

An Element-by-Element Version of Variable-step Multilevel Preconditioning Methods

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Abstract

In this paper an element-by-element implementation of the recently proposed variable-step multilevel preconditioning method for solving second-order elliptic boundary value problems is studied. A special technique based on the internal properties of the preconditioning are used for analysis of the rate of convergence. Performance results on standard test problems are presented and discussed.

Keywords: variable-step preconditioners, element-by-element preconditioners, generalized conjugate gradient method.

1 Introduction

Today, robust and efficient parallel algorithms for solving the linear system of equations resulting from large-scale problems of engineering, natural sciences, computer science, mathematics and medicine, whose rate of convergence is optimal or nearly optimal and is independent of both jumps of coefficients and anisotropy, are of considerable importance. Recently a variable-step multilevel preconditioning method for self-adjoint and positive definite elliptic problems, discretized by the finite element method, have been proposed [4]. In spite of the fact that its rate of convergence is limited by that of the exact hierarchical two-level method [1, 15], it is robust in the above sense and, in addition, has the advantage of being totally free of parameters to estimate as well as robust with respect to the regularity of the elliptic problem. However, this method has the same disadvantage as the algebraic multilevel or multigrid methods, in the respect that to get an optimal order both of the rate of convergence and of the arithmetic cost per iteration the method requires the number of nodes on the initial (coarse) grid to be sufficiently small assuming that we solve this system by a direct method. A similar requirement holds if we solve it by an element-by-element method. The latter can be a great problem in their application for solving real life problems, since in order to obtain a good approximation we must use a large number of levels. Another problem arises when solving extremely large-scale problems, for example of the order 100M – 1000M

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($10^8 - 10^9$ degrees of freedom, see e.g. [13, 14]), i.e., where the full matrix data structure cannot be located in a single memory, there is a problem of assigning or distributing data among processors. One possible way to solve it is to keep the matrices in the element level form, i.e., the original matrix and its preconditioner are then not based on the global stiffness matrix.

In the present paper a simple and efficient way to overcome these difficulties is suggested. By the combination of the new, generalized conjugate gradient method with a variable-step preconditioner, and the old, element-by-element preconditioned conjugate gradient (EBE/PCG) method, techniques we obtain an element-by-element version of the variable-step multilevel preconditioning method, which is nearly (asymptotically) optimal and in which both the original matrix and its preconditioner are stored as a set of elementary finite element stiffness matrices and their preconditioners. Note that the use of the proposed method for parallel computers is very natural since all required operations are additive and we use the local finite element data structure only.

The outline of the paper is as follows. The differential, variational and matrix problem formulations of an elliptic boundary value problem are given in Section 2. The description of the generalized conjugate gradient (GCG) method with a variable-step preconditioner is presented in Section 3. The hierarchical basis function method and its properties are described in Section 4. The algorithm of constructing the multilevel variable-step preconditioning matrix M is presented in Section 5. In Section 6 the method of building the element-by-element preconditioner is given. The element-by-element version of the multilevel variable-step preconditioning method is described and analyzed in Section 7. In the final Section of the paper experimental results on standard test problems are presented.

2 The problem formulation

Consider the elliptic boundary value problem with homogeneous boundary conditions

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) = f(x), \quad x \in \Omega, \quad u(x) = g(x), \quad x \in \partial\Omega, \quad (2.1)$$

where Ω is a polygonal domain in R^2 with boundary $\partial\Omega$, where the coefficient matrix $[a_{ij}(x)]$ is assumed to be uniformly symmetric positive definite for any $x \in \overline{\Omega}$, and $g(x)$ is a given function. Although we assume Dirichlet boundary conditions, the results of this paper are valid for general boundary conditions.

The Galerkin variational formulation of the boundary value problem (2.1) is as follows. Find $u \in H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$ such that

$$a(u, v) = (f, v) \quad (2.2)$$

for all $v \in H_0^1(\Omega)$, where the bilinear form $a(u, v)$ and the linear functional (f, v) are defined by

$$a(u, v) = \int_{\Omega} \left[\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right] d\Omega, \quad (f, v) = \int_{\Omega} f v d\Omega.$$

Let us assume that the initial (coarse) triangulation Υ_0 of the domain Ω is such that the coefficient matrix $[a_{ij}(\underline{x})]$ is constant in each element $T_i \in \Upsilon_0$ and can be discontinuous across the elements.

To obtain a sufficiently accurate solution of the problem (2.2) we make a uniform refinement procedure to construct a sequence of nested meshes (triangulations) $\Upsilon_0 \subset \Upsilon_1 \subset \dots \subset \Upsilon_{l-1} \subset \Upsilon_l$. Denote by $V_k \subset H^1(\Omega)$ the space of continuous functions, which are linear on each element from Υ_k , and denote by $A^{(k)}$ the global stiffness matrix, corresponding to standard piecewise linear functions in V_k , which is calculated as

$$A^{(k)} = \{a(\phi_i^{(k)}, \phi_j^{(k)})\}_{x_i, x_j \in N_k},$$

where $\{\phi_i^{(k)}\}_{x_i \in N_k}$ is a set of standard Lagrangian (nodal) basis functions in V_k .

Thus, using the standard finite element approach we replace the variational problem formulation by the matrix problem

$$A^{(l)} u^{(l)} = f^{(l)}, \tag{2.3}$$

where $A = A^{(l)}$ is a symmetric positive definite matrix, corresponding to the standard basis functions on the finest mesh.

3 The GCG-VS method

Now we consider the variable-step preconditioned generalized conjugate gradient (GCG-VS) method used for solving the system of linear equations

$$Ax = b,$$

where A is a symmetric and positive definite matrix of order N and $x, b \in R^N$, in the form proposed by Axelsson and Vassilevski [3].

Let M be a mapping R^N onto R^N , the action of which depends on a vector $v \in R^N$ and can be represented by a symmetric positive definite matrix M_v for each fixed vector $v \in R^N$.

Following [3], the GCG-VS method used here is defined as follows.

Let $x^{(0)}$ be an initial approximation, then $r^{(0)} = b - Ax^{(0)}$ is the initial residual and $d^{(0)} = M_{r^{(0)}} r^{(0)}$ is the corresponding initial search direction. Now, for $k = 1, 2, \dots$, the following actions have to be performed to compute the next approximation $x^{(k)}$:

- Compute $\Lambda^{(k)} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1k} \\ \lambda_{21} & & & \\ \vdots & & \Lambda^{(k-1)} & \\ \lambda_{k1} & & & \end{bmatrix},$

where $\lambda_{1j} = \lambda_{j1} = (Ad^{(k-1)}, Ad^{(k-j)}), \quad j = 1, \dots, k;$

- Compute $\gamma_1 = (r^{(k-1)}, Ad^{(k-1)}), \quad \gamma_j = 0, \quad j = 2, \dots, k;$

- Solve $\Lambda \alpha = \gamma;$

- Compute $x^{(k)} = x^{(k-1)} + \sum_{s=1}^k \alpha_s d^{(k-s)}$;
- Compute $r^{(k)} = r^{(k-1)} - \sum_{s=1}^k \alpha_s A d^{(k-s)}$;
- Check the stopping criterion;
- Solve $\tilde{r}^{(k)} = M_{r^{(k)}} r^{(k)}$;
- Compute $\beta_j = \frac{(\tilde{r}^{(k)}, d^{(j)})}{(d^{(j)}, d^{(j)})}$, $j = 0, \dots, k-1$;
- Compute $d^{(k)} = \tilde{r}^{(k)} - \sum_{s=1}^k \beta_{k-s} d^{(k-s)}$;

where $(u, v) = u^T v$ and $\|v\| = (v, v)^{1/2}$.

