On the Existence of the Composite Curve near a Degeneration Point

Siegfried Müller and Alexander Voß*

January 9, 2001

Abstract

The paper is concerned with the analysis of the composite curve. This is a special type of curve in phase space which arises in the construction of the solution of a Riemann problem for a system of strictly hyperbolic conservation laws which exhibits non–genuinely nonlinear characteristic fields. It will be shown that this curve uniquely exists near a degeneration point of the corresponding characteristic field. Furthermore it is tangentially connected to the admissible part of the rarefaction curve breaking off at this point. The proof is based on Liapunov-Schmidt Reduction which reduces the system to an equivalent scalar equation. Then this equation is solved by means of an ansatz common in bifurcation theory.

Key Words: Riemann problem, composite curve, Liapunov–Schmidt reduction.

AMS Subject Classification: 35L65, 35L67, 35Q05, 70H35, 76N10

1 Introduction

We are concerned with the Riemann problem for a system of conservation laws in 1D

$$u_t + f(u)_x = 0 \tag{1}$$

and piecewise constant initial data

$$u(0,x) = \begin{cases} u_l & , \quad x < 0 \\ u_r & , \quad x > 0 \end{cases} \tag{2}$$

for two states $u_l,\ u_r$ in the admissible phase space $\mathcal{D} \subset \mathbb{R}^n$. Here $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{D}$ denotes the vector of n conservative quantities and $f : \mathcal{D} \to \mathbb{R}^n$ is the flux vector which is supposed to be sufficiently smooth, i.e., $f \in C^3(\mathcal{D})$.

The Riemann problem has been subject of extensive research. Significant contributions concerning the construction of its solution have been made by Lax [Lax57], Gelfand [Gel59] and Liu [Liu75]. In the sequel, we consider the general frame as in [Liu75]. For this purpose, we assume that the system of conservation laws (1) is *strictly hyperbolic*, i.e., there is a complete set of eigenvalues $\lambda_k(u),\ k = 1, \ldots, n$, of the Jacobian of f such that

$$\lambda_1(u) < \ldots < \lambda_n(u) \qquad \forall\, u \in \mathcal{D}.$$

*The work of this author is supported by the DFG in the Priority Program "Analysis and Numerics of Conservation Laws".

The corresponding right eigenvectors $r_k(u)$ and left eigenvectors $l_k(u)$ satisfy

$$l_k^T(u)\, r_j(u) = \delta_{kj} \quad 1 \leq k, j \leq n, \quad \forall\, u \in \mathcal{D}.$$

In the literature, the eigenvalues λ_k are called *characteristic velocities* corresponding to the *characteristic k–field* which is characterized by the *nonlinearity factor*

$$\chi_k(u) := (\nabla_u \lambda_k(u))\, r_k(u) \qquad u \in \mathcal{D}.$$

Whenever χ_k vanishes for all $u \in \mathcal{D}$ the k–field is called *linearly degenerated.* However, if χ_k is not equal to zero in the admissible phase space, the k–field is called *genuinely nonlinear.* In the classical case the k–field is supposed to be either linearly degenerated or genuinely nonlinear. Here, we also consider the case of a *non–genuinely nonlinear field*, i.e., χ_k locally vanishes at certain points of the phase space. To be precise, we assume that there is an $(n-1)$–dimensional hypersurface $\mathcal{M} \subset \mathcal{D}$ where $\chi_k|_{\mathcal{M}} = 0$ and the trajectories defined by $u'(\frac{x}{t}) = r_k(u(\frac{x}{t}))$ intersect $\mathcal{M}$ only at isolated points. Finally, we assume that the characteristic fields are *simply degenerated* on $\mathcal{M}$, i.e.,

$$\chi'_k(u) := (\nabla_u \chi_k(u))\, r_k(u) \neq 0 \qquad u \in \mathcal{M}.$$

In this frame Liu outlined the general principles for constructing the solution of the Riemann problem, see [Liu75]. In particular, he introduced the composite curve near degeneration points of nonlinear fields. For this purpose we introduce the Hugoniot locus $\mathcal{H}(u^*)$ which is composed of all states $u \in \mathcal{D}$ such that the *Rankine–Hugoniot jump condition*

