Adaptive Source Term Iteration
– A Stable Formulation for Radiative Transfer –

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(mono-energetic) Radiative Transfer Problem

We are looking for a particle density $u \in L_2(\mathrm{D} \times \mathrm{S})$ satisfying

$$
\begin{aligned}
(\mathcal{T}_\mathrm{S} - \mathcal{K}_\mathrm{S}) u(x,s) &= f(x,s) && \text{in } \mathrm{D} \times \mathrm{S} \\
u &= g && \text{on } \Gamma_-
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{T}_\mathrm{S} u(x,s) &:= s \cdot \nabla u(x,s) + \sigma(x,s) u(x,s) \\
\mathcal{K}_\mathrm{S} u(x,s) &:= \int_\mathrm{S} k(x,s',s) u(x,s') \, \mathrm{d}s'
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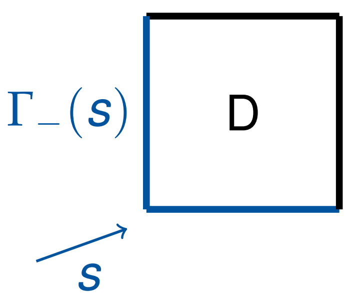
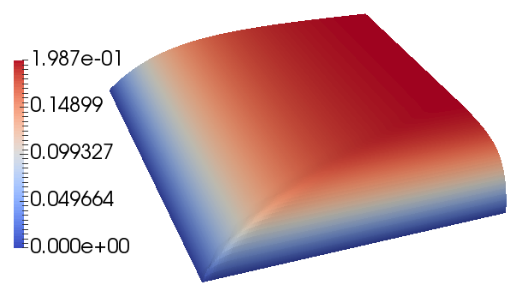
$$





(mono-energetic) Radiative Transfer Problem

Obstructions

- high dimensional ($n = 2, 3$ dimensional space + $n - 1$ dimensional transport direction)
- global scattering kernel
- reliable error estimates?
  – a priori error estimates often require unrealistic regularity assumptions on the solution

  $\rightarrow$ a posteriori estimates
    - ▶ error control of the solution
    - ▶ adaptive grid refinement

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Adaptive Source Term Iteration

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- Assuming $\|\mathcal{T}_{\mathrm{S}}^{-1}\mathcal{K}_{\mathrm{S}}\|_{\mathcal{L}(L_2,L_2)} = \rho < 1$, we have the *ideal iteration*

$$u_0 = 0$$
$$u_{n+1} = \mathcal{T}_{\mathrm{S}}^{-1}(\mathcal{K}_{\mathrm{S}} u_n + f), \quad n = 0, 1, 2, \ldots.$$

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- We need a numerical scheme to realize approximately the fixed point iterations while still guaranteing convergence of the infinite dimensional iteration.
- In the iterations:
  - Accuracy in solution of transport problems is dynamically increased across iterations $\rightarrow$ Need for tight error bounds to avoid adding unnecessary numerical effort $\rightsquigarrow$ DPG [Broersen et al., 2017]
  - Repeated application of $\mathcal{K}_S$ with increased accuracy $\rightsquigarrow$ Compression techniques.

The Henyey–Greenstein Kernel

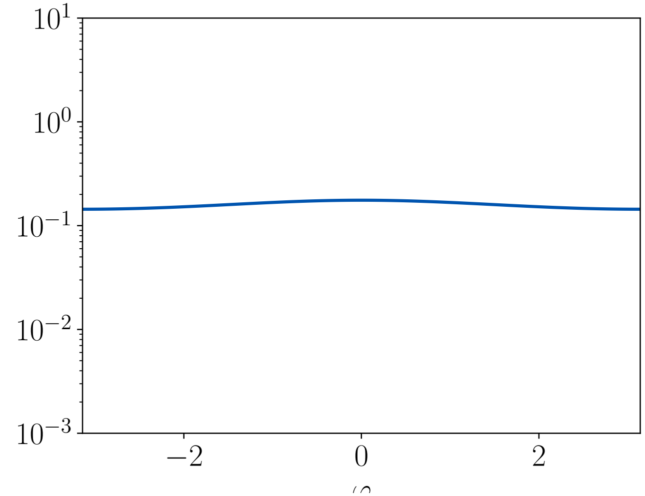
We explain how to approximate

$$\mathcal{K}v(x,s) = \int_{\mathbb{S}} k(s,s')v(x,s')\,\mathrm{d}s'$$

with the model kernel of Henyey and Greenstein [1941]

$$k(s,s') = k(\underbrace{s\cdot s'}_{=\cos\theta}) := \frac{1}{2\pi}\,\frac{1-\gamma^2}{1+\gamma^2-2\gamma s\cdot s'}$$

where $\gamma \in (-1,1)$.



$\gamma = 0.05$

The example illustrates the following issue:

- A naive approximation of $\mathcal{K}v$ with quadratures entails a quadratic cost in the number of directions.
- One can diminish this cost by exploiting sparsity.
- We illustrate how to achieve this depending on γ.

