Using a compensation principle in algebraic multilevel iteration method for finite element matrices

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Abstract

In the present paper an improved version of the algebraic multilevel iteration (AMLI) method for finite element matrices, which has been suggested in [6], is proposed. To improve the quality of the AMLI-preconditioner or, the same, speed up the rate of convergence the family of iterative parameters defined on an error compensation principle is suggested and analyzed. Performance results on standard test problems are presented and discussed.

Keywords: algebraic multilevel iteration method, preconditioned conjugate gradient method, finite element approximation.

1 Introduction

The solution of many problems in science and engineering can be reduced to solving an algebraic linear system of equations

$$Ax = b, \tag{1}$$

where A is a sparse symmetric positive definite matrix of order N, which arise as a result of finite element approximations of elliptic self-adjoint second order boundary value problems.

There are several methods to construct the *optimal* preconditioning matrix for solving the linear system (1). For such preconditioners there is an iterative method, which converges in a number of iterations independent of N, and a whole computational cost per iteration of which is proportional to N. One of the modern method for constructing of preconditioners with these properties is the algebraic multilevel iteration (AMLI) method [2, 3, 4, 5, 6, 7, 8, 9, 11].

Recently the author and Axelsson [6] are suggested and analyzed an AMLI method for finite element matrices, which can be considered as an extension of methods proposed by Axelsson and Eijkhout [4] for nine-point matrices and later generalized by Axelsson and Neytcheva [5] for the Stieltjes matrices, on a more wider class of sparse symmetric positive definite matrices. However, due to the difference between definitions of the triangular grid and of the matrix graph the AMLI method from [6] does not work properly on the isosceles

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right triangular mesh, whereas for non-regular finite-element meshes the method shows a good performance.

It is well-known that a coarsening algorithm and corresponding approximation algorithm play a key role for constructing an optimal iterative methods. In the present paper we propose an improvement of the method based on an error compensation principle for constructing the approximation of the Schur complement matrix on each level, which is avoided the above-mentioned restriction on the triangulation, whereas for convenience of presentation the coarsening algorithm remain still unchanged.

The paper is organized as follows. In Section 2 the background information for the AMLI method are recalled. The new approach for constructing the approximation of the Schur complement matrix is given and analyzed in Section 3. In the final section of the paper performance results on standard test problems are presented and discussed.

2 The background of the AMLI method

To construct the AMLI-preconditioner we have to define a sequence of matrices $\{A^{(k)}\}$, $k = k_0, k_0+1, \dots, L-1, L$ of an increasing order n_k, where $A^{(L)} = A$.

There are two different ways to define the sequence $\{A^{(k)}\}$. The first approach is based on finite element approximation on the sequence of nested meshes, for detail see [2, 3, 7, 8]. In the second one the sequence of matrices constructed by deletion of certain matrix entries and their diagonal compensation for a given two-by-two block matrix partitioning [4, 5, 6, 9, 10].

In the present section the original AMLI method for finite element matrices of second type is described as in [6].

2.1 The AMLI method

Let $\{G_k\}$ be a sequence of the matrix graphs corresponding to the sequence of matrices $\{A^{(k)}\}$, i.e. $G_k = (X_k, E_k) = G(A^{(k)})$, where $X_k = X(A^{(k)})$ is a set of vertices and $E_k = E(A^{(k)})$ is a set of edges in the matrix graph for A^k, where $(i,j) \in E_k$ if and only if $a_{ij}^{(k)} \neq 0$. We assume that the following conditions are satisfied:

(C1) $\{X_k\}_{k=k_0}^{L}$ is a sequence of nested meshes, i.e.

$$X_{k_0} \in X_{k_0+1} \in \dots \in X_k \in X_{k+1} \in \dots \in X_L,$$

(C2) the number of vertices increase in a geometric ratio, i.e.

$$\frac{n_{k+1}}{n_k} = \rho_k \geq \rho > 1, k = k_0, k_0+1, \dots, L-1.$$

(C3) the number of vertices on the coarse grid is less or equal a fixed value n_0, i.e., $n_{k_0} \leq n_0$.

Let $A^{(L)} = A$ be a sparse symmetric positive definite matrix and consider the construction of matrices $A^{(k)}$, $k = L, L-1, \dots, k_0+1, k_0$, by a recursion from the top to the bottom.

Each matrix $A^{(k+1)}$, $k > 0$, is permuted and partitioned in some way in a two by two block matrix form

$$A^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix} \begin{matrix} \} \, X_{k+1} \backslash X_k \\ \} \, X_k \end{matrix} \quad ,$$

where the first group of unknowns correspond to the vertices in X_{k+1}, which are not in X_k, and the second one corresponds to those in X_k. The selection of sets of the vertices X_k for finite element matrices will be described in Section 2.2.

Then we approximate the first pivoting block $A_{11}^{(k+1)}$ by another matrix $\overline{A_{11}^{(k+1)}}$ which is sparse, symmetric and positive definite, such that

$$A_{11}^{(k+1)} e_1^{(k+1)} = \overline{A_{11}^{(k+1)}} e_1^{(k+1)}, \tag{2}$$

where $e_1^{(k+1)} = (1, ..., 1)^T$ is a column-vector of the order $n_{k+1} - n_k$. Next we consider an intermediate matrix

$$\widetilde{A}^{(k+1)} = \begin{bmatrix} \overline{A_{11}^{(k+1)}} & A_{12}^{(k+1)} \\ & \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix}$$

and define the matrix $A^{(k)}$ as Schur's complement of $\widetilde{A}^{(k+1)}$, i.e.

$$A^{(k)} = A_{22}^{(k+1)} - A_{21}^{(k+1)} \overline{A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)}. \tag{3}$$

Note that the new matrix $A^{(k)}$ is an approximation of the Schur's complement of matrix $A^{(k+1)}$ and is a sparse symmetric positive definite matrix, which is similar to the original matrix. Further, we repeat this process until the matrix corresponding to a coarse mesh is obtained.

