# Analysis of Composite Wave Curves for Non–Convex Systems of Conservation Laws

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ABSTRACT. The paper is concerned with the analysis of the so-called composite curve. This is a special type of curve in phase space which arises in the construction of the solution of a Riemann problem for a system of strictly hyperbolic conservation laws which exhibits non–genuinely nonlinear characteristic fields. It will be shown by means of a bifurcation approach that this curve uniquely exists near a degeneration point of the corresponding characteristic field. Additionally the main result also states that it is tangentially connected to the admissible part of the rarefaction curve breaking off at this point. The proof is mainly based on Liapunov-Schmidt reduction which reduces the system to an equivalent scalar equation. The result applies to central models in mathematical continuum mechanics such as nonlinear elasticity, magnetohydrodynamics and equilibrium hydrodynamics.

## 1. Introduction

We are concerned with the Riemann problem for a system of conservation laws

$$u_t + f(u)_x = 0 \tag{1}$$

and piecewise constant initial data

$$u(0,x) = \begin{cases} u_l & , \quad x < 0 \\ u_r & , \quad x > 0 \end{cases} \tag{2}$$

for two states $u_l$, $u_r$ in the admissible phase space $\mathcal{D} \subset \mathbb{R}^n$. Here $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{D}$ denotes the vector of $n$ conservative quantities and $f : \mathcal{D} \to \mathbb{R}^n$ is the flux vector which is supposed to be sufficiently smooth, i.e., $f \in C^3(\mathcal{D})$.

The Riemann problem has been subject of extensive research. Significant contributions concerning the construction of its solution have been made by Lax [4],

This work has been supported by the DFG in the Priority Research Program "Analysis and Numerics of Conservation Laws".

Gelfand [2], Wendroff [8] and Liu [5]. In the sequel, we consider the general frame as in [5]. For this purpose, we assume that the system of conservation laws (1) is *strictly hyperbolic*, i.e., there is a complete set of eigenvalues $\lambda_k(u)$, $k = 1, \ldots, n$, of the Jacobian of $f$ such that

$$\lambda_1(u) < \ldots < \lambda_n(u) \qquad \forall\, u \in \mathcal{D}.$$

The corresponding right eigenvectors $r_k(u)$ and left eigenvectors $l_k(u)$ satisfy

$$l_k^T(u)\; r_j(u) = \delta_{kj} \quad 1 \le k, j \le n, \quad \forall\, u \in \mathcal{D}.$$

The eigenvalues $\lambda_k$ denote the *characteristic velocities* corresponding to the *characteristic k–field* which is characterized by the *nonlinearity factor*

$$\mathcal{X}_k(u) := (\nabla_u \lambda_k(u))^T\, r_k(u) \qquad u \in \mathcal{D}.$$

Whenever $\mathcal{X}_k$ vanishes for all $u \in \mathcal{D}$ the $k$–field is called *linearly degenerated.* However, if $\mathcal{X}_k$ is not equal to zero in the admissible phase space, the $k$–field is called *genuinely nonlinear.* In the classical case the $k$–field is supposed to be either linearly degenerated or genuinely nonlinear. Here, we also consider the case of a *non–genuinely nonlinear field*, i.e., $\mathcal{X}_k$ locally vanishes at certain points of the phase space. To be precise, we assume that there is an $(n-1)$–dimensional hypersurface $\mathcal{M} \subset \mathcal{D}$ such that

i) the characteristic $k$-field $\mathcal{X}_k$ vanishes on $\mathcal{M}$, i.e., $\mathcal{X}_k(u) = 0$ for all $u \in \mathcal{M}$,
ii) $\mathcal{X}_k$ is simply degenerated on $\mathcal{M}$, i.e.,

$$\mathcal{X}'_k(u) := (\nabla_u \mathcal{X}_k(u))^T\, r_k(u) \neq 0 \qquad u \in \mathcal{M}. \tag{3}$$

A state $u \in \mathcal{M}$ is called a degeneration point of the nonlinear $k$-field.

In this frame Liu [5] outlined the general principles for constructing the solution of the Riemann problem. In particular, he introduced the *composite curve* near degeneration points of nonlinear fields which is the composite of a rarefaction curve and certain states located on shock curves. In order to describe the composite curve we first introduce some notations, starting with a rarefaction curve. All states of a $k$-rarefaction curve lie on a trajectory satisfying the ordinary differential equation

$$u'(\xi) = r_k(u(\xi)) \tag{4}$$

where $\xi$ is an appropriately chosen parameter. Note that the rarefaction curve is uniquely characterized by imposing an initial value. Moreover we conclude from (3) and (4) that the rarefaction curve intersects the hypersurface $\mathcal{M}$ only at isolated points.