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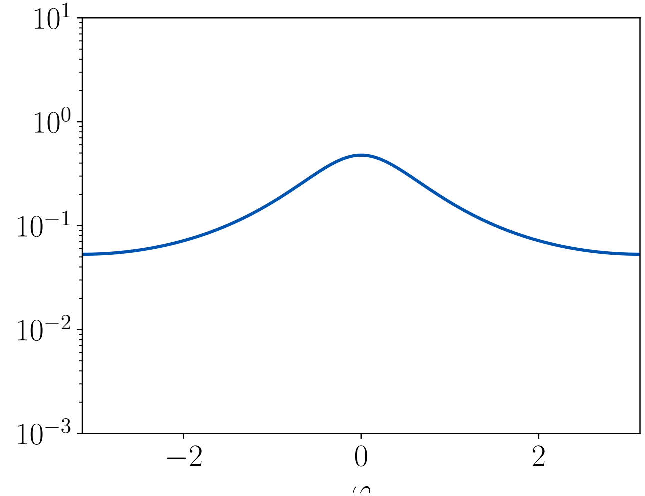

$\gamma = 0.5$

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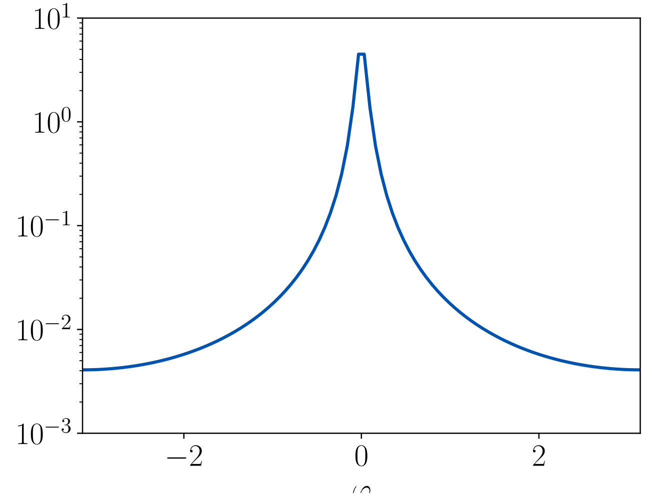

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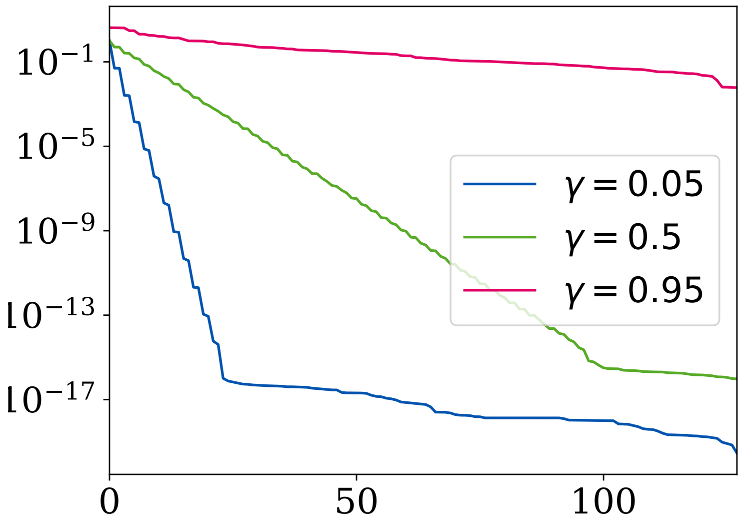
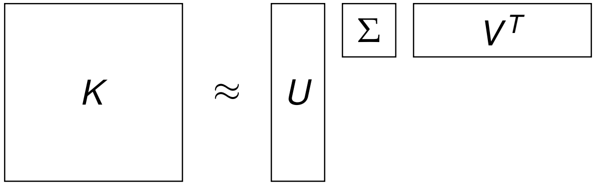
Low-Rank Approximation of the Henyey–Greenstein Kernel

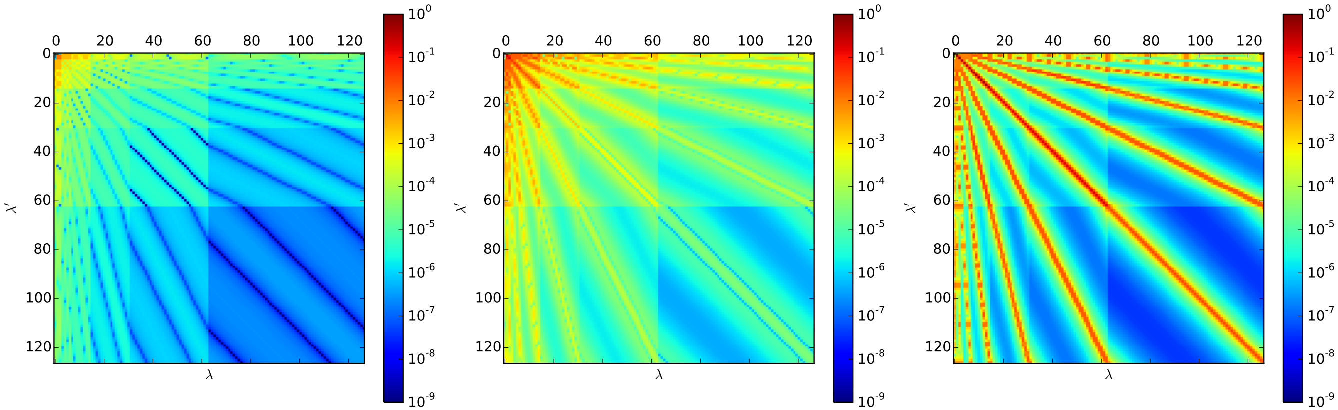


Singular values of Henyey–Greenstein kernel.



