Low rank tensor approximation

Krylov-type methods and perturbation analysis

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A few multilinear algebraic definitions I

- Inner product

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i,j,k} a_{ijk} b_{ijk}$$

- Frobenius norm

$$\|\mathcal{A}\|_F = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}$$

- Contracted tensor products

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle_{2,3} = \langle \mathcal{A}, \mathcal{B} \rangle_{-1} \qquad c_{\alpha\beta} = \sum_{j,k} a_{\alpha jk} b_{\beta jk}$$

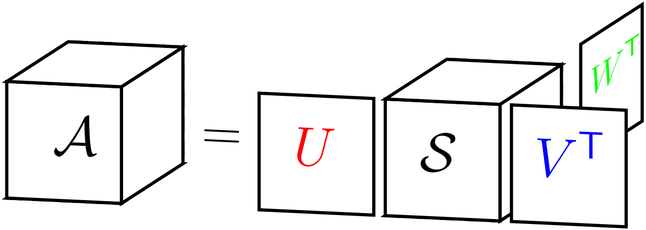
- Multilinear rank

$$\text{rank}\,(\mathcal{A}) = (r_1, r_2, r_3)$$

$$r_1 = \text{rank}(A^{(1)}) \qquad r_2 = \text{rank}(A^{(2)}) \qquad r_3 = \text{rank}(A^{(3)})$$

A few multilinear algebraic definitions II

- Multilinear tensor-matrix multiplication: U, V, W matrices



$$\mathcal{A} = (U, V, W) \cdot \mathcal{S} \qquad a_{ijk} = \sum_{\lambda,\mu,\nu} u_{i\lambda} v_{j\mu} w_{k\nu} s_{\lambda\mu\nu}$$

- convention

$$\left(U^{\mathsf{T}}, V^{\mathsf{T}}, W^{\mathsf{T}}\right) \cdot \mathcal{A} = \mathcal{A} \cdot (U, V, W)$$

- special cases

$$(U, V) \cdot A = UAV^{\mathsf{T}} \qquad \mathcal{A} \cdot (I, I, W) = \mathcal{A} \cdot (W)_3$$

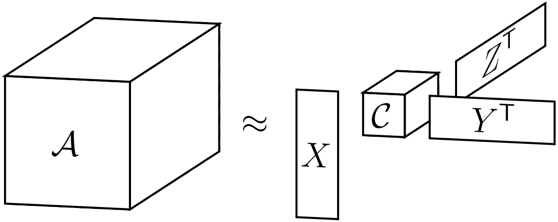
Multilinear low rank approximation

$$\min_{\mathcal{B}} \|\mathcal{A} - \mathcal{B}\| \qquad \text{rank}(\mathcal{B}) \leq (r_1, r_2, r_3)$$

With

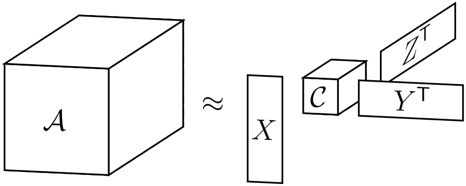
$$\mathcal{B} = (X, Y, Z) \cdot \mathcal{C}$$

$\mathcal{C}$ has dimensions $r_1 \times r_2 \times r_3$



Multilinear low rank approximation

$$\min \|\mathcal{A} - (X, Y, Z) \cdot \mathcal{C}\| \qquad \min \left\| A - XCY^{\mathsf{T}} \right\|$$



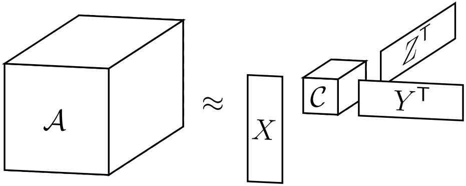
Problem overparameterized: $\mathcal{C}$ can be eliminated
Problem equivalent to

$$\max \|\mathcal{A} \cdot (X, Y, Z)\| \qquad \max \|X^{\mathsf{T}} A Y\|$$

$$X^{\mathsf{T}} X = I \qquad Y^{\mathsf{T}} Y = I \qquad Z^{\mathsf{T}} Z = I$$

Multilinear low rank approximation

$$\min \|\mathcal{A} - (X, Y, Z) \cdot \mathcal{C}\| \qquad \min \left\| A - XCY^{\mathsf{T}} \right\|$$



In addition to $\quad X^{\mathsf{T}}X = I \qquad Y^{\mathsf{T}}Y = I \qquad Z^{\mathsf{T}}Z = I$

$$\|\mathcal{A} \cdot (X, Y, Z)\| = \|\mathcal{A} \cdot (XU, YV, ZW)\|$$
$$\left\| X^{\mathsf{T}}AY \right\| = \left\| U^{\mathsf{T}}X^{\mathsf{T}}AYV \right\|$$

Grassmann manifold problem!

