Hybrid Discontinuous Galerkin methods for incompressible flow problems

Christoph Lehrenfeld, Joachim Schöberl

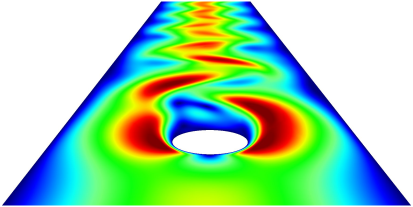
RWTH AACHEN UNIVERSITY

MathCCES

CME Computational Mathematics in Engineering

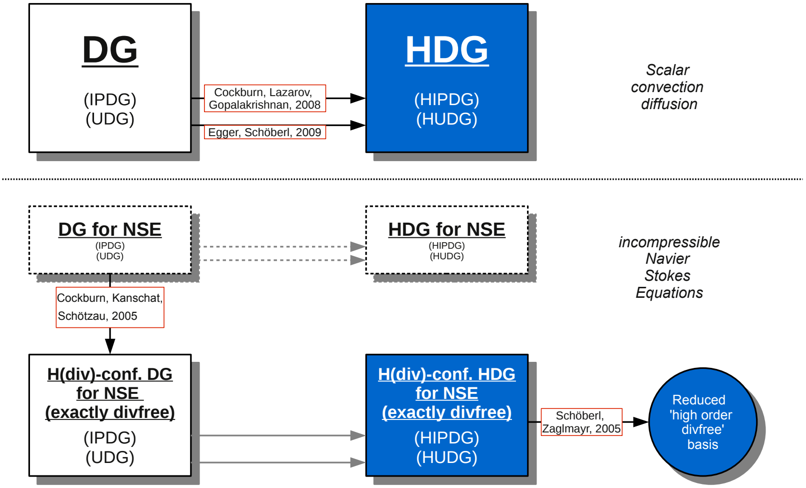
TU WIEN

DG Workshop Linz, May 31 - June 1, 2010



Contents

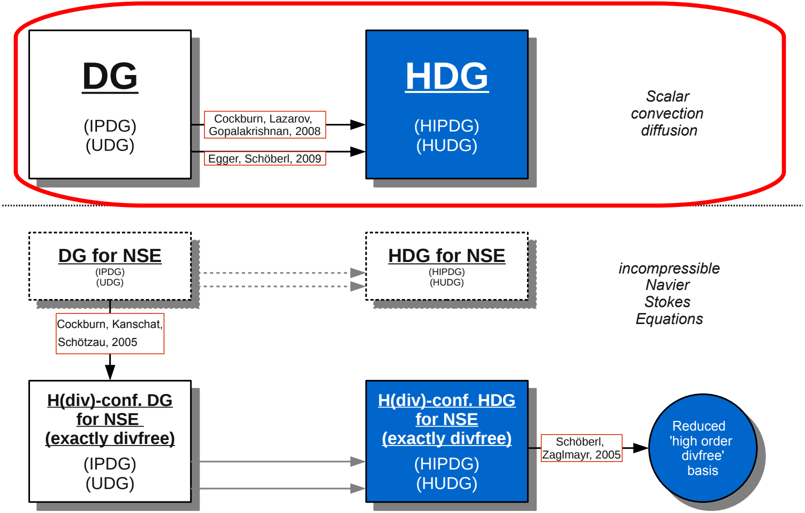
- The methods:



- Software, Examples, Conclusion

Contents

- The methods:



- Software, Examples, Conclusion

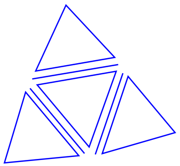
Hybrid DG for scalar convection diffusion equations

$$-\Delta u + \mathrm{div}(bu) = f$$

Discretization space

Trial functions are:

- discontinuous
- piecewise polynomials
  (on each facet and element)



with appropriate formulations we get

- more unknowns but less matrix entries
- implementation fits into standard element-based assembling
- structure allows condensation of element unknowns

HDG formulation for $-\Delta u$

Symmetric Interior Penalty Formulation ($\tau_h \sim \frac{h}{p^2}$):

$$
\begin{aligned}
B^{DG}(u,v) &= \sum_T \int_T \nabla u \nabla v \; dx \\
&+ \sum_E \left\{ -\int_E \left\{\!\!\left\{ \frac{\partial u}{\partial n} \right\}\!\!\right\} [\![v]\!] \; ds - \int_E \left\{\!\!\left\{ \frac{\partial v}{\partial n} \right\}\!\!\right\} [\![u]\!] \; ds + \int_E \tau_h [\![u]\!][\![v]\!] \; ds \right\}
\end{aligned}
$$