Low-rank approximation,
accurate for small γ.

Wavelet Approximation of the Henyey–Greenstein Kernel



(a) $\gamma = 0.05,\ M = 0$ (b) $\gamma = 0.5,\ M = 0$ (c) $\gamma = 0.95,\ M = 0$

When $\gamma \to 1$, the matrix $(k_{\lambda,\lambda'})_{\lambda,\lambda'}$ becomes quasi-sparse and we can apply compression techniques to efficiently apply the kernel.

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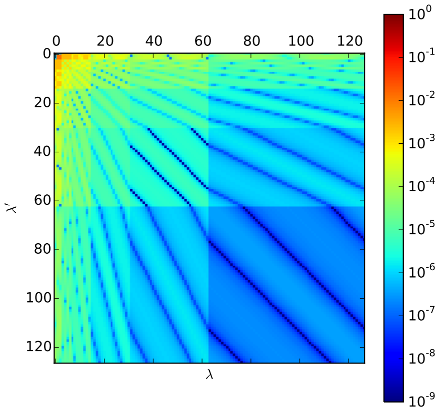

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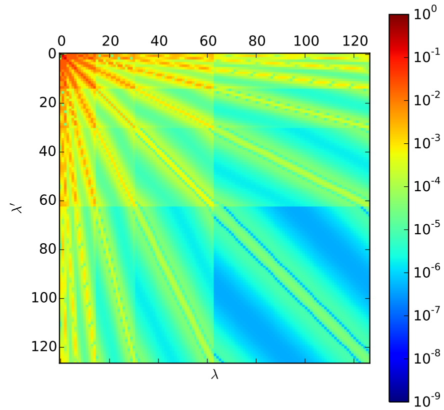

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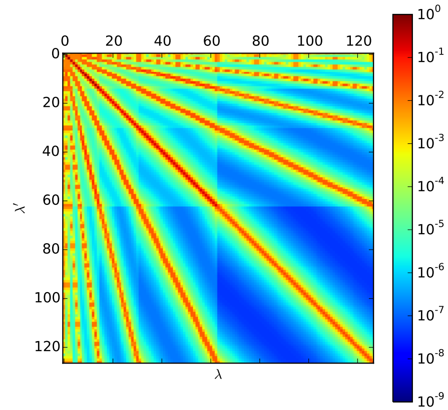

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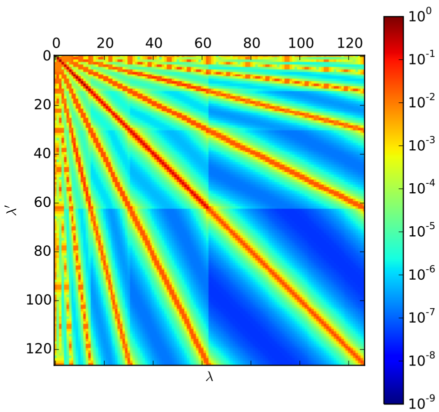

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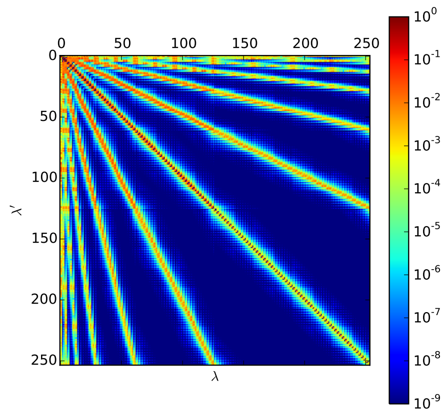

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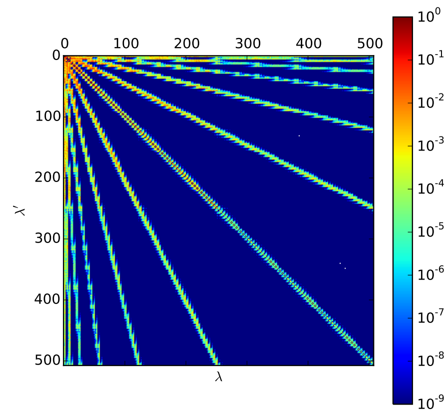

(f) $\gamma = 0.95,\ M = 3$

When $\gamma \to 1$, the matrix $(k_{\lambda,\lambda'})_{\lambda,\lambda'}$ becomes quasi-sparse and we can apply compression techniques to efficiently apply the kernel.

Solving the Transport Part

In our ideal iteration, we need to solve transport problems of the type

$$\mathcal{T}_s u = s \cdot \nabla u + \sigma u = f.$$

To obtain reliable a posteriori error estimates, we solve this equation using a Discontinuous Petrov–Galerkin method.

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Stable Variational Formulations

Given a variational formulation:

Find $u \in \mathcal{U}$ such that

$$b(u, v) = \ell(v) \qquad \text{for all } v \in \mathcal{V}$$

for Hilbert spaces $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$, continuous bilinear form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and $\ell \in \mathcal{V}'$.