Remark 1. As it is mentioned in [3], this formulation is quite general and can be used for an arbitrary mapping M . In practice the preconditioner M has to be chosen such that the action of the nonlinear operator AM is in a sense close to the identity operator.

Remark 2. As it is readily seen from the above it is necessary to store all previous search directions $d^{(k)}$ to be able to compute the parameters β_j . In practice, however, it frequently suffices to keep only a few (or even just one) previous search directions, for details see [3].

As proven in [3] the GCG-VS method cannot fail and converges monotonically with a rate given by the inequality

$$\|x^{(k)} - x\|_A \leq \sqrt{1 - \left(\frac{\delta_1}{\delta_2}\right)^2} \|x^{(k-1)} - x\|_A,$$

where $\|v\| = (v, Av)^{1/2}$ and δ_1 and δ_2 are some positive constants such that

$$(v, AM_v v) \geq \delta_1 (v, v) \quad \text{for all } v \in R^N,$$

$$\|AM_v v\| \leq \delta_2 \|v\| \quad \text{for all } v \in R^N.$$

Unfortunately, this bound gives a certain overestimation in the case of symmetric positive definite matrices as has been shown in [12], and hence, does not give a desired information about the convergence behaviour.

To obtain a more accurate estimate for this case we will analyze the rate of convergence of the GCG-VS method when $\alpha_s = 0$, $s = 0, 1, \dots, k-2$, and $\beta_j = 0$, $j = 0, \dots, k-1$. This case has only theoretical interest because the GCG-VS method converges at least faster.

It is readily seen that in this case the GCG-VS method without preconditioning is identical to the steepest descent method, the rate of convergence of which is given by

$$\|x^{(k)} - x\|_A \leq \frac{\kappa - 1}{\kappa + 1} \|x^{(k-1)} - x\|_A,$$

where κ is the condition number of the matrix A , which is defined by

$$\kappa = \text{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximal and the minimal eigenvalue of the matrix A , respectively.

Now using the standard preconditioning technique (see, for example, [2, 9]) we directly obtain that if the preconditioning matrix M is symmetric and positive definite, then the GCG-VS method with the *fixed* preconditioner cannot fail and converges monotonically with a rate given by the inequality

$$\|x^{(k)} - x\|_A \leq \frac{\kappa - 1}{\kappa + 1} \|x^{(k-1)} - x\|_A, \quad \kappa = \text{cond}(M^{1/2}AM^{1/2}).$$

On the other hand, by assumption we have that on each iteration of the GCG-VS method we generate a new preconditioning matrix $M_{r^{(k)}}$, and hence, the GCG-VS method can be considered as preconditioned GCG method with a sequence of preconditioning symmetric positive definite matrices $\{M_{r^{(k)}}\}$.

Thus, we obtain

$$\begin{aligned} \|x^{(k)} - x\|_A &\leq \frac{\kappa_k - 1}{\kappa_k + 1} \|x^{(k-1)} - x\|_A \leq \dots \\ &\leq \prod_{i=1}^k \frac{\kappa_i - 1}{\kappa_i + 1} \|x^{(0)} - x\|_A \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \|x^{(0)} - x\|_A, \end{aligned} \tag{3.1}$$

where

$$\kappa = \max_i \kappa_i = \max_i \text{cond}(M_{r^{(i)}}^{1/2}AM_{r^{(i)}}^{1/2}).$$

Hence, in order to obtain the rate of convergence of the GCG-VS method we have to find λ_{\min} and λ_{\max} such that

$$\lambda_{\min} \|v\|^2 \leq (M_v^{1/2}AM_v^{1/2}v, v) \leq \lambda_{\max} \|v\|^2 \tag{3.2}$$

for all $v \in R^N$. Note that (3.2) is a more strong condition, then is in fact required since the set of vectors $\{r^{(k)}\}$ generates a subspace of the whole space R^N .

4 The hierarchical basis function method

In this Section the hierarchical basis function method suggested by Yserentant [15] (see also [1]) is considered as a method for construction of block-diagonal preconditioners.

Let now $V_0 \subset V_1 \subset \dots \subset V_l$ be a sequence of nested finite element spaces in $H^1(\Omega)$ defined above, and let N_k be a set of nodes in V_k . Moreover, denote by n_k the number of nodes in V_k . Then together with the set of standard basis functions $\{\phi_i^{(k)}\}$ introduce a set of hierarchical basis functions $\{\hat{\phi}_i^{(k)}\}$ in the finite element space V_k , that is,

$$\{\hat{\phi}^{(k)}\}_{x_i \in N_k} = \{\phi^{(k)}\}_{x_i \in N_k \setminus N_{k-1}} \cup \{\hat{\phi}^{(k-1)}\}_{x_i \in N_{k-1}} \quad \text{and} \quad \{\hat{\phi}^{(0)}\}_{x_i \in N_0} = \{\phi^{(0)}\}_{x_i \in N_0}.$$

Hence, there is a mapping $J = J_k$, which transforms the coefficient vector $v \in V_k$

$$v = \sum_{x_i \in N_k} v_i \phi_i^{(k)}$$

to the coefficient vector $\hat{v} \in V_k$

$$v = \sum_{x_i \in N_k} \hat{v}_i \hat{\phi}_i^{(k)}$$

and is written as follows

$$v = J\hat{v} \iff \begin{bmatrix} v_l \\ \vdots \\ v_1 \\ v_0 \end{bmatrix} = \begin{bmatrix} I & & J_l & \\ & \ddots & & \vdots \\ & & I & J_1 \\ 0 & & & I \end{bmatrix} \begin{bmatrix} \hat{v}_l \\ \vdots \\ \hat{v}_1 \\ \hat{v}_0 \end{bmatrix} \begin{matrix} \} N_l \setminus N_{l-1} \\ \vdots \\ \} N_1 \setminus N_0 \\ \} N_0 \end{matrix}.$$

As it has been shown in Yserentant [15] for the case of piecewise linear functions on triangular elements, the computation of the products $J\hat{v}$ and $J^T v$ can be efficiently implemented, i.e., the number of arithmetic operations to do this is proportional to $O(n_l)$.

Thus, we can solve the equivalent system

$$\hat{A}\hat{x} = \hat{b} \tag{4.1}$$

instead of (2.3), where

$$\hat{A} = J^T A J, \quad \hat{x} = J^{-1} x, \quad \hat{b} = J^T b.$$

Note that $\hat{A} = \hat{A}^{(l)} = \{a(\hat{\phi}_i^{(l)}, \hat{\phi}_j^{(l)})\}_{x_i, x_j \in N_l}$ and, moreover, to find the solution of system (2.3) we make a simple transformation $x = J\hat{x}$.