Hybrid Symmetric Interior Penalty Formulation ($\tau_h \sim \frac{h}{p^2}$):

$$
\begin{aligned}
B\big((u,u_F),(v,v_F)\big) = \sum_T \Bigg\{ & \int_T \nabla u \; \nabla v \; dx \\
& - \int_{\partial T} \frac{\partial u}{\partial n} \, (v - v_F) \; ds - \int_{\partial T} \frac{\partial v}{\partial n} \, (u - u_F) \; ds \\
& + \int_{\partial T} \tau_h (u - u_F)(v - v_F) \; ds \Bigg\}
\end{aligned}
$$

This and other *hybridizations* of CG, mixed and DG methods were discussed in [Cockburn+Gopalakrishnan+Lazarov,'08]

HDG formulation for $-\Delta u$

Symmetric Interior Penalty Formulation ($\tau_h \sim \frac{h}{p^2}$):

$$
\begin{aligned}
B^{DG}(u,v) &= \sum_T \int_T \nabla u \nabla v \; dx \\
&+ \sum_E \left\{ -\int_E \left\{\!\!\left\{ \frac{\partial u}{\partial n} \right\}\!\!\right\} [\![v]\!] \; ds - \int_E \left\{\!\!\left\{ \frac{\partial v}{\partial n} \right\}\!\!\right\} [\![u]\!] \; ds + \int_E \tau_h [\![u]\!][\![v]\!] \; ds \right\}
\end{aligned}
$$

Hybrid Symmetric Interior Penalty Formulation ($\tau_h \sim \frac{h}{p^2}$):

$$
\begin{aligned}
B\big((u,u_F),(v,v_F)\big) = \sum_T \Big\{ &\int_T \nabla u \, \nabla v \; dx \\
&- \int_{\partial T} \frac{\partial u}{\partial n} \, (v - v_F) \; ds - \int_{\partial T} \frac{\partial v}{\partial n} \, (u - u_F) \; ds \\
&+ \int_{\partial T} \tau_h (u - u_F)(v - v_F) \; ds \Big\}
\end{aligned}
$$

This and other *hybridizations* of CG, mixed and DG methods were discussed in [Cockburn+Gopalakrishnan+Lazarov,'08]

HDG formulation for $\mathrm{div}(bu) = f \qquad (\mathrm{div}(b) = 0)$

After partial integration on each element we get

$$\int_\Omega \mathrm{div}(bu)\ v\ dx = \sum_T \left\{ - \int_T bu\ \nabla v\ dx + \int_{\partial T} b_n u^? v\ ds \right\}$$

Upwind DG $\longrightarrow$ Hybridized Upwind DG:

$$u^{UDG} = \begin{cases} u_{nb} & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases} \quad \Longrightarrow \quad u^{HUDG} = \begin{cases} u_F & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases}$$

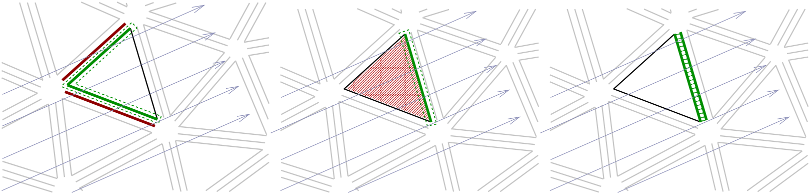
HDG formulation for $\text{div}(bu) = f \qquad (\text{div}(b) = 0)$

After partial integration on each element we get

$$\int_\Omega \text{div}(bu)\; v \; dx = \sum_T \left\{ - \int_T bu \; \nabla v \; dx + \int_{\partial T} b_n u^? v \; ds \right\}$$

Upwind DG $\longrightarrow$ Hybridized Upwind DG:

$$u^{UDG} = \begin{cases} u_{nb} & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases} \quad \Longrightarrow \quad u^{HUDG} = \begin{cases} u_F & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases}$$



$$+ \sum_T \int_{\partial T_{out}} b_n (u_F - u) v_F \; ds$$

This gives us an appropriate bilinearform $C\big((u, u_F), (v, v_F)\big)$ for the convective term. Already used in [Egger+Schöberl, '09]