Furthermore we introduce the Hugoniot locus $\mathcal{H}(u^*)$ which is composed of all states $u \in \mathcal{D}$ such that the *Rankine–Hugoniot jump condition*

$$\sigma(u - u^*) = f(u) - f(u^*) \tag{5}$$

holds. Here $\sigma = \sigma(u^*, u)$ denotes the speed of the discontinuity. In [4] Lax verified that there is a family of $n$ one–parametric smooth curves all satisfying the jump conditions (5) and $\lim_{u \to u^*} \sigma_k(u^*, u) = \lambda_k(u^*)$. Since the system is supposed to be

strictly hyperbolic, these curves are usually enumerated with increasing velocity $\lambda_k(u^*)$. The corresponding Hugoniot curve is denoted by $\mathcal{H}_k(u^*)$.
Finally the composite curve $\mathcal{C}_k$ can be determined by means of the $k$–rarefaction curve denoted by $\mathcal{R}_k$ and the Hugoniot curves $\mathcal{H}_k(u^*)$ emanating from a state on the rarefaction curve. To be precise, for any state $u^* \in \mathcal{R}_k$ near $\bar{u} \in \mathcal{M}$ we move along $\mathcal{H}_k(u^*)$ until we reach the first state $u \in \mathcal{H}_k(u^*)$ with $u \neq u^*$ satisfying the sonic condition

$$\sigma(u^*, u) = \lambda_k(u^*). \tag{6}$$

According to Liu's definition in [5] $\mathcal{C}_k$ is the set of all these states $u$. Liu proved that the shock curve $\mathcal{H}_k(u^*)$ is tangent to the composite curve $\mathcal{C}_k$ at $u$ and tangent to $\mathcal{R}_k$ at $u^*$. In particular, he verified that the shocks corresponding to the states on the composite curve satisfy the physical entropy condition. The situation is sketched in Figure 1.

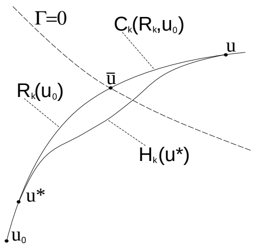


FIGURE 1. Composite: $\mathcal{R}_k \to \mathcal{C}_k$

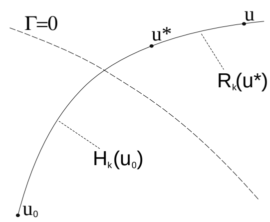


FIGURE 2. Composite: $\mathcal{H}_k \to \mathcal{R}_k$

The admissible part of a rarefaction curve is characterized by increasing wave speed $\lambda$. This guarantees the non-folding of the rarefaction fan in the $x$-$t$ plane. In case of $\lambda$ reaches a maximum a composite starts. Note that the construction of this composite curve is admissible as long as two conditions hold: first, the corresponding state $u^*$ on the rarefaction branch does not coincide with the state $u_0$, and second, $\sigma(u^*, u)$ does not equal $\lambda(u)$, or equivalent, $\sigma$ is decreasing along the corresponding shock curves. The second condition indicates a double sonic shock. Then the wave curve has to be continued with a rarefaction.

Finally, we would like to remark that there is the possibility of sonic states without having started a composite curve. To this end, we start with a shock curve. Then there might be a sonic state provided that the shock curve intersects the hypersurface $\mathcal{M}$ at an earlier state. This situation is sketched in Figure 2. Note that this configuration is obtained by reversing the roles of $u$ and $u_0$ in Figure 1.

In the present work we consider an issue that was not investigated in Liu's work for general systems, see [5]. This concerns the *unique existence* of the composite curve at a degeneration point $\overline{u}$. For convenience the following definition is introduced.