Consider the following block partitioning of the original matrix $\hat{A}^{(l)}$, which corresponds to the hierarchical decomposition of the nodes N_l :

$$\hat{A}^{(l)} = \begin{bmatrix} \hat{A}_{11}^{(l)} & & * \\ & \ddots & \\ * & & \hat{A}_{11}^{(1)} \\ & & & \hat{A}^{(0)} \end{bmatrix} \begin{matrix} \} N_l \setminus N_{l-1} \\ \vdots \\ \} N_1 \setminus N_0 \\ \} N_0 \end{matrix}$$

and also its block-diagonal part

$$\hat{D}^{(l)} = \begin{bmatrix} \hat{A}_{11}^{(l)} & & 0 \\ & \ddots & \\ 0 & & \hat{A}_{11}^{(1)} \\ & & & \hat{A}^{(0)} \end{bmatrix} \begin{matrix} \} N_l \setminus N_{l-1} \\ \vdots \\ \} N_1 \setminus N_0 \\ \} N_0 \end{matrix}.$$

Note that by the obvious equality

$$\hat{A}_{11}^{(k)} = A_{11}^{(k)}, \quad k = 1, 2, \dots, l,$$

and by the spectrally equivalent substitution of the matrix $A^{(k-1)}$ for the matrix $\hat{A}_{22}^{(k)}$ for all $k = 1, 2, \dots, l$ (for the detailed proof, see [5]), blocks of the matrix $\hat{D}^{(l)}$ can be computed using standard basis functions on all levels.

Theorem 1 [15, 16]. For $\hat{A}^{(l)}$ and $\hat{D}^{(l)}$, defined above, there are two positive constants K_1 and K_2 , which are independent of the number of levels l , such that the following inequalities for all $x \in R^{n_l}$ are valid:

$$\frac{K_1}{(l+1)^2}(\hat{D}^{(l)}x, x) \leq (\hat{A}^{(l)}x, x) \leq K_2(\hat{D}^{(l)}x, x).$$

Hence, the rate of convergence of the conjugate gradient method with the block-diagonal hierarchical basis preconditioner $\hat{D}^{(l)}$ is nearly optimal, i.e., the method requires $O(l)$ iterations. Note, however, that the above remarkable result holds only in the two-dimensional case.

5 Construction of the variable-step preconditioner

In the present Section the algorithm of construction of the variable-step preconditioner in the form proposed by Axelsson and Vassilevski [4] is given.

Let $k_0 \geq 1$ and $\nu \geq 1$ be the given integer parameters defining the number of levels after which we repeat the correction step and the number of inner iterations with the preconditioned GCG method for each correction step, respectively.

Define the variable-step preconditioner $M^{(k+k_0)}[\cdot]$ mapping $R^{n_{k+k_0}}$ onto itself as follows:

1. $M^{(0)} = A^{(0)^{-1}}$ and

$$M^{(k_0)} = \begin{bmatrix} B_{11}^{(k_0)^{-1}} & & & 0 \\ & \ddots & & \\ & & B_{11}^{(1)^{-1}} & \\ 0 & & & M^{(0)} \end{bmatrix} \begin{matrix} \} N_{k_0} \setminus N_{k_0-1} \\ \vdots \\ \} N_1 \setminus N_0 \\ \} N_0 \end{matrix},$$

where $B_{11}^{(i)}$ is a matrix which is assumed to be spectrally equivalent to $\hat{A}_{11}^{(i)}$.

2. For $s = 1, 2, \dots, \frac{l}{k_0} - 1$ and $k = sk_0$

$$M^{(k+k_0)}[\cdot] = \begin{bmatrix} B_{11}^{(k+k_0)^{-1}} & & & 0 \\ & \ddots & & \\ & & B_{11}^{(k+1)^{-1}} & \\ 0 & & & \tilde{M}_\nu^{(k)}[\cdot] \end{bmatrix} \begin{matrix} \} N_{k+k_0} \setminus N_{k+k_0-1} \\ \vdots \\ \} N_{k+1} \setminus N_k \\ \} N_k \end{matrix},$$

where $\tilde{M}_\nu^{(k)}[\cdot]$ is also a variable-step preconditioner to $\hat{A}^{(k)}$ and defined as follows:

Apply the GCG method with the variable-step preconditioner $M^{(k)}[\cdot]$ to the system

$$\hat{A}^{(k)}\hat{x} = \hat{v}.$$

We have

$$\tilde{M}_\nu^{(k)}[\hat{v}] = \hat{x}^{(\nu)},$$

where $\hat{x}^{(\nu)}$ is the ν th iterate of this GCG method beginning with $\hat{x}^{(0)} = 0$.

Remark 3. Hence the GCG method is applied in the recursively defined variable-step preconditioner after every k_0 steps. Moreover, this theory does not require exact solution of the system with matrix $\hat{A}^{(k)}$, $k = 0, 1, \dots, l$.

Theorem 2 [16]. *The GCG method with the variable-step preconditioner $M^{(l)}[\cdot]$ defined above is optimal if the following inequalities for k_0 and ν are valid:*

$$Ck_0^4 \leq \nu < 4^{k_0}$$

for some positive constant C , independent of k_0 and also of possible jumps of coefficients $[a_{ij}(x)]$.

Note that ν , the number of the internal GCG-type iterations depends on the inner stopping criterion ε . Hence, choosing ν (or ε) and assuming k_0 is sufficiently large we can derive both an optimal rate of convergence and an optimal order of computational complexity, if the above relation between ν and k_0 hold.

6 The EBE/PCG method

In the present Section, preliminary considerations about the well-known element-by-element preconditioned conjugate gradient (EBE/PCG) method, which was originally suggested in [10] and [11], are described.

If we denote by A and A^e the global and elementary (local) finite element stiffness matrices, where $A = \sum A^e$, and by $W = \text{diag}(A)$ and $W^e = \text{diag}(A^e)$ their diagonal parts, respectively, then we define the Winget regularized element matrix \bar{A}^e as follows

$$\bar{A}^e = I + W^{-1/2}(A^e - W^e)W^{-1/2}.$$

This regularization ensures \bar{A}^e to be a positive definite matrix, hence the Crout factorization

$$P^e \bar{A}^e (P^e)^T = L_p D_p U_p$$

is well defined. Here P^e is a permutation matrix whose role is to interchange rows and columns of \bar{A}^e so that they are consistent with the ‘‘local’’ nodal ordering. Define now the element-by-element preconditioner as the LU decomposition

$$B = W^{1/2} \times \prod_{e=1}^{N_{el}} L_p^e \times \prod_{e=1}^{N_{el}} D_p^e \times \prod_{e=N_{el}}^1 U_p^e \times W^{1/2},$$

where

$$L_p^e = (P^e)^T L_p P^e, D_p^e = (P^e)^T D_p P^e, U_p^e = (P^e)^T U_p P^e,$$

and N_{el} is the number of elements in the mesh. Note that the central product is simply a diagonal matrix, which is efficient to compute once and for all. The order is reversed in the final product to preserve symmetry.

