Numerical Methods for Two-Phase Flows with Mass Transport

C. Lehrenfeld,
J. Berger, P. Esser, J. Grande, S. Groß, E. Loch, X. Xu, L. Zhang, Y. Zhang,
A. Reusken

IGPM, RWTH Aachen

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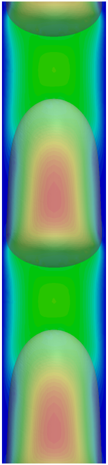
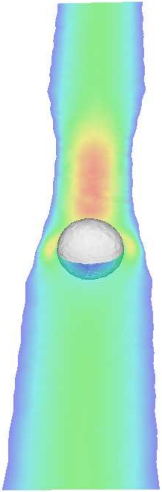
Problem classes



- Fluid dynamics for Two-Phase (immiscible) incompressible flows:
  (Navier-Stokes Eqns. + surface tension effects)
- with mass transport
  (convection-diffusion eqn. + Henry interface condition)
- and surfactant transport
  (convection-diffusion eqn. on the surface)

applications:

- falling film (gas-liquid)
- rising bubble (liquid-liquid)
- taylor flow (gas-liquid)



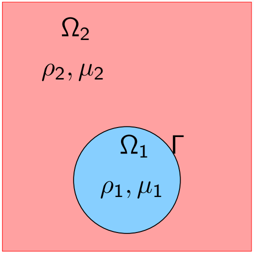
Processes at (close to) the interface!

Standard (sharp interface) model for Two-Phase flows

Domains: $\Omega_1 = \Omega_1(t)$ and $\Omega_2 = \Omega_2(t)$

Interface: $\Gamma = \Gamma(t) = \partial\Omega_1 \cap \partial\Omega_2$

ρ_i: density in Ω_i
μ_i: viscosity in Ω_i
τ: surface tension coefficient
$\mathbf{D}(\mathbf{w}) = \nabla\mathbf{w} + \nabla\mathbf{w}^T$, $\sigma = -p\,\mathbf{I} + \mu\mathbf{D}(\mathbf{w})$
κ: curvature



Navier-Stokes equations

$$\begin{cases} \rho_i\big(\mathbf{w}_t + (\mathbf{w}\cdot\nabla)\mathbf{w}\big) = \mathrm{div}\,(\sigma) + \rho_i\mathbf{g} & \text{in } \Omega_i \\ \mathrm{div}\,\mathbf{w} = 0 & \text{in } \Omega_i \end{cases} \quad \text{for } i = 1,2$$

$$[\![\sigma\mathbf{n}]\!]_\Gamma = \tau\kappa\mathbf{n} + \nabla_\Gamma\,\tau, \quad [\![\mathbf{w}]\!]_\Gamma = 0\ .$$

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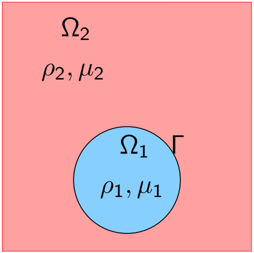

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$$[\![\sigma\mathbf{n}]\!]_\Gamma = \tau\kappa\mathbf{n} + \nabla_\Gamma\,\tau, \quad [\![\mathbf{w}]\!]_\Gamma = 0\ .$$

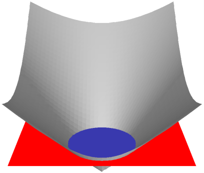
How to identify the interface?

Interface capturing: Level set approach

$\Gamma(t)$ = zero-level of a scalar function:

the level set function $\varphi(x, t)$

$$\varphi(x,t) = \begin{cases} < 0 & \text{for } x \text{ in phase } \Omega_1 \\ > 0 & \text{for } x \text{ in phase } \Omega_2 \\ = 0 & \text{at the interface} \end{cases}$$



should be an *"approximate signed distance function"*.

$$x(t) \in \Gamma(t) \quad \Rightarrow \quad \varphi(x(t), t) = 0.$$

Level set equation

$$\varphi_t + \mathbf{w} \cdot \nabla \varphi = 0 \quad \text{in } \Omega$$

Level set approach: Pros and Cons

Eulerian vs. Lagrangian

Level set description allows us to stay in a purely Eulerian framework.

Interface capturing (in contrast to interface tracking (e.g. ALE))

- + no remeshing
- + topology changes (breakups, coalescence) and large deformations
- − interface description implicitely (interface is not meshed)
- − tracking of evolution of particles at or close to interface

Level set (in contrast to VOF)

- + accuracy of (implicit) interface
- − no discrete mass conservation
- (+) signed distance property
- − signed distance property gets lost (re-initialization)

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Level set (in contrast to VOF)

- $+$ accuracy of (implicit) interface
- $-$ no discrete mass conservation (videos,JB)
- $(+)$ signed distance property
- $-$ signed distance property gets lost (re-initialization)

- no corr. (1 b.),
- single corr. (1 b.)
- single corr. (2 b.)
- multicomp. corr. (2 b.)

Model: Navier-Stokes + level set equation

Navier-Stokes equations coupled with level set equation ($\tau = const$)

$$
\begin{aligned}
\rho(\varphi)\Big(\mathbf{w}_t + (\mathbf{w}\cdot\nabla)\mathbf{w}\Big) - \operatorname{div}\Big(\mu(\varphi)\,\mathbf{D}(\mathbf{w})\Big) + \nabla p &= \rho(\varphi)\,g - \tau\,\kappa(\varphi)\,\delta_\Gamma \mathbf{n}_\Gamma \\
\operatorname{div}\mathbf{w} &= 0 \\
\varphi_t + \mathbf{w}\cdot\nabla\varphi &= 0
\end{aligned}
$$

where ρ, μ and $\kappa, \delta_\Gamma, \mathbf{n}_\Gamma$ depend on φ.

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\end{aligned}
$$

where ρ, μ and $\kappa, \delta_\Gamma, \mathbf{n}_\Gamma$ depend on φ.