The preconditioning matrix M can be recursively defined, based on a sequence of matrices $\{A^{(k)}\}$, $k = 0, 1, \ldots, L$, by the sequence of the preconditioning matrices $M^{(k)}$. They are defined as follows:

$$M^{(0)} = A^{(0)}$$
for $k = 0, 1, \ldots, L-2, L-1,$

$$M^{(k+1)} = \begin{bmatrix} \overline{A_{11}^{(k+1)}} & 0 \\ & \\ A_{21}^{(k+1)} & I \end{bmatrix} \begin{bmatrix} I & \overline{A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)} \\ & \\ 0 & Z^{(k)} \end{bmatrix}, \tag{4}$$

where the matrix $\overline{A_{11}^{(k+1)}}$ is defined above and

$$Z^{(k)} = A^{(k)} \left[I - P_{\nu_k}({M^{(k)}}^{-1} A^{(k)}) \right]^{-1}, \tag{5}$$

where $P_{\nu_k}(t)$ is a polynomial of degree ν_k, $P_{\nu_k}(0) = 1$, and which is small in the interval $I_k = [\underline{t}_k, \bar{t}_k]$ containing the eigenvalues of ${M^{(k)}}^{-1} A^{(k)}$ and is chosen as

$$P_{\nu_k}(t) = \frac{T_{\nu_k}\left(\frac{\bar{t}_k + \underline{t}_k - 2t}{\bar{t}_k - \underline{t}_k}\right) + 1}{T_{\nu_k}\left(\frac{\bar{t}_k + \underline{t}_k}{\bar{t}_k - \underline{t}_k}\right) + 1}, \tag{6}$$

where $T_\nu(t)$ are the Chebyshev polynomials of degree ν,

$$T_0(t) = 1,\ T_1(t) = t,\ T_{\nu+1}(t) = 2tT_\nu(t) - T_{\nu-1}(t).$$

The choice of the polynomial degrees plays a crucial role in AMLI methods. Indeed, if ν_k increase, then by the definition of $P_{\nu_k}(t)$ we obtain that the matrix $Z^{(k)}$ close to $A^{(k)}$ in a matrix sense, and hence, we derive an optimal rate of convergence. On the other hand, it is also evident that the whole computational complexity of the method grows when ν_k increase. Thus, the upper and lower bounds on the polynomial degrees are derived from natural conditions of an optimal order for a whole computational complexity and for a rate of convergence, respectively. The choice of the degrees of polynomials will be discussed in Sections 2.3 and 2.4. It is readily seen that the method is correct since both $A^{(k)}$ and $M^{(k)}$ are symmetric and positive definite.

2.2 Definition of sets X_k

Let a matrix $A^{(k)}$, which corresponds to the structure of a triangular finite element mesh/matrix, be given on each level.

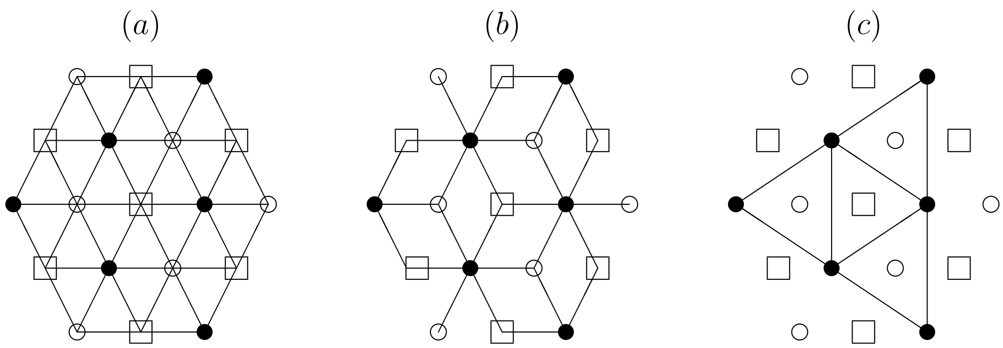

Fig.1. Transformation of the graph G_{k+1} into the graph G_k
(a) Example of coloring of a triangular mesh
(b) The triangular mesh after deleting certain couplings
(c) The new triangular mesh

First step (Partitioning the set of vertices X_{k+1}): Color the nodes of the graph in three colors: red($\circ$), blue($\square$) (the set $X_{k+1}\backslash X_k$) and green($\bullet$) (the set X_k) by a certain principle: nodes which have the same color are not connected, i.e. there are no couplings between the nodes of the same color (Fig.1(a)).

Second step (Approximation of the block $A_{11}^{(k)}$): Delete all couplings between red and blue nodes by diagonal compensation. It leads to the mesh corresponding to the intermediate matrix $\widetilde{A}^{(k+1)}$, which is shown in Fig.1(b).

Third step (Calculation of the new matrix $A^{(k)}$): Make a graph reconstruction which corresponds to Gaussian elimination of all red and blue nodes. It generates a new triangular finite element mesh G_k for the green nodes, which has the same structure and much fewer number of nodes (Fig.1(c)).

As the corresponding matrix has again the same a finite element matrix structure, the process can be repeated until a sufficiently coarse grid satisfying (C3) is reached. In this case the total computational cost to solve the corresponding linear system of equations with the matrix $A^{(k_0)}$ by the direct method is proportional to $O(n_L)$, see, for detail, [6].

2.3 Computational complexity

Let us recall that on each level we have to solve a system with the matrix $M^{(k)}$, which is a preconditioner to the matrix $A^{(k)}$. By its structure it breaks up into forward and back substitutions. More precisely, to solve a system with matrix $M^{(k)}$ we have to solve two systems with the diagonal matrix $\overline{A_{11}^{(k+1)}}$ and the system with $Z^{(k)}$, which was firstly suggested in [2] and can be written as follows

$$\begin{array}{l} \text{Solve } M^{(k)}x = a_{\nu_k} w. \\ \text{For } r = 1 \text{ step } 1 \text{ until } \nu_k - 1: \\ \qquad \text{solve } M^{(k)}x = A^{(k)}x + a_{\nu_k - r} w, \end{array}$$

where

$$P_{\nu_k}(t) = 1 - a_1 t - \ldots - a_{\nu_k} t^{\nu_k}.$$

Hence solving the system with $M^{(k+1)}$ we have to solve ν_k systems with $M^{(k)}$, which are required ν_{k-1} solutions with $M^{(k-1)}$ for each of these and so on.

To estimate the total computational complexity we shall use a recursive technique, which has been already discussed in [5]. Let $\nu, 0 \leq \nu \leq L$ be an integer parameter and let the polynomial degrees ν_k be chosen as 1 for every μ consecutive values of k, i.e.

$$\begin{array}{llllllll} \nu_L & = 1, & \nu_{L-1} & = 1, & \ldots, & \nu_{L-\mu+1} & = 1, & \nu_{L-\mu} = \nu, \\ \nu_{L-\mu-1} & = 1, & \nu_{L-\mu-2} & = 1, & \ldots, & \nu_{L-2\mu+1} & = 1, & \nu_{L-2\mu} = \nu, \\ \nu_{L-2\mu-1} & = 1, & \nu_{L-2\mu-2} & = 1, & \ldots, & \nu_{k_0} = 1. & & \end{array} \tag{7}$$

Then under condition

$$\nu < \rho^{\mu+1}, \tag{8}$$

the total computational complexity for one applications of the preconditioners is proportional to the number of nodes on the fine mesh.