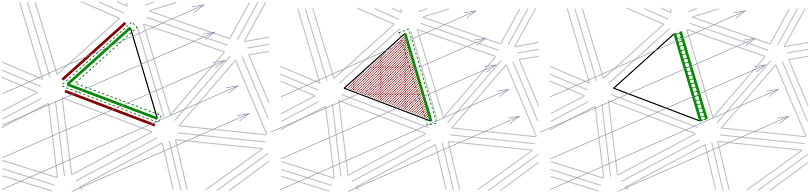
HDG formulation for $\operatorname{div}(bu) = f \qquad (\operatorname{div}(b) = 0)$

After partial integration on each element we get

$$\int_\Omega \operatorname{div}(bu)\ v\ dx = \sum_T \left\{ -\int_T bu\ \nabla v\ dx + \int_{\partial T} b_n u^? v\ ds \right\}$$

Upwind DG $\longrightarrow$ Hybridized Upwind DG:

$$u^{UDG} = \begin{cases} u_{nb} & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases} \quad \Longrightarrow \quad u^{HUDG} = \begin{cases} u_F & \text{if } b_n \leq 0 \\ u & \text{if } b_n > 0 \end{cases}$$



$$+ \sum_T \int_{\partial T_{out}} b_n (u_F - u) v_F\ ds$$

This gives us an appropriate bilinearform $C\big((u, u_F), (v, v_F)\big)$ for the convective term. Already used in [Egger+Schöberl, '09]

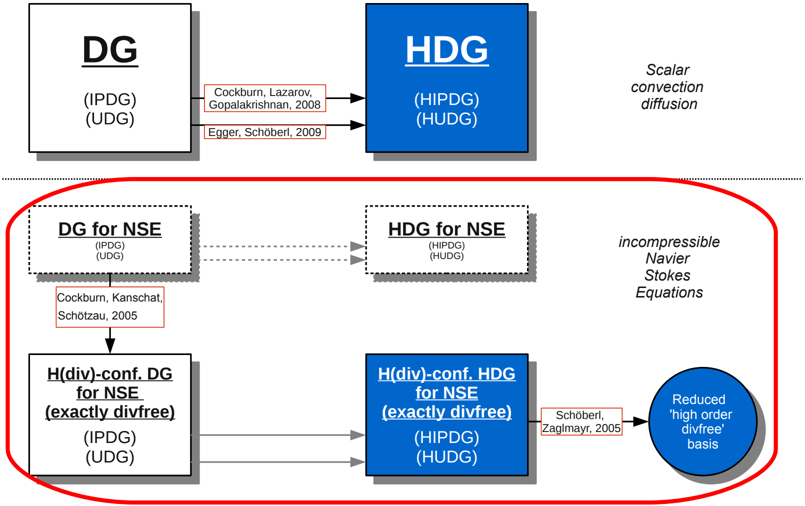
HDG formulation for $-\Delta u + \operatorname{div}(bu) = f \quad (\operatorname{div}(b) = 0)$

Now we can add the introduced bilinearforms together and easily get a formulation for the convection diffusion problem:

$$A\big((u, u_F), (v, v_F)\big) := B\big((u, u_F), (v, v_F)\big) + C\big((u, u_F), (v, v_F)\big) = (f, v)$$

Contents

- The methods:



- Software, Examples, Conclusion

Hybrid DG for incompressible Navier Stokes Equations

$$\begin{aligned} \mathrm{div}(-\nu \nabla u + u \otimes u + pI) &= f \\ \mathrm{div}\ u &= 0 \end{aligned} \quad + \quad b.c.$$

(steady) incompressible Navier Stokes Equations

variational formulation:

$$B(u, v) + C(u; u, v) + D(p, v) + D(q, u) = f(v)$$

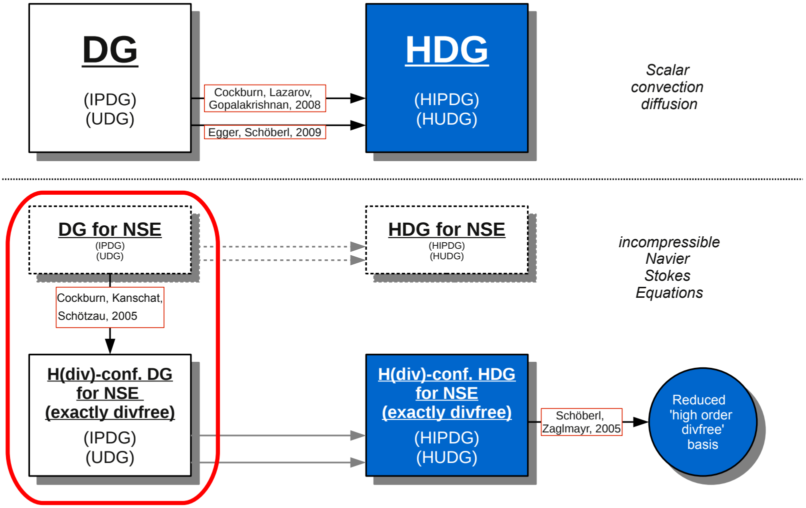
where B is a suitable bilinearform for the viscous term, D is the bilinearform for the incompressibility-constraint and the pressure term and C is the convective Trilinearform.

tasks:

- Find an appropriate discretization space
- Find appropriate bilinearforms
- Find an iterative procedure to solve the nonlinearity ($\Rightarrow$ Oseen its.)

Contents

- The methods:



- Software, Examples, Conclusion

H(div)-conforming elements for Navier Stokes

[Cockburn, Kanschat, Schötzau, 2005]:

DG

$+$

inc. Navier-Stokes

$+$

local conservation

$+$

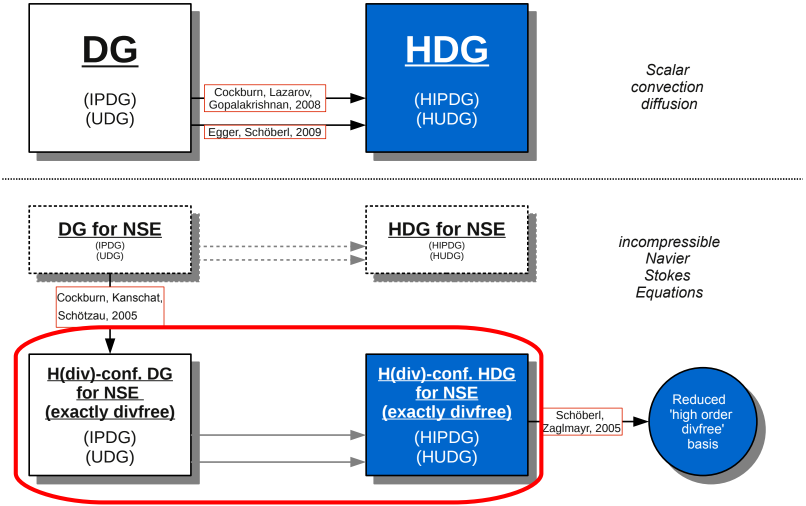
energy-stability

$\Longrightarrow$

need exactly divergence-free convective velocity for $C(u; u, v)$

Contents

- The methods:

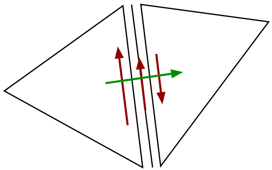


- Software, Examples, Conclusion

$H(\text{div})$-conforming elements for Navier Stokes

Trial and test functions:

- normal-cont., tangential-discont. velocity element functions, piecewise polynomial (degree k)
  u, v
- facet velocity functions for the tangential component only, piecewise polynomial (degree k)
  u_F^t, v_F^t
- discont. element pressure functions, piecewise polynomial (degree $k-1$)
  p, q



Hybrid DG Navier Stokes bilinearforms

Viscosity:

$$B\big((u,u_F),(v,v_F)\big) = \sum_T \Big\{ \int_T \nu \, \nabla u : \nabla v \; dx - \int_{\partial T} \nu \, \nabla u \cdot n \, (v^t - v_F^t) \; ds$$
$$- \int_{\partial T} \nu \, \nabla v \cdot n \, (u^t - u_F^t) \; ds + \int_{\partial T} \nu \tau_h (u^t - u_F^t) \cdot (v^t - v_F^t) \; ds \Big\}$$

Convection:

$$C\big(w;(u,u_F),(v,v_F)\big) = \sum_T \Big\{ - \int_T u \otimes w : \nabla v \; dx + \int_{\partial T} w_n u^{up} v \; ds + \int_{\partial T_{out}} w_n (u_F^t - u^t) v_F^t \; ds$$