Computing the core $\mathcal{C}$ and the low rank approximation

Given orthonormal X, Y, Z

$$\min_{\mathcal{C}} \|\mathcal{A} - (X, Y, Z) \cdot \mathcal{C}\| \qquad \min_{C} \left\| A - XCY^{\mathsf{T}} \right\|$$

Solution

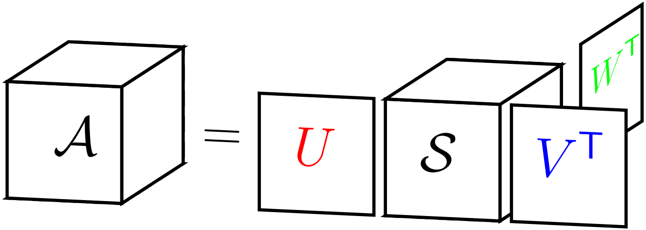
$$\mathcal{C} = \mathcal{A} \cdot (X, Y, Z) \qquad C = X^{\mathsf{T}} A Y$$

and the low rank approximation is

$$\hat{\mathcal{A}} = (X, Y, Z) \cdot \mathcal{C} = \left(XX^{\mathsf{T}}, YY^{\mathsf{T}}, ZZ^{\mathsf{T}} \right) \cdot \mathcal{A}$$

$$\hat{A} = XCY^{\mathsf{T}} = XX^{\mathsf{T}} A YY^{\mathsf{T}} = \left(XX^{\mathsf{T}}, YY^{\mathsf{T}} \right) \cdot A$$

HOSVD [De Lathauwer et al. 2000]



$$\mathcal{A} = (U, V, W) \cdot \mathcal{S} \qquad a_{ijk} = \sum_{\lambda,\mu,\nu} u_{i\lambda} v_{j\mu} w_{k\nu} s_{\lambda\mu\nu}$$

- U, V, W are orthogonal
- $\mathcal{S}$ is all-orthogonal

Methods for best low rank approximation

- Alternating minimization
  1. HOOI (Kroonenberg, De Lathauwer)
  2. "Trace maximization" (Regalia)
- Grassmann manifold approach
  1. Newton (Eldén, Savas)
  2. Trust region/Newton (Ishteva, De Lathauwer et al.)
  3. BFGS quasi-Newton (Savas, Lim)
  4. Limited memory BFGS (Savas, Lim)
  5. Symmetric tensors, without explicit representation (Morton)
- In this talk we will consider Krylov-type tensor computations.

OBS: This approach does not solve the best low rank problem

Krylov-type tensor computations

Krylov subspaces for matrices

Given $A \in \mathbb{R}^{m \times m}$ and starting vector $v \in \mathbb{R}^m$

$$\mathcal{K}_k(A, v) = \text{span}\{v, Av, A^2v, \ldots, A^{k-1}v\}$$

If $v_1 = v$

$$\mathcal{K}_k(A, v) = \text{span}\{v_1, v_2, v_3, \ldots, v_k\}$$

$$v_{i+1} = Av_i, \qquad i = 1, \ldots, k-1$$

Specifically useful for large and sparse problems

- Systems of linear equations
- Eigenvalues and eigenvectors
- Singular values and singular vectors
- Approximation of matrices and functions of matrices

A general square—The Arnoldi process

for $k = 1, 2, \ldots$ **do**

1 $h_k = U_k^{\mathsf{T}} A u_k$

2 $v = A u_k - U_k h_k$

3 $\beta_k = h_{k+1,k} = \|v\|_2$

4 $u_{k+1} = v / \beta_k$

5 $\hat{H}_k = \begin{pmatrix} \hat{H}_{k-1} & h_k \\ 0 & h_{k+1,k} \end{pmatrix}$

end for

- The Arnoldi decomposition:

$$A U_k = U_{k+1} \hat{H}_k, \qquad \hat{H}_k \sim (k+1) \times k \quad \text{Hessenberg}$$

- If A is symmetric $\Rightarrow$ Lanczos recurrence.

$A \in \mathbb{R}^{m\times n}$—Golub-Kahan bidiagonalization

- $\beta_1 v_1, \qquad u_0 = 0$

  for $j = 1, 2, \ldots, k$ **do**

  1 $\alpha_j u_j = A^{\mathsf{T}} v_j - \beta_j u_{j-1}$

  2 $\beta_{j+1} v_{j+1} = A u_j - \alpha_j v_j$

  end for

  α_j, β_j are chosen to normalize u_j, v_j
- $U_k = [u_1, \ldots, u_k]$ and $V_{k+1} = [v_1, \ldots, v_{k+1}]$ we have

$$AU_k = V_{k+1} B_{k+1}, \qquad V_k^{\mathsf{T}} V_k = I, \quad U_{k+1}^{\mathsf{T}} U_{k+1} = I$$

  and B_{k+1} is bidiagonal,
- U_k and V_k orthonormal basis for

$$\mathcal{K}_k(A^{\mathsf{T}}A, u) = \mathrm{span}\{u, (A^{\mathsf{T}}A)u, (A^{\mathsf{T}}A)^2 u, \ldots, (A^{\mathsf{T}}A)^{k-1} u\}$$
$$\mathcal{K}_k(AA^{\mathsf{T}}, v) = \mathrm{span}\{v, (AA^{\mathsf{T}})v, (AA^{\mathsf{T}})^2 v, \ldots, (AA^{\mathsf{T}})^{k-1} v\}$$