Additional equations

Surfactant transport (on the interface):

$$
s_t + \mathbf{w}\cdot\nabla_\Gamma s - \mu_\Gamma \Delta_\Gamma s = f_\Gamma \quad \text{on } \Gamma
$$

Mass transport (discontinuity across the interface):

$$
u_t + \mathbf{w}\cdot\nabla u - \alpha\Delta u = f \qquad \text{in } \Omega_i
$$

$$
[\![\alpha \frac{\partial u}{\partial \mathbf{n}}]\!] = 0, \quad [\![\beta u]\!] = 0 \quad \text{on } \Gamma
$$

Discretization background

Starting point

- Adaptive tetrahedral mesh for $\Omega = \Omega_1 \cup \Omega_2$
- Conforming finite elements:
  - P2-P1 Taylor-Hood elements (continuous)
  - P2 for level set (continuous)
  - P1 for mass/surfactants transport (continuous)
- Method of lines time integration (implicit euler, etc..)

Numerical Challenges

Key problems

- Multiscale phenomena close to the interface.
- Highly nonlinear couplings between $\mathbf{w}$, φ, f_Γ, c, c_Γ.
- Interesting modeling aspects, e.g.: $\tau = \tau(c, c_\Gamma)$ (YZ) or coalescence (JB)
- Accurate resolution of unknown interface.
  Treatment of the level set equation (EL), volume correction (JB)
- Treatment of surface tension force (Laplace-Beltrami discretization)
- Treatment of discontinuous pressure
- Treatment of kinks in velocity (SG)
- Coupling of flow + interface dynamics + transport equations.
- Discretization of transport equation with (moving) discontinuities (CL)
- Discretization of transport equation on (moving) interface (JG) + (XZ)
- Efficiency/robustness of iterative solvers.
- High complexity: parallel solvers needed, (SG) + (PE)
- Higher order time integration (space-time formulation) (PE)

Interface force balance

Interface condition

We have the condition:

$$[\![\sigma \mathbf{n}]\!]_\Gamma = [\![(\mu D(\mathbf{w}) + pI)\mathbf{n}]\!]_\Gamma = \tau \kappa \mathbf{n} \quad (+\nabla_\Gamma \tau)$$

Typically:

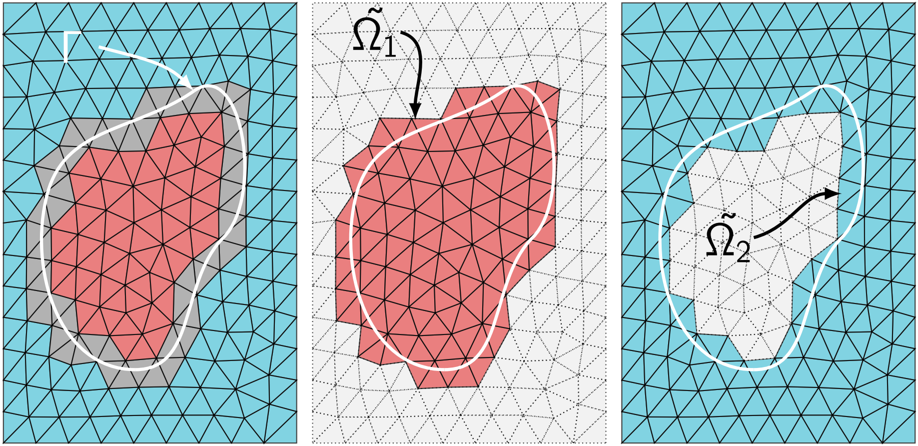
$$[\![p]\!]_\Gamma \neq 0$$

Non-aligned mesh + standard polynomial FE space Q_h
$\Rightarrow$ approximation quality reduces to

$$inf_{p_h \in Q_h} \|p - p_h\|_{L^2} \leq \mathcal{O}(\sqrt{h})$$

Handling Discontinuities: Domain-wise cont. ansatz

$$Q_h^\Gamma := Q_h(\tilde{\Omega}_1) \cdot H^\Gamma \oplus Q_h(\tilde{\Omega}_2) \cdot (1 - H^\Gamma)$$



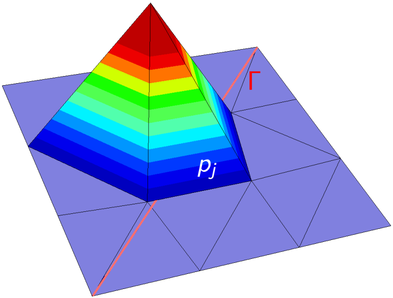
Handling Discontinuities: Add. ansatz functions (XFEM)

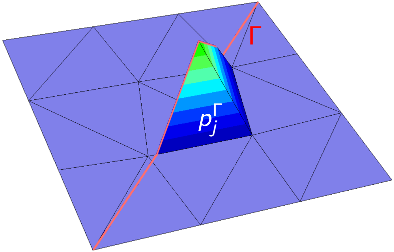
Remedy

Extend P_1 FE basis with discontinuous basis functions near Γ:

$$p_j^\Gamma := p_j \,(H_\Gamma(x) - H_\Gamma(x_j)), \quad H_\Gamma = \begin{cases} 1 & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases}$$

and use $Q_h^\Gamma = Q_h \oplus \{p_j^\Gamma\}$

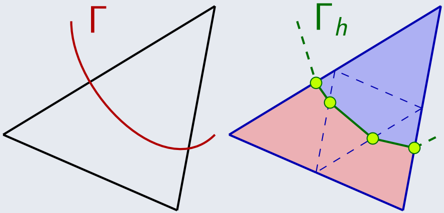




Handling (Pressure) Discontinuities: XFEM

Remarks

- In practice: Γ_h instead of Γ.
- $\dim(Q_h^\Gamma)$ depends on Γ.
- New basis functions can have very small supports.
- other applications:
  [Belytschko (1999 ->)]: elasticity,
  [Hansbo (2002 ->)]: interf. probl.,
  Two-Phase Mass Transport



P2-P1-Taylor-Hood + XFEM for pressure

Enrichment: we use a P1X finite element space for the pressure space.