Goal: Stability

[see e. g. Demkowicz and Gopalakrishnan, 2015; Barrett and Morton, 1984; Ern and Guermond, 2004]

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Given a variational formulation:

Find $u_h \in \mathcal{U}_h$ such that

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- Discrete approximation yields a best approximation (up to a constant)

$$\|u - u_h\|_{\mathcal{U}} \leq \kappa_{\mathcal{U},\mathcal{V}'}(\mathcal{B}_h) \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} \,.$$

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- Residual-based error bound

$$\|u - u_h\|_{\mathcal{U}} \simeq \|b(u_h, \cdot) - \ell\|_{\mathcal{V}'} \ .$$

[see e. g. Demkowicz and Gopalakrishnan, 2015; Barrett and Morton, 1984; Ern and Guermond, 2004]

Stable Variational Formulations

- For elliptic problems we can set $\mathcal{V} = \mathcal{U}$. Then coercivity

$$b(u, u) \geq c_B \|u\|_{\mathcal{U}}^2 \qquad \forall u \in \mathcal{U}$$

gives us stability (Lax–Milgram / Céa lemma).

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- Without coercivity Banach–Nečas–Babuška guarantees stability when we choose the test space $\mathcal{V}$ in such a way that

$$\inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{b(u, v)}{\|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}} \geq c_B > 0.$$

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  This needs to be checked for the finite dimensional discretization, as it does not automatically carry over from the infinite dimensional setting!

[see e. g. Demkowicz and Gopalakrishnan, 2015; Barrett and Morton, 1984; Ern and Guermond, 2004]

Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

[see e. g. Demkowicz and Gopalakrishnan, 2015; Barrett and Morton, 1984; Ern and Guermond, 2004]

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- Optimal test norm

$$\|\cdot\|_{\mathcal{V},opt} := \sup_{u \in \mathcal{U}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}}$$

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$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

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- This yields
  - residual based error bound

$$\|u_h - u\|_{\mathcal{U}} = \|b(u_h,\cdot) - \ell\|_{\mathcal{V}',opt}$$

  - best approximation property

$$\|u - u_h\|_{\mathcal{U}} = \inf_{w_h\in\mathcal{U}_h} \|u - w_h\|_{\mathcal{U}}$$

[see e. g. Demkowicz and Gopalakrishnan, 2015; Barrett and Morton, 1984; Ern and Guermond, 2004]

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

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We can compute an approximation for $\mathcal{V}_h$ by solving

$$\langle \hat{T}(u_h), v\rangle_{\mathcal{V},K} = b_K(u_h, v)\,, \qquad \forall\, v \in \hat{\mathcal{V}}\big|_K\,,$$

where K is a single cell and $\langle\cdot,\cdot\rangle_{\mathcal{V},K}$ and $b_K(\cdot,\cdot)$ denote restrictions to K.

[see e. g. Demkowicz and Gopalakrishnan, 2015; Broersen et al., 2017]

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A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$s \cdot \nabla u + \sigma u = f.$$

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In our ideal iteration, we need to solve transport problems of the type

$$\int_{\mathrm{D}} s \cdot \nabla u v + \sigma u v \, dx = \int_{\mathrm{D}} f v \, dx.$$

cell-wise integration by parts on a fixed grid D_h leads to

$$\int_{\mathrm{D}} \sigma u v - u s \cdot \nabla v \, dx + \int_{\partial \mathrm{D}_h} n \cdot s u [\![v]\!] \, dx = \int_{\mathrm{D}} f v \, dx.$$

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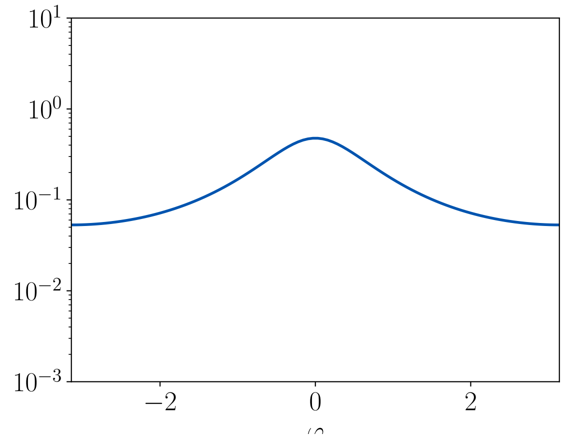
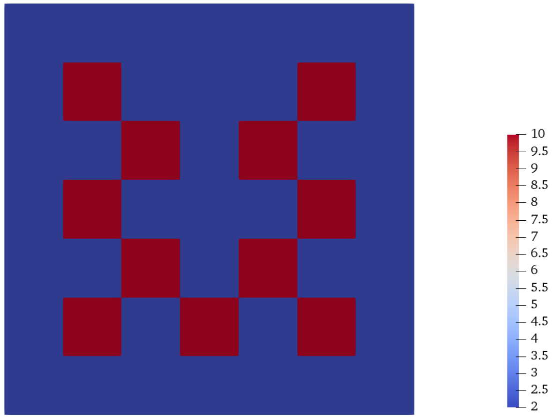
Broersen et al. [2017]: Fullfills requirements of Banach–Nečas–Babuška.