Matrix-vector and tensor-vector products

- $A \in \mathbb{R}^{m \times n}, \quad v \in \mathbb{R}^n$

$$Av = A \cdot (v)_2 \in \mathbb{R}^m$$

- $\mathcal{A} \in \mathbb{R}^{l \times m \times n}, \quad v \in \mathbb{R}^m, \quad w \in \mathbb{R}^n$

$$\mathcal{A} \cdot (w)_3 \in \mathbb{R}^{l \times m} \qquad [\mathcal{A} \cdot (w)_3]_{ij} = \sum_k a_{ijk} w_k$$

$$\mathcal{A} \cdot (v, w)_{2,3} \in \mathbb{R}^l \qquad [\mathcal{A} \cdot (v, w)_{2,3}]_i = \sum_{jk} a_{ijk} v_j w_k$$

- similarly

$$\mathcal{A} \cdot (u, w)_{1,3} \in \mathbb{R}^m \qquad \mathcal{A} \cdot (u, v)_{1,2} \in \mathbb{R}^n$$

Minimal Krylov method I

$\mathcal{A} \in \mathbb{R}^{l \times m \times n}$ and starting vectors with norm one
$u_1 \in \mathbb{R}^l$, $v_1 \in \mathbb{R}^m$, $w_1 \in \mathbb{R}^n$

$$\begin{aligned} u_{i+1} &= \mathcal{A} \cdot (v_i, w_i)_{2,3} & i &= 1, \ldots, k-1 \\ v_{i+1} &= \mathcal{A} \cdot (u_i, w_i)_{1,3} & i &= 1, \ldots, k-1 \\ w_{i+1} &= \mathcal{A} \cdot (u_i, v_i)_{1,2} & i &= 1, \ldots, k-1 \end{aligned}$$

Set

$$U_k = [u_1 \ u_2 \ \ldots \ u_k] \quad V_k = [v_1 \ v_2 \ \ldots \ v_k] \quad W_k = [w_1 \ w_2 \ \ldots \ w_k]$$

orthogonalize explicitly using Gram–Schmidt on each sequence

Minimal Krylov method II

$\mathcal{A} \in \mathbb{R}^{l \times m \times n}$ and starting vectors with norm one
$u_1 \in \mathbb{R}^l$, $v_1 \in \mathbb{R}^m$, $w_1 \in \mathbb{R}^n$

$$\begin{aligned} u_{i+1} &= \mathcal{A} \cdot (v_i, w_i)_{2,3} && i = 1, \dots, k-1 \\ v_{i+1} &= \mathcal{A} \cdot (u_{i+1}, w_i)_{1,3} && i = 1, \dots, k-1 \\ w_{i+1} &= \mathcal{A} \cdot (u_{i+1}, v_{i+1})_{1,2} && i = 1, \dots, k-1 \end{aligned}$$

Orthogonalize, set

$$U_k = [u_1 \ u_2 \ \dots \ u_k] \quad V_k = [v_1 \ v_2 \ \dots \ v_k] \quad W_k = [w_1 \ w_2 \ \dots \ w_k]$$

and approximate

$$\mathcal{A} \approx \left(U_k U_k^{\mathsf{T}}, V_k V_k^{\mathsf{T}}, W_k W_k^{\mathsf{T}} \right) \cdot \mathcal{A}$$

Krylov method on a low rank matrix

Let $A \in \mathbb{R}^{n \times n}$ with $\operatorname{rank}(A) = k$.

$$A = U_k \Sigma_k V_k^\mathsf{T}$$

then

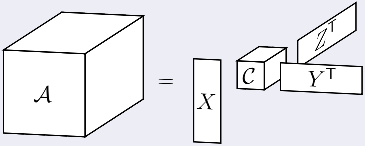
$$\mathcal{K}_k(A, u) = \mathcal{K}_{k+p}(A, u) \qquad p \geq 1$$

We only need to do k steps of Arnoldi.

Low rank tensor and minimal Krylov method

Theorem

Let $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$ *with* $\text{rank}(\mathcal{A}) = (p, q, r)$ *and assume* $p \leq q \leq r$. *Then*



$$\mathcal{A} = (X, Y, Z) \cdot \mathcal{C}$$

- *In p steps we get* U_p*, s.t.* $\text{span}(U_p) = \text{span}(X)$
- *In q steps we get* V_q*, s.t.* $\text{span}(V_q) = \text{span}(Y)$
- *In r steps we get* W_r*, s.t.* $\text{span}(W_r) = \text{span}(Z)$

using a "modified" minimal Krylov recursion.

