CERTIFIED REDUCED BASIS METHODS FOR PARAMETRIZED DISTRIBUTED ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH CONTROL CONSTRAINTS*

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Abstract. In this paper, we employ the reduced basis method for the efficient and reliable solution of parametrized optimal control problems governed by scalar coercive elliptic partial differential equations. We consider the standard linear-quadratic problem setting with distributed control *and* unilateral control constraints. For this problem class, we propose two different reduced basis approximations and associated error estimation procedures. In our first approach, we directly consider the resulting optimality system, introduce suitable reduced basis approximations for the state, adjoint, control, and Lagrange multipliers, and use a projection approach to bound the error in the reduced optimal control. For our second approach, we first reformulate the optimal control problem using a slack variable, we then develop a reduced basis approximation for the slack problem by suitably restricting the solution space, and derive error bounds for the slack based optimal control. We discuss benefits and drawbacks of both approaches and substantiate the comparison by presenting numerical results for several model problems.

Key words. Optimal control, reduced basis method, a posteriori error estimation, model order reduction, parameter-dependent systems, partial differential equations, elliptic problems, control constraints

AMS subject classifications. 49J20, 65K10, 65M15, 65M60, 93C20

1. Introduction. Optimal control problems governed by partial differential equations (PDEs) appear in a wide range of applications in science and engineering, such as heat phenomena, crystal growth, and fluid flow, see e.g. [23, 13, 22, 31, 7] for theoretical results and applications. Their solution using classical discretization techniques such as finite elements (FE) or finite volumes can be computationally expensive and time-consuming. Often, additional parameters enter the optimal control formulation either through the PDE, e.g., material or geometry parameters or boundary and initial conditions, the cost functional, e.g., regularization parameters, or the constraints, e.g., lower or upper bounds for the control variable. For a fixed set of parameters, one obtains a standard PDE-constrained optimal control problem. In many applications, however, such as a (controller) design exercise or a robust control context, the parameter themselves may vary or be allowed to vary. A bilevel optimization problem arises in such a case, where the upper-level optimization is performed over the parameters and the lower-level optimization task is to solve the optimal control problem. The surrogate model approach, where the original high-dimensional (say, finite element) approximation is replaced by a reduced order approximation, allows to speed up the low-level optimization task and thus has proven very useful in this context. In fact, various model order reduction techniques have been proposed to solve PDE-constrained optimal control problems: proper orthogonal decomposition (POD) e.g. in [20, 1, 21, 32], reduction based on inertial manifolds in [15], and reduced basis (RB) methods in [16, 8, 18, 17, 26]. However, the solution of the reduced order

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optimal control problem is generally suboptimal and reliable error estimation is thus crucial.

Previous work on reduced basis methods for optimal control problems considered distributed but unconstrained controls or constrained but scalar controls (possibly as a function of time in the parabolic case). There exists, to the best of our knowledge, no previous work providing reduced order approximations *and* associated rigorous as well as efficiently evaluable *a posteriori* error bounds for distributed optimal control problems with control constraints. Control constraints obviously pose a major challenge for reduced order approaches because (i) the problem becomes nonlinear, and thus many of the standard approaches for model reduction cannot be applied, and (ii) the construction of the optimal solution and error bound is usually based on a pointwise analysis of the optimality equation. Since this pointwise analysis takes place in the original high-dimensional approximation space, it often results in an inefficient reduced order method — besides contradicting the reduced order philosophy of using global basis functions and fully decoupling the low-dimensional approximation from the underlying high-dimensional problem.

In the present paper we aim to close this gap. We employ the reduced basis method [28, 30] as a surrogate model for the solution of distributed *and* constrained optimal control problems governed by parametrized elliptic partial differential equations. After stating the problem in section 2 we present the following contributions:

- In section 3 we extend previous work on reduced basis methods for variational inequalities in [12] to the optimal control setting. We introduce a reduced basis approximation for the first-order optimality system where we impose an additional convex cone condition for the Lagrange multiplier. Although the RB approximation is online-efficient, the reduced order optimal control is not guaranteed to be feasible. The proposed *a posteriori* error bound accounts for the constraint-violation which needs to be evaluated pointwise and is thus not fully offline-online decomposable, i.e. the online cost depends linearly on the dimension of the underlying high-dimensional control space. However, due to the only linear dependence the online cost may be acceptable in many applications.
- In section 4 we build on the recent work in [35] and propose a novel reduced basis slack approach for optimal control. More precisely, we first introduce a slack formulation for the optimal control problem which we obtain by shifting the optimal control by the control constraint. We then impose a convex cone condition on the reduced basis approximation of the slack variable, i.e., the shifted control. The overall reduced basis slack approximation is not only online-efficient, but also provides a feasible reduced order optimal control. We further propose an associated *a posteriori* error bound which exploits the feasibility of the slack approximation and satisfies a full offline-online decomposition.

In section 5 we present a variation of the standard greedy algorithm proposed originally in [33]. After presenting numerical results for three model problems – two heat phenomena and a Graetz flow problem – in section 6, we provide a detailed comparison of our approach with other approaches in section 7.

2. General problem statement and finite element discretization. In this section we introduce the parametrized linear-quadratic optimal control problem with elliptic PDE constraint and a constrained distributed control. We introduce a finite element truth discretization for the exact, i.e., continuous problem and recall the first-order necessary (and in our convex setting sufficient) optimality conditions.

2.1. Preliminaries. Let $Y_{\rm e}$ with $H_0^1(\Omega) \subset Y_{\rm e} \subset H^1(\Omega)$ be a Hilbert space over the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \{1,2,3\}$, with boundary Γ.[1] The inner product and induced norm associated with $Y_{\rm e}$ are given by $(\cdot,\cdot)_Y$ and $\|\cdot\|_Y = \sqrt{(\cdot,\cdot)_Y}$, respectively. We assume that the norm $\|\cdot\|_Y$ is equivalent to the $H^1(\Omega)$-norm and denote the dual space of $Y_{\rm e}$ by $Y_{\rm e}'$. We also introduce the control Hilbert space $U_{\rm e} = L^2(\Omega)$, together with its inner product $(\cdot,\cdot)_U$, induced norm $\|\cdot\|_U = \sqrt{(\cdot,\cdot)_U}$, and associated dual space $U_{\rm e}'$.[2] Furthermore, let $\mathcal{D} \subset \mathbb{R}^P$ be a prescribed P-dimensional compact parameter set in which our P-tuple (input) parameter $\mu = (\mu_1, \ldots, \mu_P)$ resides.

We directly consider a finite element approximation for the infinite-dimensional optimal control problem. To this end, we define two conforming finite element spaces $Y \subset Y_{\rm e}$ and $U \subset U_{\rm e}$ of dimensions $\mathcal{N}_Y = \dim(Y)$ and $\mathcal{N}_U = \dim(U)$. We shall assume that the truth spaces Y and U are sufficiently rich such that the finite element solutions guarantee a desired accuracy over the whole parameter domain $\mathcal{D}$.

We next introduce the parameter-dependent bilinear form $a(\cdot,\cdot;\mu) : Y \times Y \to \mathbb{R}$, and shall assume that $a(\cdot,\cdot;\mu)$ is continuous,

$$0 < \gamma_a(\mu) = \sup_{w \in Y\backslash\{0\}} \sup_{v \in Y\backslash\{0\}} \frac{a(w,v;\mu)}{\|w\|_Y \|v\|_Y} \le \gamma_0^a < \infty \quad \forall \mu \in \mathcal{D}, \tag{2.1}$$

and coercive,

$$\alpha_a(\mu) = \inf_{v \in Y\backslash\{0\}} \frac{a(v,v;\mu)}{\|v\|_Y^2} \ge \alpha_0^a > 0 \quad \forall \mu \in \mathcal{D}. \tag{2.2}$$

Furthermore, we introduce the parameter-dependent continuous linear functional $f(\cdot;\mu) : Y \to \mathbb{R}$. We also introduce the parameter-dependent bilinear form $b(\cdot,\cdot;\mu) : U \times Y \to \mathbb{R}$ and assume that $b(\cdot,\cdot;\mu)$ is continuous,

$$0 < \gamma_b(\mu) = \sup_{z \in U\backslash\{0\}} \sup_{v \in Y\backslash\{0\}} \frac{b(z,v;\mu)}{\|z\|_U \|v\|_Y} \le \gamma_0^b < \infty \quad \forall \mu \in \mathcal{D}. \tag{2.3}$$

Finally, in anticipation of the optimal control problem defined in subsection 2.2, we introduce the parametrized control constraint $u_a(\mu) \in U$ and desired state $y_d \in Y_D$. Here, $Y_D \subset L^2(\Omega_D)$ is a suitable FE space for the observation subdomain $\Omega_D \subset \Omega$.

The involved bilinear and linear forms as well as the control constraint are assumed to depend affinely on the parameter, i.e., for all $w, v \in Y, z \in U$ and all parameters $\mu \in \mathcal{D}$,

$$\begin{aligned} a(w,v;\mu) &= \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^q(w,v), & b(z,v;\mu) &= \sum_{q=1}^{Q_b} \Theta_b^q(\mu)\, b^q(z,v), \\ f(v;\mu) &= \sum_{q=1}^{Q_f} \Theta_f^q(\mu)\, f^q(v), & u_a(x;\mu) &= \sum_{q=1}^{Q_{ua}} \Theta_{ua}^q(\mu)\, u_a^q(x), \end{aligned} \tag{2.4}$$

[1] The subscript "e" denotes "exact".

[2] Our framework directly extends to Neumann boundary controls $U_{\rm e} = L^2(\Gamma)$ or finite dimensional controls $U_{\rm e} = \mathbb{R}^m$. Also distributed controls on a subdomain $\Omega_U \subset \Omega$ or Neumann boundary controls on a boundary segment $\Gamma_U \subset \Gamma$ are possible.

for some (preferably) small integers Q_a, Q_b, Q_f, and Q_{ua}. Here, the coefficient functions $\Theta^q_\bullet(\cdot) : \mathcal{D} \to \mathbb{R}$ are continuous and depend on μ; whereas, the continuous bilinear forms $a^q(\cdot,\cdot) : Y \times Y \to \mathbb{R}$, $b^q(\cdot,\cdot) : U \times Y \to \mathbb{R}$, as well as the continuous linear forms $f^q : Y \to \mathbb{R}$ and $u^q_a \in U$ do *not* depend on μ. Although we here choose $y_d(x)$ to be parameter-independent, our approach directly extends to an affinely parameter-dependent $y_d(x;\mu)$ [19].

For the development of the *a posteriori* error bounds we will also require the following ingredients. We assume that we are given a positive lower bound $\alpha_a^{\mathrm{LB}}(\cdot) : \mathcal{D} \to \mathbb{R}_+$ for the coercivity constant $\alpha_a(\mu)$ defined in (2.2) such that

$$0 < \alpha_0^a \le \alpha_a^{\mathrm{LB}}(\mu) \le \alpha_a(\mu) \quad \forall \mu \in \mathcal{D}. \tag{2.5}$$

Furthermore, we assume that we have upper bounds available for the constant

$$C_{\Omega_D}^{\mathrm{UB}} \ge C_{\Omega_D} \equiv \sup_{v \in Y \setminus \{0\}} \frac{|v|_{L^2(\Omega_D)}}{\|v\|_Y} \ge 0 \quad \forall \mu \in \mathcal{D}, \tag{2.6}$$

and the continuity constant of the bilinear form $b(\cdot,\cdot;\mu)$

$$\gamma_b^{\mathrm{UB}}(\mu) \ge \gamma_b(\mu) \quad \forall \mu \in \mathcal{D}. \tag{2.7}$$

It is possible to compute these constants (or their bounds) efficiently in terms of an offline-online procedure; see subsection 3.4 for details.

2.2. Abstract formulation. We consider the following finite element optimal control problem with weak formulation of the control constraint

$$\text{(P)} \qquad \begin{aligned} &\min_{y,u} J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_D)} + \frac{\lambda}{2}\|u\|^2_U \\ &\text{s.t.} \quad (y,u) \in Y \times U \quad \text{solves} \quad a(y,v;\mu) = b(u,v;\mu) + f(v;\mu) \quad \forall v \in Y, \\ &\qquad (u_a(\mu),\rho)_U \le (u,\rho)_U \quad \forall \rho \in U^+, \end{aligned}$$

where $U^+ = \{\rho \in U; \rho \ge 0 \text{ almost everywhere}\}$ defines a nonnegative convex cone and we dropped the μ-dependence of the state y and control u for the sake of readability. We note that the last line of (P) is equivalent to u being in the closed convex admissible set $U_{\mathrm{ad}} = \{u \in U; (u_a(\mu),\rho)_U \le (u,\rho)_U \ \forall \rho \in U^+\}$. In the following we call problem (P) the "primal" problem. The existence and uniqueness of the solution is standard (see e.g. Theorem 1.43 in [13]); for the boundedness of the solution we refer to [2].