(+) pressure approximation is optimal: $\mathcal{O}(h^2)$ in L^2-norm.

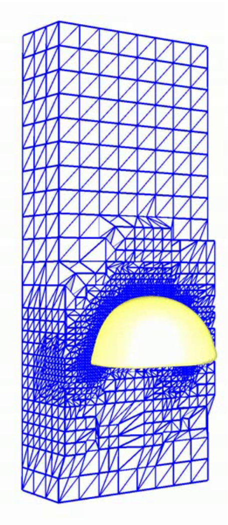
(-) no LBB-stability result for velocity-pressure pair (V_h, Q_h^Γ) any more

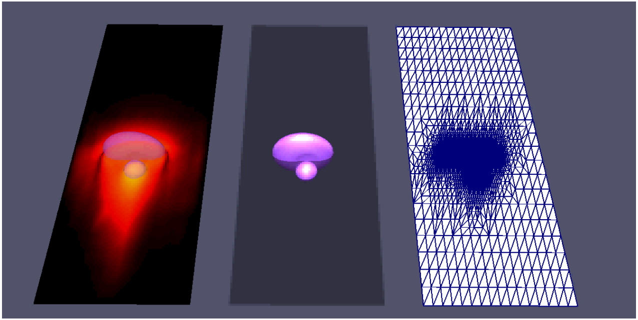
Kinks in velocity field also need a "kink-version" XFEM!

Key components in DROPS:

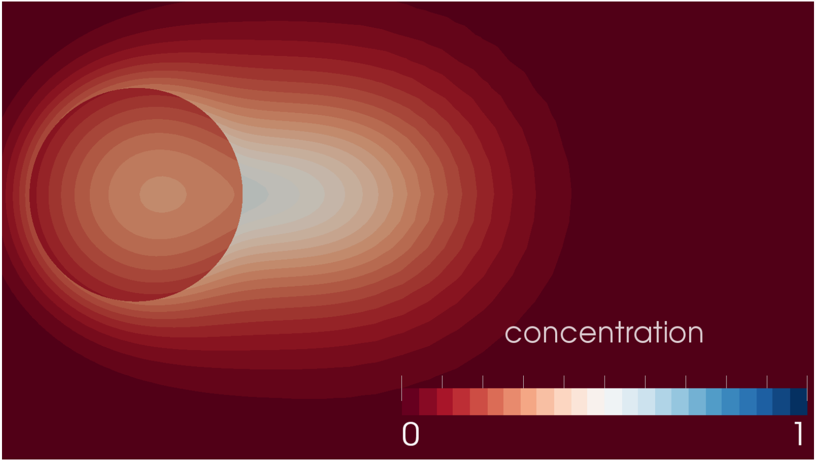
- Hybrid level set technique for interface capturing
- Adaptive multilevel triangulations (tetrahedra) + Finite Elements (not aligned to the interface)
- Special quadrature techniques (discontinuous μ, ρ)
- Discretization of localized force term f_Γ: Laplace-Beltrami
- Finite element space for discontinuous pressure: XFEM
- Special iterative solvers: robustness w.r.t. h, Δt, jumps in ρ, (μ)
- MPI based parallelization (Institute for Scientific Computing)
- Surfactant transport equation: Interface FE method
- Mass transport equation: Nitsche-XFEM method

Some DROPS movies

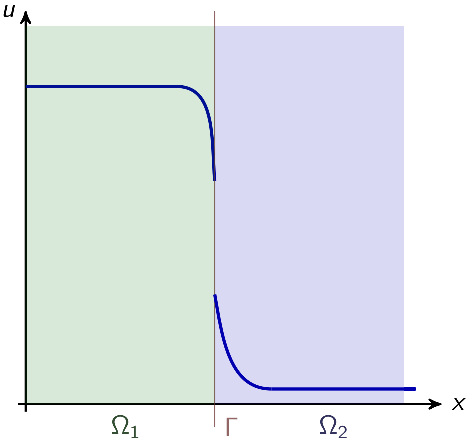




Two-Phase mass transport: A movie



Situation at the interface



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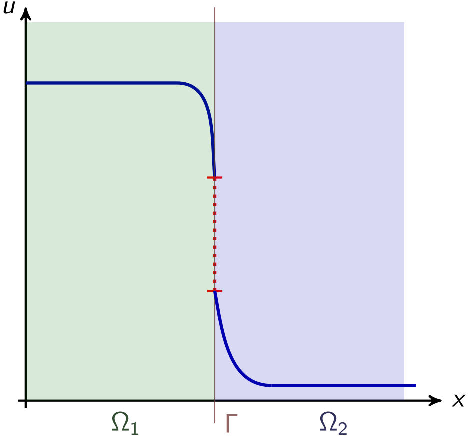

discontinuous conc.
$u_1/u_2 = \beta_2/\beta_1$

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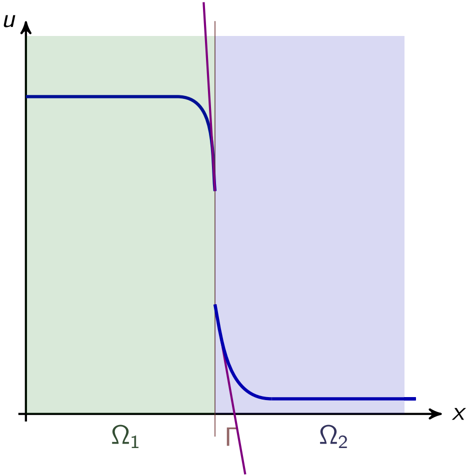

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disc. normal derivative
$\alpha_1 \frac{\partial u_1}{\partial \mathbf{n}_1} = \alpha_2 \frac{\partial u_2}{\partial \mathbf{n}_2}$

