The Discontinous Petrov–Galerkin Method for Radiative Transfer Problems

Felix Gruber

joint work with Wolfgang Dahmen and Olga Mula

IGPM, RWTH Aachen

MathCCES Lunch Seminar, June 22, 2017

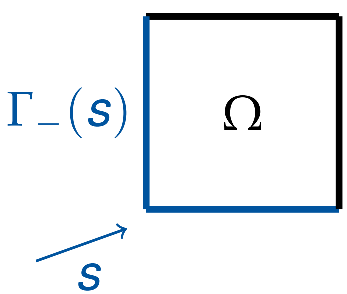
Institute for Geometry and Applied Mathematics

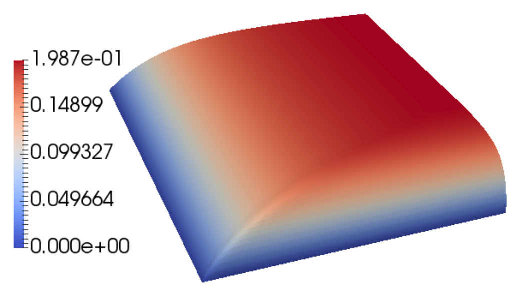
(mono-energetic) Radiative Transfer Problem

$$\begin{aligned} \mathcal{B}u(x,s) := (\mathcal{T}_S - \mathcal{K}_S)u(x,s) &= f(x,s) && \text{in } \Omega \times S \\ u &= g && \text{on } \Gamma_- \end{aligned}$$

with

$$\begin{aligned} \mathcal{T}_S u(x,s) &:= s \cdot \nabla u(x,s) + \sigma(x,s)u(x,s) \\ \mathcal{K}_S u(x,s) &:= \int_S k(x,s',s)u(x,s')\,\mathrm{d}s' \end{aligned}$$





Obstructions

- high dimensional ($n = 2, 3$ dimensional space + $n - 1$ dimensional transport direction)
- global scattering kernel
- reliable error estimates?
  - a priori error estimates often require unrealistic regularity assumptions on the solution

  $\rightarrow$ a posteriori estimates
    - error control for “outer iteration”
    - adaptive grid refinement

Roadmap

- ideal iteration $u_{n+1} = u_n + \mathcal{P}(f - \mathcal{B} u_n), \quad n = 0, 1, 2, \ldots$
  - in infinite dimensional setting
  - $\mathcal{K}_S$ only needs to be evaluated
  - mapping property, norm equivalence
  - $\mathcal{P} = \mathcal{T}_S^{-1}$ for dominating transport, i. e. $\|\mathcal{T}_S^{-1}\mathcal{K}_S\|_{\mathcal{L}(L_2,L_2)} < 1$, otherwise use inner iteration for preconditioning
  - costs of last iteration dominate

Roadmap

- ideal iteration $u_{n+1} = u_n + \mathcal{P}(f - \mathcal{B}u_n), \quad n = 0, 1, 2, \ldots$
  - in infinite dimensional setting
  - $\mathcal{K}_S$ only needs to be evaluated
  - mapping property, norm equivalence
  - $\mathcal{P} = \mathcal{T}_S^{-1}$ for dominating transport, i. e. $\|\mathcal{T}_S^{-1}\mathcal{K}_S\|_{\mathcal{L}(L_2,L_2)} < 1$, otherwise use inner iteration for preconditioning
  - costs of last iteration dominate
- numerical realization:
  - approximately solve inner iteration with prescribed accuracy; this accuracy has to be chosen to ensures convergence to the infinite dimensional solution
  - a posteriori error estimates for efficient transport solvers
    - adaptive grid refinement for non-smooth solutions
    - Discontinuous Petrov–Galerkin (DPG)

Roadmap

- ideal iteration $u_{n+1} = u_n + \mathcal{P}(f - \mathcal{B}u_n), \quad n = 0, 1, 2, \ldots$
  - in infinite dimensional setting
  - $\mathcal{K}_\mathrm{S}$ only needs to be evaluated
  - mapping property, norm equivalence
  - $\mathcal{P} = \mathcal{T}_\mathrm{S}^{-1}$ for dominating transport, i. e. $\|\mathcal{T}_\mathrm{S}^{-1}\mathcal{K}_\mathrm{S}\|_{\mathcal{L}(L_2,L_2)} < 1$, otherwise use inner iteration for preconditioning
  - costs of last iteration dominate
- numerical realization:
  - approximately solve inner iteration with prescribed accuracy; this accuracy has to be chosen to ensures convergence to the infinite dimensional solution
  - a posteriori error estimates for efficient transport solvers
    - adaptive grid refinement for non-smooth solutions
    - Discontinuous Petrov–Galerkin (DPG)
- tool: Banach–Nečas–Babuška
  - convergence of ideal iteration
  - a posteriori estimate for transport solver