2.3. First-order optimality conditions. Introducing the Lagrange functional for (P)

$$\mathcal{L}(y,u,p,\sigma;\mu) = J(y,u) + a(y,p;\mu) - b(u,p;\mu) - f(p;\mu) + (u_a(\mu) - u, \sigma)_U$$

we obtain the necessary (and here sufficient) first-order optimality system: Given $\mu \in \mathcal{D}$, the optimal solution $(y^*,p^*,u^*,\sigma^*) \in Y \times Y \times U \times U$ satisfies

$$a(y^*,\phi;\mu) = b(u^*,\phi;\mu) + f(\phi;\mu) \qquad \forall \phi \in Y, \tag{2.8a}$$

$$a(\varphi,p^*;\mu) = (y_d - y^*,\varphi)_{L^2(\Omega_D)} \qquad \forall \varphi \in Y, \tag{2.8b}$$

$$(\lambda u^*,\psi)_U - b(\psi,p^*;\mu) - (\sigma^*,\psi)_U = 0 \qquad \forall \psi \in U, \tag{2.8c}$$

$$(u_a(\mu) - u^*,\rho)_U \le 0 \qquad \forall \rho \in U^+, \tag{2.8d}$$

$$(u_a(\mu) - u^*,\sigma^*)_U = 0, \quad \sigma^* \ge 0. \tag{2.8e}$$

Note that we follow a first-discretize-then-optimize approach here, but we would obtain the same optimality conditions by employing a first-optimize-then-discretize approach for our control problem. The last line is the well-known complimentary slackness condition. We refer to Corollary 1.3 and Theorem 1.43 in [13] for a detailed derivation of (2.8) via the variational inequality $(\lambda u^*, \psi - u^*)_U - b(\psi - u^*, p^*; \mu) \geq 0 \ \forall \psi \in U_{\mathrm{ad}}$. The last inequality can be reformulated by the means of the Lagrange multiplier $\sigma \in U^+$ into (2.8c)-(2.8e).

2.4. Algebraic formulation. In this paper we assume that the state variable is discretized by P_1, i.e., continuous and piecewise linear, and the control variable by P_0, i.e., piecewise constant, finite elements. We introduce two bases for the finite element spaces Y and U, such that

$$Y = \mathrm{span}\{\phi_i^y, i = 1, \ldots, \mathcal{N}_Y\} \quad \text{and} \quad U = \mathrm{span}\{\phi_i^u, i = 1, \ldots, \mathcal{N}_U\},$$

where $\phi_i^y \geq 0$, $i = 1, \ldots, \mathcal{N}_Y$, and $\phi_i^u \geq 0$, $i = 1, \ldots, \mathcal{N}_U$, are the usual hat and bar basis functions. Using these basis functions we can express the functions $y \in Y$, $p \in Y$, $u \in U$, and $\sigma \in U$ as

$$y = \sum_{i=1}^{\mathcal{N}_Y} y_i \phi_i^y, \quad p = \sum_{i=1}^{\mathcal{N}_Y} p_i \phi_i^y, \quad u = \sum_{i=1}^{\mathcal{N}_U} u_i \phi_i^u, \text{ and } \sigma = \sum_{i=1}^{\mathcal{N}_U} \sigma_i \phi_i^u,$$

respectively. The corresponding finite element coefficient vectors are given by $\boldsymbol{y} = (y_1, \ldots, y_{\mathcal{N}_Y})^T \in \mathbb{R}^{\mathcal{N}_Y}$, $\boldsymbol{p} = (p_1, \ldots, p_{\mathcal{N}_Y})^T \in \mathbb{R}^{\mathcal{N}_Y}$, $\boldsymbol{u} = (u_1, \ldots, u_{\mathcal{N}_U})^T \in \mathbb{R}^{\mathcal{N}_U}$, and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{\mathcal{N}_U})^T \in \mathbb{R}^{\mathcal{N}_U}$. Note that by definition of U^+ and since $\phi_i^u \geq 0$, the condition $\rho \in U^+$ translates into the condition $\boldsymbol{\rho} \geq 0$ for the corresponding coefficient vector, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_{\mathcal{N}_U})^T \in \mathbb{R}^{\mathcal{N}_U}$. We also introduce the control mass matrix M_U with entries $(M_U)_{ij} = (\phi_i^u, \phi_j^u)_U$, the matrices B^q, $1 \leq q \leq Q_b$ with entries $B_{ij}^q = b^q(\phi_j^u, \phi_i^y)$, and $B(\mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu) B^q$. Note that for a P_0 control discretization M_U is a positive diagonal matrix and hence the point-wise and weak constraint formulations are equivalent: $u(x) \geq u_a(x;\mu)$ a.e. $\Leftrightarrow (u, \rho)_U \geq (u_a(\mu), \rho)_U \ \forall \rho \in U^+$. However, this is in general not true for other control discretizations, e.g. P_1. The algebraic formulation of the optimality system (2.8) is standard and thus omitted.

3. Reduced basis method for the primal problem. In this section we present the first main contribution of the paper. Based on the truth optimal control problem (P) we propose a reduced basis optimal control problem ($\mathrm{P_N}$) in the following section. Subsequently, we introduce a rigorous *a posteriori* bound for the error between the truth and reduced optimal control in Theorem 4.

3.1. Reduced basis approximation. To begin, we define the reduced basis spaces $Y_N \subset Y$, $U_N \subset U$, $\Sigma_N \subset U$, as well as the convex cone $\Sigma_N^+ \subset U^+$ as follows: given N parameter samples $\mu^1, \ldots, \mu^N$, we set

$$\begin{aligned}
Y_N &= \mathrm{span}\{\zeta_1^y, \ldots, \zeta_{N_Y}^y\} &&= \mathrm{span}\{y^*(\mu^1), p^*(\mu^1), \ldots, y^*(\mu^N), p^*(\mu^N)\},\\
U_N &= \mathrm{span}\{\zeta_1^u, \ldots, \zeta_{N_U}^u\} &&= \mathrm{span}\{u^*(\mu^1), \sigma^*(\mu^1), \ldots, u^*(\mu^N), \sigma^*(\mu^N)\},\\
\Sigma_N &= \mathrm{span}\{\zeta_1^\sigma, \ldots, \zeta_{N_\sigma}^\sigma\} &&= \mathrm{span}\{\sigma^*(\mu^1), \ldots, \sigma^*(\mu^N)\},\\
\Sigma_N^+ &= \mathrm{span}_+\{\zeta_1^\sigma, \ldots, \zeta_{N_\sigma}^\sigma\} &&= \mathrm{span}_+\{\sigma^*(\mu^1), \ldots, \sigma^*(\mu^N)\},
\end{aligned}$$

where we assume that the basis functions, $\zeta_1^\bullet, \ldots \zeta_{N_\bullet}^\bullet$, are linearly independent. Note that we employ integrated spaces for the state and adjoint as well as for the control (cf. Remarks 1 and 2). For the spaces Y_N and U_N we additionally assume that the basis functions are orthogonal, i.e., $(\zeta_i^y, \zeta_j^y)_Y = \delta_{ij}$ and $(\zeta_i^u, \zeta_j^u)_U = \delta_{ij}$, where δ_{ij} is the Kronecker delta. This orthogonality is favorable to keep the condition of the RB algebraic linear systems small [30]. For the RB space Σ_N and the RB cone Σ_N^+ we assume that $\zeta_1^\sigma, \dots \zeta_{N_\sigma}^\sigma$ are not orthogonalized, such that all elements in Σ_N^+ are nonnegative. This nonnegativity is crucial to define the reduced problem ($\mathrm{P_N}$) of (P). We describe the greedy approach to construct the RB spaces in section 5.

Given the RB spaces we introduce the reduced primal problem

$$\text{(P}_\text{N}\text{)} \qquad \min_{y_N, u_N} J(y_N, u_N) = \frac{1}{2}\|y_N - y_d\|^2_{L^2(\Omega_D)} + \frac{\lambda}{2}\|u_N\|_U^2$$
$$\text{s.t.} \quad (y_N, u_N) \in Y_N \times U_N \quad \text{solves} \quad a(y_N, v; \mu) = b(u_N, v; \mu) + f(v; \mu) \; \forall v \in Y_N,$$
$$(u_a(\mu), \rho)_U \le (u_N, \rho)_U \; \forall \rho \in \Sigma_N^+.$$

It is important to note that the last line of ($\mathrm{P_N}$) defines the admissible set for u_N: $U_{\mathrm{ad},N} = \{u_N \in U_N; (u_a(\mu), \rho)_U \le (u_N, \rho)_U \; \forall \rho \in \Sigma_N^+\}$, which is generally *not* a subset of U_{ad}. Although we thus cannot ensure feasibility of the RB control u_N^* on the FE level, i.e. $u_N^* \in U_{\mathrm{ad}}$, we still expect u_N^* to provide good approximations to the optimal FE control u^*. We also note that the definition of the RB admissible set $U_{\mathrm{ad},N}$ through the cone Σ_N^+ is the reason why we do not orthogonalize the σ-snapshots: the orthogonalization would destroy the nonnegativity of the snapshots $\zeta_1^\sigma, \dots, \zeta_{N_\sigma}^\sigma$ which is of central importance for the weak constraint formulation.

Employing the Lagrange functional for ($\mathrm{P_N}$), given by

$$\mathcal{L}_N(y_N, u_N, p_N, \sigma_N; \mu)$$
$$= J(y_N, u_N) + a(y_N, p_N; \mu) - b(u_N, p_N; \mu) - f(p_N; \mu) + (u_a(\mu) - u_N, \sigma_N)_U,$$

we obtain the necessary (and here sufficient) first-order optimality system: Given $\mu \in \mathcal{D}$, the optimal RB solution $(y_N^*, p_N^*, u_N^*, \sigma_N^*) \in Y_N \times Y_N \times U_N \times \Sigma_N$ satisfies

$$a(y_N^*, \phi; \mu) = b(u_N^*, \phi; \mu) + f(\phi; \mu) \qquad \forall \phi \in Y_N, \tag{3.1a}$$
$$a(\varphi, p_N^*; \mu) = (y_d - y_N^*, \varphi)_{L^2(\Omega_D)} \qquad \forall \varphi \in Y_N, \tag{3.1b}$$
$$(\lambda u_N^*, \psi)_U - b(\psi, p_N^*; \mu) - (\sigma_N^*, \psi)_U = 0 \qquad \forall \psi \in U_N, \tag{3.1c}$$
$$(u_a(\mu) - u_N^*, \rho)_U \le 0 \qquad \forall \rho \in \Sigma_N^+, \tag{3.1d}$$
$$(u_a(\mu) - u_N^*, \sigma_N^*)_U = 0, \quad \sigma_N^* \in \Sigma_N^+. \tag{3.1e}$$

Remark 1 (Existence, uniqueness, integrated space Y_N). *Since* ($\mathrm{P_N}$) *is a linear-quadratic optimal control problem over the closed convex admissible set* $U_{\mathrm{ad},N}$*, the existence and uniqueness of the RB optimal control* u_N^* *follows from standard arguments. We refer to Theorem 1.43 in [13]. Also note that we use a single integrated reduced basis trial and test space* Y_N *for the state and adjoint equations as one ingredient to ensure stability of the system* (3.1)*. We refer to [18, 17, 26] for further details and discussion on the use of integrated spaces for the state and adjoint equations.*

Remark 2 (Stability, integrated space U_N). *For the stability of the RB solutions we need to also show that the RB inf-sup constant*

$$\beta_N := \inf_{\psi_\sigma \in \Sigma_N} \sup_{\psi_u \in U_N} \frac{(\psi_\sigma, \psi_u)_U}{\|\psi_\sigma\|_U \|\psi_u\|_U}$$

is bounded away from zero. Although the FE inf-sup constant satisfies

$$\beta := \inf_{\psi_\sigma \in U} \sup_{\psi_u \in U} \frac{(\psi_\sigma, \psi_u)_U}{\|\psi_\sigma\|_U \|\psi_u\|_U} > 0,$$

it is well known that the inf-sup stability of the RB problem is not directly inherited from the FE problem (note that in our case $\beta = 1$). However, we can guarantee that $\beta_N \geq \beta > 0$ by enriching the RB control space with suitable supremizers [24]. To this end, we introduce the supremizing operator $T : U \to U$, defined by $(T\psi_\sigma, \psi_u)_U = (\psi_\sigma, \psi_u)_U$, $\forall \psi_\sigma, \psi_u \in U$, from which we conclude that T is the identity. Hence, as a second ingredient, we have to enrich the RB control space U_N by the Lagrange multiplier snapshots $T\sigma^*(\mu^n) = \sigma^*(\mu^n)$, $1 \leq n \leq N$, to ensure an inf-sup stable RB space.

When using integrated space Y_N and U_N one can then show boundedness and continuous dependence on the data f, u_a, and y_d for the RB solution of ($\mathrm{P_N}$). If we further assume that the functions $\Theta^q_\bullet(\mu)$ in the affine expansion (2.4) are Lipschitz-continuous with respect to the parameter μ, the RB solution then also depends Lipschitz-continuously on the parameter. We refer to [2] for the detailed proofs.