$$\sigma(u - u^*) = f(u) - f(u^*) \tag{3}$$

hold. Here $\sigma = \sigma(u^*, u)$ denotes the speed of the discontinuity. In [Lax57] Lax verified that there is a family of n one–parameteric smooth curves all satisfying the jump conditions (3) and $\lim_{u \to u^*} \sigma_k(u^*, u) = \lambda_k(u^*)$. Since the system is supposed to be strictly hyperbolic, these curves can be enumerated with increasing velocity $\lambda_k(u^*)$. The corresponding Hugoniot curve is denoted by $\mathcal{H}_k(u^*)$.
Then the composite curve can be determined by means of the k–rarefaction curve denoted by $\mathcal{R}_k$ and the Hugoniot curves $\mathcal{H}_k(u^*)$ emanating from a state on the rarefaction curve. To a state $u^* = u^*(s) \in \mathcal{R}_k$ near $\bar{u}$ we move along $\mathcal{H}_k(u^*)$ until we reach the first state $u \in \mathcal{H}_k(u^*)$ satisfying the sonic condition

$$\sigma(u^*, u) = \lambda_k(u^*). \tag{4}$$

This is the definition given in [Liu75]. Moreover, Liu showed that the shock curve $\mathcal{H}_k(u^*)$ is tangent to the mixed curve $\mathcal{C}_k$ at u and tangent to $\mathcal{R}_k$ at u^*. The situation is sketched in Figure 1.

However, there remains an open question. Existence of the composite curve at the degeneration point $\overline{u}$ has not been investigated in [Liu75]. The objective of this paper is to verify the unique existence of the composite curve in a neighborhood of $\overline{u}$. Moreover, we show that if we are approaching locally $\overline{u}$ on the rarefaction curve, then the corresponding state on the mixed curve approaches $\overline{u}$ as well.

In the following we consider a slightly different definition.

Definition 1 *Let k denote a nonlinear field that degenerates at an isolated state $\bar{u} \in \mathcal{M}$, i.e., $\chi_k(\overline{u}) = 0$. Furthermore let $u^*(s) : [-\varepsilon, 0] \to \mathbb{R}^n$ a parameterized k–rarefaction curve characterized by*

$$u^*(0) = \bar{u}, \quad u^*_s(s) = r_k(u^*(s)), \quad u \neq u^*.$$

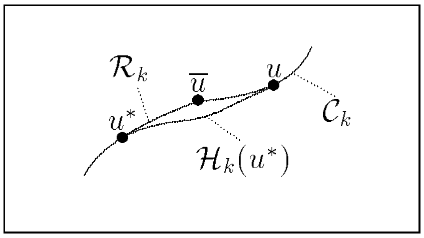

Figure 1: Composite curve

Then the composite curve is composed of all roots $u \in \mathcal{D}$ of the function

$$\Phi(u,s) := f(u) - f(u^*(s)) - \lambda(u^*(s))(u - u^*(s)) = 0. \tag{5}$$

Thus, the outline of the paper is as follows. In Section 2 we prove the local existence of the composite curve. First of all, we consider the scalar case. This will give us some inside to the underlying techniques from bifurcation theory. To this end, we reduce the n dimensional problem in $n+1$ unknowns into a scalar problem of two unknowns by means of the Liapunov–Schmidt reduction. In Section 3 we verify that our results can be applied to some physical relevant systems. In the Appendix we summarize some technical details that arise in the course of proving the result in Section 2.

2 Existence of the Composite Curve

First of all, we consider a single conservation law. Then the setting reduces to $\lambda = f'$, $r = 1$, $\chi = f''$ and $\chi' = f'''$. The most simplest example that fits into the setting is the cubic flux function $f(u) = u^3$. There is an inflection point $\overline{u} = 0$ and, in particular, $f'''(\overline{u}) \neq 0$. According to Definition 1 the composite locus is characterized by the roots of

$$\Phi(u,s) = u^3 - s^3 - 3\,s^2(u-s) = (u-s)^2\,(u+2s) = 0$$

where the rarefaction curve is the identity, i.e., $u^*(s) = s$. The double root $u = s$ is the trivial solution of (5) and there is exactly one non–trivial solution $u(s) = -2\,s$.

In order to verify the unique existence of a composite curve for more general scalar fluxes, we proof the following result which will be the key tool for proving the main theorem.