2.4 Rate of convergence

An analysis of the condition number of the matrix A to the preconditioner M can be done by comparing the condition numbers on two adjacent levels.
Note first that

$$\frac{x^T A^{(k+1)} x}{x^T M^{(k+1)} x} = \frac{x^T A^{(k+1)} x}{x^T \widetilde{A}^{(k+1)} x} \cdot \frac{x^T \widetilde{A}^{(k+1)} x}{x^T M^{(k+1)} x}.$$

By definitions the following inequalities for matrices $A^{(k+1)}$, $\widetilde{A}^{(k+1)}$ and $M^{(k+1)}$ is valid

$$\alpha_{k+1} \leq \frac{x^T A^{(k+1)} x}{x^T \widetilde{A}^{(k+1)} x} \leq \beta_{k+1}, \quad 1 - P_{\nu_k}(\underline{t}_k) \leq \frac{x^T \widetilde{A}^{(k+1)} x}{x^T M^{(k+1)} x} \leq 1, \tag{9}$$

from which the boundary points of the interval I_{k+1} to be compute as follows

$$\underline{t}_{k+1} = 1 - P_{\nu_k}(\underline{t}_k), \quad \bar{t}_{k+1} = \beta_{k+1}.$$

To obtain the final condition on the lower bound of degrees of polynomials one can use the standard technique, which is described in [9]. The condition of the optimal rate of convergence is

$$\nu > \left(\max_{\xi=1,2,\ldots L/\mu} \prod_{s=L-\xi\mu}^{L-(\xi-1)\mu} \frac{\beta_s}{\alpha_s} \right)^{\frac{1}{2}}, \tag{10}$$

where ν_k is chosen as in (7).

Thus, choosing proper degrees of the polynomial ν_k we have an optimal rate of convergence, i.e. the condition number of $M^{-1}A$ is the magnitude $O(1)$, and the whole computational complexity is $O(n_L)$, i.e. proportional to the number of nodes on the fine mesh. However, as it has been shown in [6] in a case of the isosceles right triangular mesh the value of β_k is unbounded, and hence, the method is not applied for such a mesh.

3 The improved AMLI method

¿From results of previous section one can see that to define the coefficients of polynomials we have to find the upper and lower bounds of eigenvalues of the matrix $M^{(k)^{-1}}A^{(k)}$ on each level, which are depended on the choice of degrees of polynomials and the quality of the approximation of matrix $A^{(k)}$ by the intermediate matrix $\widetilde{A}^{(k)}$, respectively.

In the present section we propose a new method for computing a robust approximation of $A^{(k+1)}$ based on a new error compensation approach. As it will be seen the new method can be considered as a generalization of AMLI methods of second (matrix) type.

3.1 Construction of a new sequence of matrices $A^{(k)}$

Due to some fill-in occurs during the factorization we have that the sparsity structure of the matrix $A^{(k)}$ on each level is *a more dense* than the sparsity structure of $A_{22}^{(k+1)}$. Moreover, by the definition of the approximation of the first pivoting block of $A^{(k+1)}$ one can ease examine that there exist two positive constants c_1 and c_2 such that

$$0 < c_1(\overline{A_{11}^{(k+1)}}x_1, x_1) \leq (A_{11}^{(k+1)}x_1, x_1) \leq c_2(\overline{A_{11}^{(k+1)}}x_1, x_1) \tag{11}$$

for all $x_1 \in R^{n_{k+1}-n_k}$.

We can try to use this fact and construct an approximation of $A_{22}^{(k+1)}$ by another matrix $\overline{A_{22}^{(k+1)}}$ with similar to the matrix $A^{(k)}$ sparsity structure. The new matrix is sparse, symmetric and positive definite, such that

$$A_{22}^{(k+1)}e_2^{(k)} = \overline{A_{22}^{(k+1)}}e_2^{(k)}, \tag{12}$$

where $e_2^{(k)} = (1, ..., 1)^T$ is a column-vector of the order n_k, and for which the following spectral inequalities are valid

$$0 < (A_{22}^{(k+1)}x_2, x_2) \leq (\overline{A_{22}^{(k+1)}}x_2, x_2) \tag{13}$$

for all $x_2 \in R^{n_k}$.

However, the aim of the second approximation is still not clear. To explain it we first consider the case of Stieltjes matrices. In this case the inequalities (11) takes the form

$$(\overline{A_{11}^{(k+1)}}x_1, x_1) \leq (A_{11}^{(k+1)}x_1, x_1)$$

for all $x_1 \in R^{n_{k+1}-n_k}$. Thus if we construct the new matrix $\overline{A_{22}^{(k+1)}}$ such that the inequality (13) holds, then we obtain

$$\frac{x^T \widetilde{A}^{(k+1)} x}{x^T A^{(k+1)} x} = \frac{x_1^T \overline{A_{11}^{(k+1)}} x_1 + x_1^T A_{12}^{(k+1)} x_2 + x_2^T A_{21}^{(k+1)} x_1 + x_2^T \overline{A_{22}^{(k+1)}} x_2}{x_1^T A_{11}^{(k+1)} x_1 + x_1^T A_{12}^{(k+1)} x_2 + x_2^T A_{21}^{(k+1)} x_1 + x_2^T A_{22}^{(k+1)} x_2}$$

$$= 1 + \frac{x_1^T \left(\overline{A_{11}^{(k+1)}} - A_{11}^{(k+1)} \right) x_1}{x^T A^{(k+1)} x} + \frac{x_2^T \left(\overline{A_{22}^{(k+1)}} - A_{22}^{(k+1)} \right) x_2}{x^T A^{(k+1)} x}$$

for all $x_1 \in R^{n_{k+1}-n_k}$, $x_2 \in R^{n_k}$. In other words we would like to *compensate* a "negative" error of the first approximation by a "positive" error of the second one.

In general case we would like to exploit the same idea to improve the quality of the preconditioner. However, as it is readily seen there are a lot of possibilities to define the matrix $\overline{A_{22}^{(k+1)}}$ satisfying (12) and (13). For an unique definition of $\overline{A_{22}^{(k+1)}}$ we have to put an additional condition on it. Here we propose the following natural condition: a minimization of the condition number of the matrix $\widetilde{A}^{(k)^{-1}} A^{(k)}$. The actual construction of $\overline{A_{22}^{(k+1)}}$, for which the chosen condition is satisfied, will be discussed later.