3.2. Algebraic formulation. We express the RB variables as

$$y_N = \sum_{i=1}^{N_Y} y_{Ni} \zeta_i^y, \quad p_N = \sum_{i=1}^{N_Y} p_{Ni} \zeta_i^y, \quad u_N = \sum_{i=1}^{N_U} u_{Ni} \zeta_i^u, \text{ and } \sigma_N = \sum_{i=1}^{N_\sigma} \sigma_{Ni} \zeta_i^\sigma,$$

and denote the corresponding coefficient vectors as $\boldsymbol{y}_N = (y_{N1}, \ldots, y_{NN_Y})^T \in \mathbb{R}^{N_Y}$, $\boldsymbol{p}_N = (p_{N1}, \ldots, p_{NN_Y})^T \in \mathbb{R}^{N_Y}$, $\boldsymbol{u}_N = (u_{N1}, \ldots, u_{NN_U})^T \in \mathbb{R}^{N_U}$, and $\boldsymbol{\sigma}_N = (\sigma_{N1}, \ldots, \sigma_{NN_\sigma})^T \in \mathbb{R}^{N_\sigma}$. For later use we also introduce an RB matrix, $Z_N^\sigma = (\boldsymbol{\zeta}_1^\sigma \mid \cdots \mid \boldsymbol{\zeta}_{N_\sigma}^\sigma) \in \mathbb{R}^{\mathcal{N}_U \times N_\sigma}$, which is associated to the above defined RB space Σ_N. We can then represent an RB function $\sigma_N \in \Sigma_N$ with RB coefficients $\boldsymbol{\sigma}_N \in \mathbb{R}^{N_\sigma}$ in terms of its FE basis coefficient vector as $Z_N^\sigma \boldsymbol{\sigma}_N \in \mathbb{R}^{\mathcal{N}_U}$. Analogously, we define the RB matrix $Z_N^y = (\boldsymbol{\zeta}_1^y \mid \cdots \mid \boldsymbol{\zeta}_{N_Y}^y) \in \mathbb{R}^{\mathcal{N}_Y \times N_Y}$.

The algebraic formulation of the RB state equation (3.1a), adjoint equation (3.1b), and optimality condition (3.1c) is standard, see e.g. [19]. We thus focus on the new ingredient, the inequality condition (3.1d) and the complimentary slackness condition (3.1e). To this end, we introduce the RB vector $U_{a,N}^\sigma(\mu) \in \mathbb{R}^{N_\sigma}$ with entries $U_{a,N}^\sigma(\mu)_i = (u_a(\mu), \zeta_i^\sigma)_U$, and RB matrix $U_N^\sigma \in \mathbb{R}^{N_\sigma \times N_U}$ with entries $(U_N^\sigma)_{ij} = (\zeta_j^u, \zeta_i^\sigma)_U$. The inequality condition, $(u_a(\mu) - u_N^*, \rho)_U \leq 0 \; \forall \rho \in \Sigma_N^+$, is then equivalent to the algebraic formulation $U_{a,N}^\sigma(\mu) - U_N^\sigma \boldsymbol{u}_N^* \leq 0$. Similarly, the complementarity condition $(u_a(\mu) - u_N^*, \sigma_N^*)_U = 0$ translates into $(U_{a,N}^\sigma(\mu) - U_N^\sigma \boldsymbol{u}_N^*)^T \boldsymbol{\sigma}_N^* = 0$. By definition of Σ_N^+ it follows that $\sigma_N^* \in \Sigma_N^+$ is equivalent to $\boldsymbol{\sigma}_N^* \geq 0$. Furthermore, the specific choice of the set Σ_N^+ for the Lagrange multiplier and the condition $\boldsymbol{\sigma}_N^* \geq 0$ ensure the nonnegativity of the approximation σ_N^*.

3.3. Primal error bound. We next propose an *a posteriori* error bound for the optimal control. The bound is based on an RB approach for variational inequalities of the first kind proposed in [12]. Before stating the main result, we define the following approximation errors of the primal RB system

$$e_y(\mu) = y^* - y_N^*, \qquad e_p(\mu) = p^* - p_N^*, \qquad e_u(\mu) = u^* - u_N^*, \qquad e_\sigma(\mu) = \sigma^* - \sigma_N^*,$$

as well as the corresponding residuals.

DEFINITION 3. *The residuals of the state equation, the adjoint equation, and the optimality equation are defined by*

$$
\begin{aligned}
r_y(\phi;\mu) &= b(u_N^*,\phi;\mu) + f(\phi;\mu) - a(y_N^*,\phi;\mu) && \forall \phi \in Y \quad \forall \mu \in \mathcal{D},\\
r_p(\varphi;\mu) &= (y_d - y_N^*,\varphi)_{L^2(\Omega_D)} - a(\varphi,p_N^*;\mu) && \forall \varphi \in Y \quad \forall \mu \in \mathcal{D},\\
r_u(\psi;\mu) &= -\lambda(u_N^*,\psi)_U + b(\psi,p_N^*;\mu) + (\sigma_N^*,\psi)_U && \forall \psi \in U \quad \forall \mu \in \mathcal{D}.
\end{aligned}
$$

THEOREM 4. *Let u^* and u_N^* be the optimal solutions of the FE optimal control problem (P) and of the reduced primal problem ($\mathrm{P_N}$), respectively. For any given parameter $\mu \in \mathcal{D}$ the error in the optimal control satisfies*

$$\|e_u(\mu)\|_U \le \Delta_N^{\mathrm{pr}}(\mu),$$

where $\Delta_N^{\mathrm{pr}}(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)}$ with nonnegative coefficients

$$c_1(\mu) = \frac{1}{2\lambda}\left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)}\|r_p\|_{Y'} + \lambda\delta_1\right), \tag{3.2a}$$

$$c_2(\mu) = \frac{1}{\lambda}\left[\frac{2}{\alpha_a^{\mathrm{LB}}(\mu)}\|r_y\|_{Y'}\|r_p\|_{Y'} + \frac{1}{4}\left(\frac{C_{\Omega_D}^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}(\mu)}\left(\|r_y\|_{Y'} + \gamma_b^{\mathrm{UB}}(\mu)\delta_1\right)\right)^2 + \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)}\left(\|r_p\|_{Y'}\right)\right)\delta_1 + \delta_2\right], \tag{3.2b}$$

and

$$\delta_1 = \|[u_a - u_N^*]_+\|_U, \qquad \delta_2 = ([u_a - u_N^*]_+, \sigma_N^*)_U.$$

Here $[\cdot]_+ = \max(\cdot,0)$ denotes the positive part. Note that we sometimes use $r_\bullet$ instead of $r_\bullet(\cdot;\mu)$ for the sake of readability.

Proof. The finite element optimal solutions y^*, p^*, u^*, σ^* satisfy the optimality conditions (2.8) and hence we obtain the following error-residual equations:

$$a(e_y,\phi;\mu) - b(e_u,\phi;\mu) = r_y(\phi;\mu) \qquad \forall \phi \in Y, \tag{3.3a}$$

$$a(\varphi,e_p;\mu) + (e_y,\varphi)_{L^2(\Omega_D)} = r_p(\varphi;\mu) \qquad \forall \varphi \in Y, \tag{3.3b}$$

$$\lambda(e_u,\psi)_U - b(\psi,e_p;\mu) - (e_\sigma,\psi)_U = r_u(\psi;\mu) \qquad \forall \psi \in U. \tag{3.3c}$$

From (3.3a) with $\phi = e_y$ and invoking (2.2) and (2.5) we obtain

$$\alpha_a^{\mathrm{LB}}(\mu)\|e_y\|_Y^2 \le a(e_y,e_y;\mu) = r_y(e_y;\mu) + b(e_u,e_y;\mu),$$

and hence

$$\|e_y\|_Y \le \frac{1}{\alpha_a^{\mathrm{LB}}(\mu)}\left(\|r_y(\cdot;\mu)\|_{Y'} + \gamma_b^{\mathrm{UB}}(\mu)\|e_u\|_U\right), \tag{3.4}$$

where we used (2.3) and (2.7). Similarly, from (3.3b) with $\varphi = e_p$ we get

$$\alpha_a^{\mathrm{LB}}(\mu)\|e_p\|_Y^2 \le a(e_p,e_p;\mu) = r_p(e_p;\mu) - (e_y,e_p)_{L^2(\Omega_D)},$$

and hence invoking (2.6)

$$\|e_p\|_Y \le \frac{1}{\alpha_a^{\mathrm{LB}}(\mu)}\left(\|r_p(\cdot;\mu)\|_{Y'} + C_{\Omega_D}^{\mathrm{UB}}\|e_y\|_{L^2(\Omega_D)}\right). \tag{3.5}$$

Choosing $\phi = e_p$, $\varphi = e_y$, $\psi = e_u$ in (3.3) we have

$$a(e_y, e_p; \mu) - b(e_u, e_p; \mu) = r_y(e_p; \mu), \tag{3.6a}$$

$$a(e_y, e_p; \mu) + (e_y, e_y)_{L^2(\Omega_D)} = r_p(e_y; \mu), \tag{3.6b}$$

$$\lambda(e_u, e_u)_U - b(e_u, e_p; \mu) - (e_\sigma, e_u)_U = r_u(e_u; \mu). \tag{3.6c}$$

Adding (3.6b) and (3.6c) and subtracting (3.6a) results in

$$\lambda(e_u, e_u)_U + (e_y, e_y)_{L^2(\Omega_D)} - (e_\sigma, e_u)_U = -r_y(e_p; \mu) + r_p(e_y; \mu) + r_u(e_u; \mu),$$

and we thus obtain the inequality

$$\begin{aligned} \lambda \|e_u\|_U^2 + \|e_y\|_{L^2(\Omega_D)}^2 \le \|r_y\|_{Y'} \|e_p\|_Y + \|r_p\|_{Y'} \|e_y\|_Y \\ + \|r_u\|_{U'} \|e_u\|_U + (e_\sigma, e_u)_U. \end{aligned} \tag{3.7}$$

We now consider the last term $(e_\sigma, e_u)_U$ and note that

$$\begin{aligned} (e_\sigma, e_u)_U &= (\sigma^* - \sigma_N^*, u^* - u_N^*)_U = (u^* - u_N^*, \sigma^*)_U + (u_N^* - u^*, \sigma_N^*)_U \\ &= (u^* - u_a, \sigma^*)_U + (u_a - u_N^*, \sigma^*)_U + (u_N^* - u_a, \sigma_N^*)_U + (u_a - u^*, \sigma_N^*)_U. \end{aligned}$$

By the complementarity relations (2.8e) and (3.1e) the first and third term on the right hand side are zero. Since $(u_a - u^*, \sigma_N^*)_U \le 0$ (by (2.8d) with $\rho = \sigma_N^*$) we obtain

$$\begin{aligned} (e_\sigma, e_u)_U &\le (u_a - u_N^*, \sigma^*)_U \le ([u_a - u_N^*]_+, \sigma^*)_U \\ &= ([u_a - u_N^*]_+, \sigma^* - \sigma_N^*)_U + ([u_a - u_N^*]_+, \sigma_N^*)_U \\ &\le \|[u_a - u_N^*]_+\|_U \|\sigma^* - \sigma_N^*\|_U + ([u_a - u_N^*]_+, \sigma_N^*)_U. \end{aligned} \tag{3.8}$$

Using the definitions of δ_1 and δ_2 we thus obtain the overall estimate

$$(e_\sigma, e_u)_U \le \|e_\sigma\|_U \ \delta_1 + \delta_2. \tag{3.9}$$

It remains to bound the term $\|e_\sigma\|_U$. Choosing $\psi = e_\sigma$ in (3.3c) we obtain the bound

$$\begin{aligned} \|e_\sigma\|_U^2 &= -r_u(e_\sigma; \mu) + \lambda(e_u, e_\sigma)_U - b(e_\sigma, e_p; \mu) \\ &\le \left(\|r_u\|_{U'} + \lambda \|e_u\|_U + \gamma_b^{\mathrm{UB}}(\mu) \|e_p\|_Y\right) \|e_\sigma\|_U. \end{aligned}$$

Invoking again (3.4) and (3.5) we thus have

$$\|e_\sigma\|_U \le \|r_u\|_{U'} + \lambda \|e_u\|_U + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)} \Big(\|r_p\|_{Y'} + C_{\Omega_D}^{\mathrm{UB}} \|e_y\|_{L^2(\Omega_D)} \Big). \tag{3.10}$$

Next, we use the inequalities (3.4), (3.5), (3.9), and (3.10) in (3.7) to obtain:

$$\begin{aligned} \lambda \|e_u\|_U^2 + \|e_y\|_{L^2(\Omega_D)}^2 \le\ & \|e_u\|_U \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_p\|_{Y'} + \lambda \delta_1 \right) \\ & + \|e_y\|_{L^2(\Omega_D)} \frac{C_{\Omega_D}^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}(\mu)} \left(\|r_p\|_{Y'} + \gamma_b^{\mathrm{UB}}(\mu) \delta_1 \right) \\ & + \frac{2}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_y\|_{Y'} \|r_p\|_{Y'} + \delta_1 \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_p\|_{Y'} \right) + \delta_2. \end{aligned} \tag{3.11}$$

