence rates, IGPM Report # 204, RWTH Aachen, July 2001.

[DHU] S. Dahlke, R. Hochmuth, K. Urban, Adaptive wavelet methods for saddle point problems, Math. Model. Numer. Anal. (M2AN) 34(5) (2000), 1003-1022.

[DKS] W. Dahmen, A. Kunoth, R. Schneider, Wavelet least squares methods for boundary value problems, IGPM Report # 175, RWTH Aachen, Sep. 1999, to appear in SIAM J. Numer. Anal.

[Do] W. Dörfler, A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal., **33** (1996), 1106–1124.

[EEHJ] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, Introduction to adaptive methods for differential equations, Acta Numerica **4**, Cambridge University Press, (1995), 105–158.

[N] E. Novak, On the power of adaptation, J. Complexity, **12**(1996), 199–237.

[TW] J. Traub and H. Woźniakowski, *A General Theory of Optimal Algorithms*, Academic Press, New York, 1980.