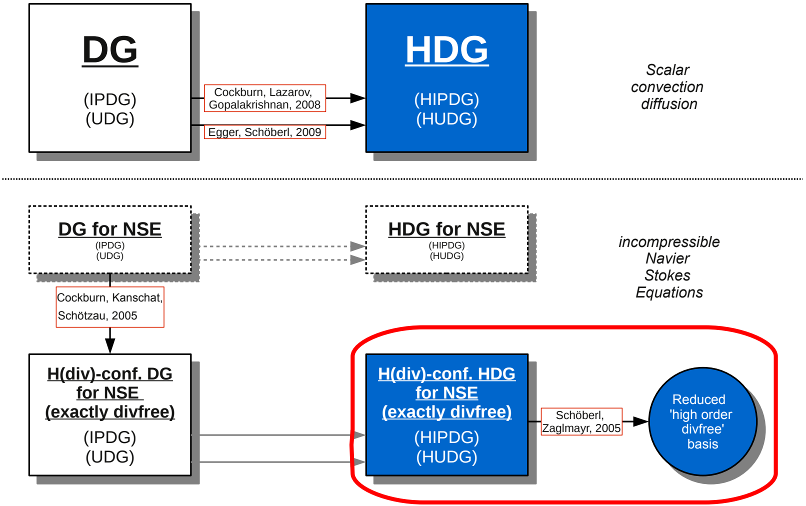
pressure / incompressibility constraint:

$$D\big((u,u_F),q\big) = \sum_T \int_T \operatorname{div}(u) q \; dx$$

$\Rightarrow$ weak incompressibility ($+$ $H(\text{div})$-conformity) $\Rightarrow$ exactly divergence-free solutions

Contents

- The methods:



- Software, Examples, Conclusion

Reduced basis $H(\mathrm{div})$-conforming Finite Elements

We use the construction of high order finite elements of [Schöberl+Zaglmayr, '05, Thesis Zaglmayr '06]:
It is founded on the **de Rham** complex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & S_h & \xrightarrow{\mathrm{div}} & Q_h
\end{array}
$$

$$1+3+a \qquad a+3+3+b \qquad b+3+1+c \qquad c+1$$

Leading to a natural separation of the space (for higher order)

$$
\begin{aligned}
W_{hp} &= W_{\mathcal{L}_1} + \mathrm{span}\{\varphi^W_{h.o.}\} \\
V_{hp} &= V_{\mathcal{N}_0} + \mathrm{span}\{\nabla\varphi^W_{h.o.}\} + \mathrm{span}\{\varphi^V_{h.o.}\} \\
S_{hp} &= S_{\mathcal{R}T_0} + \mathrm{span}\{\mathrm{curl}\,\varphi^V_{h.o.}\} + \mathrm{span}\{\varphi^S_{h.o.}\} \\
Q_{hp} &= Q_{\mathcal{P}_0} + \mathrm{span}\{\mathrm{div}\,\varphi^S_{h.o.}\}
\end{aligned}
$$

Reduced basis $H(\mathrm{div})$-conforming Finite Elements

We use the construction of high order finite elements of
[Schöberl+Zaglmayr, '05, Thesis Zaglmayr '06]:
It is founded on the **de Rham** complex

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \cup & & \cup & & \cup & & \cup \\ W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & S_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$

$$1+3+a \qquad a+3+3+b \qquad b+3+1+c \qquad c+1$$

Leading to a natural separation of the space (for higher order)

$$\begin{aligned} W_{hp} &= W_{\mathcal{L}_1} + \mathrm{span}\{\varphi^W_{h.o.}\} \\ V_{hp} &= V_{\mathcal{N}_0} + \mathrm{span}\{\nabla\varphi^W_{h.o.}\} + \mathrm{span}\{\varphi^V_{h.o.}\} \\ S_{hp} &= S_{\mathcal{R}T_0} + \mathrm{span}\{\mathrm{curl}\,\varphi^V_{h.o.}\} + \mathrm{span}\{\varphi^S_{h.o.}\} \\ Q_{hp} &= Q_{\mathcal{P}_0} + \mathrm{span}\{\mathrm{div}\,\varphi^S_{h.o.}\} \end{aligned}$$

Reduced basis $H(\mathrm{div})$-conforming Finite Elements

We use the construction of high order finite elements of
[Schöberl+Zaglmayr, '05, Thesis Zaglmayr '06]:
It is founded on the **de Rham** complex