It thus follows from applying Young's inequality to the second line of (3.11) that

$$
\begin{aligned}
\lambda \|e_u\|_U^2 \leq & \|e_u\|_U \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_p\|_{Y'} + \lambda \delta_1 \right) \\
& + \frac{1}{4} \left(\frac{C_{\Omega_D}^{\mathrm{UB}}}{\alpha_a^{\mathrm{LB}}(\mu)} \left(\|r_p\|_{Y'} + \gamma_b^{\mathrm{UB}}(\mu) \delta_1 \right) \right)^2 \\
& + \frac{2}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_y\|_{Y'} \|r_p\|_{Y'} + \delta_1 \left(\|r_u\|_{U'} + \frac{\gamma_b^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)} \|r_p\|_{Y'} \right) + \delta_2.
\end{aligned} \tag{3.12}
$$

Finally, we can formulate the last inequality as a quadratic inequality in $\|e_u\|_U$,

$$
\|e_u\|_U^2 - 2c_1(\mu)\|e_u\|_U - c_2(\mu) \leq 0,
$$

and hence $\|e_u\|_U$ is bounded by the larger root given by $\Delta_N^{\mathrm{pr}}(\mu)$. ∎

We note that most of the ingredients of the error bound $\Delta_N^{\mathrm{pr}}(\mu)$ introduced in Theorem 4 are standard, i.e., the dual norms of state, adjoint, and control residuals as well as coercivity and continuity constants respectively their lower and upper bounds. The only non-standard terms are δ_1 and δ_2, which measure the constraint-violation of the RB optimal control u_N^*: recall that $U_{\mathrm{ad},N} \not\subset U_{\mathrm{ad}}$ and thus u_N^* may not be feasible on the FE level. As a result, the online computational cost to evaluate δ_1 and δ_2 – and hence the error bound $\Delta_N^{\mathrm{pr}}(\mu)$ – depends on the FE control dimension $\mathcal{N}_U$: first, we need to evaluate the positive part $[u_a - u_N^*]_+$ on the FE level; and second, we have to represent σ_N^* in the FE basis to compute δ_2. The most expensive operations are the matrix-vector products $Z_N^u \boldsymbol{u}_N^*$ and $Z_N^\sigma \boldsymbol{\sigma}_N^*$ requiring $\mathcal{O}((N_U + N_\sigma)\mathcal{N}_U)$ operations.

An interesting observation is that the error bound $\Delta_N^{\mathrm{pr}}(\mu)$ turns into the *a posteriori* control error bound proposed in [19] for unconstrained optimal control problems if the reduced problem is unconstrained: The terms δ_1 and δ_2 vanish by feasibility of the RB optimal control, $u_N^*(\mu) \in U_{\mathrm{ad}}$, and since $\sigma_N^*(\mu) = 0$ holds, the residual $r_u(\cdot;\mu)$ reduces to the unconstrained case, i.e., without the term $(\sigma_N^*, \cdot)_U$.

Remark 5. *The expressions for δ_1 and δ_2 given in Theorem 4 exploit the fact that M_U is a positive diagonal matrix (cf. (3.8)). If we consider control discretizations different from P_0, say P_1, we have to change the expressions for δ_1 and δ_2 in the error bound $\Delta_N^{\mathrm{pr}}(\mu)$ to*

$$
\begin{aligned}
\delta_1 &= \left([M_U(\boldsymbol{u}_a - Z_N^u \boldsymbol{u}_N^*)]_+^T M_U^{-1} [M_U(\boldsymbol{u}_a - Z_N^u \boldsymbol{u}_N^*)]_+ \right)^{\frac{1}{2}}, \\
\delta_2 &= [M_U(\boldsymbol{u}_a - Z_N^u \boldsymbol{u}_N^*)]_+^T Z_N^\sigma \boldsymbol{\sigma}_N^*,
\end{aligned}
$$

which are considerably more expensive to evaluate.

3.4. Computational procedure. We briefly comment on the computational procedure for the primal error bound. The evaluation of $\Delta_N^{\mathrm{pr}}(\mu)$ requires the following ingredients:

- the dual norm of the residuals $\|r_y\|_{Y'}$, $\|r_p\|_{Y'}$, and $\|r_u\|_{U'}$;
- the lower and upper bounds for the constants $\alpha_a^{\mathrm{LB}}(\mu)$, $\gamma_b^{\mathrm{UB}}(\mu)$, and $C_{\Omega_D}^{\mathrm{UB}}$;
- the terms δ_1 and δ_2.

The evaluation of δ_1 and δ_2 was already discussed in the last section and incurs an $\mathcal{N}_U$-dependent online cost. For the construction of the coercivity constant lower bound $\alpha_a^{\mathrm{LB}}(\mu)$ various recipes exist [14, 28, 34]. The specific choices for our numerical examples are stated in section 6. Simple (yet for our examples effective) upper bounds $C_{\Omega_D}^{\mathrm{UB}}$ and $\gamma_b^{\mathrm{UB}}(\mu)$ can be computed by solving $1 + Q_b$ generalized eigenvalue problems in the offline stage and then assembled in $\mathcal{O}(1 + Q_b)$ operations online. In general (arbitrarily tight) upper bounds can be obtained by applying the successive constraint method (SCM) [14].

The offline-online evaluation of $\|r_y\|_{Y'}$ and $\|r_p\|_{Y'}$ is standard and therefore omitted [30]; also see [19] for a summary of the computational cost in the optimal control context. For the computation of the optimality equation residual we employ the Riesz-representation $\hat{r}_u(\mu)$ of $r_u(\cdot;\mu)$ given by

$$\begin{aligned} r_u(\psi;\mu) = (\hat{r}_u(\mu), \psi)_U &= -\lambda(u_N^*, \psi)_U + b(\psi, p_N^*;\mu) + (\sigma_N^*, \psi)_U \\ &= (-\lambda u_N^* + \mathcal{B}^*(\mu)p_N^* + \sigma_N^*, \psi)_U. \end{aligned}$$

Here $\mathcal{B}^*(\mu) : Y \to U$ is the adjoint operator of the operator $\mathcal{B}(\mu) : U \to Y'$ induced by the bilinear form $b(\cdot,\cdot;\mu)$ and thus satisfies

$$\langle \mathcal{B}(\mu)\psi, \phi\rangle_{Y',Y} = b(\psi, \phi;\mu) = (\psi, \mathcal{B}^*(\mu)\phi)_U.$$

Furthermore, we can directly and explicitly identify $\hat{r}_u(\mu) = -\lambda u_N^* + \mathcal{B}^*(\mu)p_N^* + \sigma_N^*$ (without solving any Poisson problems) to get

$$\begin{aligned} \|r_u(\cdot;\mu)\|_{U'}^2 = \|\hat{r}_u(\mu)\|_U^2 &= \|-\lambda u_N^* + \mathcal{B}^*(\mu)p_N^* + \sigma_N^*\|_U^2 \\ &= \lambda^2(u_N^*, u_N^*)_U + (\mathcal{B}^*(\mu)p_N^*, \mathcal{B}^*(\mu)p_N^*)_U + (\sigma_N^*, \sigma_N^*)_U \\ &\quad - 2\lambda(u_N^*, \mathcal{B}^*(\mu)p_N^*)_U - 2\lambda(u_N^*, \sigma_N^*)_U + 2(\mathcal{B}^*(\mu)p_N^*, \sigma_N^*)_U \\ &= \lambda^2(u_N^*, u_N^*)_U + (\mathcal{B}^*(\mu)p_N^*, \mathcal{B}^*(\mu)p_N^*)_U + (\sigma_N^*, \sigma_N^*)_U \\ &\quad - 2\lambda b(u_N^*, p_N^*;\mu) - 2\lambda(u_N^*, \sigma_N^*)_U + 2b(\sigma_N^*, p_N^*;\mu). \end{aligned}$$

All terms except for $(\mathcal{B}^*(\mu)p_N^*, \mathcal{B}^*(\mu)p_N^*)_U$ are standard and are thus not discussed here. Let $\boldsymbol{\chi}(\mu)$ be the FE coefficient vector of $\chi(\mu) = \mathcal{B}^*(\mu)\phi$. From the definition of the adjoint operator we obtain the algebraic formulation

$$\boldsymbol{\psi}^T M_U \boldsymbol{\chi}(\mu) = (B(\mu)\boldsymbol{\psi})^T \boldsymbol{\phi} \quad\Leftrightarrow\quad M_U\boldsymbol{\chi}(\mu) = B(\mu)^T\boldsymbol{\phi} \quad\Leftrightarrow\quad \boldsymbol{\chi}(\mu) = M_U^{-1}B(\mu)^T\boldsymbol{\phi},$$

from which it follows that

$$(\mathcal{B}^*(\mu)\phi, \mathcal{B}^*(\mu)\phi)_U = \boldsymbol{\chi}(\mu)^T M_U \boldsymbol{\chi}(\mu) = \boldsymbol{\phi}^T B(\mu) M_U^{-1} B(\mu)^T \boldsymbol{\phi}.$$

Finally, for $\phi = p_N^*$ we obtain

$$\boldsymbol{p}_N^T (Z_N^y)^T B(\mu) M_U^{-1} B(\mu)^T Z_N^y \boldsymbol{p}_N = \boldsymbol{p}_N^T \left(\sum_{q=1}^{Q_b} \sum_{p=1}^{Q_b} \Theta_b^q(\mu)\Theta_b^p(\mu)(Z_N^y)^T B^q M_U^{-1} (B^p)^T Z_N^y \right) \boldsymbol{p}_N,$$

which allows an offline-online decomposition. In the offline phase we construct (and save) Q_b^2 (symmetric) matrices $(Z_N^y)^T B^q M_U^{-1} (B^p)^T Z_N^y$, $q = 1,\dots,Q_b$, $p = 1,\dots,Q_b$, of dimension $N_Y \times N_Y$; in the online phase we assemble the matrix in the parentheses at cost $Q_b^2 N_Y^2$ and perform the inner product at cost N_Y^2.

4. Slack problem and the primal-slack error bound. In section 6 we will present numerical results for several test problems showing that the primal bound proposed in the last section is reasonably sharp. As pointed out above, however, there are two problems intrinsic to the formulation: first, the RB optimal control is not guaranteed to be feasible on the FE level; and second, the online cost to evaluate the bound therefore depends on the FE dimension $\mathcal{N}_U$[3]. We also recall that the primal bound is based on a result for variational inequalities proposed in [12] which suffers from the same drawbacks.

In a recent note [35], the authors propose an improvement to [12] by introducing a slack formulation and associated *a posteriori* error bounds for variational inequalities. The slack approach does not only guarantee feasibility of the reduced optimal solution with respect to the FE admissible set, but also provides an *a posteriori* error bound with associated computational cost independent of the FE dimension. Moreover, the slack-based error bound is considerably sharper than the primal error bound.

Following the idea presented in [35] we proceed as follows: we first reformulate the original optimization problem (P) by replacing the control variable with a slack variable. We will therefore call the reformulation given in (S) below the "slack problem." Second, we use snapshots of the slack variable to construct an associated positive convex cone. Third, we derive a reduced slack problem by restricting the RB slack variable to this positive convex cone. And finally, we propose an *a posteriori* error bound for the error in a combined primal-slack variable approximation in Theorem 7.

4.1. Finite element slack problem. We consider the finite element optimization problem (P) and introduce the slack variable $s \in U^+$ given by $s = u - u_a$ together with the corresponding FE coefficient vector $\boldsymbol{s} = \boldsymbol{u} - \boldsymbol{u}_a$. Here, we omit the explicit dependence on the parameter μ. We note that, by construction, the feasibility of u is equivalent to $M_U \boldsymbol{s} \geq 0$, which in turn is equivalent to $\boldsymbol{s} \geq 0$ since we are using P_0 elements and hence M_U is a positive diagonal matrix.

The optimization problem (P) can then be recast as follows

$$\text{(S)} \qquad \min_{y,s} J_s(y,s) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_D)} + \frac{\lambda}{2}\|s + u_a\|^2_U$$
$$\text{s.t.} \quad (y,s) \in Y \times U^+ \quad \text{solves} \quad a(y,v;\mu) = b(s + u_a, v; \mu) + f(v;\mu) \quad \forall v \in Y.$$

Since the slack problem is equivalent to the original problem formulation, it directly inherits the well-posedness discussed in the previous sections. Also the associated optimality system is equivalent to (2.8) if we replace u by $s + u_a$.

4.2. Reduced basis slack approximation. As pointed out in the last section, the original finite element problem formulation (P) and the slack problem formulation (S) are equivalent. We may thus directly use the RB space Y_N introduced in subsection 3.1 for the state and adjoint variables for our RB slack approximation. Furthermore, for the RB approximation of the slack variable we introduce an RB slack space S_N by shifting the control snapshots by the control constraint $u_a(\mu)$, i.e., we define

$$S_N = \text{span}\{\zeta_1^s, \ldots, \zeta_{N_S}^s\} = \text{span}\{u^*(\mu^1) - u_a(\mu^1), \ldots, u^*(\mu^M) - u_a(\mu^M)\},$$

and also the positive convex cone

$$S_N^+ = \text{span}_+\{\zeta_1^s, \ldots, \zeta_{N_S}^s\} \subset U^+.$$

[3] We do want to point out that the online evaluation of the primal error bound is actually still more efficient than the solution of the reduced optimal control problem itself for our test problems and chosen discretizations. Of course, this observation strongly depends on the (FE) problem size and solvers; see section 6 for details.