igpm Institute for Geometry and Applied Mathematics | RWTH AACHEN UNIVERSITY

Roadmap

- ideal iteration $u_{n+1} = u_n + \mathcal{P}(f - \mathcal{B}u_n), \quad n = 0, 1, 2, \ldots$
  - in infinite dimensional setting
  - $\mathcal{K}_\mathrm{S}$ only needs to be evaluated
  - mapping property, norm equivalence
  - $\mathcal{P} = \mathcal{T}_\mathrm{S}^{-1}$ for dominating transport, i. e. $\|\mathcal{T}_\mathrm{S}^{-1}\mathcal{K}_\mathrm{S}\|_{\mathcal{L}(L_2,L_2)} < 1$, otherwise use inner iteration for preconditioning
  - costs of last iteration dominate
- numerical realization:
  - approximately solve inner iteration with prescribed accuracy; this accuracy has to be chosen to ensures convergence to the infinite dimensional solution
  - a posteriori error estimates for efficient transport solvers
    - adaptive grid refinement for non-smooth solutions
    - Discontinuous Petrov–Galerkin (DPG)
- tool: Banach–Nečas–Babuška
  - convergence of ideal iteration
  - a posteriori estimate for transport solver
- approximately applying $\mathcal{K}_\mathrm{S}$
  - operator compression
  - low-rank approximation

Nested Iteration DPG

- Assuming $\|\mathcal{T}_S^{-1}\mathcal{K}_S\|_{\mathcal{L}(L_2,L_2)} = \rho < 1$, we choose $\mathcal{P} = \mathcal{T}_S^{-1}$ in the *ideal iteration* and obtain

$$
\begin{aligned}
u_0 &= 0 \\
u_{n+1} &= u_n + \mathcal{T}_S^{-1}(f - \mathcal{B}u_n) \\
&= \mathcal{T}_S^{-1}(\mathcal{K}_S u_n + f), \quad n = 0, 1, 2, \dots .
\end{aligned}
$$

Nested Iteration DPG

- Assuming $\|\mathcal{T}_{\mathrm{S}}^{-1}\mathcal{K}_{\mathrm{S}}\|_{\mathcal{L}(L_2,L_2)} = \rho < 1$, we choose $\mathcal{P} = \mathcal{T}_{\mathrm{S}}^{-1}$ in the *ideal iteration* and obtain

$$
\begin{aligned}
u_0 &= 0 \\
u_{n+1} &= u_n + \mathcal{T}_{\mathrm{S}}^{-1}(f - \mathcal{B}u_n) \\
&= \mathcal{T}_{\mathrm{S}}^{-1}(\mathcal{K}_{\mathrm{S}}u_n + f), \quad n = 0, 1, 2, \ldots .
\end{aligned}
$$

- We need a numerical scheme to realize approximately the fixed point iterations while still guaranteing contraction of the infinite dimensional iteration.

Nested Iteration DPG

- Assuming $\|\mathcal{T}_{\mathrm{S}}^{-1}\mathcal{K}_{\mathrm{S}}\|_{\mathcal{L}(L_2,L_2)} = \rho < 1$, we choose $\mathcal{P} = \mathcal{T}_{\mathrm{S}}^{-1}$ in the *ideal iteration* and obtain

$$\begin{aligned} u_0 &= 0 \\ u_{n+1} &= u_n + \mathcal{T}_{\mathrm{S}}^{-1}(f - \mathcal{B}u_n) \\ &= \mathcal{T}_{\mathrm{S}}^{-1}(\mathcal{K}_{\mathrm{S}}u_n + f), \quad n = 0, 1, 2, \ldots. \end{aligned}$$

- We need a numerical scheme to realize approximately the fixed point iterations while still guaranteing contraction of the infinite dimensional iteration.
- In the iterations:
  - Accuracy in solution of transport problems is dynamically increased across iterations $\rightarrow$ Need for tight error bounds to avoid adding unnecessary numerical effort $\rightsquigarrow$ DPG [Broersen et al., 2016]
  - Repeated application of $\mathcal{K}_{\mathrm{S}}$ with increased accuracy $\rightsquigarrow$ Compression techniques.