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & S_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$

$1+3+a$ $\quad a+3+3+b$ $\quad b+3+1+c$ $\quad c+1$

Leading to a natural separation of the space (for higher order)

$$\begin{array}{rcl} W_{hp} & = & W_{\mathcal{L}_1} + \mathrm{span}\{\varphi^W_{h.o.}\} \\ V_{hp} & = & V_{\mathcal{N}_0} + \mathrm{span}\{\nabla\varphi^W_{h.o.}\} + \mathrm{span}\{\varphi^V_{h.o.}\} \\ S_{hp} & = & S_{\mathcal{R}T_0} + \mathrm{span}\{\mathrm{curl}\,\varphi^V_{h.o.}\} + \mathrm{span}\{\varphi^S_{h.o.}\} \\ Q_{hp} & = & Q_{\mathcal{P}_0} + \mathrm{span}\{\mathrm{div}\,\varphi^S_{h.o.}\} \end{array}$$

Reduced basis $H(\mathrm{div})$-conforming Finite Elements

We use the construction of high order finite elements of
[Schöberl+Zaglmayr, '05, Thesis Zaglmayr '06]:
It is founded on the **de Rham** complex

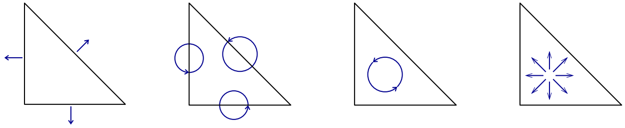
$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & S_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$

$1+3+a$ $\qquad a+3+3+b$ $\qquad b+3+1+c$ $\qquad c+1$

Leading to a natural separation of the space (for higher order)

$$\begin{aligned} W_{hp} &= W_{\mathcal{L}_1} + \mathrm{span}\{\varphi^W_{h.o.}\} \\ V_{hp} &= V_{\mathcal{N}_0} + \mathrm{span}\{\nabla\varphi^W_{h.o.}\} + \mathrm{span}\{\varphi^V_{h.o.}\} \\ {S_{hp}}^* &= S_{\mathcal{R}T_0} + \mathrm{span}\{\mathrm{curl}\,\varphi^V_{h.o.}\} \\ Q_{hp} &= Q_{\mathcal{P}_0} + \mathrm{span}\{\mathrm{div}\,\varphi^S_{h.o.}\} \end{aligned}$$

space separation in 2D



	RT_0 DOF		higher order edge DOF		higher order div.-free DOF		higher order DOF with nonz. div
	RT_0 shape fncs.		curl of H^1-edge fncs.		curl of H^1-el. fncs.		remainder
$\mathrm{div}(\Sigma_h^k) =$	$\bigoplus_T \mathcal{P}^0(T)$	$\oplus$	$\{0\}$	$\oplus$	$\{0\}$	$\oplus$	$\bigoplus_T [\mathcal{P}^{k-1} \cap \mathcal{P}^{0\perp}](T)$
#DOF $=$	3	$+$	$3k$	$+$	$\frac{1}{2}k(k-1)$	$+$	$\frac{1}{2}k(k+1)-1$

discrete functions have only piecewise constant divergence
$\Rightarrow$ only piecewise constant pressure necessary for exact incompressibility

Summary of ingredients and properties

So, we have presented a new Finite Element Method for Navier Stokes, with

- $H(\text{div})$-conforming Finite Elements
- Hybrid Discontinuous Galerkin Method for viscous terms
- Upwind flux (in HDG-sence) for the convection term

leading to solutions, which are

- locally conservative
- energy-stable ($\frac{d}{dt}\|u\|_{L_2}^2 \leq \frac{C}{\nu}\|f\|_{L_2}^2$)
- exactly incompressible
- static condensation
- standard finite element assembly is possible
- less matrix entries than for std. DG approaches
- reduced basis possible

Software, Examples and Conclusion

Software

Netgen

- Downloads: http://sourceforge.net/projects/netgen-mesher/
- Help & Instructions: http://netgen-mesher.wiki.sourceforge.net

NGSolve (including the presented scalar HDG methods)

- Downloads: http://sourceforge.net/projects/ngsolve/
- Help & Instructions: http://sourceforge.net/apps/mediawiki/ngsolve

NGSflow (including the presented methods)