Similar to before, we assume that the snapshots $\zeta_1^s, \dots, \zeta_{N_s}^s$ are linearly independent. Note that we can construct S_N from the control snapshots of the FE primal problem without having to solve the FE slack problem (S) in computational practice. More importantly, we notice that the positive cone S_N^+ satisfies $S_N^+ \subset U^+$. If we now require the RB slack approximation s_N of the slack variable s to lie in S_N^+, it follows that $s_N \geq 0$, which results in an FE feasible control approximation. Note that $s_N \in S_N^+$ is equivalent to $\boldsymbol{s}_N \geq 0$, where $\boldsymbol{s}_N = (s_{N1}, \dots, s_{NN_S})^T \in \mathbb{R}^{N_S}$ are the RB coefficients of the expansion $s_N = \sum_{i=1}^{N_S} s_{Ni}\zeta_i^s$.

We then obtain the RB optimization problem

$$\text{(S}_\text{N}\text{)} \qquad \min_{y_N^s, s_N} J_s(y_N^s, s_N) = \frac{1}{2}\|y_N^s - y_d\|_{L^2(D)}^2 + \frac{\lambda}{2}\|s_N + u_a\|_U^2$$

$$\text{s.t. } (y_N^s, s_N) \in Y_N \times S_N^+ \text{ solves } a(y_N^s, v; \mu) = b(s_N + u_a, v; \mu) + f(v; \mu)\ \forall v \in Y_N.$$

We again omit the associated optimality system since it directly follows from (3.1). We denote the associated optimal state and slack RB solutions by $(y_N^{s,*}, s_N^*) \in Y_N \times S_N^+$. From the first-order optimality system we also obtain the optimal adjoint RB solution $p_N^{s,*}$ and the optimal Lagrange multiplier $\boldsymbol{\sigma}_N^{s,*} \in \mathbb{R}_+^{N_S}$ for the inequality condition $\boldsymbol{s}_N^* \geq 0$. We like to stress, however, that we do not need $\boldsymbol{\sigma}_N^{s,*}$ for the error bound presented below, but instead we will use the primal RB Lagrange multiplier approximation σ_N^*.

We also denote the RB control approximation from (S_N) by

$$u^s(\mu) = s_N^*(\mu) + u_a(\mu), \tag{4.1}$$

and conclude from $s_N^* \geq 0$ that u^s is an FE feasible approximation for the control, i.e. $u^s \in U_{\text{ad}}$.

In contrast to the RB primal problem (P_N), it can be shown that the RB slack problem (S_N) is well-posed even without supremizer enrichment of the RB slack space S_N; we refer to [2] for details.

We note that the nonnegativity condition imposed by $s_N \in S_N^+$ in (S_N) is more restrictive than $u_N \in U_{\text{ad},N}$ in (P_N) in terms of the control approximation as it permits only conical linear combinations (i.e., with only nonnegative coefficients, $\boldsymbol{s}_N \geq 0$). Consequently, the RB slack approach — in contrast to the RB primal approach — approximates the feasible set U_{ad} from the inside and thus guarantees the feasibility of the control approximation. However, in the case of small or empty active sets (i.e. unconstrained solutions), this restriction can lead to larger approximation errors since negative coefficients cannot be employed gainfully. We also note that the condition $s_N \in S_N^+$ precludes orthogonalization of the basis functions ζ_i^s.

4.3. Primal-slack error bound. We turn to the derivation of an *a posteriori* error bound for the error $u^* - u^s$. To this end, we extend the approach presented in [35] for variational inequalities to our setting, i.e., we advantageously combine the RB approximations of the primal and slack problems and exploit the FE feasibility of u^s. More precisely, we consider the primal-slack approximation $(y_N^{s,*}, p_N^{s,*}, u^s, \sigma_N^*) \in Y_N \times Y_N \times U_{\text{ad}} \times \Sigma_N^+$ and define the corresponding errors

$$e_y^s(\mu) = y^* - y_N^{s,*}, \qquad e_p^s(\mu) = p^* - p_N^{s,*}, \qquad e_u^s(\mu) = u^* - u^s.$$

Recall that we already introduced $e_\sigma(\mu)$ in the beginning of subsection 3.3. We also require the definition of the corresponding residuals.

DEFINITION 6. *The residuals of the state equation, the adjoint equation, and the optimality equation are defined by*

$$
\begin{aligned}
r_y^s(\phi;\mu) &= b(u^s,\phi;\mu) + f(\phi;\mu) - a(y_N^{s,*},\phi;\mu) && \forall \phi \in Y \quad \forall \mu \in \mathcal{D},\\
r_p^s(\varphi;\mu) &= (y_d - y_N^{s,*},\varphi)_{L^2(\Omega_D)} - a(\varphi, p_N^{s,*};\mu) && \forall \varphi \in Y \quad \forall \mu \in \mathcal{D},\\
r_u^s(\psi;\mu) &= -\lambda(u^s,\psi)_U + b(\psi,p_N^{s,*};\mu) + (\sigma_N^*,\psi)_U && \forall \psi \in U \quad \forall \mu \in \mathcal{D}.
\end{aligned}
$$

We state the main result in the next theorem.

THEOREM 7. *Let u^* and s_N^* be the optimal solutions of the FE optimal control problem (P) and the reduced slack problem ($\mathrm{S_N}$), respectively. For any given parameter $\mu \in \mathcal{D}$ the error in the optimal control satisfies*

$$\|e_u^s(\mu)\|_U \le \Delta_N^{\text{pr-sl}}(\mu),$$

where $\Delta_N^{\text{pr-sl}}(\mu) := c_1^s(\mu) + \sqrt{c_1^s(\mu)^2 + c_2^s(\mu)}$ with nonnegative coefficients

$$c_1^s(\mu) = \frac{1}{2\lambda}\left(\|r_u^s\|_{U'} + \frac{\gamma_b^{\text{UB}}(\mu)}{\alpha_a^{\text{LB}}(\mu)}\|r_p^s\|_{Y'}\right), \tag{4.2a}$$

$$c_2^s(\mu) = \frac{1}{\lambda}\left(\frac{2}{\alpha_a^{\text{LB}}(\mu)}\|r_y^s\|_{Y'}\|r_p^s\|_{Y'} + \frac{(C_{\Omega_D}^{\text{UB}})^2}{4(\alpha_a^{\text{LB}}(\mu))^2}\|r_y^s\|_{Y'}^2 + (s_N^*,\sigma_N^*)_U\right). \tag{4.2b}$$

Before proceeding with the proof, we immediately observe that the primal-slack bound $\Delta_N^{\text{pr-sl}}(\mu)$ is similar to the primal bound defined in Theorem 4 but does not contain the two terms δ_1 and δ_2 anymore. Instead, it contains the nonnegative term $(s_N^*,\sigma_N^*)_U$ which is the only non-standard ingredient. Since s_N^* is an approximation of $u^* - u_a$, the term $(s_N^*,\sigma_N^*)_U$ measures how well the reduced basis can approximate the FE complementarity condition $(u^* - u_a, \sigma^*)_U$. Also recall that the $\mathcal{N}_U$-dependent online computational cost to evaluate the primal bound was due to δ_1 and δ_2. The online cost to evaluate the primal-slack bound, in contrast, is *independent* of $\mathcal{N}_U$ and only depends on the dimension of the reduced basis approximation. Finally, we note that the term $(s_N^*,\sigma_N^*)_U$ and the residual r_u^s incorporate the primal variable σ_N^* in addition to the variables from the RB slack problem. Hence the bound uses information from both the primal and slack RB problem and we therefore call it primal-slack error bound.

Proof. The finite element optimal solutions y^*, p^*, u^*, σ^* satisfy the first-order optimality conditions (2.8) and hence we obtain the error-residual equations

$$a(e_y^s,\phi;\mu) - b(e_u^s,\phi;\mu) = r_y^s(\phi;\mu) \qquad \forall \phi \in Y, \tag{4.3a}$$

$$a(\varphi,e_p^s;\mu) + (e_y^s,\varphi)_{L^2(\Omega_D)} = r_p^s(\varphi;\mu) \qquad \forall \varphi \in Y, \tag{4.3b}$$

$$\lambda(e_u^s,\psi)_U - b(\psi,e_p^s;\mu) - (e_\sigma,\psi)_U = r_u^s(\psi;\mu) \qquad \forall \psi \in U. \tag{4.3c}$$

We follow the same steps as in the proof of Theorem 4. We use the inequalities (3.4), (3.5), (3.7) and involve Young's inequality to obtain

$$
\begin{aligned}
\lambda\|e_u^s\|_U^2 \le{}& \|r_u^s\|_{U'}\|e_u^s\|_U + \frac{2}{\alpha_a^{\text{LB}}(\mu)}\|r_y^s\|_{Y'}\|e_p^s\|_Y\\
&+ \frac{\gamma_b^{\text{UB}}(\mu)}{\alpha_a^{\text{LB}}(\mu)}\|r_p^s\|_{Y'}\|e_u^s\|_U + \frac{(C_{\Omega_D}^{\text{UB}})^2}{4(\alpha_a^{\text{LB}}(\mu))^2}\|r_y^s\|_{Y'}^2 + (e_\sigma,e_u^s)_U.
\end{aligned}
$$

In contrast to the proof of Theorem 4, we can bound $(e_\sigma, e_u^s)_U$ differently. Since $u^s = s_N^* + u_a$ (cf. (4.1)) we have

$$\begin{aligned}(e_\sigma, e_u^s)_U &= (\sigma^* - \sigma_N^*, u^* - u^s)_U = -(u^s - u^*, \sigma^*)_U + (u^s - u^*, \sigma_N^*)_U \\ &= -(u_a - u^*, \sigma^*)_U - (s_N^*, \sigma^*)_U + (u_a - u^*, \sigma_N^*)_U + (s_N^*, \sigma_N^*)_U.\end{aligned}$$

By the complementarity condition (2.8e), the nonnegativity of $(s_N^*, \sigma^*)_U$, and since $(u_a - u^*, \sigma_N^*)_U \le 0$ (by (2.8d) with $\rho = \sigma_N^*$) we obtain

$$(e_\sigma, e_u^s)_U \le (s_N^*, \sigma_N^*)_U.$$

We thus obtain a quadratic inequality in $\|e_u^s\|_U$, $\|e_u^s\|_U^2 - 2c_1^s(\mu)\|e_u^s\|_U - c_2^s(\mu) \le 0$, whose larger root is given by $\Delta_N^{\text{pr-sl}}(\mu)$. ∎

We make two remarks. First, the only ingredient used from the primal RB problem for the primal-slack approximation is σ_N^*, which is needed to evaluate the error bound. The slack control approximation u^s, however, is completely independent of the primal RB problem. If we are only interested in an RB approximation to the optimal control u^*, we may thus either solve the primal problem for u_N^* or the slack problem for u^s — the computational effort is roughly equivalent.

Second, we note that — similar to the primal bound — the error bound $\Delta_N^{\text{pr-sl}}(\mu)$ recovers the *a posteriori* control error bound proposed in [19] for unconstrained optimal control problems if the primal RB problem ($\mathrm{P_N}$) is unconstrained: if $\sigma_N^*(\mu) = 0$ holds, then (1) the residual $r_u^s(\cdot;\mu)$ reduces to the unconstrained case, i.e., without the term $(\sigma_N^*, \cdot)_U$, and (2) the term $(s_N^*, \sigma_N^*)_U$ vanishes. Although the error bound is then identical to the one presented in [19], we stress that the RB approximation itself is different as it is obtained from the RB slack problem ($\mathrm{S_N}$) instead of the RB primal problem ($\mathrm{P_N}$).

We omit the details on the computational procedure for the primal-slack error bound since it is very similar to the primal error bound. Indeed, we need to compute the same lower and upper bounds for the constants $\alpha_a^{\text{LB}}(\mu)$, $\gamma_b^{\text{UB}}(\mu)$, and $C_{\Omega_D}^{\text{UB}}$. Furthermore, the evaluation of the dual norms of the residuals $\|r_y^s\|_{Y'}$, $\|r_p^s\|_{Y'}$, and $\|r_u^s\|_{U'}$ is completely analogous to the primal problem. The only difference is the term $(s_N^*, \sigma_N^*)_U$, which can be computed online-efficiently by precomputing the matrix $(Z_N^s)^T M_U Z_N^\sigma \in \mathbb{R}^{N_s \times N_\sigma}$, where $Z_N^s = (\boldsymbol{\zeta}_1^s \mid \cdots \mid \boldsymbol{\zeta}_{N_s}^s) \in \mathbb{R}^{\mathcal{N}_U \times N_s}$.