**Definition 1.1.** *Let $k$ denote a nonlinear field that degenerates at an isolated state $\bar{u} \in \mathcal{M}$, i.e., $\chi_k(\overline{u}) = 0$. Furthermore let $u^*(s) : [-\varepsilon, 0] \to \mathbb{R}^n$ a parameterized*

*$k$–rarefaction curve characterized by the initial value problem*

$$u^*_s(s) = r_k(u^*(s)), \quad u^*(0) = \bar{u}.$$

*Then the composite curve is composed of all roots $u \in \mathcal{D}$ of the function*

$$\Phi(u,s) := f(u) - f(u^*(s)) - \lambda(u^*(s))(u - u^*(s)) = 0 \tag{7}$$

*with*

$$u \neq u^*(s) \quad \textit{for} \quad s \neq 0. \tag{8}$$

Now the main result of this paper reads:

**Theorem 1.2.**
*Let $\overline{u} \in \mathcal{M}$ be a simple degeneration point. Then there uniquely exists a composite curve in a local neighborhood of $\overline{u}$. In particular, if we are approaching locally $\overline{u}$ on the rarefaction curve, then the corresponding state on the composite curve approaches $\overline{u}$ as well.*

The proof of this theorem is given in Section 2. First of all, we consider the scalar case. This will give us some inside to the underlying techniques from bifurcation theory. For systems of conservation laws we reduce the $n$ dimensional problem in $n+1$ unknowns to a scalar problem of two unknowns by means of the Liapunov–Schmidt reduction.

Finally, we would like to remark that several models in mathematical continuum mechanics such as nonlinear elasticity and equilibrium hydrodynamics, fit into the frame of strictly hyperbolic conservation laws with non–genuinely nonlinear characteristic fields, cf. [9]. Another example is related to magnetohydrodynamics, see [7]. A general class of systems is considered in [1].

## 2. Existence of the Composite Curve

### 2.1. The Scalar Case

First of all, we consider a single conservation law. In the scalar case the setting reduces to $\lambda = f'$, $r = 1$, $\chi = f''$ and $\chi' = f'''$. The simplest example that fits into the setting is the cubic flux function $f(u) = u^3$. There is an inflection point $\overline{u} = 0$ and, in particular, $f'''(\overline{u}) \neq 0$. According to Definition 1.1 the composite locus is characterized by the roots of

$$\Phi(u,s) = u^3 - s^3 - 3\,s^2(u - s) = (u - s)^2\,(u + 2s) = 0 \tag{9}$$

where the rarefaction curve is the identity, i.e., $u^*(s) = s$. The double root $u = s$ is the trivial solution of (7) which is of no interest because of (8). There is exactly one non–trivial solution $u(s) = -2\,s$. We note that the derivatives of the function $\Phi$ satisfy the following conditions at the point $(x,s) = (0,0)$ denoted by a bar:

$$\begin{aligned} &\bar{\Phi} = \bar{\Phi}_x = \bar{\Phi}_s = 0, \quad \bar{\Phi}_{xx} = \bar{\Phi}_{xs} = \bar{\Phi}_{ss} = 0, \\ &\bar{\Phi}_{xxx} = 6,\ \bar{\Phi}_{xxs} = 0,\ \bar{\Phi}_{xss} = -6,\ \bar{\Phi}_{sss} = 12. \end{aligned} \tag{10}$$

We like to remark that this setting is characteristic for so-called recognition problems. Unfortunately, problem (9) is not strongly equivalent to some of the normal forms for singularities, given, for instance, in [3]. In particular, it is not strongly equivalent to the well-known pitchfork bifurcation $x^3 - sx$. Some difficulties arising in the bifurcation analysis stem from the fact that (9) has a double root. The reader might find an extensive discussion on recognition problems in general also in [3], Chapter V.5.

In order to verify the unique existence of a composite curve for more general scalar fluxes, we now proof the following result motivated by the above example. In particular, it is the key tool for proving Theorem 1.2.

**Lemma 2.1.**
*Let be $g(x,s) : B \to \mathbb{R}$, $B \subset \mathbb{R} \times \mathbb{R}$ containing $(0,0)$, a function in $C^4$ satisfying*

$$\bar{g} = \bar{g}_x = \bar{g}_s = 0, \quad \bar{g}_{xx} = \bar{g}_{xs} = \bar{g}_{ss} = 0, \tag{11}$$
$$\bar{g}_{xxx} = c,\ \bar{g}_{xxs} = 0,\ \bar{g}_{xss} = -c,\ \bar{g}_{sss} = 2c$$