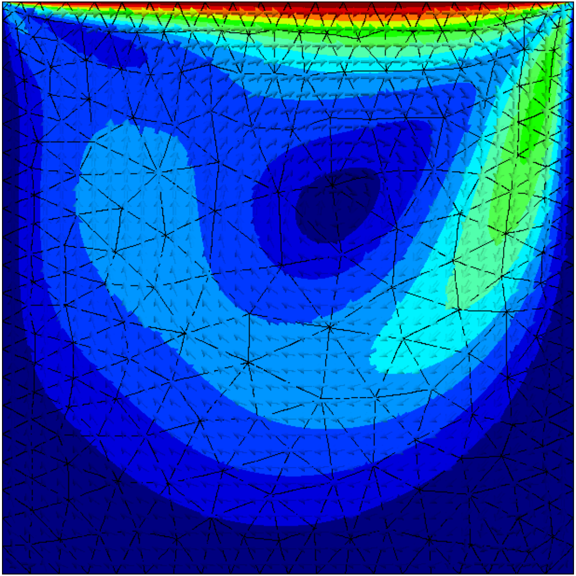
- Downloads: http://sourceforge.net/projects/ngsflow/
- Help & Instructions: http://sourceforge.net/apps/mediawiki/ngsflow

NGSflow is the flow solver add-on to NGSolve.
It includes:

- The presented Navier Stokes Solver
- A package solving for heat (density) driven flow

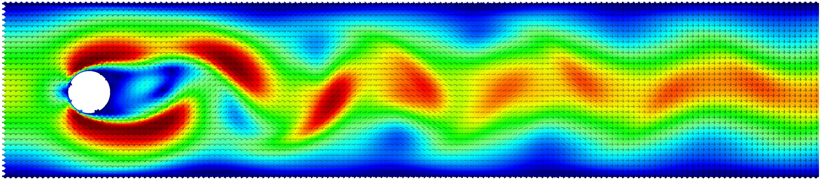
Examples

2D Driven Cavity ($u_{top} = 0.25$, $\nu = 10^{-3}$)

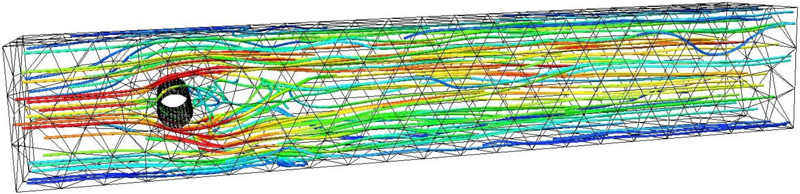


Examples

2D laminar flow around a disk (Re=100):



3D laminar flow around a cylinder (Re=100):



Heat driven flow

changes in density are small:

- incompressibility model is still acceptable
- changes in density just cause some buoyancy forces

Boussinesq-Approximation:

Incompressible Navier Stokes Equations are just modified at the force term:

$$f = g \quad \to \quad (1 - \beta(T - T_0))g \qquad \beta: \text{ heat expansion coefficient}$$

Convection-Diffusion-Equation for the temperature

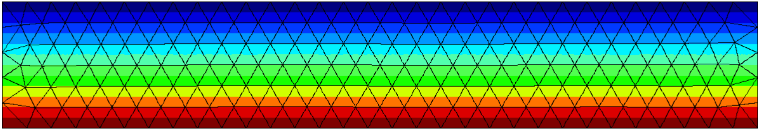
$$\frac{\partial T}{\partial t} + \mathrm{div}(-\lambda \nabla T + u \cdot T) = q$$

- Discretization of Convection Diffusion Equation with HDG method
- Weak coupling of unsteady Navier Stokes Equation and unsteady Convection Diffusion Equation
- Higher Order ($p \in 1, .., 5$) in time with IMEX schemes

Examples

Benard-Rayleigh example:
Top temperature: constant $20°C$
Bottom temperature: constant $20.5°C$

Initial mesh and initial condition ($p = 5$):



Conclusion and ongoing work

Conclusion:
efficient methods for

- scalar convection diffusion equations
- Navier Stokes Equations

using

- Hybrid DG
- $H(\mathrm{div})$-conforming Finite Elements
- reduced basis

related/future work:

- heat driven flow (Boussinesq Equations)
- time integration (IMEX schemes, ...)
- 3D Solvers (Preconditioners, ...)

Thank you for your attention!