4.4. Primal vs. primal-slack approach. We comment on the main differences between the primal(-only) approach and the primal-slack approach; also see [35] for a thorough discussion in the context of variational inequalities. We first summarize the different feasibility relations:

primal:	$U_N^\sigma \boldsymbol{u}_N^* \ge U_{a,N}^\sigma$	$\Leftrightarrow$	$u_N^* \in U_{\text{ad},N}$	$\not\Rightarrow u_N^* \in U_{\text{ad}}$
slack:	$\boldsymbol{s}_N^* \ge 0$	$\Leftrightarrow$	$s_N^* \in S_N^+ \subset U^+$	$\Rightarrow u^s \in U_{\text{ad}}$.

The FE feasibility of the RB slack approximation s_N^* provides us with an FE feasible control approximation u^s, whereas the control approximation u_N^* of the RB primal problem is not necessarily feasible. As mentioned above, the online costs to compute u^s and u_N^* are roughly the same. The different feasibility relations, however, determine if the *a posteriori* control error bounds are online-efficiently computable and — as we shall see in section 6 — affect their sharpness. The primal bound $\Delta_N^{\text{pr}}(\mu)$ requires only the solution of the RB primal problem ($\mathrm{P_N}$), i.e., $(y_N^*, p_N^*, u_N^*, \sigma_N^*)$, whereas the primal-slack bound $\Delta_N^{\text{pr-sl}}(\mu)$ requires ingredients from both the RB primal problem (P_N), i.e., σ_N^*, and the RB slack problem (S_N), i.e., $(y_N^{s,*}, p_N^{s,*}, u^s)$. Given these optimal solutions, the online computational cost for the dual norms of the residuals depends on the dimensions of the RB spaces N_Y, N_U, N_S, N_σ and the number of terms in the affine expansions (2.4). The cost is roughly the same for both approaches and independent of the FE dimensions $\mathcal{N}_Y$ and $\mathcal{N}_U$. For the primal-slack bound, however, we additionally need to evaluate the terms δ_1 and δ_2, which requires $\mathcal{O}(\mathcal{N}_U(N_U + N_\sigma))$ operations and is thus not online-efficient.

We compare the sharpness of the two bounds for several numerical test problems in section 6.

Finally, we note that – for both approaches – the lower control constraint can of course be replaced by an upper control constraint (also see the numerical results in section 6). For an extension to bilateral control constraints we refer to [2].

5. Greedy algorithm. The reduced basis spaces introduced in subsections 3.1 and 4.2 are constructed using the greedy sampling procedure outlined in Algorithm 1. A survey of parametric model reduction methods is presented in [3], one aspect of which are different sampling strategies in parameter space (e.g., structured or random sampling, or sampling via local sensitivity analysis); we thus refer to [3] for an overview of these strategies. Since our objective is to reduce the error in the optimal control approximation, we employ the proposed *a posteriori* error bounds in the greedy search. Here, $\Xi_\text{train} \subset \mathcal{D}$ is a finite but suitably large parameter training sample, $\mu^1 \in \Xi_\text{train}$ is the initial parameter value, N_max the maximum number of greedy iterations, $\epsilon_{\text{tol},\min} > 0$ is a prescribed desired error tolerance, and $\Delta_N^\bullet(\mu)$, $\bullet \in \{\text{pr}, \text{pr-sl}\}$, is the primal or primal-slack error bound and $u_N^\bullet \in \{u_N, u^s\}$.

In the following we like to point out the major differences to the standard greedy algorithm [33]. First, we construct integrated RB spaces Y_N (state and adjoint snapshots) in step 5 and U_N (control and Lagrange multiplier snapshots) in step 6 as discussed previously. Second, we point out that in steps 5 to 8 we need to check if the new snapshots are already contained in the reduced basis spaces and consequently discard linearly dependent snapshots. It may thus happen in our setting that some RB spaces are not enriched in every greedy step. Consider the following case: let u_a be parameter-independent and, say, $a(\cdot,\cdot,\mu)$ parameter-dependent. Assume that the greedy algorithm previously picked a parameter μ^N with $u^*(\mu^N) \equiv u_a$, i.e., the constraint is fully active, and we thus have $u_a \in U_N$. If the greedy search now picks another parameter μ^{N+1} where again $u^*(\mu^{N+1}) \equiv u_a$, we do not enrich U_N in step 6 to avoid linearly dependent basis functions. Note that such a scenario is indeed possible since the parameter enters through $a(\cdot,\cdot,\mu)$.

Finally, we comment on two special cases: (1) if the control constraint is fully active in each greedy step, i.e. we have $u^*(\mu^n) = u_a,\ n = 1,\dots,N$, we set $s_N^* = 0$; and (2) if the control constraint is never active, i.e., for all snapshots $\sigma^*(\mu^n) = 0,\ n = 1,\dots,N$, we set $\sigma_N^* = 0$.

6. Numerical results. In this section we consider three numerical examples: (i) a thermal block problem with one conductivity parameter and a fixed lower bound, (ii) a thermal block problem with one conductivity parameter and a parametrized upper bound and (iii) a Graetz flow problem with varying Péclet number and a parametrized domain. Our goal is to demonstrate the different properties of the approximations and their error bounds.

Before we discuss the three examples in detail we introduce another well-known error bound $\Delta_N^{\text{TV}}(\mu)$ in order to compare it with the error bounds proposed in this

Algorithm 1 Greedy Sampling Procedure

Choose $\Xi_{\text{train}} \subset \mathcal{D}$, $\mu^1 \in \Xi_{\text{train}}$ (arbitrary), $N_{\max}$, and $\epsilon_{\text{tol,min}} > 0$

Set $N \leftarrow 1, \quad Y_0 \leftarrow \{0\}, \quad U_0 \leftarrow \{0\}, \quad S_0 \leftarrow \{0\}, \quad \Sigma_0 \leftarrow \{0\}$

Set $\Delta_N^\bullet(\mu^N) \leftarrow \infty$

while $\Delta_N^\bullet(\mu^N)/\|u_N^\bullet(\mu^N)\|_U > \epsilon_{\text{tol,min}}$ **and** $N \leq N_{\max}$ **do**

  $Y_N \leftarrow Y_{N-1} \oplus \text{span}\{\, y^*(\mu^N), p^*(\mu^N)\,\}$

  $U_N \leftarrow U_{N-1} \oplus \text{span}\{\, u^*(\mu^N), \sigma^*(\mu^N)\,\}$

  $S_N \leftarrow S_{N-1} \oplus \text{span}\{\, s^*(\mu^N)\,\}$

  $\Sigma_N \leftarrow \Sigma_{N-1} \oplus \text{span}\{\, \sigma^*(\mu^N)\,\}$

  $\mu^{N+1} \leftarrow \underset{\mu \in \Xi_{\text{train}}}{\arg\max}\ \Delta_N^\bullet(\mu)/\|u_N^\bullet(\mu)\|_U$

  $N \leftarrow N + 1$

end while

paper. The bound $\Delta_N^{\text{TV}}(\mu)$ has been proposed in [32], where the authors estimate the distance between a suboptimal control $\tilde{u}$ and the unknown optimal control u^* using a perturbation argument. Although the bound has been proposed in the context of proper orthogonal decomposition, it can be applied to any feasible control $\tilde{u} \in U_{\text{ad}}$. In fact, we note that the bound has been used successfully for other model order reduction methods, see [4] for an overview. However, as pointed out in [32], the computation of $\Delta_N^{\text{TV}}(\mu)$ requires a solution of the FE state and adjoint equation for the given control $\tilde{u}$. Hence, the evaluation of the error bound is computationally expensive as it depends on the FE dimension $\mathcal{N}_Y$. Also note that the error bound $\Delta_N^{\text{TV}}(\mu)$ can only be computed for P_0 control discretizations, since its construction is based on a pointwise analysis of the optimality equation.

In principle, we can evaluate the error bound $\Delta_N^{\text{TV}}(\mu)$ for both the primal optimal control $u_N^*(\mu)$ and the slack optimal control $u^s(\mu)$. However, recall that for the slack approach we can guarantee $u^s \in U_{\text{ad}}$ and hence directly evaluate the bound $\Delta_N^{\text{TV}}(\mu)$, whereas u_N^* is not necessarily feasible. We therefore need to first project u_N^* onto U_{ad} to obtain $u_N^{\text{proj}} \in U_{\text{ad}}$ and then evaluate $\Delta_N^{\text{TV}}(\mu)$ for u_N^{proj}. In this case we encounter rather poor results for all our examples, i.e., a relative bound of approximately 0.1 for all N. We therefore evaluate $\Delta_N^{\text{TV}}(\mu)$ and present results only for the slack optimal control $u^s(\mu)$. We also note that [32] did not consider a reduction of the control space and the feasibility issue of the "reduced" control thus did not appear.

The following computations are performed with MATLAB (R2014b) on an Intel Core i5-4570R 2.7 GHz processor and 16 GB RAM. To solve the FE optimal control problems we use the primal-dual active set method, see [31] for details.

6.1. Thermal block: Lower bound. We consider a linear-quadratic optimal control problem governed by a steady heat conduction problem in a two-dimensional domain. The spatial domain is given by $\Omega = [0,1]^2$ and is subdivided into the two subdomains $\Omega_1 = [0, 0.5] \times [0,1]$ and $\Omega_2 = [0.5, 1] \times [0,1]$ with thermal conductivities μ_1 and 1, respectively.

We impose homogeneous Dirichlet boundary conditions for the state on Γ. The amount of heat supply in the domain Ω is regulated by the distributed control function

$u \in U \subset U_\mathrm{e} \equiv L^2(\Omega)$. The parametrized optimal control problem is then

$$\min_{y \in Y, u \in U} J(y,u;\mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \quad \mu_1 \int_{\Omega_1} \nabla y \cdot \nabla v \,\mathrm{d}x + 1 \int_{\Omega_2} \nabla y \cdot \nabla v \,\mathrm{d}x = \int_\Omega u\, v \,\mathrm{d}x \quad \forall v \in Y,$$

$$(u_a, \rho)_U \le (u, \rho)_U \quad \forall \rho \in U^+,$$

where $Y \subset Y_\mathrm{e} = H^1_0(\Omega)$ and $u_a(x) = 2+2\cdot(x_1-0.5)$ is fixed. The parameter μ_1 satisfies $\mu_1 \in \mathcal{D} = [0.5, 3]$. We set the desired state to $y_d = 1$ in Ω and the regularization parameter to $\lambda = 0.01$. We choose the inner product $(w,v)_Y = \mu_1^\mathrm{ref} \int_{\Omega_1} \nabla w \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_2} \nabla w \cdot \nabla v \,\mathrm{d}x$ for $\mu_1^\mathrm{ref} = 1$ and hence get $\alpha_a^\mathrm{LB}(\mu) = \min(\mu_1/\mu_1^\mathrm{ref}, 1)$ by the min-theta approach. For the control space U we use the standard L^2-norm and L^2-inner product. The considered problem satisfies the affine representation (2.4) with $Q_a = 2$, $Q_b = Q_f = Q_{ua} = Q_{yd} = 1$.

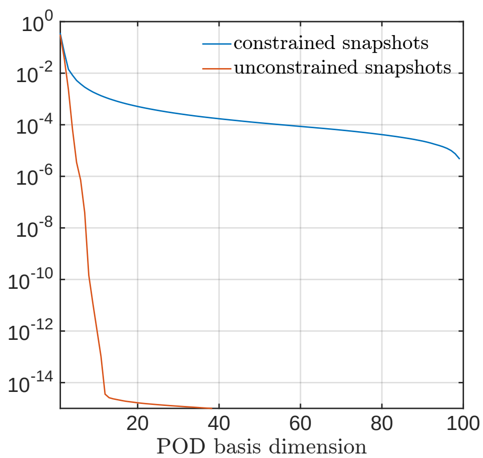

Fig. 1a: POD error for unconstrained and constrained training snapshots.

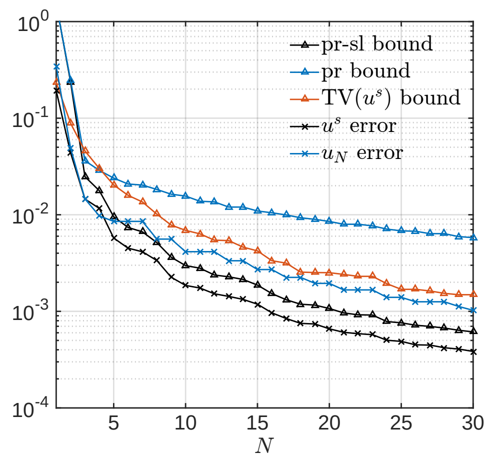

Fig. 1b: Maximal relative control errors and bounds over the number of greedy iterations.

For the truth discretization we consider a piecewise linear finite element approximation space $Y \subset Y_\mathrm{e}$ for the state and adjoint variables, and a piecewise constant FE approximation space $U \subset U_\mathrm{e}$ for the control variable. We use a quasi-uniform unstructured mesh on Ω, leading to $\dim(Y) = \mathcal{N}_Y \approx 8{,}300$ and $\dim(U) = \mathcal{N}_U \approx 17{,}000$. The associated finite element discretization error is roughly 1%.