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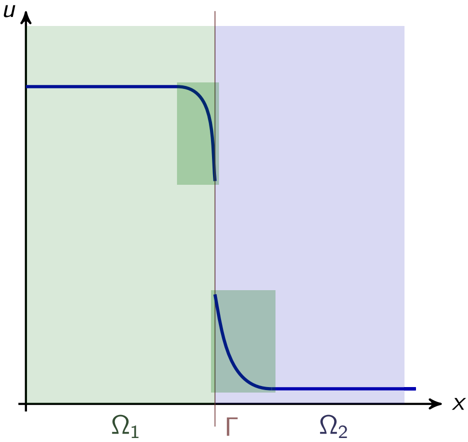

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sharp layers

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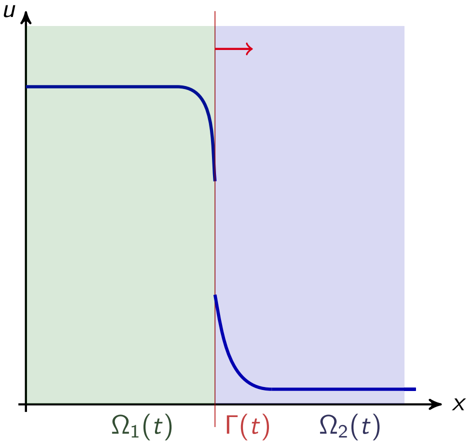

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sharp layers

moving interface $\Gamma(t)$

Mass transport

Mass transport equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u - \alpha \Delta u &= 0 && \text{in } \Omega_1 \cup \Omega_2, \\
[\![-\alpha \nabla u]\!] \cdot \mathbf{n} &= 0 && \text{on } \Gamma, \\
[\![\beta u]\!] &= 0 && \text{on } \Gamma.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V} \cdot \mathbf{n} &= \mathbf{w} \cdot \mathbf{n} && \text{on } \Gamma. \\
\operatorname{div}(\mathbf{w}) &= 0 && \text{in } \Omega
\end{aligned}
$$

u: concentration,
α: piecewise constant diffusion coefficients,
β: piecewise constant Henry coefficients,
$\mathbf{w}$: convection velocity (from Navier Stokes)
$\mathcal{V}$: the interface velocity
Henry condition: discontinuity in u.

Numerical Aspects

Mass transport equation

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Numerical Challenges

- Level set to capture the interface
  $\Rightarrow$ Interface is not aligned with the mesh (might depend on time)
- concentration has discontinuities (approximation)
- time integration for (non-matched) moving interfaces
- problem is typically highly convection dominated (stability)

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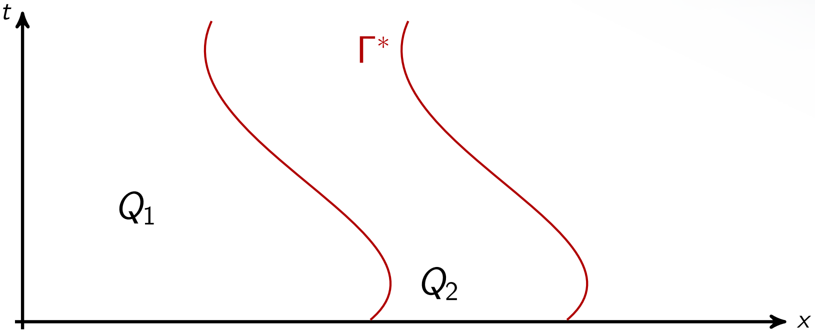
Numerical Approaches

- Extended Finite Element space (XFEM)
- Space-time formulation on each time slab
- Nitsche-type technique to enforce Henry's law in a weak sense
- (Space-time Streamline Diffusion) Stabilization

Space-time weak formulation I



Anisotropic Spaces; $Q = Q_1 \cup Q_2$

$$V_\beta = \left\{ u \in L^2(Q) \mid u_i \in H^{1,0}(Q_i),\ i = 1,2,\ u_{|\partial\Omega} = 0,\ [\![\beta u]\!]_{\Gamma_*} = 0 \right\}$$

$$W_\beta = \left\{ v \in V_\beta \mid \frac{\partial v}{\partial t} \in H_0^{1,0}(Q)' \right\}.$$

Space-time weak formulation II

Well-posed weak formulation [Gross/Reusken 11]

Determine $u \in W_\beta$ with $u(\cdot, 0) = 0$ such that

$$\frac{\partial u}{\partial t}(v) - \int_Q u\mathbf{w} \cdot \nabla v \, dx \, dt + \sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla v \, dx \, dt = \int_Q fv \, dx \, dt$$

for all $v \in H_0^{1,0}(Q) \neq V_\beta$

Remarks:

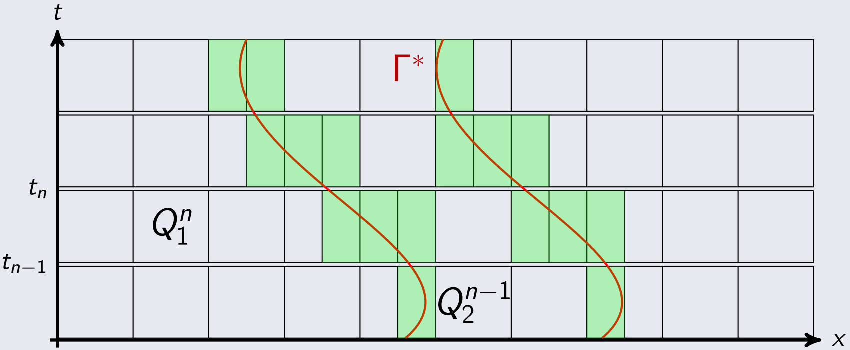
- Space-time (n+1 dimensional) formulation
- Trial functions are discontinuous across Γ_*, test functions are not
- Condition $[\![\beta u]\!]_\Gamma = 0$ essential condition in space W_β