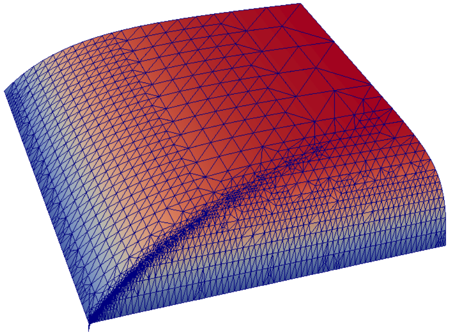
Nested Iteration DPG

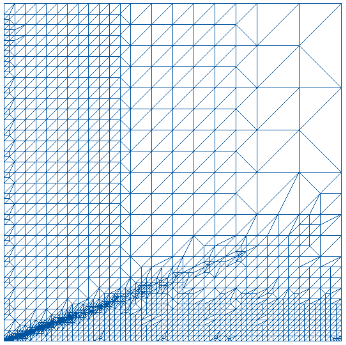
Algorithm 1 NI-DPG$[\mathcal{T}_\mathrm{S}, \mathcal{K}_\mathrm{S}, f, \varepsilon] \to u_\varepsilon$

$u \leftarrow 0$
err $\leftarrow$ err_0
while err $> \varepsilon$ **do**
  Approximatively compute $w = \mathcal{K}_\mathrm{S} u$.
  repeat
    Solve $\mathcal{T}_\mathrm{S} u = f + w$ as a set of transport problems using DPG.
    Refine grid.
  until target accuracy is reached
  err $\leftarrow$ combined error of computing w and u
  Increase target accuracy.
end while
$u_\varepsilon \leftarrow u$

Adaptive Grid Refinement

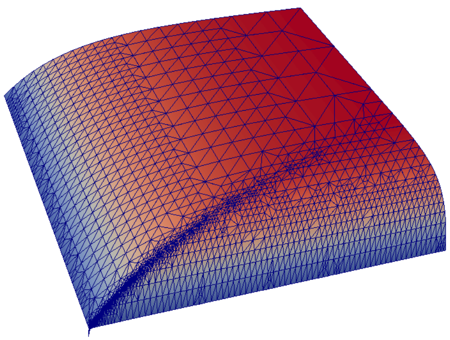
Solutions to the transport problems are smooth in large parts $\rightarrow$ adaptive grid refinements keep the number of DoFs minimal.

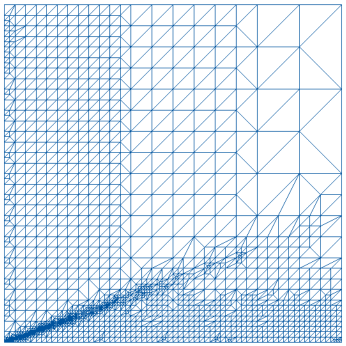




Adaptive Grid Refinement

Solutions to the transport problems are smooth in large parts $\rightarrow$ adaptive grid refinements keep the number of DoFs minimal.





For the computation of the scattering integral, we need to merge the adaptively refined grids from the transport solutions.
But the more expensive task of computing transport solutions can be done on the smaller adaptively refined grids.

Stable Variational Formulations

Given a variational formulation:

Find $u \in \mathcal{U}$ such that

$$b(u, v) = \ell(v) \qquad \text{for all } v \in \mathcal{V}$$

for Hilbert spaces $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$, continuous bilinear form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and $\ell \in \mathcal{V}'$.

Goal: Stability

Stable Variational Formulations

Given a variational formulation:

Find $u_h \in \mathcal{U}_h$ such that

$$b(u_h, v_h) = \ell(v_h) \qquad \text{for all } v_h \in \mathcal{V}_h$$

for Hilbert spaces $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$, continuous bilinear form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and $\ell \in \mathcal{V}'$.

Goal: Stability

- Discrete approximation yields a best approximation (up to a constant)

$$\|u - u_h\|_{\mathcal{U}} \leq \kappa_{\mathcal{U},\mathcal{V}'}(\mathcal{B}_h) \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} \, .$$

Stable Variational Formulations

Given a variational formulation:

Find $u_h \in \mathcal{U}_h$ such that

$$b(u_h, v_h) = \ell(v_h) \qquad \text{for all } v_h \in \mathcal{V}_h$$

for Hilbert spaces $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$, continuous bilinear form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and $\ell \in \mathcal{V}'$.