Now, we construct a new intermediate matrix

$$\widetilde{A}^{(k+1)} = \begin{bmatrix} \overline{A_{11}^{(k+1)}} & A_{12}^{(k+1)} \\ & \\ A_{21}^{(k+1)} & \overline{A_{22}^{(k+1)}} \end{bmatrix}$$

which is a sparse symmetric positive definite matrix due to (11) and (13), then we define as usual the matrix $A^{(k)}$ as the Schur's complement of $\widetilde{A}^{(k+1)}$, i.e.,

$$A^{(k)} = \overline{A_{22}^{(k+1)}} - A_{21}^{(k+1)} \overline{A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)}. \qquad (14)$$

Note that to preserve the sparsity structure of $A^{(k)}$ similar to the original matrix we have to properly define the sparsity structure of $\overline{A_{11}^{(k+1)}}$ and $\overline{A_{22}^{(k+1)}}$. In contrary case we need to do the second approximation step, i.e. excessive entries of $A^{(k)}$ are deleted and compensated by a diagonal matrix [9]. It can be readily seen that the sequence of matrices $A^{(k)}$ is also a sequence of sparse symmetric positive definite matrices.

The definition of the preconditioning matrix M is similar to one as in Section 2.1 and the upper and lower bounds for degrees of polynomials are also derived from conditions of the optimality of the computational complexity and the rate of convergence, respectively. More precisely, as it is readily seen from the construction of the new sequence of matrices $A^{(k)}$ the lower bound is equal to (8) and the upper one is analogous to (10), but with another constants α_k and β_k, which will be estimated in Section 3.4.

3.2 Method to approximate the matrix block $A_{22}^{(k+1)}$

As it has been mentioned above we would like to approximate the sparse matrix block $A_{22}^{(k+1)}$ by the more dense matrix $\overline{A_{22}^{(k+1)}}$ and for which the conditions (12) and (13) are valid.

The simplest way to approximate $A_{22}^{(k+1)}$ is to use the matrix $A_{22}^{(k+1)}$ itself, but it has been considered earlier in [6]. More generally, we can approximate $A_{22}^{(k+1)}$ as follows:

Let $R_{22}^{(k+1)}$ be a sparse symmetric positive semidefinite matrix such that

(A1) $R_{22}^{(k+1)}e^{(k)} = 0$;

(A2) the sparsity structure of $R_{22}^{(k+1)}$ is equal or less (more sparse) then the sparsity structure of $A^{(k)}$.

Then for the next matrix
$$\overline{A_{22}^{(k+1)}} = A_{22}^{(k+1)} + R_{22}^{(k+1)}$$
the assumptions (12) and (13) are satisfied.

However, there are still a lot of possibilities for computing of entries of $R_{22}^{(k+1)}$. The simplest sparse symmetric positive semidefinite matrix, for which the assumption (A1) is valid, is

$$R_{ij} = r_{ij} \cdot \begin{array}{c} \begin{matrix} & & & i & & & & j & & & \end{matrix} \\ \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & & & & \vdots & & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{1} & 0 & \dots & 0 & \mathbf{-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & & & & \vdots & & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{-1} & 0 & \dots & 0 & \mathbf{1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & & & & \vdots & & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}$$

where i,j are indexes of rows/columns and r_{ij} is a non-negative value. Now, we define

$$R_{22}^{(k+1)} = \sum_{(i,j)\in\mathcal{S}_k} R_{ij}$$

where $\mathcal{S}_k$ is a sparse pattern of $A^{(k)}$. Note that by the definition of $R_{22}^{(k+1)}$ the properties (A1) and (A2) are valid. Below we consider a local method to compute the values of r_{ij}. However, at first for completeness we would like to consider a graph transformation of G_{k+1} into G_k caused by the above modification.

3.3 Transformation of graph G_{k+1} into G_k

It is evident that the approximation of $A_{22}^{(k+1)}$ defined above corresponds to a graph transformation. More precisely, a set of new edges corresponding to the set of new entries r_{ij} is added to the old one. This will be shown now.

Let us consider the matrix $A^{(k+1)}$, which corresponding a triangular finite element mesh on that level.

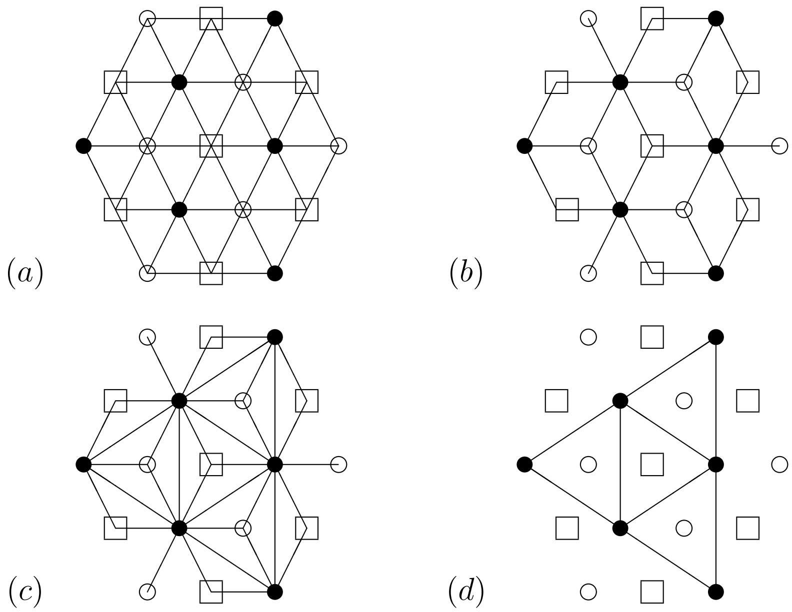

Fig.2. Transformation of the graph G_{k+1} into the graph G_k
(a) Example of coloring of a triangular mesh
(b) The triangular mesh after deleting certain couplings
(c) The triangular mesh after adding certain couplings
(d) The new triangular mesh

First step (Partitioning the set of vertices X_{k+1}): Color the nodes of the graph in three colors as in Section 2.2: red($\circ$), green($\bullet$) and blue($\square$) (Fig.2(a)).

Second step (Approximation of the block $A_{11}^{(k)}$): Delete all couplings between red and blue nodes by diagonal compensation as in Section 2.2 (Fig.2(b)).

Third step (Approximation of the block $A_{22}^{(k)}$): Add new couplings between green nodes, which occurs during Gaussian elimination of all red and blue nodes, by the diagonal compensation (Fig.2(c)). (This is a main point where the present method differs from the method in [6]).

Fourth step (Calculation of the new matrix $A^{(k)}$): Eliminate all red and blue nodes. It leads to a new triangular finite element mesh for green points (Fig.2(d)).

As this one is again a finite-element matrix, the process can be repeat until a sufficiently coarse grid has been reached.

3.4 Local analysis

In this section we suggest and analyze a local method for calculating of values r_{ij}, or the same τ, which are minimized the ratio β_k/α_k.

The following lemma shows that the global analysis of the corresponding generalized eigenvalue problem on each level

$$A^{(k)}v = \lambda \widetilde{A}^{(k)}v$$

can be reduced to the local analysis by a construction of element matrices, which are associated with the global ones.