Lemma 1
Let be $g(x,s): B \to \mathbb{R}$, $B \subset \mathbb{R} \times \mathbb{R}$ containing $(0,0)$, a function in C^4 satisfying

$$\begin{aligned} &\bar{g} = \bar{g}_x = \bar{g}_s = 0, \quad \bar{g}_{xx} = \bar{g}_{xs} = \bar{g}_{ss} = 0, \\ &\bar{g}_{xxx} = c,\ \bar{g}_{xxs} = 0,\ \bar{g}_{xss} = -c,\ \bar{g}_{sss} = 2c \end{aligned} \tag{6}$$

with a constant $c \in \mathbb{R}$, $c \neq 0$. Here the bar denotes the evaluation at $(0,0)$ and the index is the derivation with respect to x or s.
Then there is a unique solution $h = h(s)$ near $(0,0)$ such that

$$g(h(s),s) = 0, \quad s \text{ sufficiently small},$$

and, in particular,

$$h_s(0) = h'(0) = -2.$$

Proof: The Taylor expansion of g yields[1]

$$\begin{aligned} g(x,s) &= \bar{g} + \bar{g}_x\, x + \bar{g}_s\, s + \frac{1}{2}\bar{g}_{xx}\, x^2 + \bar{g}_{xs}\, xs + \frac{1}{2}\bar{g}_{ss}\, s^2 \\ &+ \frac{1}{6}\bar{g}_{xxx}\, x^3 + \frac{1}{2}\bar{g}_{xxs}\, x^2 s + \frac{1}{2}\bar{g}_{xss}\, xs^2 + \frac{1}{6}\bar{g}_{sss}\, s^3 \\ &+ \sum_{|k|=4} \chi'_k(x,s)\,(x,s)^k. \end{aligned}$$

Incorporating the relations (6) this expansion simplifies to

$$g(x,s) = \frac{1}{6}c\underbrace{(x^3 - 3xs^2 + 2s^3)}_{(x-s)^2(x+2s)} + \sum_{|k|=4} \chi'_k(x,s)\,(x,s)^k. \tag{7}$$

Note, that this is exactly the situation for $f(u) = u^3$ where, in particular, all fourth order terms vanish.

Now we use an ansatz that is common in bifurcation theory. To this end, we consider the function

$$\hat{g}(x,s) := \frac{1}{s^3} g(xs, s).$$

By means of (7) this function can be written as

$$\hat{g}(x,s) = \frac{1}{6}c\underbrace{(x^3 - 3x + 2)}_{(x-1)^2(x+2)} + s\left(\sum_{|k|=4} \chi'_k\,(xs,s)\,(x,1)^k\right).$$

From this representation we deduce

$$\hat{g}(-2,0) = 0, \quad \hat{g}_x(-2,0) = \frac{3}{2}c \neq 0.$$

Then the Implicit Function Theorem, see Section 4.5, shows that there is a ball around $s = 0$ and a unique function $x = x(s)$ such that

$$\hat{g}(x(s), s) = 0 \quad \text{and} \quad x(0) = -2.$$

Hence $h(s) := x(s)s$ solves $g(h(s), s) = 0$. In particular, the derivative of h satisfies

$$h'(s) = x'(s)s + x(s), \qquad h'(0) = x(0) = -2.$$

■

In order to motivate the main result, we consider $f(u) = u^5$. Then Φ is determined by

$$\Phi(u,s) := u^5 - s^5 - s^4(u-s) = (u^3 + 2su^2 + 3s^2u + 4s^3)(u-s)^2 = 0.$$

Again we obtain the trivial solution as a double root and three non–trivial solutions, one real and two conjugate complex. The only interesting solution is $u = a\,s$, $a \approx -1.650629192$ with $u(0) = 0 = \overline{u}$. Since the physical relevant systems we consider in Section 3 are only simply degenerated, we exclude saddle points in our analysis.

Here is the main result.

[1] $k = (k_1, k_2)$ is a multiindex with $|k| = k_1 + k_2$ and $(x,s)^k = x^{k_1} s^{k_2}$.