Nitsche-DG-XFEM discretization

Space-time FE

$I_n = (t_{n-1}, t_n]$, $Q^n = \Omega \times I_n$. V_n : standard FE space on Ω.

$$W_n := \{\, v : Q^n \to \mathbb{R} \mid v(x,t) = \phi_0(x) + t\phi_1(x), \quad \phi_0, \phi_1 \in V_n \,\}$$
$$W := \{\, v : Q \to \mathbb{R} \mid v_{|Q^n} \in W_n \,\} \quad \text{(space-time FE).}$$

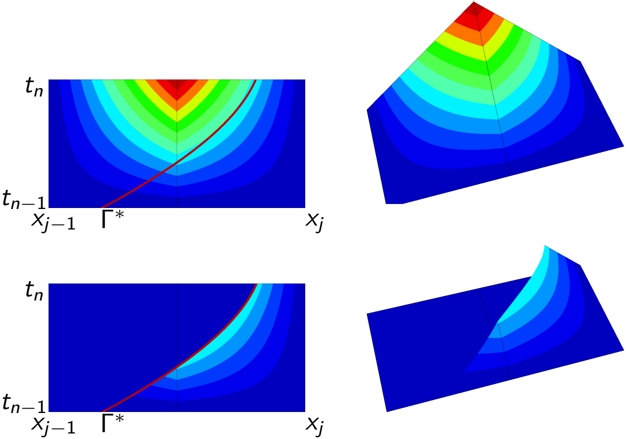


Approximating discontinuities

Space-time FE ⇒ Space-time XFEM

$$Q_i^n := \cup_{t \in I_n} \Omega_i(t), \quad R_i^n : \quad \text{restriction to } Q_i^n$$

$$W_n^\Gamma := R_1^n W_n \oplus R_2^n W_n, \quad W^{\Gamma_*} := \{\, v : Q \to \mathbb{R} \mid v_{|Q^n} \in W_n^\Gamma \,\}$$



Bilinear forms (within time slab Q^n) I

Conforming part (strong form + part. int. on diffusion)

$$a^n(u,v) = \sum_{i=1}^{2} \int_{Q_i^n} \Big(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i\Big)\beta_i v_i + \alpha_i \beta_i \nabla u_i \cdot \nabla v_i \, dx \, dt$$

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Discontinuous Galerkin Upwind w.r.t. time:

$$d^n(u,v) = \int_{\Omega} \beta(\cdot, t_n)[u]^{n-1} v_+^{n-1} \, dt$$

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Discontinuous Galerkin Upwind w.r.t. time:

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Henry interface condition

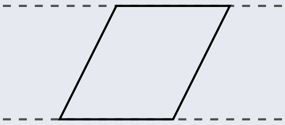
$$W_n^{\Gamma} \not\subset W_\beta \quad \text{(non-conformity)}$$

$\Rightarrow$ enforce condition $[\![\beta u]\!]_{\Gamma^*} = 0$ only weakly (Nitsche)

Bilinear forms (within time slab Q^n) II

Nitsche method for Henry condition:

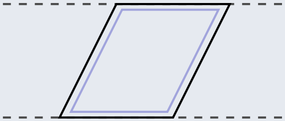
$$-\int_{Q^n} \beta \mathrm{div}(\alpha \nabla u) v \, dx$$



Bilinear forms (within time slab Q^n) II

Nitsche method for Henry condition:

$$-\int_{Q^n} \beta \mathrm{div}(\alpha \nabla u) v \, dx = \int_{Q^n} \alpha\beta \nabla u \nabla v \, dx - \int_{\partial Q^n} \begin{pmatrix} \alpha \nabla u \\ 0 \end{pmatrix} \cdot (\mathbf{n}^*) \, v \, ds$$

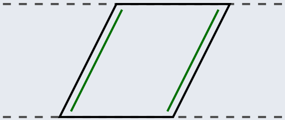


Bilinear forms (within time slab Q^n) II

Nitsche method for Henry condition:

$$-\int_{Q^n} \beta \operatorname{div}(\alpha \nabla u) v \, dx = \int_{Q^n} \alpha\beta \nabla u \nabla v \, dx - \int_{\partial Q^n} \begin{pmatrix} \alpha \nabla u \\ 0 \end{pmatrix} \cdot (\mathbf{n}^*) \, v \, ds$$

$$= \quad \ldots - \int_{\Gamma_*^n} \nu \alpha \nabla u \cdot \mathbf{n} \beta v \, ds$$

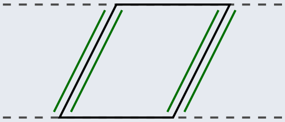


$\nu = 1/\sqrt{1 + (\mathbf{w} \cdot \mathbf{n})^2}$: $\int_{t^{n-1}}^{t^n} \int_{\Gamma^n} f \, ds \, dt = \int_{\Gamma_*^n} \nu f \, ds,$

Bilinear forms (within time slab Q^n) II

Nitsche method for Henry condition:

$$
\begin{aligned}
& -\int_{Q^n} \beta \mathrm{div}(\alpha \nabla u) v \, dx = \int_{Q^n} \alpha\beta \nabla u \nabla v \, dx - \int_{\partial Q^n} \begin{pmatrix} \alpha \nabla u \\ 0 \end{pmatrix} \cdot (\mathbf{n}^*) \, v \, ds \\
= \quad & \ldots - \int_{\Gamma_*^n} \nu \alpha \nabla u \cdot \mathbf{n} \beta v \, ds \\
\rightarrow \quad & \ldots - \sum_{i=1}^{2} \int_{\Gamma_*^n} \nu \{\alpha \nabla u \cdot \mathbf{n}\}_{\Gamma_*^n} \beta_i v \, ds \\
= \quad & \ldots - \int_{\Gamma_*^n} \nu \{\alpha \nabla u \cdot \mathbf{n}\}_{\Gamma_*^n} [\beta v]_{\Gamma_*^n} \, ds \quad \text{A}
\end{aligned}
$$



with $\{\cdot\}_{\Gamma_*^n}$ suitable volume weighted average.