Lemma 3.1 *[6] Let $\{A_i\}_{i=1}^n$ and $\{M_i\}_{i=1}^n$ be sequences of symmetric positive semidefinite matrices, $A = \sum_{i=1}^n A_i$, $M = \sum_{i=1}^n M_i$ such that $\mathcal{N}(A_i) = \mathcal{N}(M_i)$, where $\mathcal{N}(C)$ is a nullspace of C. Then, if for some positive constants α_i and $\beta_i \geq \alpha_i$ and for all $x \in R^n$*

$$\underline{\lambda}_i \, x^T M_i x \leq x^T A_i x \leq \overline{\lambda}_i \, x^T M_i x$$

holds, then

$$\underline{\lambda} \, x^T M x \leq x^T A x \leq \overline{\lambda} \, x^T M x,$$

where $\underline{\lambda} = \min_i \underline{\lambda}_i$, $\overline{\lambda} = \max_i \overline{\lambda}_i$.

Consider a superelement, which is consist of two triangular elements with a common edge (see Fig.3). It is well-known that after having assembled those finite element matrices defined for linear basis function on arbitrary triangles we derive the following superelement matrix for the original grid

$$K = \begin{bmatrix} \beta_1+\gamma_1+\beta_2+\gamma_2 & -\gamma_1-\gamma_2 & -\beta_1 & -\beta_2 \\ -\gamma_1-\gamma_2 & \alpha_1+\gamma_1+\alpha_2+\gamma_2 & -\alpha_1 & -\alpha_2 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1 & 0 \\ -\beta_2 & -\alpha_2 & 0 & \beta_2+\alpha_2 \end{bmatrix}.$$

where $\alpha_i, \beta_i, \gamma_i, i = 1, 2$, are cotangents of corresponding angles of triangles $\widehat{\alpha_i}, \widehat{\beta_i}, \widehat{\gamma_i}, i = 1, 2$. Despite of that on the later grids these relations are not valid one can assign a similar elementary matrix for each triangular element of the graph $G_k, 0 < k < L$, where α_i, β_i, γ_i are a half values of corresponding off-diagonal entries of $A^{(k)}$, see [6].

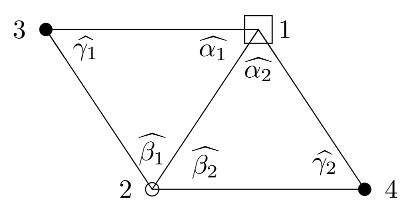

Fig.3. The superelement

Following [1] we assume that all elements of triangulation satisfies:

- *maximal angle condition*: There is a constant $\phi_* < \pi$ (independent of the meshsize h) such that the maximal interior angle ϕ of any element e is bounded by ϕ_*, i.e., $\phi \leq \phi_*$.
- *coordinate system condition*: The angle ψ between the longest side E of the triangle e and x-axis is bounded by $|\sin\psi| \leq h_2/h_1$, where h_1 is the length of E and h_2 is the thickness of e perpendicularly to E, i.e., there are no triangles with *two* sharp angles.

Then taking into account the positive definiteness of the preconditioner the following relations are satisfied

$$\alpha_1 + \alpha_2 > 0, \quad \beta_1 + \beta_2 > 0. \tag{15}$$

Since two nodes, which are lying on common edge, are red and blue, and the other nodes are green we have to delete the coupling between red(1) and blue(2) nodes and add the coupling between green nodes(3 and 4). In result of it we obtain the local preconditioning matrix B, which is

$$B = \begin{bmatrix} \beta_1+\beta_2 & 0 & -\beta_1 & -\beta_2 \\ 0 & \alpha_1+\alpha_2 & -\alpha_1 & -\alpha_2 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1+\tau & -\tau \\ -\beta_2 & -\alpha_2 & -\tau & \beta_2+\alpha_2+\tau \end{bmatrix},$$

where τ is a non-negative value, which have to be chosen from the minimization condition for the condition number of $B^{-1}K$ or, the same, $\widetilde{A}^{(k)^{-1}}A^{(k)}$. Now without loss of generality we assume that

$$\gamma_1+\gamma_2 \neq 0. \tag{16}$$

since if $\gamma_1+\gamma_2=0$, then for $\tau=0$ we obtain $B=K$, which is the best approximation to K.

Lemma 3.2 *[6] Assume that no entries α_i, i=1,2, and β_i, i=1,2, of the matrix B are equal to zero. Then all the entries of its Schur's complement are nonzero too.*

Lemma 3.2 shows that if the matrix graph of $A^{(k)}$ is identical to the triangular finite element grid, then the matrix graphs on the remaining levels are also identical to the triangular finite element grids.

Now we have to solve the local generalized eigenvalue problem

$$Kw = \lambda Bw. \tag{17}$$

As it is readily seen from Lemma 3.1 the essential point to conclude from the local generalized eigenvalue problem on the superelement level to the whole problem lies in having a common null space or not. Set a nonsingular matrix G as follows

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One can easily see that multiplying (17) by G^T from the left and extract G from the right we will obtain the equivalent eigenproblem

$$\hat{K}w = \lambda\hat{B}w, \qquad \hat{K} = G^TKG, \qquad \hat{B} = G^TBG, \qquad w = G^{-1}v, \tag{18}$$

where (up to redundant zero rows and columns)

$$\hat{K} = \begin{bmatrix} \beta_1+\gamma_1+\beta_2+\gamma_2 & -\gamma_1-\gamma_2 & -\beta_1 \\ -\gamma_1-\gamma_2 & \alpha_1+\gamma_1+\alpha_2+\gamma_2 & -\alpha_1 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1 \end{bmatrix}$$

and

$$\hat{B} = \begin{bmatrix} \beta_1+\beta_2 & 0 & -\beta_1 \\ 0 & \alpha_1+\alpha_2 & -\alpha_1 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1+\tau \end{bmatrix}.$$

Thus, we obtain $\lambda_1 = 1$ and the corresponding eigenspace is spanned onto the common nullspace $\mathcal{N} = \mathcal{N}(K) = \mathcal{N}(B)$.

Lemma 3.3 *The matrix $\hat{K}$ is positive definite.*

Proof: Follows directly from the fact that K is a symmetric positive semidefinite matrix with a one-dimensional nullspace as an elementary submatrix of $A^{(k)}$. ■

Lemma 3.4 *There exists $\tau > 0$ such that the matrix $\hat{B}$ is positive definite.*

Proof: The second matrix $\hat{B}$ has the determinant

$$det(\hat{B}) = \alpha\beta\left(\tau + \frac{\alpha_1\alpha_2}{\alpha} + \frac{\beta_1\beta_2}{\beta}\right),$$

where $\alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2$. Since from (15) $\alpha\beta > 0$ we immediately obtain that choosing τ sufficiently large we can always guarantee the positive definiteness of $\hat{B}$. ■

Moreover, using $w = (1, 1, 0)^T$ we can see that we already have the second eigenvalue $\lambda_2 = 1$ in common, and thus the remaining two eigenvalues can be expressed as roots of a quadratic polynomial. From Lemmata 3.3 and 3.4 it follows that these two eigenvalues must be positive. In what follows we will compute these eigenvalues explicitly.