Low rank tensors + noise and minimal Krylov method

Theorem

Let $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$ *with* $\operatorname{rank}(\mathcal{A}) = (p, q, r)$ *and add noise* $\rho\mathcal{E}$, *again assume* $p \leq q \leq r$. *Then*

$$\mathcal{A} = (X, Y, Z) \cdot \mathcal{C} + \rho\mathcal{E}$$

- *In* p *steps we get* U_p, *s.t.* $\operatorname{span}(U_p) \approx \operatorname{span}(X)$ *within level of noise*
- *In* q *steps we get* V_q, *s.t.* $\operatorname{span}(V_p) \approx \operatorname{span}(Y)$ *within level of noise*
- *In* r *steps we get* W_r, *s.t.* $\operatorname{span}(W_p) \approx \operatorname{span}(Z)$ *within level of noise*

using a "modified" minimal Krylov recursion.

Maximal Krylov Method

Generate all possible combinations at each step

$$
\begin{aligned}
\{u_1\} \times \{v_1\} &\longrightarrow w_1 \\
\{v_1\} \times \{w_1\} &\longrightarrow u_2 \\
\{u_1, u_2\} \times \{w_1\} &\longrightarrow \{v_2\ v_3\} \\
\{u_1, u_2\} \times \{v_1, v_2, v_3\} &\longrightarrow \{w_1, w_2, w_3, w_4, w_5, w_6\} \\
\{v_1, v_2, v_3\} \times \{w_1, w_2, \ldots, w_6\} &\longrightarrow \{u_2, u_3, \ldots, u_{19}\} \\
\{u_1, u_2, \ldots, u_{19}\} \times \{w_1, w_2, \ldots, w_6\} &\longrightarrow \{v_2, v_3, v_4, \ldots, v_{115}\}
\end{aligned}
$$

Krylov factorization for maximal recursion

Theorem (Tensor Krylov factorizations)

After a complete u-loop:

$$\mathcal{A} \cdot (V_k, W_l)_{2,3} = (U_j)_1 \cdot \mathcal{H}_{jkl}.$$

After a complete v-loop:

$$\mathcal{A} \cdot (U_j, W_l)_{1,3} = (V_m)_2 \cdot \mathcal{H}_{jml}.$$

After a complete w-loop:

$$\mathcal{A} \cdot (U_j, V_m)_{1,2} = (W_n)_3 \cdot \mathcal{H}_{jmn}.$$

Example of a maximal Krylov method

1. $\{u_1\} \times \{v_1\} \longrightarrow w_1 \Rightarrow \mathcal{A} \cdot (u_1, v_1)_{1,2} = (w_1)_3 \cdot \mathcal{H}_{111}$
2. $\{v_1\} \times \{w_1\} \longrightarrow u_2 \Rightarrow \mathcal{A} \cdot (v_1, w_1)_{2,3} = ([u_1 \ u_2])_1 \cdot \mathcal{H}_{211}$
3. $\{u_1, u_2\} \times \{w_1\} \longrightarrow \{v_2 \ v_3\}$

   $$\Rightarrow \mathcal{A} \cdot ([u_1 \ u_2], w_1)_{1,3} = ([v_1 \ v_2 \ v_3])_2 \cdot \mathcal{H}_{231}$$

4. $\{u_1, u_2\} \times \{v_1, v_2, v_3\} \longrightarrow \{w_1, w_2, w_3, w_4, w_5, w_6\}$

   $$\Rightarrow \mathcal{A} \cdot ([u_1 \ u_2], [v_1 \ v_2 \ v_3])_{1,2} = ([w_1 \ \cdots \ w_6])_2 \cdot \mathcal{H}_{236}$$

5. ....
6. the bad: dimension of subspaces explode

Krylov subspaces of contracted tensor products

Recall: for a matrix $A \in \mathbb{R}^{m\times n}$ we have

$$AA^\mathsf{T} = \langle A, A\rangle_{-1} \in \mathbb{R}^{m\times m}, \qquad A^\mathsf{T}A = \langle A, A\rangle_{-2} \in \mathbb{R}^{n\times n}$$

$\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $u \in \mathbb{R}^m, v \in \mathbb{R}^n, w \in \mathbb{R}^l$ and consider

$$\begin{aligned}
\langle \mathcal{A}, \mathcal{A}\rangle_{-1} &= A^{(1)}(A^{(1)})^\mathsf{T} \in \mathbb{R}^{m\times m} & \qquad \mathcal{K}_k(\langle \mathcal{A}, \mathcal{A}\rangle_{-1}, u)\\
\langle \mathcal{A}, \mathcal{A}\rangle_{-2} &= A^{(2)}(A^{(2)})^\mathsf{T} \in \mathbb{R}^{n\times n} & \qquad \mathcal{K}_k(\langle \mathcal{A}, \mathcal{A}\rangle_{-2}, v)\\
\langle \mathcal{A}, \mathcal{A}\rangle_{-3} &= A^{(3)}(A^{(3)})^\mathsf{T} \in \mathbb{R}^{l\times l} & \qquad \mathcal{K}_k(\langle \mathcal{A}, \mathcal{A}\rangle_{-3}, w)
\end{aligned}$$

Symmetric matrices! Apply the Lanczos recurrence.