It is well known that the rate of convergence of the EBE/PCG method is of order $O(h^{-1})$, where h is a characteristic mesh size, i.e., the same order as the unpreconditioned conjugate gradient method [10, 11, 17]. However, the results of those papers show that the EBE preconditioner is always better than the simpler diagonal scaling.

7 The EBE/GCG-VS method

7.1 Preliminaries and description

To formulate now the element-by-element implementation of the GCG-VS method we consider the following tasks:

1. choose the stabilization levels;
2. define spectrally equivalent preconditioners to $\hat{A}_{11}^{(k)}$, $k = 1, 2, \dots, l$;
3. define an element-by-element method for solving a linear system of equations corresponding to the initial (coarse) grid;

Usually, if we want to create an algorithm for solving real life problems, we cannot use a large number of stabilized levels. Hence, the dimension of the linear system of equations on the coarse grid defined from the geometry of the domain Ω is still large. In practice, we make two or three refinements only. In this situation, as it seems to us, an optimal or at least, close to optimal, choice from a practical point of view will be the choice where we use only one level for stabilization: the coarse level, i.e., we will use an (internal) iterative method as a coarse grid solver with a weaker stopping criterion compared with the stopping criterion for the outer iteration process.

We define now a spectrally equivalent preconditioner to $\hat{A}_{11}^{(k)}$, which can be performed on the element level, as follows:

Applying the conjugate gradient method with zero initial guess to solve the system

$$\hat{A}_{11}^{(k)} x_1 = v_1$$

we obtain $x_{\varepsilon_{11}}$ as an approximation of x_1 , for which the following inequality for an arbitrarily chosen positive constant ε_{11}

$$\|\hat{A}_{11}^{(k)} B_{11}^{(k)-1} [v_1] - v_1\| \leq \varepsilon_{11} \|v_1\|$$

holds.

As has been shown in [1] the condition number of $\hat{A}_{11}^{(k)}$ is $O(1)$. Hence, the number of iterations of this method is independent of the dimension of the system $n_k - n_{k-1}$. Thus, the arithmetic cost for solving the system with the matrix $\hat{A}_{11}^{(k)}$ with the prescribed accuracy ε_{11} is proportional to its dimension.

Remark 4. The proposed procedure is not a good technique for strong anisotropic problems, where we can use instead the preconditioned conjugate gradient method with a spectrally equivalent preconditioner $\hat{B}_{11}^{(k)}$ to $\hat{A}_{11}^{(k)}$ proposed by Axelsson and Padiy [7], for which

$$\kappa(\hat{B}_{11}^{(k)-1} \hat{A}_{11}^{(k)}) \leq \frac{1 + \sqrt{\frac{7}{15}}}{1 - \sqrt{\frac{7}{15}}} \approx 5.31$$

holds. However, this method can not be directly implemented on the element level.

Similarly, for solving a system with the matrix $\hat{A}^{(0)}$ we make use of the EBE/PCG method with zero initial guess and the following inequality should be valid for an arbitrarily chosen positive constant ε_0

$$\|\hat{A}^{(0)}\tilde{M}^{(0)}[v] - v\| \leq \varepsilon_0\|v\|.$$

As has been mentioned above the number of iterations for the coarse grid solver is proportional to $O(\sqrt{n_0})$. Hence, the algebraic system with the matrix $\hat{A}^{(0)}$ can be solved with the prescribed accuracy ε_0 with the arithmetic cost proportional to $O(n_0\sqrt{n_0})$.

Now we can present a complete description of the element-by-element variable-step preconditioner for the GCG-VS method.

Define the variable-step preconditioner $M[\cdot]$ to the matrix \hat{A} as follows:

$$M[\cdot] = M^{(l)}[\cdot] = \begin{bmatrix} B_{11}^{(l)-1}[\cdot] & & & 0 \\ & \ddots & & \\ & & B_{11}^{(1)-1}[\cdot] & \\ 0 & & & \tilde{M}^{(0)}[\cdot] \end{bmatrix} \begin{matrix} \} N_l \setminus N_{l-1} \\ \vdots \\ \} N_1 \setminus N_0 \\ \} N_0 \end{matrix},$$

where $B_{11}^{(k)}[\cdot]$, $k = 1, \dots, l$, and $\tilde{M}^{(0)}[\cdot]$ are defined above.

7.2 The rate of convergence

It is well known that applying ν steps of the conjugate gradient method for solving a linear system of equations we produce a matrix polynomial on the original matrix of degree $\nu - 1$, the coefficients of which depend on the initial guess and the right-hand side of the system. This polynomial can be represented as a symmetric positive definite matrix, if the original matrix is symmetric and positive definite. Hence, by the definition of solvers for $\hat{A}_{11}^{(k)}$ and $\hat{A}^{(0)}$ on each iteration we have a symmetric positive definite matrix, which depends on the right-hand side and a zero initial approximation.

Thus, on each outer iteration of the GCG-VS method we generate a new preconditioning matrix $M^{(k)}$, and hence, the nonlinear operator M can be considered as a sequence of symmetric positive definite matrices $\{M^{(k)}\}$.

In order to investigate the rate of convergence of the GCG-VS method we follow the analysis given in [3] and use the formulae (3.1) and (3.2).

Let now v be an arbitrary chosen vector from R^n , then the action of the nonlinear operator $M[\cdot]$ is fixed and can be represented by the symmetric positive definite matrix denoted by M_v , i.e., for all $v \in R^n$ we have

$$M[v] \stackrel{def}{=} M_v v.$$

To derive the corresponding values λ_{min} and λ_{max} we then first consider the action of $M_v^{1/2} A M_v^{1/2}$ on v in the following two-level form

$$M_v^{1/2} A M_v^{1/2} v = \begin{bmatrix} M_{11}[\cdot] & 0 \\ 0 & M_{22}[\cdot] \end{bmatrix}^{1/2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} M_{11}[\cdot] & 0 \\ 0 & M_{22}[\cdot] \end{bmatrix}^{1/2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad (7.1)$$

where $v_1 \in V_1 = V_l \setminus V_{l-1}$, $v_2 \in V_2 = V_{l-1}$, $M_{11}[\cdot]$ and $M_{22}[\cdot]$ are matrix polynomials on A_{11} and A_{22} with the coefficients depending on v_1 and v_2 respectively and also

$$\begin{aligned} \|A_{11}M_{11}[v_1] - v_1\| &\leq \varepsilon_1 \|v_1\| \quad \text{for all } v_1 \in V_1, \\ \|A_{22}M_{22}[v_2] - v_2\| &\leq \varepsilon_2 \|v_2\| \quad \text{for all } v_2 \in V_2, \end{aligned} \tag{7.2}$$

where $\varepsilon_1, \varepsilon_2$ are prescribed small positive numbers. Here for simplicity we omit the indexes of the number of a level. Consider also the corresponding block-diagonal part of matrix A

$$D = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

which is symmetric and positive definite. Further we need the following strengthened Cauchy-Schwarz-Bunyakowski (CBS) inequality, proved in Axelsson and Gustafsson [1]:

There is a constant $\gamma \in [0, 1)$, independent of the characteristic mesh size, such that

$$(v_1, A_{12}v_2) \leq \gamma(v_1, A_{11}v_1)^{1/2}(v_2, A_{22}v_2)^{1/2}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$.