Goal: Stability

- Discrete approximation yields a best approximation (up to a constant)

$$\|u - u_h\|_{\mathcal{U}} \leq \kappa_{\mathcal{U},\mathcal{V}'}(\mathcal{B}_h) \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} .$$

- Residual-based error bound

$$\|u - u_h\|_{\mathcal{U}} \simeq \|b(u_h, \cdot) - \ell\|_{\mathcal{V}'} .$$

Institute for Geometry and Applied Mathematics
RWTH AACHEN UNIVERSITY

Stable Variational Formulations

- For elliptic problems we can set $\mathcal{V} = \mathcal{U}$. Then coercivity

$$b(u,u) \geq c_B \|u\|_{\mathcal{U}}^2 \qquad \forall u \in \mathcal{U}$$

gives us stability (Lax–Milgram / Céa lemma).

Stable Variational Formulations

- For elliptic problems we can set $\mathcal{V} = \mathcal{U}$. Then coercivity

$$b(u, u) \geq c_B \|u\|_{\mathcal{U}}^2 \qquad \forall u \in \mathcal{U}$$

gives us stability (Lax–Milgram / Céa lemma).

- Without coercivity Banach–Nečas–Babuška guarantees stability when we choose the test space $\mathcal{V}$ in such a way that

$$\inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{b(u, v)}{\|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}} \geq c_B.$$

Institute for Geometry and Applied Mathematics
RWTH AACHEN UNIVERSITY

Stable Variational Formulations

- For elliptic problems we can set $\mathcal{V}_h = \mathcal{U}_h$. Then coercivity
$$b(u,u) \geq c_B \|u\|_{\mathcal{U}}^2 \qquad \forall u \in \mathcal{U}$$
gives us stability (Lax–Milgram / Céa lemma).
- Without coercivity Banach–Nečas–Babuška guarantees stability when we choose the test space $\mathcal{V}_h$ in such a way that
$$\inf_{u_h \in \mathcal{U}_h} \sup_{v_h \in \mathcal{V}_h} \frac{b(u_h, v_h)}{\|u_h\|_{\mathcal{U}_h} \|v_h\|_{\mathcal{V}_h}} \geq c_B.$$
This needs to be checked for the finite dimensional discretization, as it does not automatically carry over from the infinite dimensional setting!

Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

- Optimal test norm

$$\|\cdot\|_{\mathcal{V},opt} := \sup_{u \in \mathcal{U}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}}$$

Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

- Optimal test norm

$$\|\cdot\|_{\mathcal{V},opt} := \sup_{u \in \mathcal{U}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}}$$

- Optimal test space $\mathcal{V}_h = B^{-*} R_{\mathcal{U}} \mathcal{U}_h$, where $R_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}'$ is the Riesz isomorphism, that is $\mathcal{V}_h := T(\mathcal{U}_h)$ where

$$\langle T(u_h), v \rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

- Optimal test norm

$$\|\cdot\|_{\mathcal{V},opt} := \sup_{u \in \mathcal{U}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}}$$

- Optimal test space $\mathcal{V}_h = B^{-*} R_{\mathcal{U}} \mathcal{U}_h$, where $R_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}'$ is the Riesz isomorphism, that is $\mathcal{V}_h := T(\mathcal{U}_h)$ where

$$\langle T(u_h), v \rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This yields
  - residual based error bound

$$\|u_h - u\|_{\mathcal{U}} = ||b(u_h, \cdot) - \ell||_{\mathcal{V}',opt}$$

  - best approximation property

$$\|u - u_h\|_{\mathcal{U}} = \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}}$$

igpm Institute for Geometry and Applied Mathematics
RWTH AACHEN UNIVERSITY

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v \rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This is infinite dimensional

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This is infinite dimensional
  - Projection approach: Replace $\mathcal{V}$ by some good enough finite dimensional test search space $\hat{\mathcal{V}} \subset \mathcal{V}$

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This is infinite dimensional
  - Projection approach: Replace $\mathcal{V}$ by some good enough finite dimensional test search space $\hat{\mathcal{V}} \subset \mathcal{V}$
- We cannot (afford to) compute the optimal test norm and solve global problems

Institute for Geometry and Applied Mathematics

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This is infinite dimensional
  - Projection approach: Replace $\mathcal{V}$ by some good enough finite dimensional test search space $\hat{\mathcal{V}} \subset \mathcal{V}$
- We cannot (afford to) compute the optimal test norm and solve global problems
  - replace the optimal test norm by an equivalent but localizable and computable norm
  - use an ultra-weak variational formulation