We construct the reduced basis spaces using the greedy sampling procedure described in section 5. To this end, we employ a training sample $\Xi_\mathrm{train} \subset \mathcal{D}$ consisting of $n_\mathrm{train} = 100$ equidistant points on a log-scale and stop the greedy enrichment after 30 steps, i.e., we set $N_\mathrm{max} = 30$ resulting in a relative pr-sl error bound of $\approx 1\,\mathrm{E}{-3}$. Note that here and for the following examples we generate two separate RB spaces for the primal (pr) and the primal-slack (pr-sl) approach. The first one uses the primal bound $\Delta_N^\mathrm{pr}(\mu)$ in the greedy search and the second one uses the pr-sl bound $\Delta_N^\text{pr-sl}(\mu)$.

In Figure 1a we present the POD error decay of the control snapshots over Ξ_train for the constrained and corresponding unconstrained problem, i.e., setting $u_a = -\infty$ in the problem formulation.[4] Note that POD serves as an indicator for how many

[4]The error for a POD basis of size N_POD is given by $(\sum_{i=N_\mathrm{POD}+1}^{n_\mathrm{train}} \sigma_i^2)^{1/2}$, where σ_i, $1 \le i \le$

snapshots we need in our RB basis to approximate the control with a desired accuracy for all $\mu \in \mathcal{D}$. We immediately observe that the decay behavior is extremely different: the unconstrained problem can be approximated by roughly 12 snapshots in the RB basis to a precision of $1\text{E}-14$, whereas for the constrained problem the POD error decays much slower and hence we need a larger RB basis to reach a certain desired accuracy. Although POD is only optimal in the L^2-sense over the snapshot set, it serves as a good indicator of the error one can expect from our reduced basis approach (also see the discussion in Section 8.1.4 in [30]). Note that the POD results reflect only the control dynamics since only control snapshots are used in the basis construction. In contrary, the RB approach is based on an approximation of the full optimality system, and hence snapshots of the state, adjoint, control, and Lagrange multiplier variable are required. Nevertheless, in our numerical examples the control dynamics seem to dominate and thus the results for POD and RB are comparable.

In Figure 1b we present, as a function of the number of greedy iterations N, the maximum relative errors and bounds over a test sample $\Xi_{\text{test}} \subset \mathcal{D}$ of size $n_{\text{test}} = 125$ with logarithmically distributed parameter points in $[0.503, 2.99] \subset \mathcal{D}$. Here, the errors and bounds are defined as follows: the pr-sl bound is the maximum of $\Delta_N^{\text{pr-sl}}(\mu)/\|u^*(\mu)\|_U$ over Ξ_{test}, the primal bound is the maximum of $\Delta_N^{\text{pr}}(\mu)/\|u^*(\mu)\|_U$ over Ξ_{test}, the $\text{TV}(u^s)$ bound is the maximum of $\Delta_N^{\text{TV}}(\mu)/\|u^*(\mu)\|_U$ over Ξ_{test} evaluated for u^s, and the u^s and u_N errors are the maxima of $\|u^*(\mu) - u^s(\mu)\|_U/\|u^*(\mu)\|_U$ and $\|u^*(\mu) - u_N^*(\mu)\|_U/\|u^*(\mu)\|_U$ over Ξ_{test}, respectively. We observe that both the primal and pr-sl error converge, but also that the primal error is approximately three times larger than the pr-sl error. The pr-sl error reaches $\approx 3\text{E}-4$ at $N = 30$, which is in good agreement with the POD results in Figure 1a. Furthermore, the pr-sl error bound is considerably sharper than the primal bound and even the TV bound. For $N = 30$ we obtain an average effectivity, i.e., the ratio of the bound and the error, of 5 for the primal bound and 1.6 for the pr-sl bound.

6.2. Thermal block: Upper bound. We consider the same problem setting as in the last subsection, but replace the lower constraint u_a with a parametrized upper constraint $u_b(x; \mu_2)$, given by the FE interpolant of $5 + \mu_2 \sin(3\pi x_1) \sin(3\pi x_2) \exp(-x_1)$. As in the previous example we consider the conductivity as a parameter such that $(\mu_1, \mu_2)^T \in \mathcal{D} = [0.5, 3] \times [-2, 2]$. We also consider the same truth discretization as before and obtain a discretization error of roughly 2% for this problem. In Figure 2 we illustrate the behavior of the solution $u(\mu)$ and the active sets for different values of μ. Note that changing μ_2 influences $u_b(x; \mu_2)$ which in turn leads to different constrained controls $u(\mu)$; see lower part of Figures 2a and 2b. In the lower part of Figure 2c we discover that $\mu_1 = 3$ leads to an unconstrained $u(\mu)$. We observe in the upper row of Figure 2 that the active set is changing considerably and is empty for some parameters.

We construct the reduced basis spaces using the greedy sampling procedure described in section 5. We employ a training sample with $50 \cdot 50 = 2500$ equidistant parameter points (on a log-scale in μ_1 and a lin-scale in μ_2) and stop the greedy enrichment after 30 steps. We also introduce a parameter test sample with $16 \cdot 16 = 256$ (log $\times$ lin) parameter points in $[0.503, 2.99] \times [-1.98, 1.9] \subset \mathcal{D}$.

We again present results for the POD error and the RB errors and bounds in

n_{train}, are the singular values (in decreasing order) of $\frac{1}{\sqrt{n_{\text{train}}}} \mathbb{U}^{1/2} S$. Here, $\mathbb{U}$ is the finite element matrix associated with the inner product $(\cdot, \cdot)_U$, and $S \in \mathbb{R}^{\mathcal{N}_U \times n_{\text{train}}}$ is the snapshot matrix of optimal controls $u^*(\mu)$ for all $\mu \in \Xi_{\text{train}}$. See also [29].

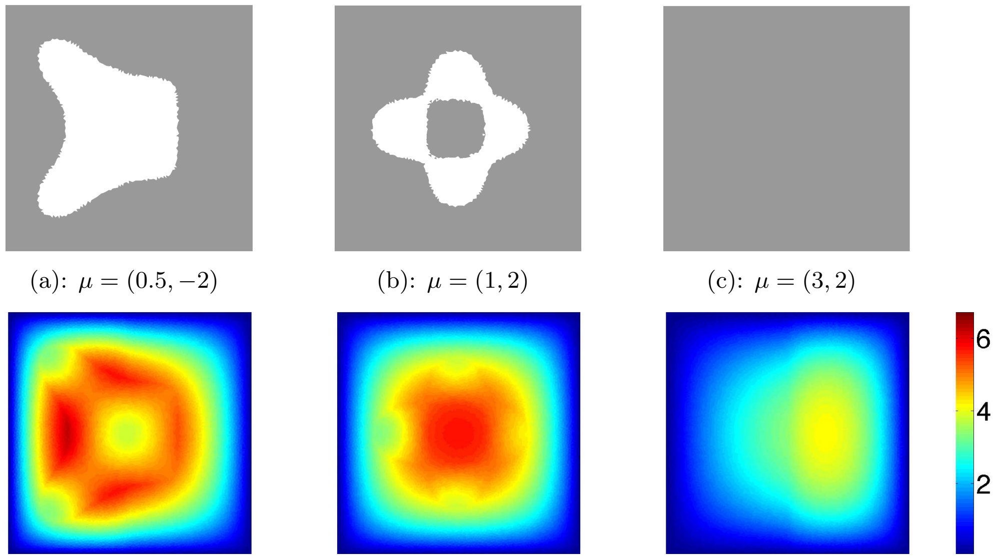

Fig. 2: Snapshots of active sets (upper row) and optimal control (lower row) for different parameters. The active (inactive) sets are displayed in white (gray).

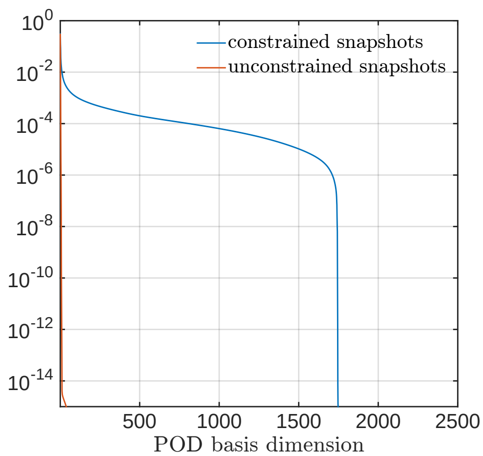

Fig. 3a: POD error for unconstrained and constrained training snapshots.

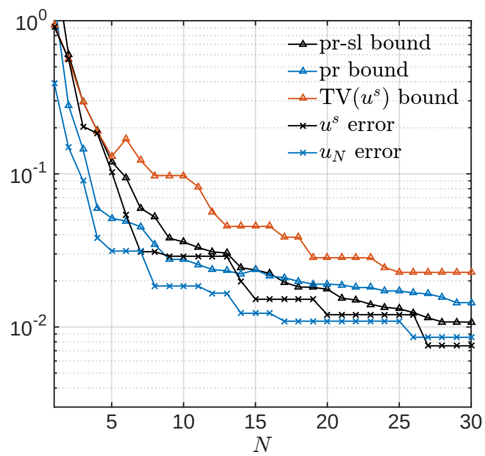

Fig. 3b: Maximal relative control errors and bounds over the number of greedy iterations.

Figures 3a and 3b, respectively. All quantities are defined as in the last example. We first observe that the POD error of the unconstrained problem again converges very fast (note the much larger training sample). For the constrained problem, however, the POD error decay is initially similar to the previous example, but then also drops to zero at a POD dimension of approx. 1750. The reason for this is that the control constraint is not active for approximately one third of the parameter domain and hence the problem reduces to an unconstrained problem for these parameters. We can expect for a POD basis of size 30 an error of $\approx 4\,\text{E}-3$, which again roughly matches the results in Figure 3b.

We observe a similar convergence of the primal and slack error, but this time the primal error is slightly smaller (except for $N \geq 27$). In contrast to the previous example, the overall convergence rate is smaller due to the parametrized upper bound resulting in more diverse solutions (also visible from the POD convergence and the variation in the active sets). We also note that the pr-sl error bound is again much sharper than the primal bound and also the TV bound. Thus, although the slack error is larger than the primal error, the smaller effectivity of the pr-sl bound still leads to a better overall *a posteriori* certificate for the control error.

Finally, we briefly report the computational timings: the solution of the FE optimization problem takes ≈ 2 seconds (for a discretization error of 2%) and the solution of the RB optimization problem using the primal (slack) approach, for $N = 25$, takes ≈ 0.027 (≈ 0.022) seconds. The RB slack problem can be solved slightly faster since we do not have to enrich S_N by supremizers and $\dim(S_N)$ is thus roughly only half the size of $\dim(U_N)$. Furthermore, the evaluation of the $\mathrm{TV}(u^s)$ bound takes 0.053 seconds, whereas the evaluation of the primal bound takes 0.0075 seconds and the pr-sl bound, given σ_N^*, takes 0.0045 seconds. Overall, we achieve an online speed-up of $\approx 70-90$ for the RB solution without certification and $\approx 40-60$ for the RB solution with certification. Finally, we note that the offline phase for the primal (primal-slack) approach takes ≈ 1900 seconds (≈ 2700 seconds), which is mainly due to the fairly large training set including 2500 parameter points.

6.3. Graetz flow: Constant lower bound and geometry parametrization. We consider a linear-quadratic optimal control problem governed by a steady Graetz flow in a two-dimensional domain inspired by the numerical examples in [26]. The problem is parametrized by a varying Péclet number $\mu_1 \in [5, 18]$ and a geometry parameter $\mu_2 \in [0.8, 1.2]$. Hence, the parameter domain is $\mathcal{D} = [5, 18] \times [0.8, 1.2]$. The parametrized geometry is given by $\Omega(\mu) = [0, 1.5 + \mu_2] \times [0, 1]$ and is subdivided into the three subdomains $\Omega_1(\mu) = [0.2\mu_2, 0.8\mu_2] \times [0.3, 0.7]$, $\Omega_2(\mu) = [\mu_2 + 0.2, \mu_2 + 1.5] \times [0.3, 0.7]$, and $\Omega_3(\mu) = \Omega(\mu) \setminus \{\Omega_1(\mu) \cup \Omega_2(\mu)\}$. A sketch of the domain is shown in Figure 4. We impose boundary conditions of homogeneous Neumann and of non-homogeneous Dirichlet type: $y_n \equiv 0$ on $\Gamma_N(\mu)$, and $y \equiv 1$ on $\Gamma_D(\mu)$. Thus the trial space is given by $Y(\mu) \subset Y_{\mathrm{e}}(\mu) \equiv \{v \in H^1(\Omega(\mu)); v|_{\Gamma_D(\mu)} \equiv 1\}$. The amount of heat supply in the whole domain $\Omega(\mu)$ is regulated by the distributed control $u \in U(\mu) \subset U_{\mathrm{e}}(\mu) \equiv L^2(\Omega(\mu))$ and bounded by the lower constraint $u_a \equiv -0.5$. The observation domain is $\Omega_D(\mu) = \Omega_1(\mu) \cup \Omega_2(\mu)$ and the desired state is given by $y_d \equiv 0.5$ on $\Omega_1(\mu)$ and $y_d \equiv 2$ on $\Omega_2(\mu)$.