Using now the strengthened CBS inequality we can prove that for all $v \in V$,

$$(1 - \gamma)(Dv, v) \leq (Av, v) \leq (1 + \gamma)(Dv, v),$$

for details, see [5], which we rewrite as follows

$$(1 - \gamma)\|v\|^2 \leq (D^{-1/2}AD^{-1/2}v, v) \leq (1 + \gamma)\|v\|^2. \tag{7.3}$$

Denoting by

$$\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$$

and taking into account the inequality (7.2) in the following form

$$\|DM[v] - v\| \leq \varepsilon\|v\|, \tag{7.4}$$

we can now prove the next theorem.

Theorem 3. *Let $M_v^{1/2}AM_v^{1/2}$ be defined by (7.1), with $M_{11}[\cdot]$ and $M_{22}[\cdot]$ satisfying (7.2), and ε be a sufficiently small positive value. Then*

$$(1 - \gamma)(1 - \varepsilon)\|v\|^2 \leq (M_v^{1/2}AM_v^{1/2}v, v) \leq (1 + \gamma)(1 + \varepsilon)\|v\|^2$$

for all $v \in R^n$.

Proof. Due to (7.4) and the triangle inequality we obtain

$$\| \|DM[v] - \|v\| \| \leq \|DM[v] - v\| \leq \varepsilon\|v\|,$$

from which it follows

$$(1 - \varepsilon)\|v\| \leq \|DM[v]\| \leq (1 + \varepsilon)\|v\|.$$

Moreover, inequality (7.4) shows that

$$\|DM[v]\|^2 - 2(DM[v], v) + \|v\|^2 \leq \varepsilon^2 \|v\|^2.$$

Hence

$$\begin{aligned} 2(DM[v], v) &\geq \|DM[v]\|^2 + (1 - \varepsilon^2)\|v\|^2 \\ &\geq (1 - \varepsilon)^2\|v\|^2 + (1 - \varepsilon^2)\|v\|^2 \\ &= 2(1 - \varepsilon)\|v\|^2. \end{aligned} \tag{7.5}$$

On the other hand, by the Cauchy-Bunyakowski inequality we obtain

$$(DM[v], v) \leq \|DM[v]\| \cdot \|v\| \leq (1 + \varepsilon)\|v\|^2. \tag{7.6}$$

and moreover, by the definition of M_{11} and M_{22} the following property

$$DM_v = M_v D$$

holds. Further we need the following chain of equalities

$$\begin{aligned} (M_v^{1/2} A M_v^{1/2} v, v) &= (M_v^{1/2} D^{1/2} D^{-1/2} A D^{-1/2} D^{1/2} M_v^{1/2} v, v) = \\ &= \frac{(D^{1/2} A D^{1/2} w, w)}{(w, w)} (D^{1/2} M_v^{1/2} v, D^{1/2} M_v^{1/2} v) = \\ &= \frac{(D^{1/2} A D^{1/2} w, w)}{(w, w)} (M_v^{1/2} D M_v^{1/2} v, v) = \\ &= \frac{(D^{1/2} A D^{1/2} w, w)}{(w, w)} \left(\frac{1}{2} (D M_v + M_v D) v, v \right) = \\ &= \frac{(D^{1/2} A D^{1/2} w, w)}{(w, w)} (D M_v v, v), \end{aligned}$$

from which using (7.3), (7.5) and (7.6) we obtain

$$(1 - \gamma)(1 - \varepsilon)\|v\|^2 \leq (M_v^{1/2} A M_v^{1/2} v, v) \leq (1 + \gamma)(1 + \varepsilon)\|v\|^2.$$

■

Now from (3.1) and the results of Theorem 3 we directly obtain

$$\|r^{(k)}\| \leq \frac{\frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{1+\gamma}{1-\gamma} - 1}{\frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{1+\gamma}{1-\gamma} + 1} \|r^{(k-1)}\| = \frac{\gamma + \varepsilon}{1 + \varepsilon\gamma} \|r^{(k-1)}\|.$$

Note that ε must be less than 1 because when $\varepsilon = 1$ we can not be sure that the method has converged.

On the other hand, it is well known that the condition number of the preconditioned matrix in the case of the two-level hierarchical basis function method [1] (see (7.3)) is the following

$$\kappa = \frac{1 + \gamma}{1 - \gamma},$$

from which it follows

$$\|r^{(k)}\| \leq \gamma \|r^{(k-1)}\|.$$

Now an elementary computation shows that

$$\frac{\gamma + \varepsilon}{1 + \varepsilon\gamma} \geq \gamma,$$

i.e., the hierarchical variable-step block-diagonal preconditioning method in the two-by-two form is limited by the hierarchical two-level block-diagonal preconditioning method [1] with respect to the rate of convergence.

Now due to the fact that the inequality (7.4) include in an explicit form the maximum over all the diagonal blocks we can directly extend the results of Theorem 3 for the multilevel case. Thus, the rate of convergence of the GCG-VS method is similar to the rate of convergence of the hierarchical basis function method and depends on the maximum of the values ε_{11} and ε_0 as mentioned in (7.4) with ε defined by

$$\varepsilon = \max\{\varepsilon_{11}, \varepsilon_0\}.$$

We now collect the convergence results in a theorem.

Theorem 4. *The rate of convergence of the EBE/GCG-VS method is nearly optimal, i.e., independent of the number of nodes on the finest level, and depends on the number of levels used and the maximum of the values ε_{11} and ε_0 as mentioned above.*

7.3 Computational complexity

In order to investigate the whole computational complexity of the EBE/GCG-VS method we recall that the solution of the system with the variable-step preconditioner requires the largest computational costs. By its definition, such a solution breaks up into a set of problems with the matrices $\hat{A}_{11}^{(k)}$ on all intermediate levels and with the matrix $\hat{A}^{(0)}$ on the first (coarse) level.

Let w denote the whole computational complexity for the solution of the system with a variable-step preconditioner and let C denote the upper bound of the arithmetic work per meshpoint defined by the local iterative processes with matrices $\hat{A}_{11}^{(k)}$ and $\hat{A}^{(0)}$. Then

$$\begin{aligned} w &\leq C(n_l - n_{l-1}) + C(n_{l-1} - n_{l-2}) + \dots + C(n_1 - n_0) + Cn_0\sqrt{n_0} \\ &= C(n_l - n_0) + Cn_0\sqrt{n_0} \leq C(n_l + n_0\sqrt{n_0}). \end{aligned}$$

If we now assume that n_0 grows with n_l , for instance as

$$n_0 = n_l^\xi \tag{7.7}$$

for some $\xi < 1$, then

$$w \leq C(n_l^{\frac{3}{2}\xi} + n_l) \leq 2C \max\{n_l^{\frac{3}{2}\xi}, n_l\}, \quad (7.8)$$

from which we can find the condition on the number of nodes on the last level

$$\xi < \frac{2}{3}, \quad (7.9)$$

under which the computational complexity is proportional to n_l . Moreover, if we assume that the number of nodes increases in a geometrical ratio with a factor ρ defined by

$$\frac{n_{k+1}}{n_k} = \rho_k \geq \rho > 1, \quad k = 0, 1, \dots, l-1,$$

then $n_l = n_0 \cdot \rho^l$ and taking (7.7) into account we rewrite the condition (7.9) as follows

$$\rho^l = n_l^{1-\xi} > n_l^{1/3},$$

and hence

$$l > \frac{1}{3} \log_\rho n_l. \quad (7.10)$$

Thus, the optimal order of the computational complexity can be attained by increasing the number of levels used.