Partitioning (17) in a two by two block matrix form

$$\begin{bmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda \begin{bmatrix} B_1 & K_{12} \\ K_{21} & B_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and considering the case $\lambda_{3,4} \neq 1$ we can rewrite (17) as follows

$$\begin{bmatrix} B_1 - K_1 & 0 \\ 0 & B_2 - K_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mu \begin{bmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \mu = \frac{1-\lambda}{\lambda},\ \mu \neq 0.$$

Expressing w_1 through w_2 as follows

$$w_1 = \mu(B_1 - K_1 - \mu K_1)^{-1} A_{12} w_2,$$

and substituting it in the second matrix equality we obtain for $w_2 \notin \mathcal{N}$

$$(B_2 - K_2 - \mu K_2 - \mu^2 A_{21}(B_1 - K_1 - \mu K_1)^{-1} A_{12}) w_2 = 0. \tag{19}$$

Now from (19) we will define the rest eigenvalues of (17) as functions of τ. Denoting by

$$\gamma = \gamma_1 + \gamma_2, \qquad \Delta = (\gamma + \mu(\gamma + \alpha))(\gamma + \mu(\gamma + \beta)) - [(1 + \mu)\gamma]^2$$

and doing some elementary calculation in (19) we obtain

$$\tau \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} w_2 = \frac{\mu\theta}{\Delta} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} w_2,$$

where

$$\theta = \mu(1+\mu)\gamma(\alpha_1+\beta_1)(\alpha_2+\beta_2) + \mu^2(\alpha_1\alpha_2\beta + \beta_1\beta_2\alpha).$$

Note that $\mu \neq 0$ since $\lambda \neq 1$. Hence, due to this fact and after some simplifications we obtain

$$\mathbf{a}\mu^2 - \mathbf{b}\mu - \mathbf{c} = 0, \tag{20}$$

where

$$\begin{aligned} \mathbf{a} &= \gamma(\alpha_1+\beta_1)(\alpha_2+\beta_2) + \alpha_1\alpha_2\beta + \beta_1\beta_2\alpha, \\ \mathbf{b} &= \tau\gamma(\alpha+\beta) + \tau\alpha\beta - \gamma(\alpha_1+\beta_1)(\alpha_2+\beta_2), \\ \mathbf{c} &= \tau\gamma(\alpha+\beta). \end{aligned}$$

¿From above arguments it follows immediately that two real positive roots of (20) exists and they are

$$\mu_{1,2} = \frac{\mathbf{b} \pm \sqrt{\mathbf{D}}}{2\mathbf{a}}, \quad \mathbf{D} = \mathbf{b}^2 + 4\mathbf{ac}. \tag{21}$$

Recalling the definition of λ we obtain

$$\lambda = \frac{1}{1+\mu} \Longrightarrow \lambda_{3,4} = \frac{2\mathbf{a}}{\mathbf{b} + 2\mathbf{a} \pm \sqrt{\mathbf{D}}}.$$

Now an elementary calculation shows that

$$\lambda_3 = \frac{2\mathbf{a}}{\mathbf{b} + 2\mathbf{a} + \sqrt{\mathbf{D}}} < 1, \qquad \lambda_4 = \frac{2\mathbf{a}}{\mathbf{b} + 2\mathbf{a} - \sqrt{\mathbf{D}}} > 1.$$

Hence, we have

$$\lambda_{max}(B^{-1}K) = \lambda_4(\tau), \qquad \lambda_{min}(B^{-1}K) = \lambda_3(\tau).$$

Now we have to find the value τ, for which the condition number of $B^{-1}K$ defined as

$$cond(B^{-1}K) = \frac{\lambda_{max}(B^{-1}K)}{\lambda_{min}(B^{-1}K)} = \frac{\mathbf{b} + 2\mathbf{a} + \sqrt{\mathbf{D}}}{\mathbf{b} + 2\mathbf{a} - \sqrt{\mathbf{D}}} = \kappa(\tau)$$

is a minimum. The derivative of function $\kappa(\tau)$ is

$$\frac{d}{d\tau}\kappa(\tau) = \frac{\mathbf{D}'_\tau(\mathbf{b} + 2\mathbf{a}) - 2\mathbf{b}'_\tau\mathbf{D}}{\sqrt{\mathbf{D}}(\mathbf{b} + 2\mathbf{a} - \sqrt{\mathbf{D}})^2}.$$

As it is well known to find now the extremum point we have to solve

$$\frac{d}{d\tau}\kappa(\tau) = 0$$

or, the same,

$$\mathbf{D}'_\tau(\mathbf{b} + 2\mathbf{a}) - 2\mathbf{b}'_\tau\mathbf{D} = 0.$$

The latter can be considered as an equation with respect to τ in the form

$$(\tau(C_3+C_4) - C_1 + 2(C_1+C_2))C_3 + (\tau(C_3+C_4) - C_1 - 2C_3\tau)(C_3+C_4) = 0,$$

where

$$\begin{aligned} C_1 &= \gamma(\alpha_1+\beta_1)(\alpha_2+\beta_2), \\ C_2 &= \alpha_1\alpha_2\beta + \beta_1\beta_2\alpha, \\ C_3 &= \gamma(\alpha+\beta), \\ C_4 &= \alpha\beta, \end{aligned}$$

which has a unique solution

$$\tau = \frac{C_1C_4 - 2C_2C_3}{C_4(C_3 + C_4)}.$$

Rewriting it in terms of initial entries of the superelement matrix we obtain

$$\tau_{opt} = \frac{\alpha\beta\gamma(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) - 2\gamma(\alpha + \beta)(\alpha_1\alpha_2\beta + \beta_1\beta_2\alpha)}{\alpha\beta(\gamma(\alpha + \beta) + \alpha\beta)}.$$

In result of the simple interval analysis we can easily obtain that this point is a point of minimum of the function $\kappa(\tau)$. Thus, choosing the proper value of the iterative parameter τ for each superelement we can always minimize the value of β_{k+1}/α_{k+1} in the estimate of (10).