Computation of the Optimal Test Space

We have to compute $\mathcal{V}_h$ by solving

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

- This is infinite dimensional
  - Projection approach: Replace $\mathcal{V}$ by some good enough finite dimensional test search space $\hat{\mathcal{V}} \subset \mathcal{V}$
- We cannot (afford to) compute the optimal test norm and solve global problems
  - replace the optimal test norm by an equivalent but localizable and computable norm
  - use an ultra-weak variational formulation

We can compute an approximation for $\mathcal{V}_h$ by solving

$$\langle \hat{T}(u_h), v\rangle_{\mathcal{V},K} = b_K(u_h, v)\,, \qquad \forall\, v \in \hat{\mathcal{V}}\big|_K\,,$$

where K is a single cell and $\langle\cdot,\cdot\rangle_{\mathcal{V},K}$ and $b_K(\cdot,\cdot)$ denote restrictions to K.

Choosing a Good Test Search Space

- δ-proximality (test search space $\hat{\mathcal{V}}$ close enough to optimal test space $\mathcal{V}_h$):

$$\forall 0 \neq v_h \in \mathcal{V}_h \, \exists \hat{v}_h \in \hat{\mathcal{V}} \text{ such that } \|v_h - \hat{v}_h\|_{\mathcal{V}} \leq \delta \|v_h\|_{\mathcal{V}}.$$

Dahmen et al. [2012]: This guarantees inf-sup constant $\geq (1 - \delta)$.

Choosing a Good Test Search Space

- δ-proximality (test search space $\hat{\mathcal{V}}$ close enough to optimal test space $\mathcal{V}_h$):

$$\forall 0 \neq v_h \in \mathcal{V}_h \, \exists \hat{v}_h \in \hat{\mathcal{V}} \text{ such that } \|v_h - \hat{v}_h\|_{\mathcal{V}} \leq \delta \|v_h\|_{\mathcal{V}}.$$

  Dahmen et al. [2012]: This guarantees inf-sup constant $\geq (1 - \delta)$.
- hard to verify
- for the transport problems that we solve, Broersen et al. [2016] show that $\hat{\mathcal{V}}$ can be chosen as a finite element space that has a slightly higher polynomial degree than the trial space and lives on a subgrid.

A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$s \cdot \nabla u + cu = f.$$

A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$\int_\Omega s \cdot \nabla u v + c u v \, dx = \int_\Omega f v \, dx.$$

cell-wise integration by parts on a fixed grid Ω_h leads to

$$\int_\Omega c u v - u s \cdot \nabla v \, dx + \int_{\partial\Omega_h} n \cdot s u [\![v]\!] \, dx = \int_\Omega f v \, dx.$$

A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$\int_\Omega s \cdot \nabla u v + c u v \, dx = \int_\Omega f v \, dx.$$

cell-wise integration by parts on a fixed grid Ω_h leads to

$$\int_\Omega c u v - u s \cdot \nabla v \, dx + \int_{\partial\Omega_h} n \cdot s \theta [\![v]\!] \, dx = \int_\Omega f v \, dx.$$

Ultra-weak formulation with $u \in L_2$.

Institute for Geometry and Applied Mathematics

A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$\int_\Omega s \cdot \nabla u v + c u v \, dx = \int_\Omega f v \, dx.$$

cell-wise integration by parts on a fixed grid Ω_h leads to

$$\int_\Omega c u v - u s \cdot \nabla v \, dx + \int_{\partial\Omega_h} n \cdot s \theta [\![v]\!] \, dx = \int_\Omega f v \, dx.$$

Ultra-weak formulation with $u \in L_2$.
Broken test space with broken norm

$$\|v\|_{H(s;\Omega_h)} = \left(\sum_{T \in \Omega_h} \|v\|^2_{H(s;K)} \right)^{1/2}.$$

A Stable DPG Formulation of Linear Transport

In our ideal iteration, we need to solve transport problems of the type

$$\int_\Omega s \cdot \nabla uv + cuv\,dx = \int_\Omega fv\,dx.$$

cell-wise integration by parts on a fixed grid Ω_h leads to

$$\int_\Omega cuv - us \cdot \nabla v\,dx + \int_{\partial\Omega_h} n \cdot s\theta [\![v]\!]\,dx = \int_\Omega fv\,dx.$$

Ultra-weak formulation with $u \in L_2$.
Broken test space with broken norm

$$\|v\|_{H(s;\Omega_h)} = \left(\sum_{T \in \Omega_h} \|v\|^2_{H(s;K)} \right)^{1/2}.$$

Broersen et al. [2016]: Fullfills requirements of Banach–Nečas–Babuška.