Remark 6. Condition (7.10) is similar to the previous one [6].

Unfortunately, from a practical point of view we cannot considerably increase the number of levels, and hence, the condition (7.10) can be violated, i.e.,

$$l < \frac{1}{3} \log_\rho n_l.$$

Thus, in this case due to the fact that

$$\xi = 1 - l \cdot (\log_\rho n_l)^{-1}$$

from (7.8) we obtain

$$w \leq 2C n_l^{1+\alpha},$$

where

$$\alpha = \max \left\{ 0, \frac{1}{2} - \frac{3}{2} l \cdot (\log_\rho n_l)^{-1} \right\} = \max \left\{ 0, \frac{1}{2} - \frac{3}{2} \cdot \frac{l}{l + \log_\rho n_0} \right\}.$$

Here, if the size of the coarse level problem is fixed, i.e., n_0 is fixed, then the value of α as a function of l tends to zero when the number of levels used increases. Hence, the arithmetic cost per iteration step is nearly (asymptotically) optimal, i.e., proportional to $O(n_l^{1+\alpha})$, where α is positive and small.

We now collect the results of this subsection in a theorem.

Theorem 5. *The total computational costs of the EBE/GCG-VS method per iteration step is nearly (asymptotically) optimal and depends on the number of levels used.*

8 Numerical results

In this Section experimental results for various test problems are presented. First we test the above-described method on the Laplace equation

$$-\Delta u = 0$$

in the domain $\Omega = [0, 1]^2$ with various boundary conditions on the boundary Γ .

Problem 1. Consider the homogeneous Dirichlet boundary conditions

$$u|_{\Gamma} = 1, \quad \text{where } \Gamma = \partial\Omega.$$

Problem 2. Consider the Neumann boundary conditions on the boundary Γ except the origin, where we used the Dirichlet boundary condition

$$u|_{(0,0)} = 1.$$

The coefficient matrix on the finest level was computed using the standard piecewise linear finite elements on an isosceles right triangular mesh. The right-hand side in the system of equations was always chosen so that $u = 1$ is the solution. The following vector was always taken as an initial approximation [8]

$$u_{ij}^0 = 2 + 100 \sin^2\left(\frac{i\pi}{n+1}\right) \sin^2\left(\frac{j\pi}{n+1}\right), \quad i, j = 1, \dots, n.$$

Iterations were repeated until the condition

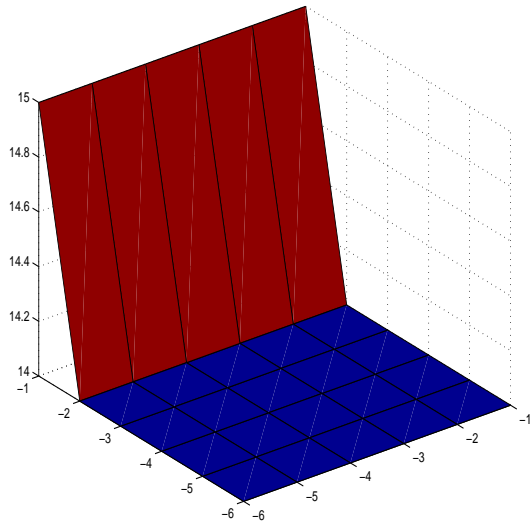
$$\frac{\|u^n - u\|_A}{\|u^0 - u\|_A} \leq 10^{-6},$$

was satisfied.

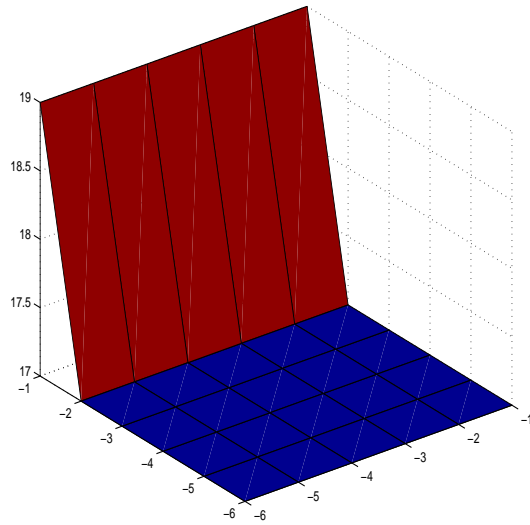
All calculations were performed on Sun SPARCstation-10 (Sun4m, 85 MHz) in double precision.

To illustrate the quality of the preconditioner used we shall show that the rate of convergence as of the outer iteration process is independent on the number of unknowns on the finest mesh and is slightly dependent both on the number of levels used and on the choice of ε_{11} and ε_0 .

The first group of numerical experiments were performed for various combinations of the values of ε_{11} and ε_0 , when the number of unknowns on the finest mesh denoted by N is fixed. The behavior both of the number of outer iterations and of the time of the whole iterative process with respect to ε_{11} and ε_0 are shown in Figures 1 and 2, respectively. Here, the value of ε_{11} and ε_0 are changed in a range from 10^{-1} to 10^{-6} , and the power of 10 are given along the left axis for ε_0 and along the right axis for ε_{11} . Moreover, $N = 3969$ for Problem 1 and $N = 4224$ for Problem 2.

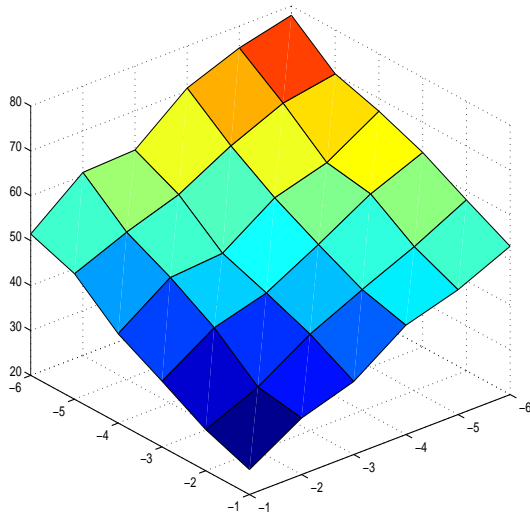


Problem 1

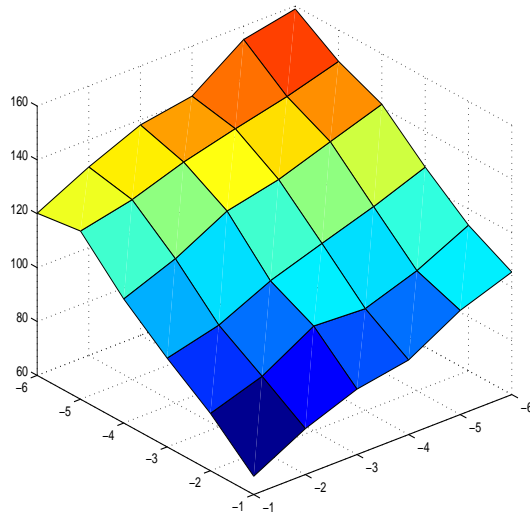


Problem 2

Fig. 1. The number of EBE/GCG-VS iterations vs. ε_{11} and ε_0 .