All computations are implemented using

$$A \cdot (v, w)_{2,3} \qquad A \cdot (u, w)_{1,3} \qquad A \cdot (u, v)_{1,2}$$

Optimal subspaces give truncated HOSVD

Optimized tensor-Krylov approach

Let $U_i = [u_1 \ \cdots \ u_i]$, $V_i = [v_1 \ \cdots \ v_i]$, and $W_{i-1} = [w_1 \ \cdots \ w_{i-1}]$.
Find θ and η that give optimal $\widehat{w}$

$$\widehat{w} = \mathcal{A} \cdot (U_i\theta, V_i\eta)_{1,2} \, ,$$

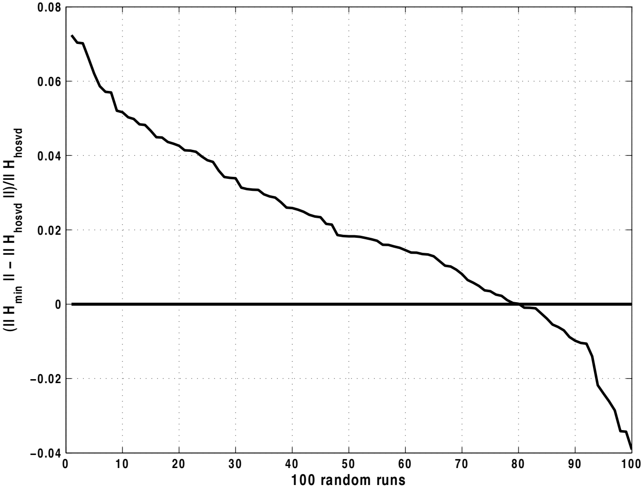
$$\max_{\theta,\eta} \|\widehat{w}\|, \quad \text{s.t.} \quad \widehat{w} \perp W_{i-1}, \|\theta\| = \|\eta\| = 1, \quad \theta, \eta \in \mathbb{R}^i.$$

Solution: best rank-$(1,1,1)$ approximation

$$(\theta, \eta, \omega) \cdot \mathcal{S}_{111} \approx A \cdot \left(U_i, V_i, I - W_{i-1} W_{i-1}^{\mathsf{T}} \right) .$$

[Goreinov, Oseledets and Savostyanov 2010] and [Savas and Eldén 2010]

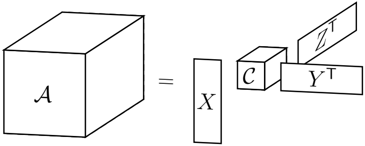
Experiments: Minimal Krylov vs truncated HOSVD



Difference between $\|\hat{\mathcal{A}}_{\text{min}}\|$ and $\|\hat{\mathcal{A}}_{\text{hosvd}}\|$, of a $50 \times 60 \times 40$ tensor $\mathcal{A}$. Rank of approximation is $(10, 10, 10)$.

HOSVD using the minimal tensor Krylov method

Let $\mathcal{A}$ have exact low rank



with HOSVD $\mathcal{A} = (U, V, W) \cdot \mathcal{S}$

Alternative I

1. SVD of $A^{(1)}$ gives U
2. SVD of $A^{(1)}$ gives V
3. SVD of $A^{(1)}$ gives W
4. Compute $S = A \cdot (U, V, W)$

Alternative II

1. Minimal Krylov method on $\mathcal{A}$ gives U_p, V_q, and W_r
2. Compute $\mathcal{C} = \mathcal{A} \cdot (U_p, V_q, W_r)$
3. Compite HOSVD of $\mathcal{C} = \left(\bar{U}, \bar{V}, \bar{W}\right) \cdot \mathcal{S}$
4. Change basis: $U = U_p \bar{U}$, $V = V_q \bar{V}$, $W = W_r \bar{W}$

Experiments: Applied on the Natflix data

Tensor of size

- User mode: 480189 — number of users
- Movie mode: 17770 — number of movies
- Time mode: 2243 — number of days

Time for computing a rank-$(100, 100, 100)$ approximation: 17 hours
Bottleneck:

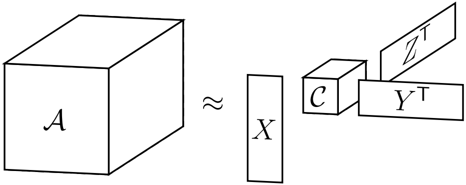
$$u = \mathcal{A} \cdot (v, w)_{2,3}$$

- Efficient storage schemes are needed
- Do not compute $\mathcal{A} \cdot (w)_3$ as it will be dense