$\nu = 1/\sqrt{1 + (\mathbf{w} \cdot \mathbf{n})^2}$: $\int_{t^{n-1}}^{t^n} \int_{\Gamma^n} f \, ds \, dt = \int_{\Gamma_*^n} \nu f \, ds$,

Bilinear forms (within time slab Q^n) II

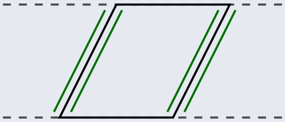
Nitsche method for Henry condition:

$$-\int_{Q^n} \beta \mathrm{div}(\alpha \nabla u) v \, dx = \int_{Q^n} \alpha\beta \nabla u \nabla v \, dx - \int_{\partial Q^n} \begin{pmatrix} \alpha \nabla u \\ 0 \end{pmatrix} \cdot (\mathbf{n}^*) \, v \, ds$$

$$= \quad \ldots - \int_{\Gamma_*^n} \nu \alpha \nabla u \cdot \mathbf{n} \beta v \, ds$$

$$\rightarrow \quad \ldots - \sum_{i=1}^{2} \int_{\Gamma_*^n} \nu \{\alpha \nabla u \cdot \mathbf{n}\}_{\Gamma_*^n} \beta_i v \, ds$$

$$= \quad \ldots - \int_{\Gamma_*^n} \nu \{\alpha \nabla u \cdot \mathbf{n}\}_{\Gamma_*^n} [\beta v]_{\Gamma_*^n} \, ds \quad \text{A}$$



$$\longrightarrow \quad - \int_{\Gamma_*^n} \nu \{\alpha \nabla v \cdot \mathbf{n}\}_{\Gamma_*^n} [\beta u]_{\Gamma_*^n} \, ds \quad \text{B} \quad + \lambda h_n^{-1} \int_{\Gamma_*^n} \nu [\beta u]_{\Gamma_*^n} [\beta v]_{\Gamma_*^n} \, ds, \quad \text{C}$$

with $\{\cdot\}_{\Gamma_*^n}$ suitable volume weighted average. $\lambda > 0$: stabilization parameter

and $\nu = 1/\sqrt{1 + (\mathbf{w} \cdot \mathbf{n})^2}$: $\int_{t^{n-1}}^{t^n} \int_{\Gamma^n} f \, ds \, dt = \int_{\Gamma_*^n} \nu f \, ds$,

$$N_{\Gamma_*}^n(c, v) := \text{A} + \text{B} + \text{C}$$

Nitsche-DG-XFEM variational problem

global bilinearforms

$$a(u, v) = \sum_{n=1}^{N} a^n(u, v), \quad \text{similarly :} \quad d(u, v), \; N_{\Gamma_*}(u, v).$$

Discrete problem

Determine $U \in W^{\Gamma_*}$ such that

$$B(U, V) = f(V) \quad \text{for all} \quad V \in W^{\Gamma_*},$$
$$B(U, V) := a(U, V) + d(U, V) + N_{\Gamma_*}(U, V).$$

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Still allows for time stepping, i.e. solving time slab by time slab!

Error analysis (results)

Theorem

Error analysis for linear (space+time) FE

$$\|(u-U)(\cdot,t_N)\|_{L^2(\Omega^N)} \leq c(h^2+\Delta t^2).$$

Error analysis (results)

Theorem

Error analysis for linear (space+time) FE

$$\|(u-U)(\cdot,t_N)\|_{L^2(\Omega^N)} \leq c(h^2+\Delta t^2).$$

Remark:

for standard space-time DG [V. Thomee] (no Nitsche, no XFEM):

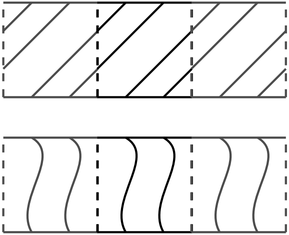
$$\|(u-U)(\cdot,t_N)\|_{L^2(\Omega^N)} \leq c(h^2+\Delta t^3)$$

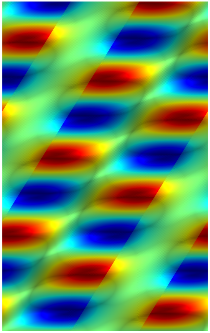
Concept of V. Thomee needs tensor product decomposition of the spaces which we don't have for moving interfaces.

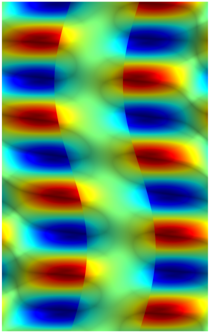
Numerical experiment

Numerical example in (1+1)D

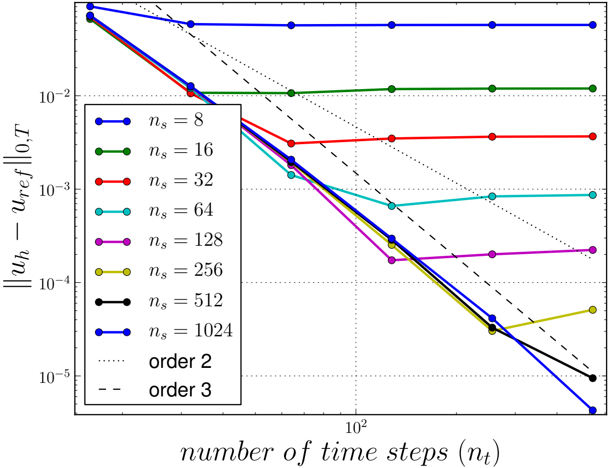
Diffusion dominates, periodic boundary conditions, artificial source terms