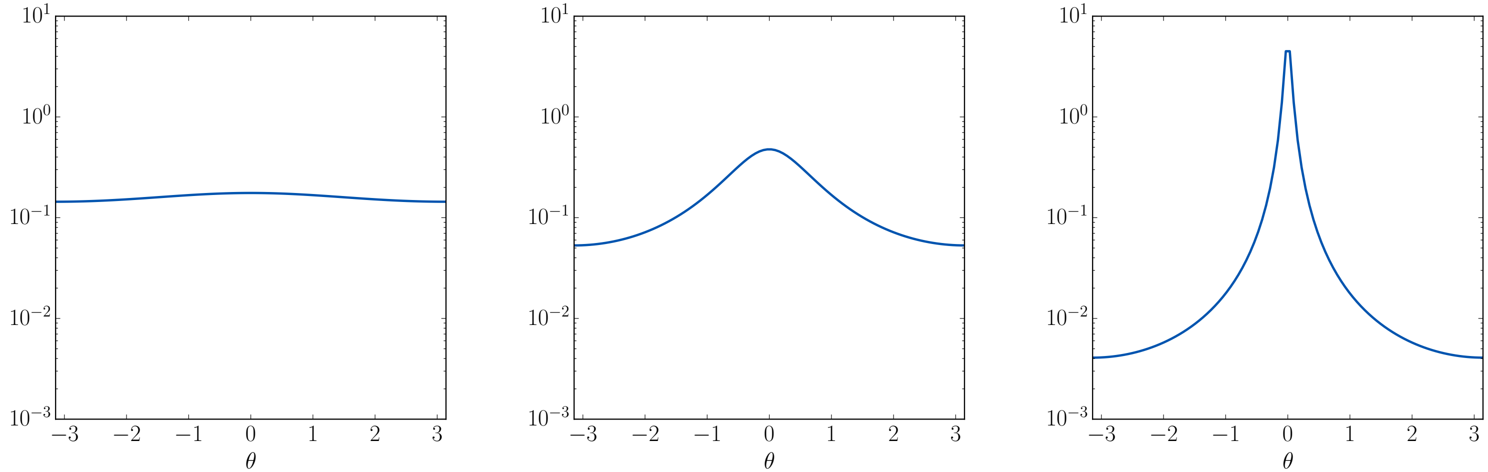
The Henyey–Greenstein Kernel

[Henyey and Greenstein, 1941]

$$k(s, s') := \frac{1}{2\pi} \frac{1 - \gamma^2}{1 + \gamma^2 - 2\gamma \cos(s - s')} \quad \text{with } \gamma \in (-1, 1).$$

- When $\gamma \geq 0$, the scattering is forward peaked and K is positive semi-definite.
- Negative γ models backward scattering.
- Originally used to model light scattering in the milky way.
- Nowadays used in a wide variety of applications including nuclear physics [Sanchez and McCormick, 2004].

The Henyey–Greenstein Kernel

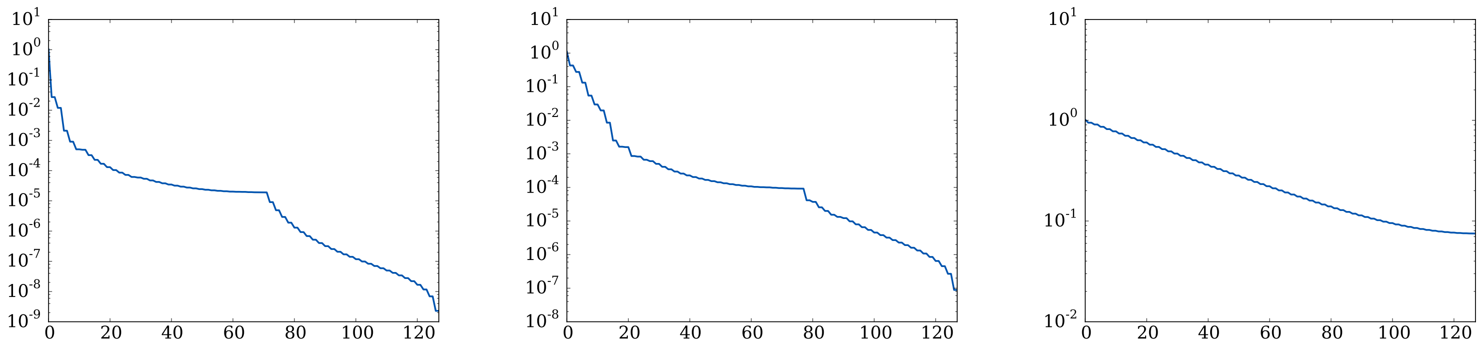


(a) $\gamma = 0.05$ (b) $\gamma = 0.5$ (c) $\gamma = 0.95$

Figure: Henyey–Greenstein kernel as a function of $\theta = \angle(s, s')$

Larger γ leads to more concentrated forward-scattering.