Problem 1



Problem 2

Fig. 2. The time of the whole iterative process vs. ε_{11} and ε_0 .

From the presented numerical results we see that the number of outer iterations depends only slightly on the values of ε_{11} and ε_0 . However, there is a strong dependence of the computing time on the values of ε_{11} and ε_0 , i.e., the computing time of the whole iterative process increases when the values of ε_{11} and ε_0 are decreased. Hence, in practice, the optimal choice of the iterative parameters seems to be $\varepsilon_{11} = \varepsilon_0 = 10^{-1}$ or close to these values. Note that in all above experiments we used only two levels: the last as a finest level and the previous one as a coarse level.

The second group of numerical experiments were performed for various problem sizes and for different values of the number of level used, when ε_{11} and ε_0 are optimal in above

mentioned sense and are fixed. The number of iterations and the time of the whole iterative process for various values of l with respect to the number of unknowns on the finest mesh are given in Figures 3 and 4, respectively.

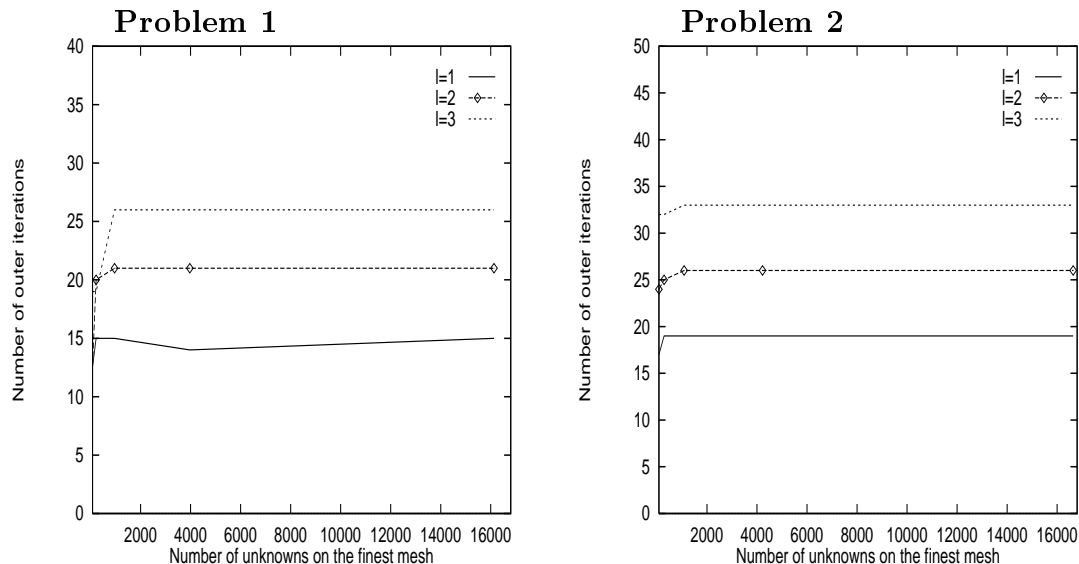


Fig. 3. The number of outer iterations for different values of the number of levels used.

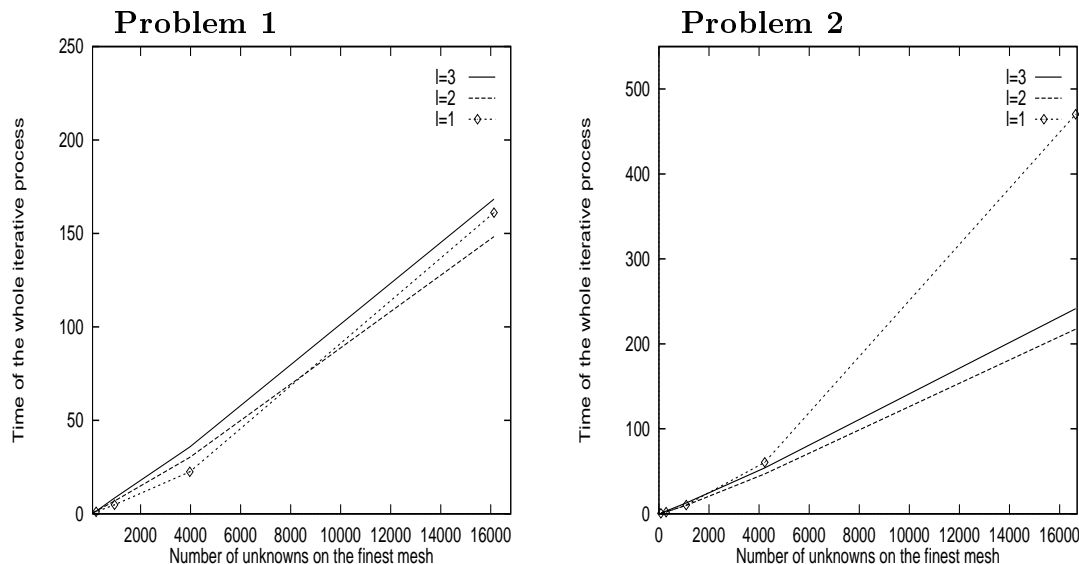


Fig. 4. The time of the whole iterative process with respect to the number of unknowns on the finest mesh for Problem 1 and 2.

Based on the presented numerical results we can see that the EBE/GCG-VS method has an nearly optimal rate of convergence, i.e., the number of iterations does not depend on the number of nodes and does only slightly depend on the number of level used; see Figure 3. As it is readily seen from Figure 4 the computing time of the whole iterative process essentially

depends on the number of level used. For example, in a case of $l = 1$ the computing time of the whole iterative process grows faster than a linear function, i.e., the computational costs per iteration step increases faster than the number of unknowns on the finest mesh. On the other hand, in if $l = 2$ or $l = 3$ the computing time of the whole iterative process grows almost linearly. However, this difference in a behaviour is of no surprise, since in the case $l = 1$ the arithmetic cost per outer iteration is not optimal, inasmuch as the optimal condition on the number of levels used is not satisfied. Indeed in this case we must satisfy

$$1 = l > \frac{1}{3} \log_{\rho} n_l = \frac{1}{3} \log_4 n_l,$$

from which follows that the computational costs per outer iteration are optimal if and only if $n_l < 4^3 = 64$ holds. For the other cases we have

$$2 > \frac{1}{3} \log_4 n_l \implies n_l < 4^6 = 4096,$$

$$3 > \frac{1}{3} \log_4 n_l \implies n_l < 4^9 = 262144.$$

Moreover, the results of the Table show that the number of iterations for the coarse mesh solver depends on the number of nodes as $O(\sqrt{n_0})$. Indeed in these tests we have $n_0 = n_{l-1} \approx n_l/4$. Hence, $\rho = 4$. At the same time the number of EBE/PCG iterations grows with factor 2.