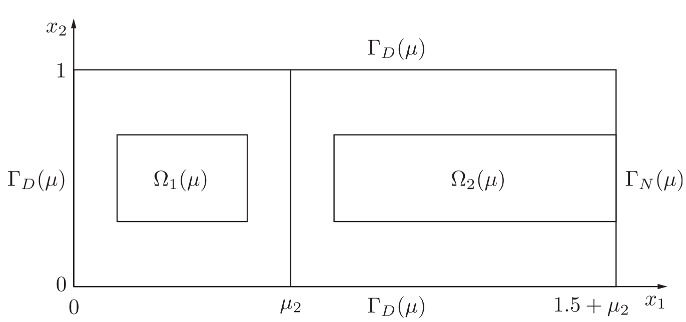

Fig. 4: Domain $\Omega(\mu)$ for the Graetz flow problem with distributed control.

Overall, the parametrized optimal control problem is given by

$$\begin{aligned}
&\min_{y\in Y(\mu),u\in U(\mu)} J(y,u;\mu) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega_D(\mu))} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega(\mu))} \\
&\text{s.t.} \quad \frac{1}{\mu_1}\int_{\Omega(\mu)} \nabla y\cdot\nabla v \,\mathrm{d}x + \int_{\Omega(\mu)} \beta(x)\cdot\nabla y\, v \,\mathrm{d}x = \int_{\Omega(\mu)} u\, v \,\mathrm{d}x \quad \forall v\in Y(\mu), \\
&\qquad (u_a,\rho)_{U(\mu)} \le (u,\rho)_{U(\mu)} \quad \forall \rho\in U(\mu)^+,
\end{aligned}$$

for the given parabolic velocity field $\beta(x) = (x_2(1-x_2), 0)^T$. The regularization parameter λ is fixed to 0.01.

After recasting the problem to a reference domain $\Omega = \Omega(\mu^{\text{ref}}) = [0,2.5]\times[0,1]$ for $\mu^{\text{ref}} = (5,1)^T$, and introducing suitable lifting functions that take into account the non-homogeneous Dirichlet boundary conditions, we can reformulate the problem in terms of the parameter-independent FE spaces $Y \subset Y_{\text{e}} = H^1_0(\Omega)$ and $U \subset U_{\text{e}} = L^2(\Omega)$ [30]. We then obtain the affine representation (2.4) of all involved quantities with $Q_a = Q_f = 4$, $Q_b = Q_d = Q_u = Q_{yd} = 2$, and $Q_{ua} = 1$. The inner product is given by $(w,v)_Y = \frac{1}{\mu_1^{\text{ref}}}\int_\Omega \nabla w\cdot\nabla v \,\mathrm{d}x + \frac{1}{2}(\int_\Omega \beta(x)\cdot\nabla w\, v\,\mathrm{d}x + \int_\Omega \beta(x)\cdot\nabla v\, w\,\mathrm{d}x)$ and we obtain a lower bound $\alpha_a^{\text{LB}}(\mu)$ for the coercivity constant by the min-theta approach. Note that for the control space we obtain a parameter-dependent inner product $(\cdot,\cdot)_{U(\mu)}$ from the affine geometry parametrization. Hence the control error is measured in the parameter-dependent energy norm $\|\cdot\|_{U(\mu)}$. The derivations of the primal and primal-slack error bounds remain the same in this case and they bound the control error in the energy norm. We also note that an offline-online decomposition for a domain parametrization only works for the chosen P_0 discretization of the control. In this case the control mass matrix $M_U(\mu)$ is diagonal and hence its inverse is easily computable and satisfies an affine expansion.

We choose the same type of discretization as in the previous examples (P_1 for the state and adjoint, P_0 for the control) to obtain $\dim(Y) = \mathcal{N}_Y \approx 11{,}000$ and $\dim(U) = \mathcal{N}_U \approx 22{,}000$. The chosen discretization induces a discretization error of roughly 2%. In Figure 5 we present control snapshots and associated active sets for two different parameters. Although here the control constraint u_a is parameter-independent, we again observe strongly varying control solutions and active sets.

We construct the reduced basis spaces using the greedy algorithm by employing an equidistant training sample $\Xi_{\text{train}} \subset \mathcal{D}$ of size $30\cdot 30 = 900$ (log-scale in μ_1 and lin-scale in μ_2) and stop the greedy enrichment after 30 steps. We also introduce a test sample with $10\cdot 5$ (log $\times$ lin) equidistant parameter points in $[5.2, 17.5]\times[0.82, 1.17] \subset \mathcal{D}$.

We plot the POD errors in Figure 6a and observe a very similar behavior as in the first example: the unconstrained problem converges much faster than the constrained one, and the control constraint is active for all snapshots in the training set. In Figure 6b we present, as a function of N, the resulting energy norm errors and bounds over the test sample. The decay of the errors and bounds is slightly slower than in the previous examples as a results of the strongly varying controls and active sets. The primal and pr-sl errors are very close to each other, but the pr-sl bound is considerably sharper than the primal bound. In contrast to the previous examples the $\text{TV}(u^s)$ bound tracks the error as well as the pr-sl bound.

As with the previous example, we briefly report the computational timings: the solution of the FE optimization problem takes ≈ 4 seconds (for a discretization error of 2%) and the solution of the RB optimization problem using the primal (slack) approach, for $N = 25$, takes ≈ 0.035 (≈ 0.024) seconds. The evaluation of the $\text{TV}(u^s)$ bound takes 0.12 seconds, whereas the evaluation of the primal bound takes

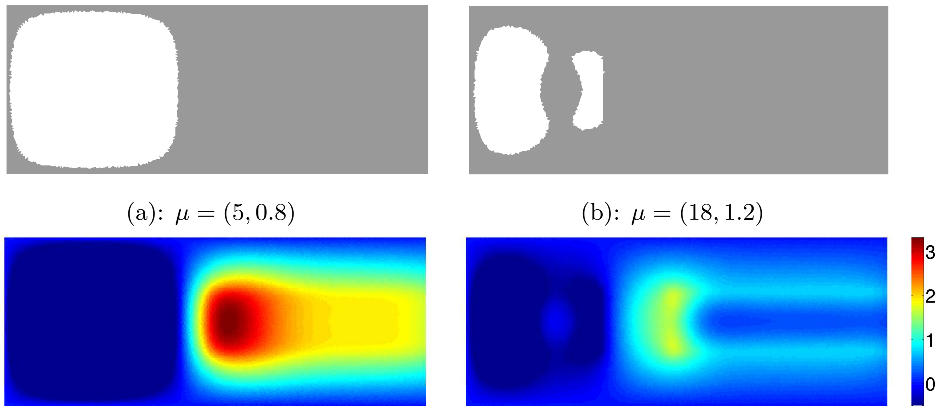

Fig. 5: Snapshots of active sets (upper row) and optimal control (lower row) on the reference domain. The active (inactive) sets are displayed in white (gray).

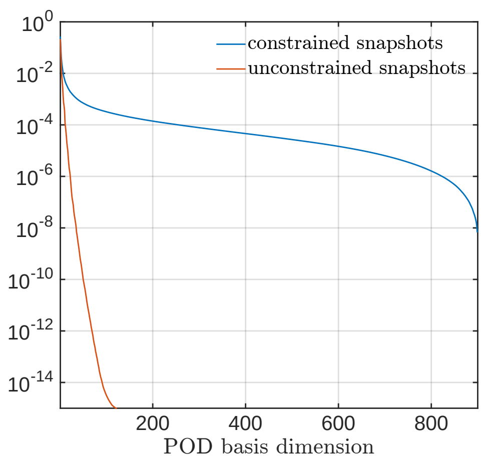

Fig. 6a: POD error for unconstrained and constrained training snapshots.

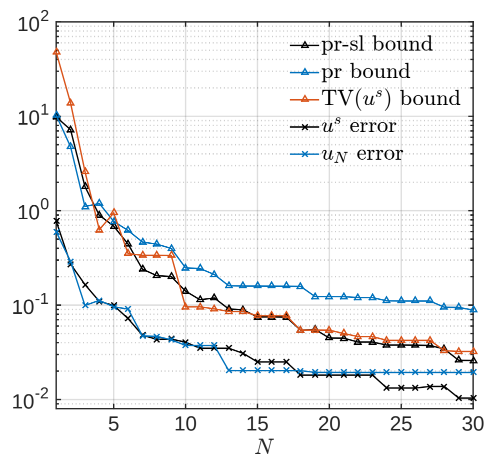

Fig. 6b: Maximal relative control errors and bounds over the number of greedy iterations.

0.008 seconds and the pr-sl bound, given σ_N^*, takes 0.005 seconds. Overall, we achieve an online speed-up of $\approx 110 - 170$ for the RB solution without certification and $\approx 60 - 90$ for the RB solution with certification. The offline phase for the primal (primal-slack) approach takes ≈ 900 seconds (≈ 1300 seconds).

7. Related works. In this section, we place our work in the context of already published contributions combining reduced order methods and *a posteriori* error estimation for optimal control problems. We note that, besides the reduced basis method, there exists a large amount of literature on parametric model reduction for control and optimization based on other model reduction approaches, e.g. balanced truncation. Since we focus on reduced basis methods and *a posteriori* error estimation in this paper, we do not intend to provide an extensive review of all the other works and thus refer to the excellent review articles [3] and [4] for further references on other approaches.

Elliptic optimal control problems with distributed control have been considered recently in [26, 25]. The proposed error bound is based on the Banach-Nečas-Babuška (BNB) theory applied to the first order optimality system and directly follows from previous work on reduced basis methods for noncoercive problems [33]. The approach thus provides a combined bound for the error in the state, adjoint, and control variable, but it is only applicable to problems *without* control constraints. Furthermore, the bound requires the computation of (a lower bound to) the parameter-dependent Babuška inf-sup constant of the first-order optimality system, which is usually computed via the SCM. Since the required offline computations are expensive or — as stated in [25] — even unaffordable, the authors in [25] propose to replace the rigorous SCM lower bound by a heuristic interpolant surrogate, thus turning the error bound into an error estimate.

Based on the ideas presented in [32], we proposed rigorous *and* online-efficient control error bounds for reduced basis approximations of scalar elliptic optimal control problems in [11, 18]. In [19] we extended these ideas to distributed control problems, proposed a new control error bound which is similar in spirit to the bounds proposed in this paper, and compared these two approaches with the BNB bound from [26].

As already mentioned in the beginning of section 6, POD *a posteriori* error bounds for elliptic *and* parabolic optimal control problems have been proposed in [32]. The approach provides rigorous and often very sharp control error bounds including control constraints. The evaluation of these bounds, however, requires a solution of the underlying high-dimensional state and adjoint equations and, as pointed out in [32], is thus computationally expensive.

Reduced basis *a posteriori* error bounds for parabolic optimal control problems including control constraints have been derived in [8, 9] and [17, 27]. In [8, 9], the author proposes an estimate for the error in the optimal value of the cost functional which is efficient to evaluate but not a rigorous bound for the error. Our approach presented in [17] is again based on [32], but provides a rigorous and efficiently evaluable bound for the error in the control and associated cost functional. We note that a pointwise analysis of the optimality equations is still feasible and online-efficient in the parabolic case as long as the controls are scalar functions of time. The online complexity of a pointwise analysis only depends on the number of time steps, which is inherent in any time-stepping discretization scheme. The approach presented in [19] is extended to a space-time reduced basis approach in [27].

Finally, although we considered a purely deterministic problem here, the input parameters could also be considered random inputs. We note that the approach presented here can be gainfully employed in such a stochastic setting, see for example [10] or [5]. For a more detailed comparison between the reduced basis method and stochastic collocation methods we refer to [6].

8. Conclusions. We proposed two novel certified reduced basis approaches for distributed elliptic optimal control problems with control constraints: a primal and a primal-slack approach. We observed in the numerical results that the approximation error is roughly the same for both approaches. The *a posteriori* error bound of the pr-sl approach, however, turned out to be considerably sharper than the bound of the primal approach, thus allowing a better *a posteriori* certification for the control error. Concerning the computational cost, the reduced basis approximation and error bound of the pr-sl approach can be evaluated efficiently using the standard offline-online decomposition. We do, however, need to solve two reduced order optimization problems to evaluate the pr-sl bound. The reduced basis approximation of the primal approach is also online-efficient, but the primal bound does not satisfy a full offline-online decomposition, i.e., the evaluation of the constraint-violation requires a search over the degrees of freedom of the underlying FE control space and thus depends linearly on its dimension $\mathcal{N}_U$.

In summary, the proposed methods allow an efficient and reliable online solution of parametrized optimal control problems involving control constraints. Such methods have the potential to greatly reduce the solution time while providing a rigorous certificate on the approximation quality. Thus, given a demand for real-time or repeated solutions, and the pervasive appearance of optimal control problems in engineering and science, we believe that the approach presented here and — more generally — reduced order methods for optimal control problems could prove beneficial in a large variety of real-world applications.

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