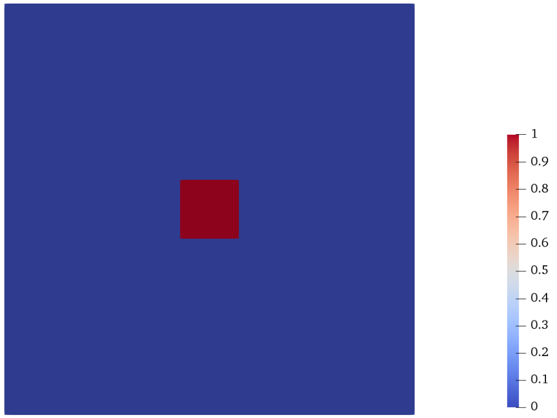
Checkerboard Example



Henyey–Greenstein scattering with $\gamma = 0.5$



absorbtion coefficient σ



source term f

Data of the checkerboard benchmark problem

Checkerboard Example

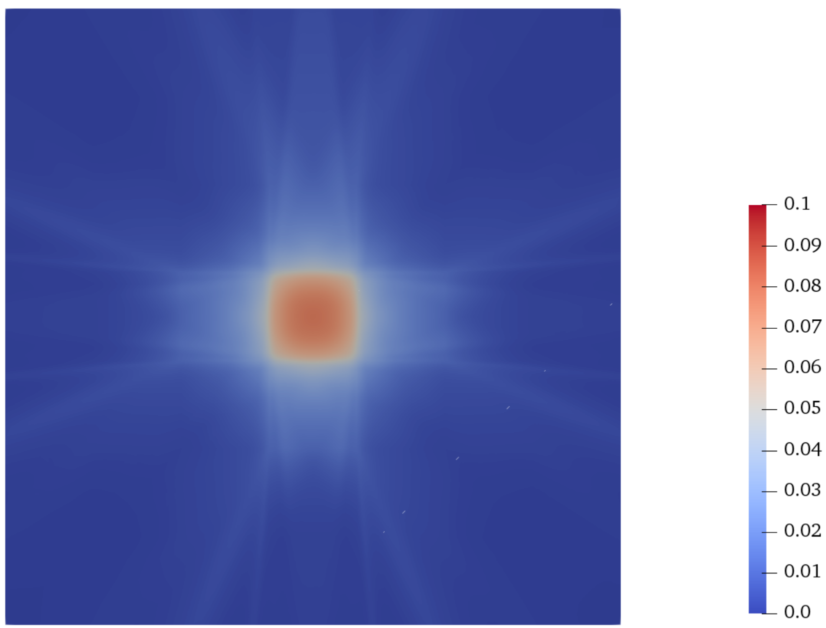
To reach target accuracy η in a step of the Source Term Iteration, we have some freedom in distributing the discretization errors:

- $\kappa_1\eta$: accuracy for kernel application
- $\kappa_2\eta$: accuracy for evaluation of source term f
- $\kappa_3\eta$: accuracy for transport solver

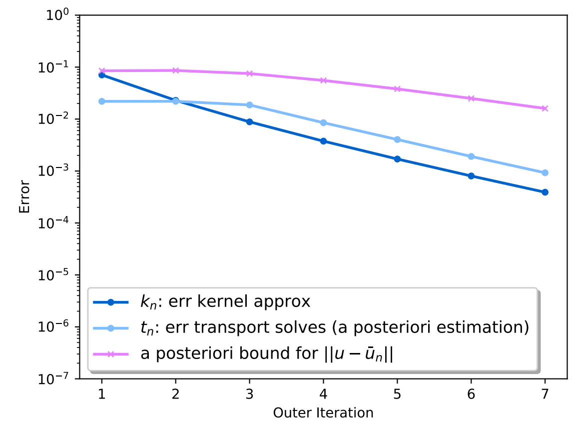
To guarantee convergence, the following restriction holds:

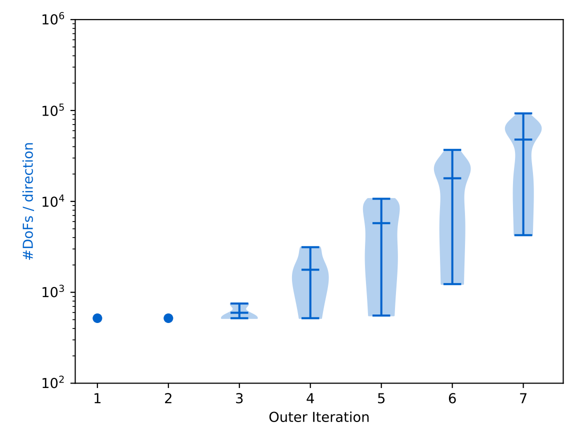
$$C_{\mathcal{T}}\kappa_1 + C_{\mathcal{T}}\kappa_2 + 2\kappa_3 \leq 1.$$

