Geometrische Mechanik

7. In this exercise we study co-ordinate transformations in one degree of freedom.

   Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a given smooth function, with corresponding Hamiltonian vector field X_H. Here we use the standard symplectic structure on $\mathbb{R}^2$. Moreover, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism. Consider both the function $K := H \circ g^{-1}$, together with the associated Hamiltonian vector field X_K, and the transformed vector field $g_\star(X_H)$, defined by $g_\star(X_H)(g(p)) := D_p g X_H(p)$. Show that

   $$g_\star(X_H) \quad = \quad \det(Dg) \cdot X_K \ .$$

   Hint: exploit a coordinate free formulation of the fact that X_H is the Hamiltonian vector field corresponding to H. Discuss the implication for the integral curves of $g_\star(X_H)$ and X_K. Also consider the time-parametrisation of these curves. What happens in the special case that g is canonical?

8. A matrix $A \in M_{n\times n}(\mathbb{R})$ defines a linear differential equation $\dot{x} = Ax$, write $\mathcal{L}_A$ for the corresponding vector field. Next to the commutator $[A, B] = AB - BA$ of two matrices there is also the Lie bracket $[\mathcal{L}_A, \mathcal{L}_B]$ defined by how vector fields act as derivations on smooth functions f, given as

   $$[\mathcal{L}_A, \mathcal{L}_B](f) \quad := \quad \mathcal{L}_A(\mathcal{L}_B(f)) \; - \; \mathcal{L}_B(\mathcal{L}_A(f)) \ .$$

   Show that

   $$[\mathcal{L}_A, \mathcal{L}_B] \quad = \quad -\mathcal{L}_{[A,B]} \ .$$

   Furthermore show that

   $$\varphi_t \circ \psi_s \; - \; \psi_s \circ \varphi_t \quad = \quad s\,t\,[A, B] \; + \; \mathcal{O}\left((s^2 + t^2)^{\frac{3}{2}}\right)$$

   for the flows φ of $\mathcal{L}_A$ and ψ of $\mathcal{L}_B$. Hence, $[\mathcal{L}_A, \mathcal{L}_B] = 0$ if these two flows commute.

9. In this exercise we study some properties of the Poisson bracket on a symplectic manifold.

   Show that if two functions F, G are integrals of H then so is $\{F, G\}$.

   Show that if $h : \mathcal{P} \longrightarrow \mathcal{P}$ is symplectic then $\{F, G\} \circ h = \{F \circ h, G \circ h\}$. What is the meaning of this equation?