Optimality conditions

Recall: Multilinear low rank approximation

$$\min \|\mathcal{A} - (X, Y, Z) \cdot \mathcal{C}\|$$



Problem overparameterized, nonlinear, and equivalent to

$$\max f(X, Y, Z), \quad \text{s.t.} \quad X^\mathsf{T} X = I \qquad Y^\mathsf{T} Y = I \qquad Z^\mathsf{T} Z = I$$

where $f(X, Y, Z) = \|\mathcal{A} \cdot (X, Y, Z)\|_F^2$

First order optimality conditions

- Objective function $f(X, Y, Z) = \|A \cdot (X, Y, Z)\|_F^2$
- At a stationary point (X, Y, Z) we have $\nabla f = 0$

$$\nabla f_x = \langle \mathcal{A} \cdot (X, Y, Z), \mathcal{A} \cdot (X_\perp, Y, Z) \rangle_{-1} = 0$$
$$\nabla f_y = \langle \mathcal{A} \cdot (X, Y, Z), \mathcal{A} \cdot (X, Y_\perp, Z) \rangle_{-2} = 0$$
$$\nabla f_z = \langle \mathcal{A} \cdot (X, Y, Z), \mathcal{A} \cdot (X, Y, Z_\perp) \rangle_{-3} = 0$$

- $[X \ X_\perp]$, $[Y \ Y_\perp]$, $[Z \ Z_\perp]$ are orthogonal
- For the matrices we have $f(X, Y) = \|X^{\mathsf{T}} A Y\|_F^2$

$$\langle X^{\mathsf{T}} A X, X_\perp^{\mathsf{T}} A Y \rangle_{-1} = X^{\mathsf{T}} A X (X_\perp^{\mathsf{T}} A Y)^{\mathsf{T}} = 0$$

- Let $[U_k \ U_\perp]$ and $[V_k \ V_\perp]$ contain left and right singular vectors, then

$$[U_k \ U_\perp]^{\mathsf{T}} A [V_k \ V_\perp] = \begin{bmatrix} U_k^{\mathsf{T}} A V_k & U_k^{\mathsf{T}} A V_\perp \\ U_\perp^{\mathsf{T}} A V_k & U_\perp^{\mathsf{T}} A V_\perp \end{bmatrix} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & \Sigma_\perp \end{bmatrix}$$

WOOW..., what is that??

$$\langle \mathcal{A} \cdot (X, Y, Z), \mathcal{A} \cdot (X_\perp, Y, Z) \rangle_{-1} = 0$$

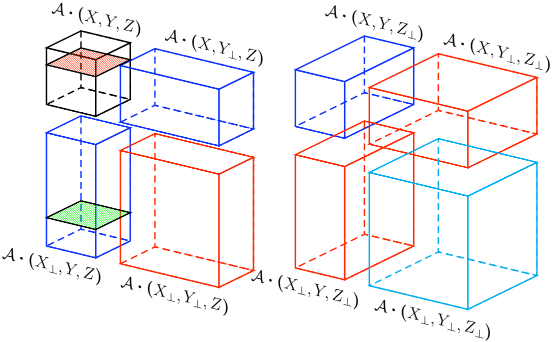


Figure visualizes the generalization of

$$[U_k \ U_\perp]^\mathsf{T} A [V_k \ V_\perp] = \begin{bmatrix} U_k^\mathsf{T} A V_k & U_k^\mathsf{T} A V_\perp \\ U_\perp^\mathsf{T} A V_k & U_\perp^\mathsf{T} A V_\perp \end{bmatrix} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & \Sigma_\perp \end{bmatrix}$$

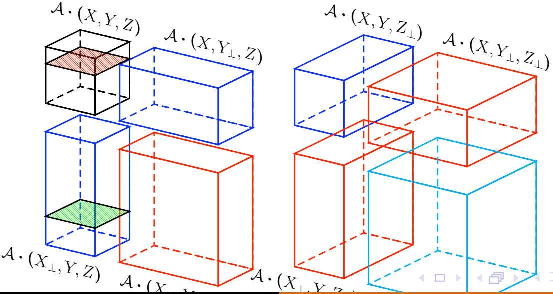
Second order optimality conditions ⇒ ordering

- (X, Y, Z) is a local max of $f(X,Y,Z) = \|\mathcal{A} \cdot (X, Y, Z)\|_F^2$ if the Hessian is negative definite.

Theorem

Let (X, Y, Z) be a local min and* $\mathcal{B} = \begin{bmatrix} \mathcal{A} \cdot (X, Y, Z) \\ \mathcal{A} \cdot (X_\perp, Y, Z) \end{bmatrix}$ *then*

$$\|\mathcal{B}(1,:,:)\| \geq \cdots \geq \|\mathcal{B}(r_1,:,:)\| > \|\mathcal{B}(r_1+1,:,:)\| \geq \cdots \geq \|\mathcal{B}(I,:,:)\|$$



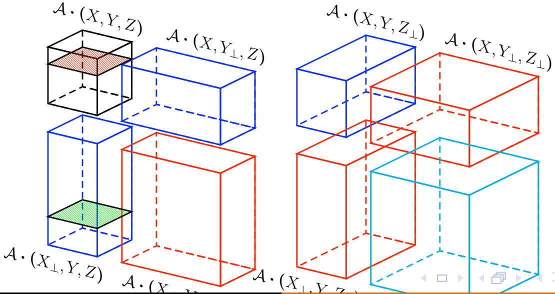
We can also get orthogonality

- (X, Y, Z) is a local max of $f(X, Y, Z) = \| \mathcal{A} \cdot (X, Y, Z) \|_F^2$ if the Hessian is negative definite.