Checkerboard Example



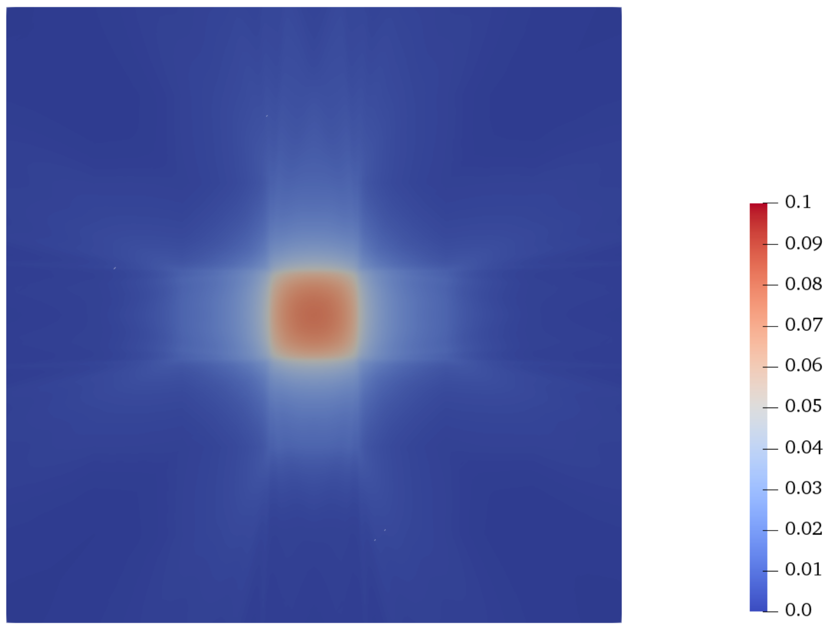
integrated solution



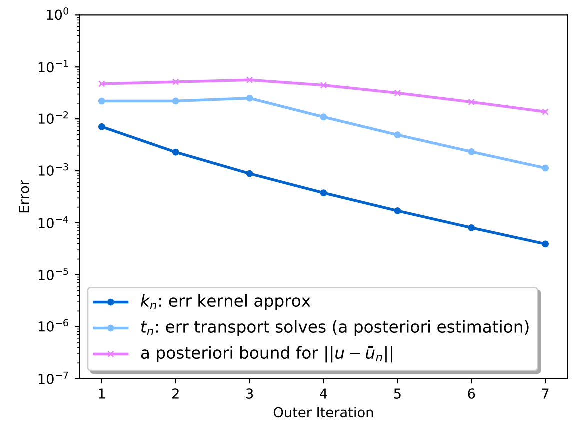


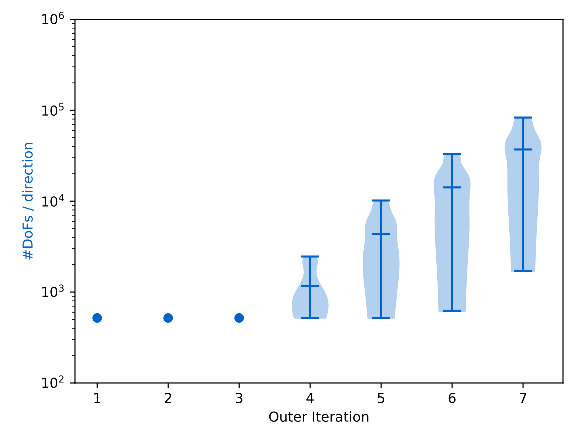
$$\kappa_1 = \theta / C_{\mathcal{T}},\ \kappa_2 = 0,\ \kappa_3 = (1-\theta)/2,\ \theta = 0.2$$