Table. The average number of EBE/PCG iterations for the coarse mesh solver per one inner iteration with respect to the number of unknowns on the coarse mesh for Problems 1 and 2 ($l = 1$).

Number of unknowns	9	49	225	961	Number of unknowns	24	80	288	1088
Problem 1	2.80	4.89	7.53	14.39	Problem 2	6.59	12.64	23.85	45.06

Finally, we emphasize that the results of the numerical experiments are in a good agreement with the theoretical results.

Now to illustrate the efficiency of the techniques used and to compare the method presented here with the earlier ones the experiments with the GCG, EBE/GCG and HB-GCG methods were performed. Here GCG denotes the unpreconditioned generalized conjugate gradient method applied to the standard (nodal) finite element system (2.3), EBE/GCG denotes element-by-element preconditioned generalized conjugate gradient method applied to the same system (2.3), HB-GCG denotes the GCG method applied to the hierarchical basis function system (4.1) and EBE/GCG-VS denotes the method used with $\varepsilon_{11} = \varepsilon_0 = 10^{-1}$ and $l = 1$. The time of the whole iterative process for GCG, EBE/GCG, HB-GCG and EBE/GCG-VS methods for Problems 1 and 2 are given in Figure 5.

It can be seen from Figure 5 that the EBE/GCG-VS method with optimal values of iterative parameters ε_{11} and ε_0 performs more efficient than the other ones even in a case of only two levels used ($l = 1$).

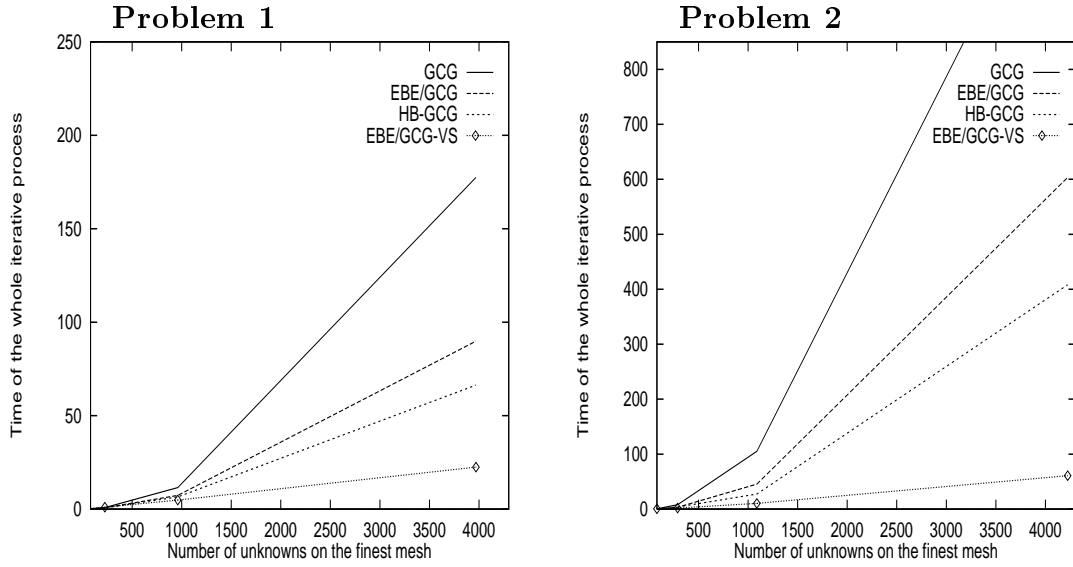


Fig. 5. The time of the whole iterative process with respect to the number of unknowns on the finest mesh for Problem 1 and 2.

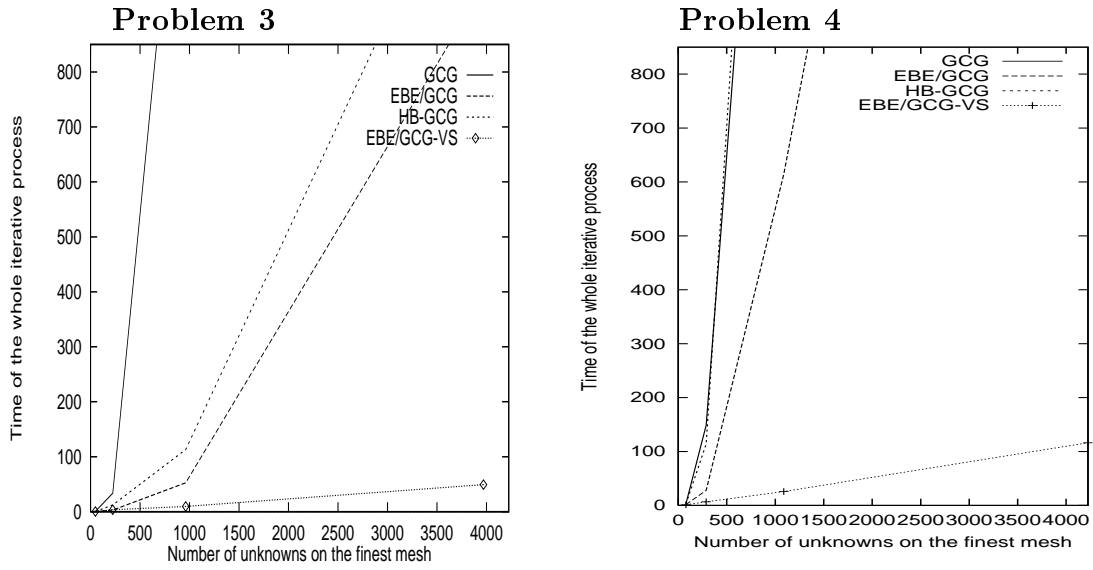


Fig. 6. The time of the whole iterative process with respect to the number of unknowns on the finest mesh for Problem 3 and 4.

The final experiments are made for the elliptic boundary value problem with discontinuous coefficients

$$-\nabla a \nabla u = 0,$$

where

$$a = \begin{cases} 100, & \text{in } (3/8, 5/8) \times (3/8, 5/8) \\ 1, & \text{otherwise.} \end{cases}$$

in the domain $\Omega = [0, 1]^2$ with various boundary conditions on the boundary Γ .

Problem 3. Consider the homogeneous Dirichlet boundary conditions

$$u|_{\Gamma} = 1, \quad \text{where } \Gamma = \partial\Omega.$$

Problem 4. Consider the Neumann boundary conditions on the boundary Γ except the origin, where we used the Dirichlet boundary condition

$$u|_{(0,0)} = 1.$$

From Figure 6 it can be seen that the time of the whole iterative process for the EBE/GCG-VS method with optimal values of iterative parameters ε_{11} and ε_0 grows linearly with the number of nodes on the finest level.

Hence, the EBE/GCG-VS method has all the advantages of both the variable-step multilevel preconditioning methods and the element-by-element technique, i.e., the proposed method is an *efficient* "black box" solver with a *minimal* requirement of amount of memory. In addition, it has an optimal order of computational complexity.

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