*with a constant $c \in \mathbb{R}$, $c \neq 0$. Here the bar again denotes the evaluation at $(0,0)$ and the index is the derivation with respect to $x$ or $s$.*
*Then there is a unique non–trivial solution $h = h(s) \neq s$ near $(0,0)$ such that*

$$g(h(s),s) = 0, \quad s \text{ sufficiently small}, \tag{12}$$

$$h(0) = 0,\ h_s(0) = h'(0) = -2. \tag{13}$$

**Proof:** First of all, we verify that there exists a solution near $(0,0)$ such that (12) and (13) hold. For this purpose, we expand $g$ in a Taylor series[1]

$$\begin{aligned} g(x,s) &= \bar{g} + \bar{g}_x\, x + \bar{g}_s\, s + \frac{1}{2}\bar{g}_{xx}\, x^2 + \bar{g}_{xs}\, xs + \frac{1}{2}\bar{g}_{ss}\, s^2 + \\ &\quad \frac{1}{6}\bar{g}_{xxx}\, x^3 + \frac{1}{2}\bar{g}_{xxs}\, x^2 s + \frac{1}{2}\bar{g}_{xss}\, xs^2 + \frac{1}{6}\bar{g}_{sss}\, s^3 + \sum_{|k|=4} \gamma_k(x,s)\,(x,s)^k. \end{aligned}$$

Incorporating the relations (11) this expansion simplifies to

$$g(x,s) = \frac{1}{6}c\,\underbrace{(x^3 - 3xs^2 + 2s^3)}_{(x-s)^2(x+2s)} + \sum_{|k|=4} \gamma_k(x,s)\,(x,s)^k. \tag{14}$$

Now we use an ansatz that is common in bifurcation theory. To this end, we consider the function

$$\hat{g}(x,s) := \frac{1}{s^3} g(xs, s). \tag{15}$$

By means of (14) this function can be written as

$$\hat{g}(x,s) = \frac{1}{6}c\,\underbrace{(x^3 - 3x + 2)}_{(x-1)^2(x+2)} + s\left(\sum_{|k|=4} \gamma_k(xs,s)\,(x,1)^k\right).$$

[1] $k = (k_1, k_2)$ is a multiindex with $|k| = k_1 + k_2$ and $(x,s)^k = x^{k_1} s^{k_2}$.

From this representation we deduce

$$\hat{g}(-2,0) = 0, \quad \hat{g}_x(-2,0) = \frac{3}{2}c \neq 0.$$

Then the Implicit Function Theorem, see [6], p. 59, shows that there is a ball around $s = 0$ and a unique function $x = x(s)$ such that

$$\hat{g}(x(s), s) = 0 \quad \text{and} \quad x(0) = -2.$$

Hence $h(s) := x(s)s$ locally solves (12). In particular, $h(0) = 0$ and the derivative of $h$ satisfies

$$h'(s) = x'(s)s + x(s), \qquad h'(0) = x(0) = -2.$$

Finally, we have to verify that the function $h$ is the only solution. To this end, we assume that there exists yet another function $\tilde{h} \not\equiv h$ with $g(\tilde{h}(s), s) = 0$ for $s$ sufficiently small and $\tilde{h}(0) = 0$, $\tilde{h}'(0) = -2$. Introducing $\tilde{x}(s) = \tilde{h}(s)/s$ we conclude from (15)

$$0 = g(\tilde{x}(s)s, s) = \hat{g}(\tilde{x}(s), s)\ s^3.$$

Again the Implicit Function Theorem implies uniqueness for the solution $x(s)$ and therefore we obtain locally

$$\tilde{h}(s) = \tilde{x}(s)s = x(s)s = h(s).$$

In particular, the condition $\tilde{h}(0) = h(0)$ holds by assumption. This proves the assertion. ■

## 2.2. The System Case

By means of the Lemma 2.1 we are now able to prove Theorem 1.2 which is presented here once more but in terms of Definition 1.1.

**Theorem 2.1:***

*Consider the setting in Definition 1.1. In particular, $\overline{u} \in \mathcal{M}$ is a simple degeneration point of the nonlinear field, i.e., $\chi_k(\overline{u}) = 0$ and ${\chi'}_k(\overline{u}) \neq 0$. Furthermore, let $r \in C^2$ and $\lambda \in C^3$. Then there is a unique non–trivial solution $u = u(s) \neq u^*(s)$ in a ball around $s = 0$ that solves (7). In particular,*

$$u_s(0) = -2u^*_s(0) = -2r(\overline{u}).$$

**Proof:** Without loss of generality we may assume that $k = 1$ and

$$\bar{u} = 0,\ \lambda(\bar{u}) = 0,\ r(\bar{u}) = (1, 0, \ldots, 0)^T,\ f_u(\bar{u}) = \operatorname{diag}(0, \lambda_2(\bar{u}), \ldots, \lambda_n(\bar{u})). \tag{16}$$