The Henyey–Greenstein Kernel

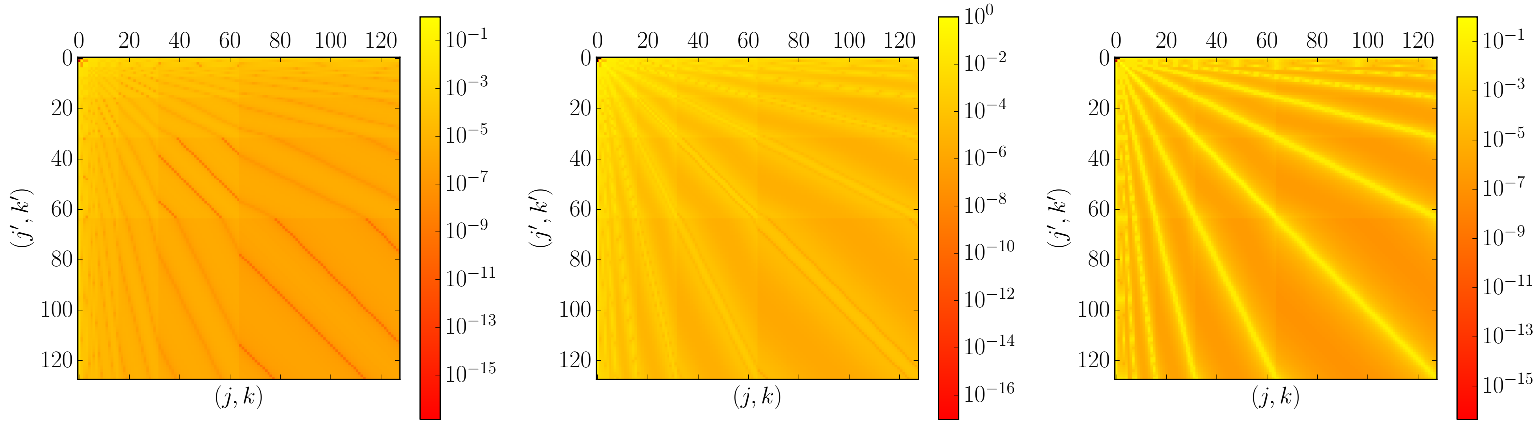


(a) $\gamma = 0.05$ (b) $\gamma = 0.5$ (c) $\gamma = 0.95$

Figure: singular values of Henyey–Greenstein kernel

For small γ fast decay of singular values.
$\rightarrow$ Kernel can be approximated by low rank matrix.

Institute for Geometry and Applied Mathematics



(a) $\gamma = 0.05$ (b) $\gamma = 0.5$ (c) $\gamma = 0.95$

Figure: wavelet representation of Henyey–Greenstein kernel

- Behaviour similar to boundary integral problems.
- For γ close to 1 we can see the typical “finger” structure which hints at good wavelet-compressibility.

Numerical Results

Nested Iteration DPG for

$$\begin{aligned} (\mathcal{T}_S - \mathcal{K}_S)u(x,s) &= 1 \quad \text{in } \Omega \times S \\ u &= 0 \quad \text{on } \Gamma_- \end{aligned}$$

with

$$\begin{aligned} \mathcal{T}_S u(x,s) &:= s \cdot \nabla u(x,s) + 5u(x,s) \\ \mathcal{K}_S u(x,s) &:= \int_S k(x,s',s)u(x,s')\,\mathrm{d}s' \end{aligned}$$

where k is a Henyey–Greenstein scattering kernel with $\gamma = 0.9$.