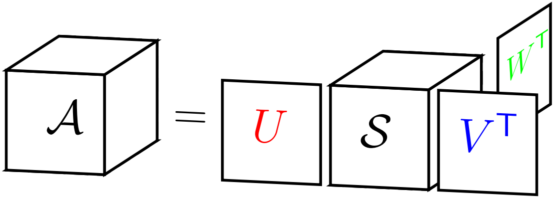
Theorem

Let (X, Y, Z) *be a local min** *and* $\mathcal{B} = \begin{bmatrix} \mathcal{A} \cdot (X, Y, Z) \\ \mathcal{A} \cdot (X_\perp, Y, Z) \end{bmatrix}$ *then*

$$\langle \mathcal{B}(i, :, :), \mathcal{B}(j, :, :) \rangle = 0, \quad i \neq j$$



Compare with the HOSVD ...



- U, V, W are orthogonal
- $\mathcal{S}$ is all-orthogonal:

$$\langle \mathcal{S}(:,:,i), \mathcal{S}(:,:,j) \rangle = 0 \quad i \neq j$$

$$\|\mathcal{S}(:,:,1)\| \geq \|\mathcal{S}(:,:,2)\| \geq \|\mathcal{S}(:,:,3)\| \geq \cdots$$

- Properties hold in all three modes simultaniously

Perturbation theory and concept of "gap" for tensors

Perturbation analysis: setup

- (X, Y, Z) representative of a stationary point for

$$\|\mathcal{A} - (X, Y, Z) \cdot \mathcal{C}\|_F^2$$

- Now perturbe $\mathcal{A}$ with a small $\mathcal{E}$

$$\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{E}$$

- What are the first order perturbation $(\tilde{X}, \tilde{Y}, \tilde{Z})$ of the stationary point?

$$\tilde{X} = X + \delta X \qquad \tilde{Y} = Y + \delta Y \qquad \tilde{Z} = Z + \delta Z$$

- We want to bound δX, δY, and δZ in terms of some properties of $\mathcal{A}$

First order optimality condition

- The perturbed point $(\tilde{X}, \tilde{Y}, \tilde{Z})$ has to satisfy the first order optimality conditions, i.e. $\nabla f = (\nabla f_x, \nabla f_y, \nabla f_z) = 0$

$$\nabla f_x = \left\langle \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right), \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}_\perp, \widetilde{Y}, \widetilde{Z}\right) \right\rangle_{-1} = 0$$

$$\nabla f_y = \left\langle \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right), \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}, \widetilde{Y}_\perp, \widetilde{Z}\right) \right\rangle_{-2} = 0$$

$$\nabla f_z = \left\langle \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right), \widetilde{\mathcal{A}} \cdot \left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}_\perp\right) \right\rangle_{-3} = 0$$

First order optimality condition: "shake the gradient"...

- ..., and we get the Hessian! In operator form we have

$$\begin{bmatrix} H_{xx} & H_{xy} & H_{xz} \\ H_{yx} & H_{yy} & H_{yz} \\ H_{zx} & H_{zy} & H_{zz} \end{bmatrix} \begin{bmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{bmatrix}$$

- The Hessian in the LHS and some "other things" on the RHS

$$\begin{aligned} \mathcal{H}_{xx}(\delta X) + \mathcal{H}_{xy}(\delta Y) + \mathcal{H}_{xz}(\delta Z) &= -(\langle \mathcal{F}_{\perp}^{x}, \mathcal{E}_0 \rangle_{-1} + \langle \mathcal{E}_x, \mathcal{F} \rangle_{-1}) \\ \mathcal{H}_{yx}(\delta X) + \mathcal{H}_{yy}(\delta Y) + \mathcal{H}_{yz}(\delta Z) &= -\left(\langle \mathcal{F}_{\perp}^{y}, \mathcal{E}_0 \rangle_{-2} + \langle \mathcal{E}_y, \mathcal{F} \rangle_{-2}\right) \\ \mathcal{H}_{zx}(\delta X) + \mathcal{H}_{zy}(\delta Y) + \mathcal{H}_{zz}(\delta Z) &= -(\langle \mathcal{F}_{\perp}^{z}, \mathcal{E}_0 \rangle_{-3} + \langle \mathcal{E}_z, \mathcal{F} \rangle_{-3}) \end{aligned}$$

- This looks messy....
- BUT: these equations will give us the "gap"!