Checkerboard Example



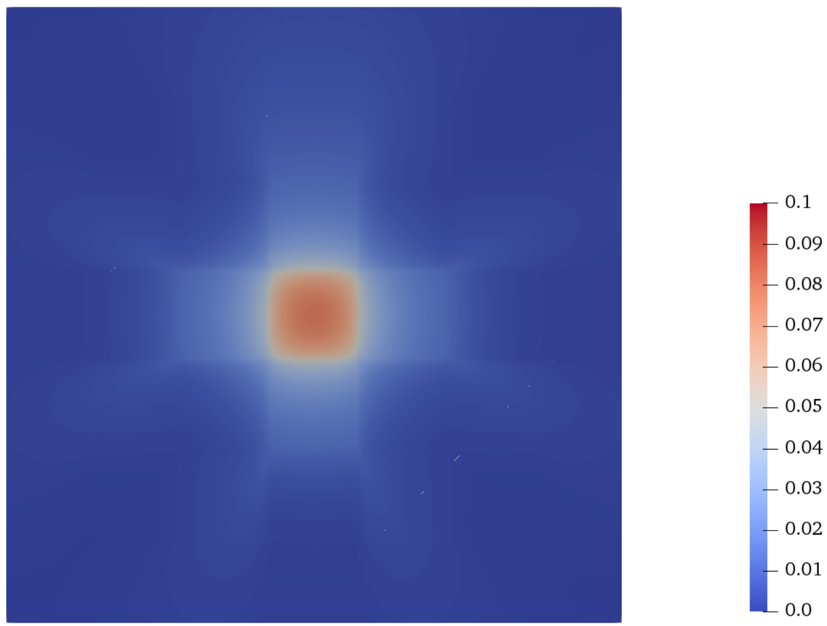
integrated solution



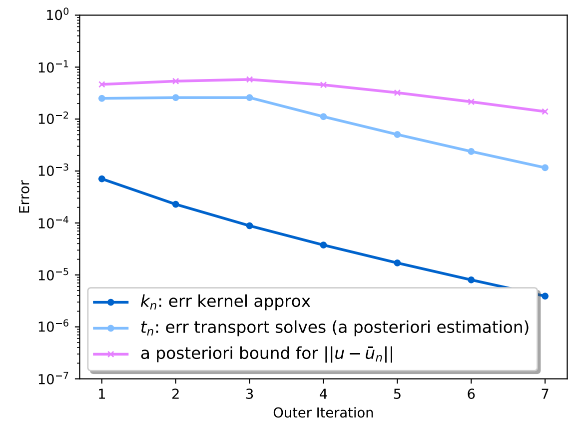


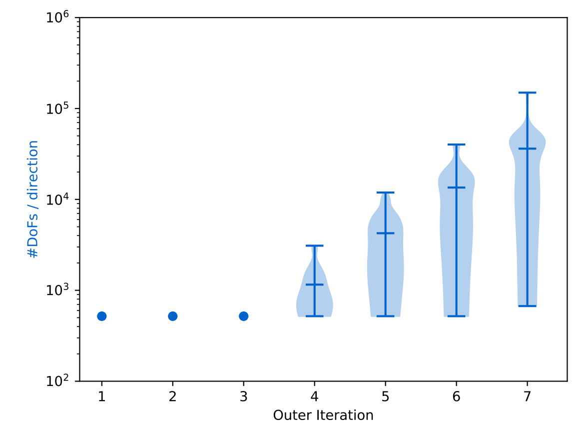
$$\kappa_1 = \theta / C_{\mathcal{T}},\ \kappa_2 = 0,\ \kappa_3 = (1-\theta)/2,\ \theta = 0.02$$

Checkerboard Example



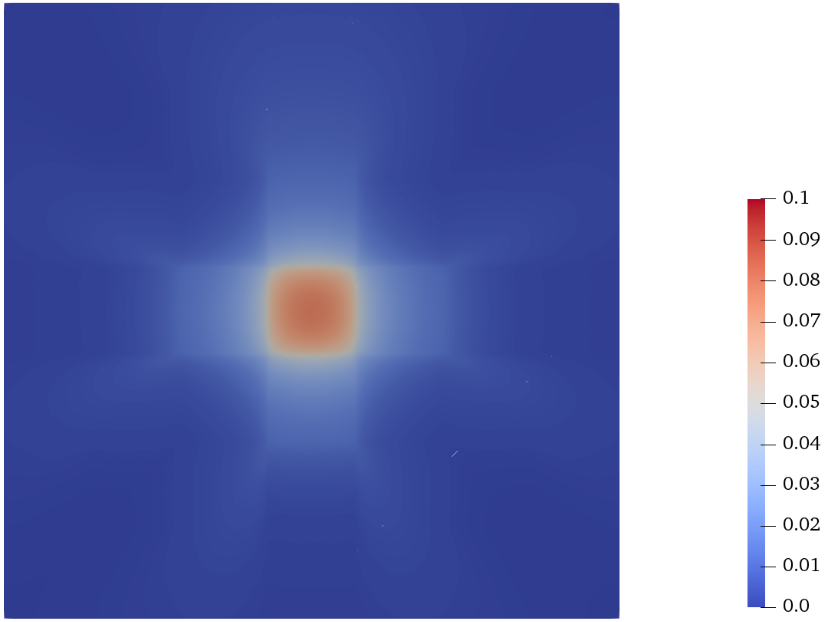
integrated solution



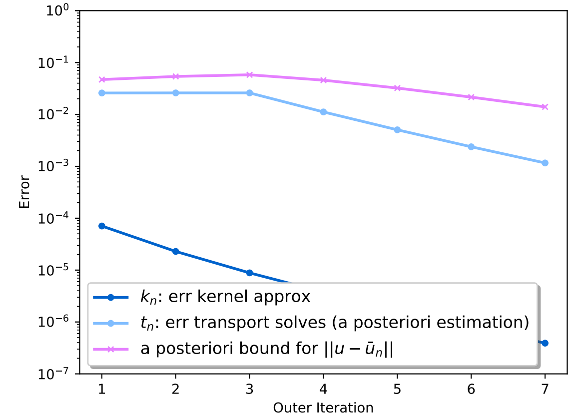


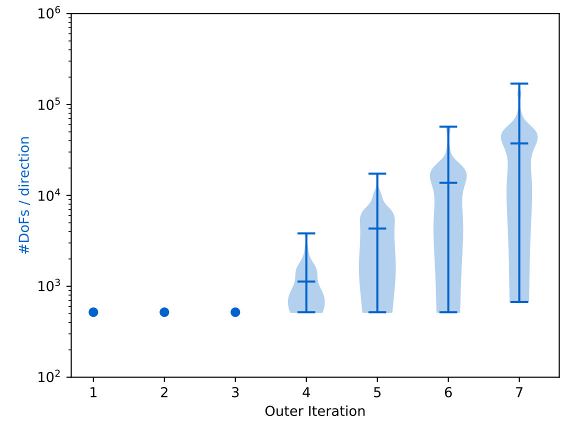
$$\kappa_1 = \theta / C_{\mathcal{T}},\ \kappa_2 = 0,\ \kappa_3 = (1-\theta)/2,\ \theta = 0.002$$

Checkerboard Example



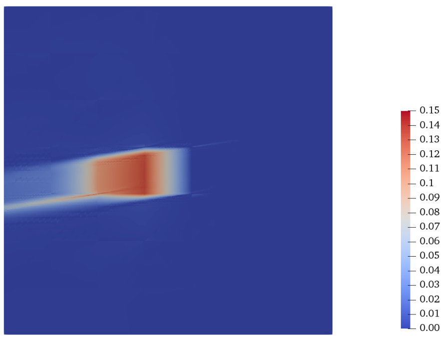
integrated solution



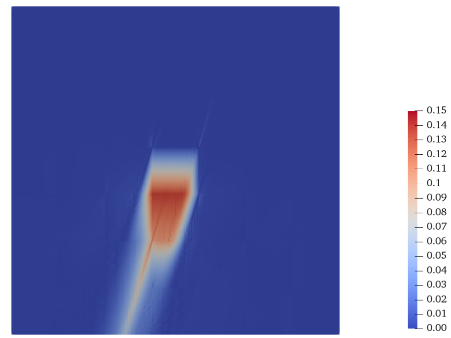


$\kappa_1 = \theta / C_{\mathcal{T}}$, $\kappa_2 = 0$, $\kappa_3 = (1 - \theta)/2$, $\theta = 0.0002$