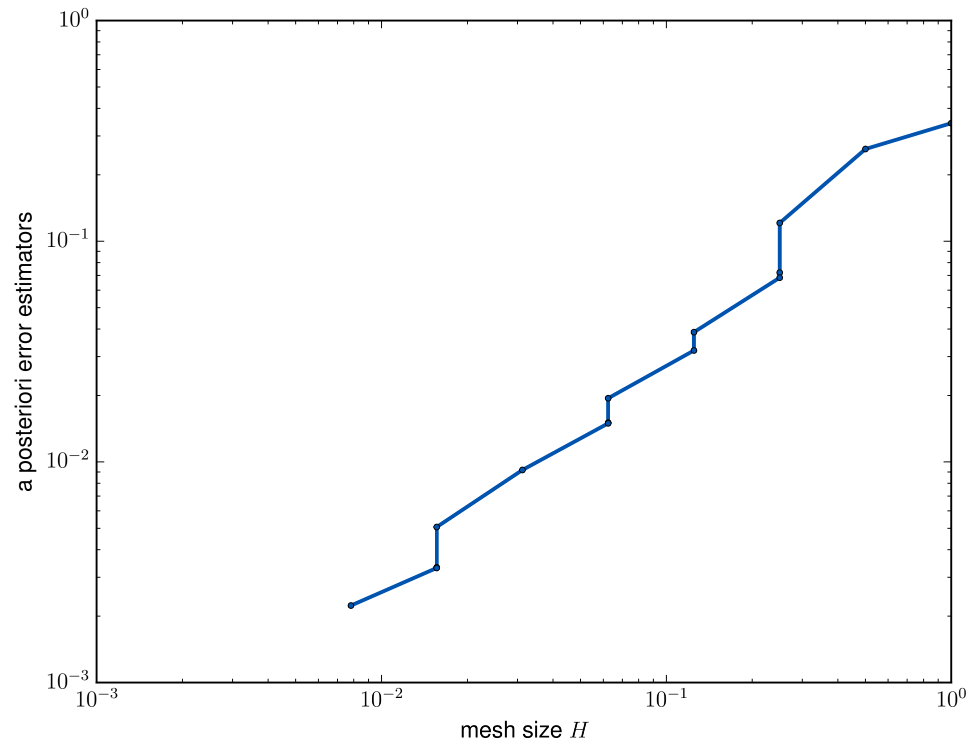


Figure: a posteriori errors of the adaptive sourceterm iteration

A Generic Library for DPG: dune-dpg

Joint work with Angela Klewinghaus and Olga Mula [Gruber et al., 2017], based upon the Dune finite element library [Blatt et al., 2016] (`https://dune-project.org/`).

- Suitable for different types of problems, e. g.
  - radiative transport (O. Mula, F. Gruber)
  - convection–diffusion (A. Klewinghaus)
  - optimal control problems with transport constraints (A. Klewinghaus)
  - soon: porous media (V. König)
- a posteriori estimation
- capable of adaptive *h* refinement
- free software (GPL 2 with runtime-exception)
  available at `https://gitlab.dune-project.org/felix.gruber/dune-dpg`

References

M. Blatt, A. Burchardt, A. Dedner, C. Engwer, J. Fahlke, B. Flemisch, C. Gersbacher, C. Gräser, F. Gruber, C. Grüninger, D. Kempf, R. Klöfkorn, T. Malkmus, S. Müthing, M. Nolte, M. Piatkowski, and O. Sander. The Distributed and Unified Numerics Environment, version 2.4. *Archive of Numerical Software*, 4(100):13–29, May 2016. doi: 10.11588/ans.2016.100.26526.

D. Broersen, W. Dahmen, and R. P. Stevenson. On the stability of DPG formulations of transport equations. *Mathematics of Computation*, Nov. 2016. doi: 10.1090/mcom/3242. Accepted for publication.

W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive Petrov–Galerkin methods for first order transport equations. *SIAM Journal on Numerical Analysis*, 50(5):2420–2445, 2012. doi: 10.1137/110823158.

References

F. Gruber, A. Klewinghaus, and O. Mula. The DUNE-DPG library for solving PDEs with Discontinuous Petrov–Galerkin finite elements. *Archive of Numerical Software*, 5(1):111–128, 6 Mar. 2017. doi: 10.11588/ans.2017.1.27719.

L. G. Henyey and J. L. Greenstein. Diffuse radiation in the galaxy. *The Astrophysical Journal*, 93:70–83, 1941.

R. Sanchez and N. J. McCormick. Discrete ordinates solutions for highly forward peaked scattering. *Nuclear Science and Engineering*, 147(3): 249–274, 2004. URL http://www.ans.org/pubs/journals/nse/a_2432.