Example 1: Matrix case

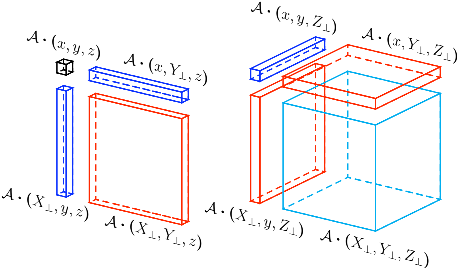
1. $$\begin{bmatrix} -(\Sigma_1^2 \otimes I) & \Sigma_1 \otimes \Sigma_2 \\ \Sigma_1 \otimes \Sigma_2 & -(\Sigma_1^2 \otimes I) \end{bmatrix} \begin{bmatrix} \text{vec}(\delta X) \\ \text{vec}(\delta Y) \end{bmatrix} = - \begin{bmatrix} (\Sigma_1 \otimes I)\,\text{vec}(E_x) \\ (\Sigma_1 \otimes I)\,\text{vec}(E_y) \end{bmatrix}$$
2. We can uncouple into 2×2 block systems. The worst is
$$\begin{bmatrix} -\sigma_r^2 & \sigma_r \sigma_{r+1} \\ \sigma_r \sigma_{r+1} & -\sigma_r^2 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = - \begin{bmatrix} \sigma_r e_x \\ \sigma_r e_y \end{bmatrix}$$
3. Solution:
$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \frac{1}{(\sigma_r - \sigma_{r+1})(\sigma_r + \sigma_{r+1})} \begin{bmatrix} \sigma_r e_x + \sigma_{r+1} e_y \\ \sigma_{r+1} e_x + \sigma_r e_y \end{bmatrix}$$
4. Bounding gives:
$$|\delta x| \leq \frac{1}{\sigma_r - \sigma_{r+1}}(|e_x| + |e_y|) \qquad |\delta y| \leq \frac{1}{\sigma_r - \sigma_{r+1}}(|e_x| + |e_y|)$$
5. The quantity $(\sigma_r - \sigma_{r+1})$ is called the gap!

Example 2: Tensor case with rank-$(1, 1, 1)$ approximation

- Let (x, y, z) be a stationary point of $f(x, y, z) = \|\mathcal{A} \cdot (x, y, z)\|_F^2$
- With $c = \mathcal{A} \cdot (x, y, z)$ the perturbation equations become

$$\begin{bmatrix} -c^2 I & cF_\perp^{xy} & cF_\perp^{xz} \\ cF_\perp^{yx} & -c^2 I & cF_\perp^{yz} \\ cF_\perp^{zx} & cF_\perp^{zy} & -c^2 I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} ce_x \\ ce_y \\ ce_z \end{bmatrix}$$

- Where $F_\perp^{xy} = \mathcal{A} \cdot (X_\perp, Y_\perp, z)$, ...



- Due to orthogonality blue vectors are zero!

Example 2: Tensor case with rank-$(1, 1, 1)$ approximation

- The perturbation satisfy

$$\begin{bmatrix} -c^2 I & cF_\perp^{xy} & cF_\perp^{xz} \\ cF_\perp^{yx} & -c^2 I & cF_\perp^{yz} \\ cF_\perp^{zx} & cF_\perp^{zy} & -c^2 I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} ce_x \\ ce_y \\ ce_z \end{bmatrix}$$

- We get the bound

$$\left\| \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \right\| \leq \|G^{-1}\| \cdot \left\| \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \right\| \qquad G = -c \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & F_\perp^{xy} & F_\perp^{xz} \\ F_\perp^{yx} & 0 & F_\perp^{yz} \\ F_\perp^{zx} & F_\perp^{zy} & 0 \end{bmatrix}$$

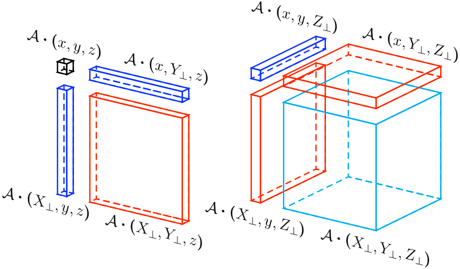
- What can we say about $\|G^{-1}\|$?

Example 2: Tensor case with rank-$(1,1,1)$ approximation

- We have that $\|G^{-1}\| \leq \frac{1}{|\lambda_{\min}(G)|}$

$$G = -c \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & F_\perp^{xy} & F_\perp^{xz} \\ F_\perp^{yx} & 0 & F_\perp^{yz} \\ F_\perp^{zx} & F_\perp^{zy} & 0 \end{bmatrix} \equiv -cI + F$$

- The "gap" becomes $|\lambda_{\min}(G)| = c - \mu_{\max}$ where $\mu_{\max}$ is the largest eigenvalue of F

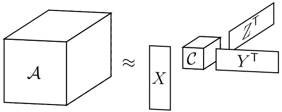


$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \Sigma_\perp \end{bmatrix}$$

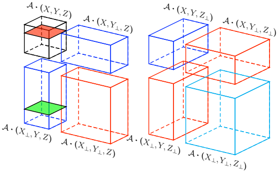
- Compare with the matrix case

Summary

1. We considered the multilinear low rank approximation of a tensor



2. Presented several Krylov-type procedures that generate low rank approximations
3. Interpreted first and second order optimality conditions: ordering and orthogonality



4. Generalized the "gap" from matrix sensitivity analysis to tensors

Thank you for your time!

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