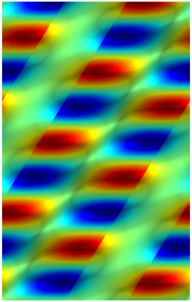






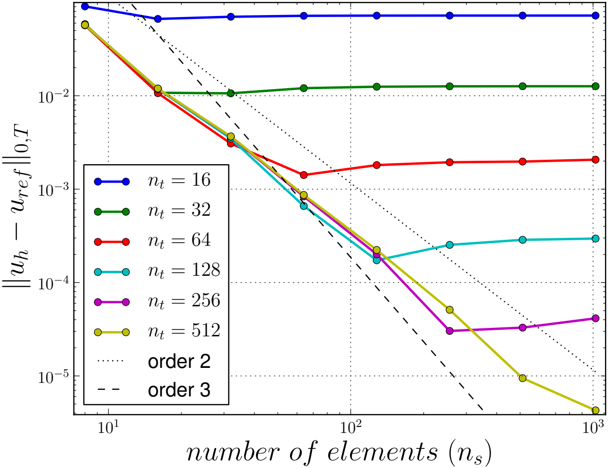
Discretization error (temporal convergence)

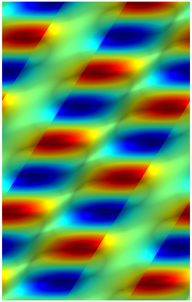




This indicates: $\|(U-u)(\cdot,t_N)\|_{L^2(\Omega)} \sim \Delta t^3$ **if** h **sufficiently small.**

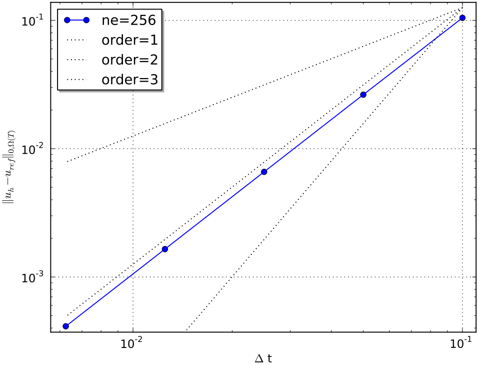
Discretization error (spatial convergence)

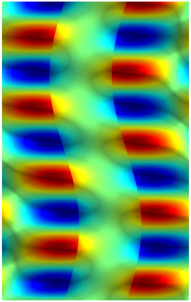




This indicates: $\|(U-u)(\cdot, t_N)\|_{L^2(\Omega)} \sim h^2$ if Δt sufficiently small.

Numerical example (continued): Non-planar interface





This indicates: $\|(U-u)(\cdot,t_N)\|_{L^2(\Omega)} \sim \cancel{\Delta t^3}\ \Delta t^2$ **if** h **sufficiently small.**

With sufficiently fine quadrature we reobtain $\|(U-u)(\cdot,t_N)\|_{L^2(\Omega)} \sim \Delta t^3$**.**

Remark on 3+1 dimensions

Quadrature in (3+1)D

- Tringulation consists of prism-4 elements
- Decompose prism-4 into four pentatopes (simplex-4)
- Decompose cut-pentatopes into uncut pentatopes (approximation of Γ_*)
- 4D quadrature on uncut pentatopes

(!) For realistic problems quadrature limits convergence to $\mathcal{O}(\Delta t^2)$

Implementation and numerical results

- First implementation in 3+1 dimesions exists
- At least second order convergence is validated
- For simple examples even third order is observed

Conclusion

Interface capturing and consequences

- Level-set approach, Eulerian frame
- Non-matching interface needs special care for space discretization
- Moving discontinuities needs special care for time discretization

Discretization concepts

- (Domain-)piecewise continuous approximation + XFEM
- Nitsche(-XFEM) for interface conditions
- Space-time FEM with (P1) Discontinuous Galerkin in time.
  [**space-time integrals** $\Rightarrow$ *Composite quadrature in (n+1) D*]

Conclusion

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  [**space-time integrals** $\Rightarrow$ *Composite quadrature in (n+1) D*]

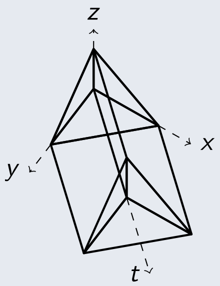
Thank you for your attention!

Decomposition into simplices

The reference element can be decomposed into simplices

$$
\begin{array}{llcl}
(1+1): & \hat{Q} \text{ is a square} & \rightarrow & 2 \text{ triangles} \\
(2+1): & \hat{Q} \text{ is a (regular) prism} & \rightarrow & 3 \text{ tetrahedra} \\
(3+1): & \hat{Q} \text{ is a (regular) prism-4} & \rightarrow & 4 \text{ pentatopes}
\end{array}
$$

The reference prism-4 $\hat{Q}$



Decomposition into simplices

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$(1+1)$:	$\hat{Q}$ is a square	$\rightarrow$	2 triangles
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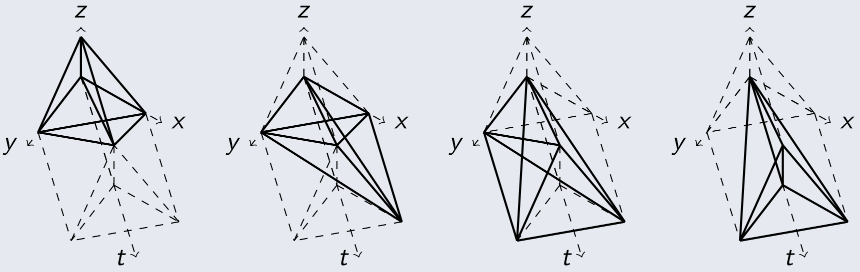
The reference prism-4 $\hat{Q}$ and it's decomposition