Otherwise we first transform the system of conservation laws as outlined in [9] such that the type of the characteristic fields are invariant under this transformation. In order to employ Lemma 2.1 we have to reduce problem (7) for $n$ equations in $n+1$ unknowns to a scalar equation of two unknowns such that the roots remain invariant. For this purpose, we perform a Liapunov-Schmidt reduction, cf. [3]. First of all, we introduce the matrices

$$J := \Phi_u|_{0,0} = f_u(\bar{u}),\ E := \operatorname{diag}(0, 1, \ldots, 1)$$

and choose vector spaces $M$ and $N$ which are the complements of ker $J$ and range $J$, i.e.,

$$\mathbb{R}^n = \ker J \oplus M, \quad \mathbb{R}^n = N \oplus \text{range } J. \tag{17}$$

Note that due to our settings we have in particular

$$\ker J = \mathbb{R} \times \{0\}^{n-1}, \quad \text{range } J = \{0\} \times \mathbb{R}^{n-1} = \text{range } E, \quad \text{kern } E = N.$$

Then we may rewrite (7) equivalently as

$$E\Phi(u,s) = 0, \tag{18}$$

$$(I-E)\Phi(u,s) = 0. \tag{19}$$

We now apply the Implicit Function Theorem to (18) and show that $n-1$ of the $n+1$ unknowns depend on the two remaining ones. Then we can substitute the $n-1$ variables into (19) and obtain an equation in two unknowns. For this purpose, we first define $F : M \times (\ker J \times \mathbb{R}) \to \text{range } J$ by

$$F(w,v,s) = E\Phi(v+w,s), \quad w \in M,\ v \in \ker J,$$

which is feasible because of the decomposition (17). Differentiation of $F$ with respect to $w$ yields

$$F_w(w,v,s)|_{0,0,0} = E\,\Phi_u|_{0,0} = E\,J = J.$$

If we restrict to $M$, the map $J : M \to$ range J is invertible. This implies that there is a unique solution of (18) for $w$ near 0. We denote this solution by

$$w = W(v,s), \quad W : \ker J \times \mathbb{R} \to M.$$

In particular, it satisfies

$$E\Phi(v + W(v,s), s) = 0 \quad \text{near } 0 \quad \text{and} \quad W(0,0) = 0.$$

We now introduce the mapping $\phi : \ker J \times \mathbb{R} \to N$ defined by

$$\phi(v,s) = (I-E)\Phi(v + W(v,s), s).$$

Then we note that the roots of $\Phi$ and $\phi$ coincide, i.e.,

$$\phi(v,s) = 0 \quad \Longleftrightarrow \quad \Phi(v + W(v,s), s) = 0.$$

So far $\phi$ is a mapping with two parameters $v$ and $s$ where $v$ is an element of a one-dimensional vector space. However, for our purpose it is more convenient to work with a scalar function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ determined by two scalar parameters. To this end, we finally perform a coordinate transformation by which $\phi$ is reduced to such a scalar function $g$. Therefore we choose $v_0 \in \ker J$ and $v_0^* \in (\text{range } J)^\perp$ and define

$$\begin{aligned} g(x,s) &:= \ < v_0^*, \phi(xv_0, s) > \, = \, < v_0^*, (I-E)\Phi(xv_0 + W(xv_0,s), s) > \\ &= \ < v_0^*, \Phi(xv_0 + W(xv_0,s), s) > . \end{aligned}$$

In particular, we may choose $v_0 = v_0^* = r(\bar{u}) = (1,0,\ldots,0)^T$. Since $\phi \in N$, we conclude

$$g(x,s) = 0 \quad \Longleftrightarrow \quad \phi(xv_0, s) = 0$$

and, moreover, the zeros of $g$ correspond one-to-one to the solutions of (7).

Note, that the function $\phi$ is implicitly determined because it depends on the function $W$ which is known to exist but without explicit representation. In order to apply Lemma 2.1 we need to determine the derivatives of $g$. To this end, the derivatives of $\Phi$ and $W$ have to be computed. The technical details of their computation can be found in [9]. Finally, from evaluating the derivatives at $(0,0)$ we verify (11) with constant $c = \mathcal{X}'(\overline{u}) \neq 0$. Therefore there is a unique solution of (7). Finally we have to determine the derivative of $u$ at the degeneration point. For this purpose we consider the representation of $u$ resulting from the Liapunov–Schmidt reduction and the function $h(s)$ according to the proof of Lemma 2.1

$$u(s) = u(h(s), s) = h(s)v_0 + W(h(s)v_0, s).$$

From this we conclude

$$u_s(0) = ((v_0 + W_x v_0)h' + W_s)(0) = v_0 h'(0) = -2r.$$

■

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