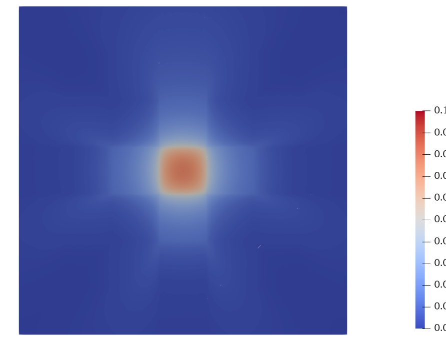
Checkerboard Example



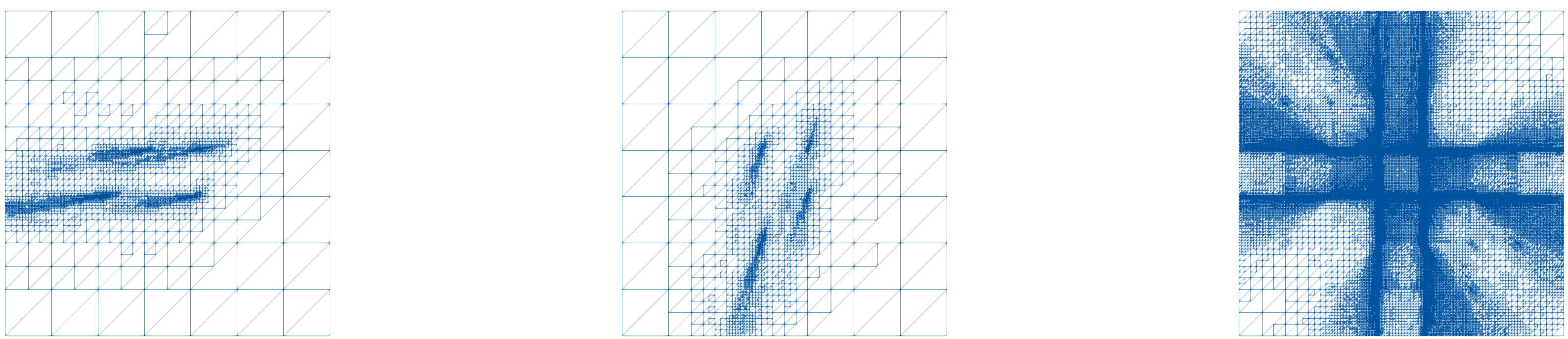
directional solution



directional solution



integrated solution



Grids for two different transport solutions and the integrated solution.

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A Generic Library for DPG: dune-dpg

Joint work with Angela Klewinghaus and Olga Mula [Gruber et al., 2017], based upon the Dune finite element library [Blatt et al., 2016] (`https://dune-project.org/`).

- Suitable for different types of problems, e. g.
  - radiative transport (O. Mula, F. Gruber)
  - convection–diffusion (A. Klewinghaus)
  - optimal control problems with transport constraints (A. Klewinghaus)
  - porous media (V. König)
- a posteriori estimation
- capable of adaptive *h* refinement
- free software (GPL 2 with runtime-exception) available at `https://gitlab.dune-project.org/felix.gruber/dune-dpg`

References

J. W. Barrett and K. W. Morton. Approximate symmetrization and Petrov–Galerkin methods for diffusion–convection problems. *Computer Methods in Applied Mechanics and Engineering*, 45(1–3):97–122, 1984. doi: 10.1016/0045-7825(84)90152-X.

M. Blatt, A. Burchardt, A. Dedner, C. Engwer, J. Fahlke, B. Flemisch, C. Gersbacher, C. Gräser, F. Gruber, C. Grüninger, D. Kempf, R. Klöfkorn, T. Malkmus, S. Müthing, M. Nolte, M. Piatkowski, and O. Sander. The Distributed and Unified Numerics Environment, version 2.4. *Archive of Numerical Software*, 4(100):13–29, May 2016. doi: 10.11588/ans.2016.100.26526.

D. Broersen, W. Dahmen, and R. P. Stevenson. On the stability of DPG formulations of transport equations. *Mathematics of Computation*, 2017. doi: 10.1090/mcom/3242.

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References

W. Dahmen, F. Gruber, and O. Mula. An adaptive nested source term iteration for radiative transfer equations. in preparation.

L. Demkowicz and J. Gopalakrishnan. Discontinuous Petrov–Galerkin (DPG) method. ICES Report 15-20, ICES, UT Austin, Oct. 2015. URL `https://www.ices.utexas.edu/media/reports/2015/1520.pdf`.

A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer, 2004. doi: 10.1007/978-1-4757-4355-5.

F. Gruber, A. Klewinghaus, and O. Mula. The DUNE-DPG library for solving PDEs with Discontinuous Petrov–Galerkin finite elements. *Archive of Numerical Software*, 5(1):111–128, 6 Mar. 2017. doi: 10.11588/ans.2017.1.27719.

L. G. Henyey and J. L. Greenstein. Diffuse radiation in the galaxy. *The Astrophysical Journal*, 93:70–83, 1941.