Decomposition into simplices

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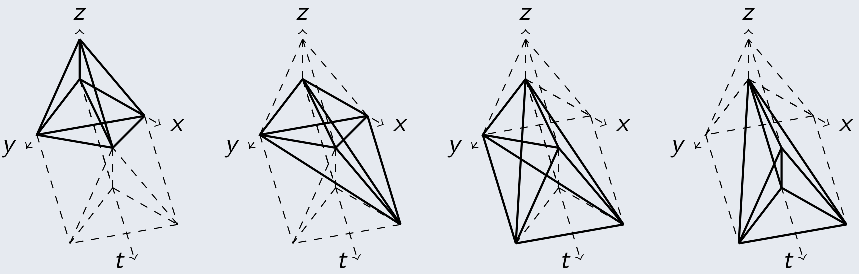
$$\begin{array}{lllll} (1+1): & \hat{Q} \text{ is a square} & \rightarrow & 2 \text{ triangles} \\ (2+1): & \hat{Q} \text{ is a (regular) prism} & \rightarrow & 3 \text{ tetrahedra} \\ (3+1): & \hat{Q} \text{ is a (regular) prism-4} & \rightarrow & 4 \text{ pentatopes} \end{array}$$

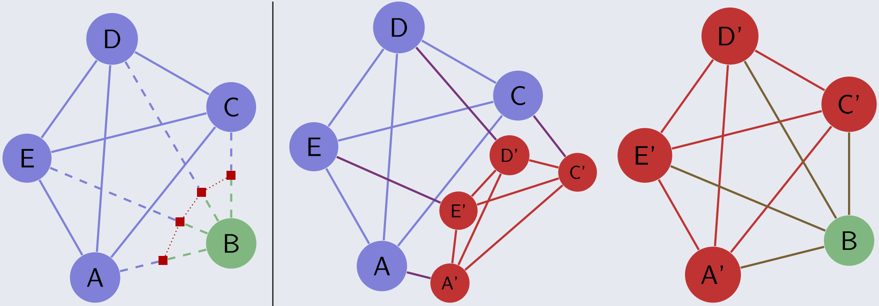
The reference prism-4 $\hat{Q}$ and it's decomposition



Consider the quadrature problem on (n+1)-simplex!

Slicing the pentatope

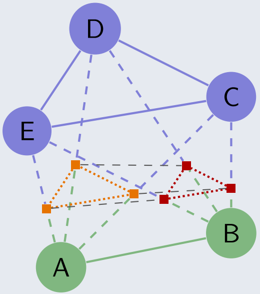
(Non-degenerated) Case 1:

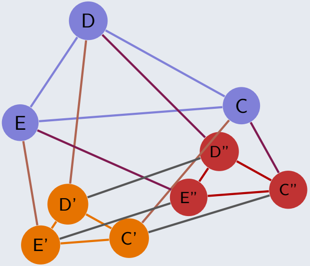


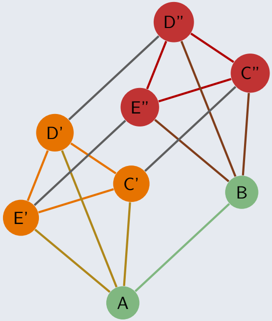
cut pentatope	(irregular) prism-4	+	pentatope
	↓ (decomposition)		↓
	4 pentatopes	+	1 pentatope
interface	1 tetrahedra		

Slicing the pentatope

(Non-degenerated) Case 2:



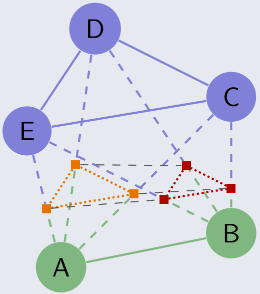


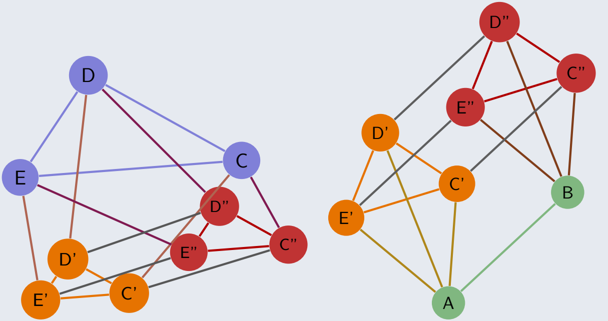


cut pentatope	(irregular) hypertriangle ↓ (decomposition) 6 pentatopes	+ +	(irregular) prism-4 ↓ 4 pentatopes
interface	prism-3 → 3 tetrahedra		

Slicing the pentatope

(Non-degenerated) Case 2:



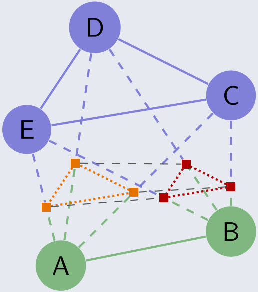


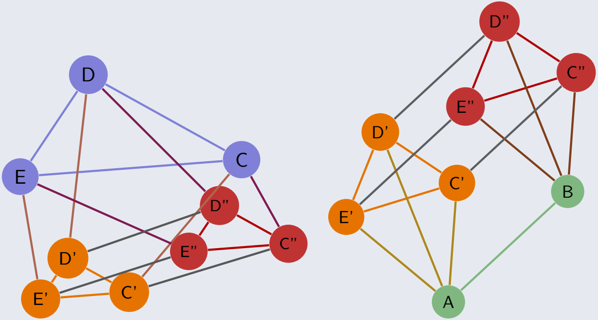
cut pentatope	(irregular) hypertriangle ↓ (decomposition) 6 pentatopes	+ (irregular) prism-4 ↓ + 4 pentatopes
interface	prism-3 → 3 tetrahedra	

Step 4: Decomposition into one-phase (n+1)-simplices and n-simplices!

Slicing the pentatope

(Non-degenerated) Case 2:





cut pentatope	(irregular) hypertriangle + (irregular) prism-4 ↓ (decomposition) ↓ 6 pentatopes + 4 pentatopes
interface	prism-3 → 3 tetrahedra

Step 4: Decomposition into one-phase (n+1)-simplices and n-simplices!

Decomposition of the hypertriangle into 6 pentatopes