Based on above results we can formulate

Theorem 3.1 *There is an optimal value of iterative parameter τ, $\tau \geq 0$, which minimized the condition number of $(\widetilde{A}^{(k)})^{-1}A^{(k)}$, or the same the ratio of β_k/α_k.*

To illustrate the efficiency of the new method we present two "extreme" examples:

Example 3.1 *The bad case of the AMLI method in [6] was isoscele right triangular mesh. Consider two possible situations for the location of two rightangle triangles (see Fig.4 and Fig.5).*

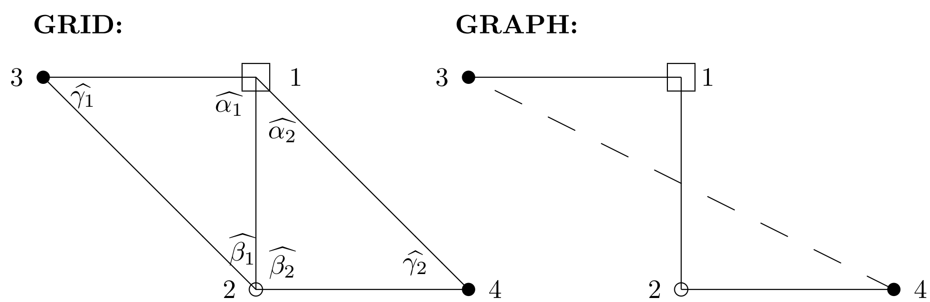

Fig.4. *The superelement with $\widehat{\alpha_1} = \widehat{\beta_2} = 90°$ and $\widehat{\beta_1} = \widehat{\alpha_2} = \widehat{\gamma_1} = \widehat{\gamma_2} = 45°$ and the corresponding graph with $a_{23} = a_{14} = 0$.*

In this case we have $\alpha_1 = \beta_2 = 0$ and $\beta_1 = \alpha_2 = \gamma_1 = \gamma_2 = 1$, and hence, we have

$$\tau_{opt} = \frac{1 \cdot 1 \cdot 2 \cdot (0+1) \cdot (1+0) - 2 \cdot 2 \cdot (1+1) \cdot (0+0)}{1 \cdot 1 \cdot (2 \cdot (1+1) + 1 \cdot 1)} = \frac{2}{5}.$$

Now we can compute the corresponding value of the condition number of $B^{-1}K$. Indeed, we have

$$\begin{aligned}
\mathbf{a} &= 2 \cdot (0+1) \cdot (1+0) + 0 + 0 = 2, \\
\mathbf{b} &= \frac{2}{5} \cdot 2 \cdot (1+1) + \frac{2}{5} \cdot 1 \cdot 1 - 2 \cdot (0+1) \cdot (1+0) = 0, \\
\mathbf{c} &= \frac{2}{5} \cdot 2 \cdot (1+1) = \frac{6}{5}, \\
\mathbf{D} &= \mathbf{b}^2 + 4\mathbf{ac} = 0 + 4 \cdot 2 \cdot \frac{6}{5} = \frac{48}{5},
\end{aligned}$$

and hence,

$$cond(B^{-1}K) = \frac{\mathbf{b} + 2\mathbf{a} + \sqrt{\mathbf{D}}}{\mathbf{b} + 2\mathbf{a} - \sqrt{\mathbf{D}}} = \frac{0 + 2 \cdot 2 + \sqrt{\dfrac{48}{5}}}{0 + 2 \cdot 2 - \sqrt{\dfrac{48}{5}}} \approx 7.8729833.$$

Next consider the following disposition for two rightangle triangles (see Fig.5).

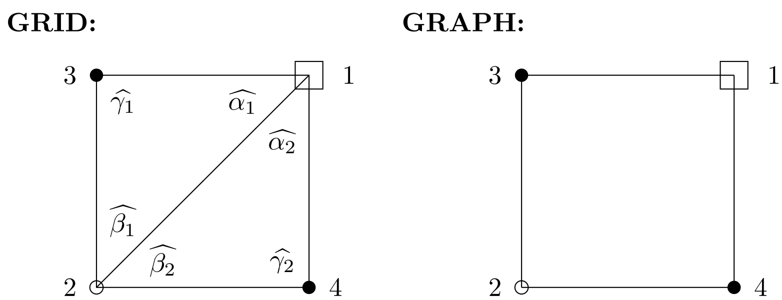

Fig.5. *The superelement with* $\widehat{\gamma_1} = \widehat{\gamma_2} = 90°$ *and* $\widehat{\alpha_1} = \widehat{\alpha_2} = \widehat{\beta_1} = \widehat{\beta_2} = 45°$ *and the corresponding graph with* $a_{12} = 0$.

In this case we have $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$ *and* $\gamma_1 = \gamma_2 = 0$, *and hence,* $\tau_{opt} = 0$. *Now we can compute the corresponding value of the condition number of* $B^{-1}K$. *Indeed, we have*

$$\begin{aligned} \mathbf{a} &= 4, \\ \mathbf{b} &= 0, \\ \mathbf{c} &= 0, \\ \mathbf{D} &= \mathbf{b}^2 + 4\mathbf{ac} = 0, \end{aligned}$$

and hence,

$$cond(B^{-1}K) = 1.$$

Example 3.2 *The best case of the AMLI method in [6] was equilateral mesh. Consider two equilateral triangles (see Fig.6).*

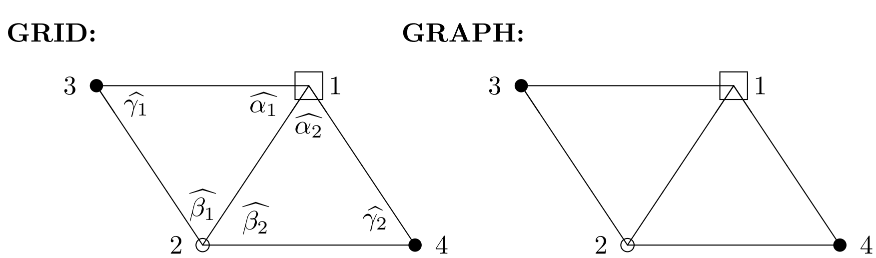

Fig.6. *The superelement with* $\widehat{\alpha_1} = \widehat{\beta_1} = \widehat{\gamma_1} = \widehat{\alpha_2} = \widehat{\beta_2} = \widehat{\gamma_2} = 60°$ *and the corresponding graph.*

In this case we have $\alpha_1 = \beta_1 = \gamma_1 = \alpha_2 = \beta_2 = \gamma_2 = (\sqrt{3})^{-1}$*, and hence, we have*

$$\tau_{opt} = \frac{2 \cdot 2 \cdot 2 \cdot (1+1) \cdot (1+1) \cdot (\sqrt{3})^{-5} - 2 \cdot 2 \cdot (2+2) \cdot (2+2) \cdot (\sqrt{3})^{-5}}{2 \cdot 2 \cdot (2 \cdot (2+2) + 2 \cdot 2) \cdot (\sqrt{3})^{-4}} = 0.$$

