Geometrische Mechanik

19. Let $\mathfrak{gl}(n,\mathbb{R})$ be the set of all real $n\times n$–matrices. Further define

$$\begin{aligned} Gl(n,\mathbb{R}) &= \{S\in\mathfrak{gl}(n,\mathbb{R}) \mid \det S\neq 0\} \\ O(n,\mathbb{R}) &= \{S\in\mathfrak{gl}(n,\mathbb{R}) \mid S^TS=\mathrm{id}\} \\ \mathfrak{o}(n,\mathbb{R}) &= \{A\in\mathfrak{gl}(n,\mathbb{R}) \mid A^T=-A\} \\ Sym(n,\mathbb{R}) &= \{A\in\mathfrak{gl}(n,\mathbb{R}) \mid A^T=A\}. \end{aligned}$$

   (a) Show that $\mathfrak{o}(n,\mathbb{R})$ and $Sym(n,\mathbb{R})$ are real vector spaces. Give their dimensions.

   (b) Show that $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{o}(n,\mathbb{R})$ form Lie algebras with respect to the product given by the commutator.

   (c) Show that $Gl(n,\mathbb{R})$ is an n^2–dimensional manifold. Show how $\mathfrak{gl}(n,\mathbb{R})$ can be regarded as the tangent space $T_{\mathrm{id}}Gl(n,\mathbb{R})$.

   (d) Let $F:\mathfrak{gl}(n,\mathbb{R})\longrightarrow\mathfrak{gl}(n,\mathbb{R})$ be defined by $F(S)=S^TS$. Show that F is a smooth map and that the image of F is a subset of $Sym(n,\mathbb{R})$.

   (e) Show that the derivative[1] $D_{\mathrm{id}}F:\mathfrak{gl}(n,\mathbb{R})\longrightarrow\mathfrak{gl}(n,\mathbb{R})$ is given by $D_{\mathrm{id}}F(B)=B^T+B$. What is the rank of this derivative?

   (f) Show that the rank of the derivative $D_SF:\mathfrak{gl}(n,\mathbb{R})\longrightarrow\mathfrak{gl}(n,\mathbb{R})$ is independent of $S\in O(n,\mathbb{R})$. *Hint:* Use the fact that $O(n,\mathbb{R})$ is a group.

   (g) Show that $O(n,\mathbb{R})$ is a manifold and determine its dimension. In what sense can $\mathfrak{o}(m,\mathbb{R})$ be regarded as the tangent space $T_{\mathrm{id}}O(n,\mathbb{R})$?

20. Verify that the identification of the Lie algebra $(\mathfrak{so}(3),[..,..])$ with $(\mathbb{R}^3,\times)$ turns the adjoint action $\mathrm{Ad}:(g,A)\mapsto T_e(R_{g^{-1}}L_g)\cdot A$ of $SO(3)$ on its Lie algebra into the standard representation of $SO(3)$ on $\mathbb{R}^3$.

21. Consider a particle in a central force field in $\mathbb{R}^3$. The (matrix) Lie group $SO(3)$ acts by simultaneous[2] rotation on the phase space $\mathcal{P}=T^*\mathbb{R}^3\cong\mathbb{R}^6$. Compute the momentum mapping and reduce the $SO(3)$–symmetry.

22. Study the geodesic flow on a Lie group G with respect to a bi-invariant metric. Show that $G\times G$ is a symmetry group, compute the momentum mapping and explain the details of symmetry reduction. Use the reduced system to obtain as much information on the geodesic flow as you can. Consider in particular the case $G=SO(3)$ of a free rigid body and give an interpretation of your results.

[1] In another notation, $D_{\mathrm{id}}F=F_{*,\mathrm{id}}$.
[2] of what?