Geometrische Mechanik

- 6. Let Λ be the exterior algebra of an *n*-dimensional vector space V with basis $\{e_1, \ldots, e_n\}$.
 - (i) Explain that the $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ together with 1 form a basis of Λ .
 - (*ii*) Check that for all subsets $\{i_1 < \ldots < i_k\}, \{j_1 < \ldots < j_\ell\} \subseteq \{1, \ldots, n\}$ one has

$$e_{j_1} \wedge \ldots \wedge e_{j_\ell} \wedge e_{i_1} \wedge \ldots \wedge e_{i_k} = (-1)^{kl} e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge e_{j_1} \wedge \ldots \wedge e_{j_\ell}$$

- (*iii*) Show that $\{v_1, \ldots, v_m\} \subseteq V$ is linear dependent if and only if $v_1 \wedge \ldots \wedge v_m = 0$.
- (iv) Let $\{u_1, \ldots, u_m\}, \{v_1, \ldots, v_m\} \subseteq V$ with $\{u_1, \ldots, u_m\}$ linear independent. Prove that $\langle u_1, \ldots, u_m \rangle = \langle v_1, \ldots, v_m \rangle$ have the same linear span if and only if $u_1 \wedge \ldots \wedge u_m$ is a scalar multiple of $v_1 \wedge \ldots \wedge v_m$.

Define $\Lambda^k V := \langle v_1 \wedge \ldots \wedge v_k \mid v_i \in V \rangle$, the *k*th exterior power of *V*.

k

(v) Show that $\Lambda^k V = 0$ for k > n and that for $k \le n$ the $e_{i_1} \land \ldots \land e_{i_k}$, $i_1 < \ldots < i_k$ with the now fixed index k form a basis of $\Lambda^k V$. Conclude dim $\Lambda^k V = \binom{n}{k}$. Identify $V = \Lambda^1 V$ and check

$$\Lambda = \bigoplus_{k=0}^{n} \Lambda^{k} V$$

and

$$\bigwedge_{\ell \in \mathbb{N}_0} \quad \Lambda^k V \ \land \ \Lambda^\ell V = \quad \Lambda^{k+\ell} V$$

where $A \wedge B := \langle a \wedge b \mid a \in A, b \in B \rangle$.

- (vi) For $\varphi: V \longrightarrow \Lambda(V)$ and $\psi: W \longrightarrow \Lambda(W)$ show that for every $f \in L(V, W)$ there is a unique 1-preserving algebra-homomorphism $\Lambda(f): \Lambda(V) \longrightarrow \Lambda(W)$ satisfying $\Lambda(f) \circ \varphi = \psi \circ f$.
- (vii) Prove that $\Lambda(f \circ g) = \Lambda(f) \circ \Lambda(g)$ if $g \in L(U, V)$ for some other vector space U with exterior algebra $\Lambda(U)$.
- (viii) Check that $\Lambda(f)(\Lambda^k V) \subseteq \Lambda^k W$, yielding $\Lambda^k f \in L(\Lambda^k V, \Lambda^k W)$ and the splitting $\Lambda(f) = \bigoplus \Lambda^k(f)$. How is $\Lambda^n(f)$ for $f \in L(V)$ related to det(f)?
 - (*ix*) Explain why $v_1 \wedge \ldots \wedge v_m$ can be interpreted as the *m*-dimensional parallelepiped with sides v_1, \ldots, v_m .