Now we can compute the corresponding value of the condition number of $B^{-1}K$*. Indeed, we have*

$$\begin{aligned}
\mathbf{a} &= 2(\sqrt{3})^{-1} \cdot 2(\sqrt{3})^{-1} \cdot 2(\sqrt{3})^{-1} + 2(\sqrt{3})^{-3} + 2(\sqrt{3})^{-3} = 12(\sqrt{3})^{-3}, \\
\mathbf{b} &= 0 + 0 - 2(\sqrt{3})^{-1} \cdot 2(\sqrt{3})^{-1} \cdot 2(\sqrt{3})^{-1} = -8(\sqrt{3})^{-3}, \\
\mathbf{c} &= 0, \\
\mathbf{D} &= \mathbf{b}^2 + 4\mathbf{ac} = 64(\sqrt{3})^{-6},
\end{aligned}$$

and hence,

$$cond(B^{-1}K) = \frac{\mathbf{b} + 2\mathbf{a} + \sqrt{\mathbf{D}}}{\mathbf{b} + 2\mathbf{a} - \sqrt{\mathbf{D}}} = \frac{-8(\sqrt{3})^{-3} + 24(\sqrt{3})^{-3} + 8(\sqrt{3})^{-3}}{-8(\sqrt{3})^{-3} + 24(\sqrt{3})^{-3} - 8(\sqrt{3})^{-3}} = 3.$$

These examples shows that the new method use all advantages of the old one (see Example 3.2) and improve them when it is needed (see Example 3.1).

Finally, we collect the above results from Section 2 and 3 in the following theorem.

Theorem 3.2 *The modified algebraic multilevel iteration method for the finite element matrices, based on the sequence of the matrices* $\{A^{(k)}\}$ *defined with (2), (12) and (14), and the sequence of their preconditioners* $\{M^{(k)}\}$*, recursively defined with (4), (5) and (6), has an optimal order of computational complexity iff*

$$\rho^{\mu+1} > \nu > \left(\max_{k=1,2,\dots L/\mu} \prod_{s=L-k\mu}^{L-(k-1)\mu} \frac{\beta_s}{\alpha_s} \right)^{\frac{1}{2}}$$

where α_s *and* β_s *are the above defined constants by Theorem 3.1,* ρ *is a coefficient of a geometric progression,* ν *is the degree of the matrix polynomials used in (7) and* $\nu > 1$ *at every* $(\mu+1)$*th step.*

Unfortunately, we could not guarantee that even for optimal value of τ the ratio of β_s/α_s is small enough, and hence, to verify the Theorem 3.2 we made the numerical experiments.

4 Numerical experiments

Due to the fact that the AMLI method, developed in [6], shown a good performance for non-uniform triangular meshes and does not work on the isosceles right triangular mesh we will test the new AMLI method on that mesh for model boundary value problems. Consider the Laplace equation

$$-\Delta u = 0$$

in the domain $\Omega = [0,1]^2$ with the homogeneous Dirichlet boundary conditions

$$u\Big|_{\Gamma} = 1, \quad \text{where } \Gamma = \partial\Omega.$$

The coefficient matrix on the finest level was derived using standard piecewise linear finite elements on the triangular mesh. The right-hand side in the system of equations was chosen so that the solution has the form

$$u = x(1-x)y(1-y)e^{xy}.$$

The solution method is the preconditioned conjugate gradient method with a preconditioner M defined by (4), (5), (6) and (7). The initial approximation was always taken as the zero vector. The following stopping criterion was used

$$\frac{r_i^T M^{-1} r_i}{r_1^T M^{-1} r_1} < 10^{-12},$$

where r_1 and r_i are the initial and the current residuals, respectively.

Table 1. The number of iterations for the modified AMLI method

	$\mu=0$			$\mu=1$			$\mu=2$		
N	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=1$	$\nu=2$	$\nu=3$
7	7	6	4	-	-	-	-	-	-
15	14	9	5	14	12	12	-	-	-
31	31	12	6	31	25	23	31	29	30
63	62	14	7	62	33	32	62	57	55
127	124	15	7	124	36	34	124	83	79

In Table 1 the results of the experiments for the preconditioned conjugate gradient method for different choices of polynomial degrees and grids are given. Moreover, ν is the degree of the matrix polynomials used and $\nu > 1$ at every $(\mu+1)$th level. For example, in a case $\nu = 2,\ \mu = 0$ we have a two-fold W-cycle.

Based on the results of experiments it can be recommended to use the W-cycle version $(\mu = 0, \nu = 3)$ of the suggested method as it gives still a reasonable number of iterations and the total computational cost is proportional to the number of unknowns on the fine mesh. Indeed, in this case we have $\rho = 3$, and hence, the following equality for the upper bound on polynomial degrees is valid

$$\nu = 3 = \rho^{\mu+1}.$$

On the other hand, analyzing the square root of corresponding condition numbers one can see that it is less than the corresponding value of ν. For example,

$$\begin{array}{lll}
N = 7, & (\max\{2.251, 6.811\})^{1/2} = (6.811)^{1/2} & = 2.6097893 < 3,\\
N = 15, & (\max\{2.385, 6.815, 7.503\})^{1/2} = (7.503)^{1/2} & = 2.7391605 < 3,\\
N = 31, & (\max\{2.662, 6.911, 7.778\})^{1/2} = (7.778)^{1/2} & = 2.7889066 < 3,\\
N = 63, & (\max\{2.877, 6.955, 7.781, 7.816\})^{1/2} = (7.816)^{1/2} & = 2.7957110 < 3,\\
N = 127, & (\max\{2.881, 6.959, 7.810, 7.834\})^{1/2} = (7.834)^{1/2} & = 2.7989284 < 3.
\end{array}$$

Now to illustrate the efficiency of the techniques used and to compare the method presented here with the earlier ones the corresponding experiments were performed (see Table 2). Here AMLI denotes the AMLI method from [5] with optimal parameters $(\mu = 1, \nu = 2)$, AMLI-IF denotes the algebraic multilevel incomplete factorization method suggested in [9] also with optimal parameters $(\mu = 1, \nu = 3)$ and AMLI-FE denotes the new method used with $\mu = 0$ and $\nu = 3$. Similar behaviour for all methods shows that suggested modification make the old AMLI method for finite elment matrices also robust for isosceles right triangular meshes.

Table 2. The number of iterations for different multilevel methods

N	AMLI	AMLI-IF	AMLI-FE
7	5	4	4
15	5	6	5
31	6	5	6
63	6	4	7
127	7	4	7

As a final conclusion, we have shown that the improved version of algebraic multilevel iteration method, applied to the finite-element matrices, leads to the iterative method of a nearly optimal order of the computational complexity and rate of convergence. The new method combines the favorable optimal behavior of the AMLI method on quasi-uniform irregular grids with the adaptive stabilization on the element level in extreme cases.

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