Theorem 1
Consider the setting in Definition 1. In particular, $\overline{u} \in \mathcal{M}$ is a simple degeneration point of the nonlinear field, i.e., $\chi_k(\overline{u}) = 0$ and $\chi'_k(\overline{u}) \neq 0$. Furthermore, let $r \in C^2$ and $\lambda \in C^3$. Then there is a unique solution $u = u(s) \neq u^*(s)$ in a ball around $s = 0$ that solves (5). In particular,

$$u_s(0) = -2u_s^*(0) = -2r(\overline{u}).$$

Proof: Without loss of generality we assume that

$$k = 1, \quad \bar{u} = 0, \quad \lambda(\bar{u}) = 0, \quad r(\bar{u}) = (1, 0, \dots, 0)^T, \tag{8}$$

$$f_u(\bar{u}) = \begin{pmatrix} 0 & & & 0 \\ & \lambda_2(\bar{u}) & & \\ & & \ddots & \\ 0 & & & \lambda_n(\bar{u}) \end{pmatrix}. \tag{9}$$

Otherwise we have to transform the system of conservation laws as outlined in Section 4.1 such that the type of the characteristic fields are invariant under these transformation. The idea is to employ Lemma 1. For this purpose, we have to transform problem (5) for n equations in $n+1$ unknowns into a scalar equation of two unknowns such that the roots remain invariant under this transformation. By means of the Liapunov–Schmidt reduction a scalar function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be constructed

$$\begin{aligned} g(x, s) &:= \ < v_0^*, \phi(xv_0, s) > \\ &= \ < v_0^*, \Phi(xv_0 + W(xv_0, s), s) > \end{aligned}$$

with appropriate functions ϕ, W and vectors v_0 and v_0^* such that

$$g = 0 \iff \Phi = 0.$$

The details of the construction are postboned to Section 4.2. In particular, we obtain

$$v_0 = v_0^* = r(\bar{u}) = (1, 0, \dots, 0)^T,$$

$$\ker J = \mathbb{R} \times \{0\}^{n-1}, \quad \text{range } J = \{0\} \times \mathbb{R}^{n-1}$$

$$J = f_u(\bar{u}) = \begin{pmatrix} 0 & & & 0 \\ & \lambda_2(\bar{u}) & & \\ & & \ddots & \\ 0 & & & \lambda_n(\bar{u}) \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \quad I - E = \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}.$$

Note, that the function ϕ is implicitly determined because it depends on the function W which is known to exist but there is no explicit representation. In order to apply Lemma 1 we need to determine the derivatives of g. To this end, the derivatives of Φ and W have to be computed. The technical details of their computation are summarized in the appendix, see Section 4.3. Finally, from evaluating the derivatives at $(0,0)$ we obtain

$$\begin{aligned}
\bar{g} &= \bar{g}_x = \bar{g}_s = 0, \\
\bar{g}_{xs} &= \bar{g}_{ss} = 0, \\
\bar{g}_{xx} &= < v_0^*, \Phi_{uu} v_0 v_0 >= 0, \\
\bar{g}_{xxx} &= < v_0^*, v_0 \lambda_{ss}|_{s=0} >= c, \\
\bar{g}_{xxs} &= 0, \\
\bar{g}_{xss} &= < v_0^*, -\lambda_{ss}|_{s=0} >= -c, \\
\bar{g}_{sss} &= < v_0^*, 2\lambda_{ss}|_{s=0} >= 2c
\end{aligned}$$

with the constant $c = \chi'(\overline{u}) \neq 0$. Since g satisfies (6), there is a unique solution of (5). Finally we have to determine the derivative of u at the degeneration point. To this end we consider the representation of u comming from the Liapunov–Schmidt reduction and the function $h(s)$ according to the proof of Lemma 1

$$u(s) = u(h(s), s) = h(s)v_0 + W(h(s)v_0, s).$$

From this we conclude

$$u_s(0) = ((v_0 + W_x v_0)h' + W_s)(0) = v_0 h'(0) = -2r.$$

$\blacksquare$

3 Applications

In the sequel, we present some physically relevant applications that fit into the frame of strictly hyperbolic conservation laws with non–genuinely nonlinear characteristic fields. These are arising from elasticity and continuum mechanics, respectively.

3.1 Elasticity

A model for nonlinear plane shear waves in viscoelastic media is determined by the evolution of the strain τ and the velocity v that can be written in conservation form (1) with $u = (\tau, v)^T$ and $f(u) = (-v, \sigma(\tau))^T$. Here σ denotes the stress which is assumed to be a strictly monotonously increasing function of the stress, i.e., $\sigma' > 0$, but not necessarily convex. However, we exclude multiple roots of the curvature σ''. For this system the eigenvalues and right eigenvectors are determined by

$$\lambda_k(u) = \varepsilon_k \sqrt{\sigma'(\tau)}, \quad r_k(u) = (1/\lambda_k(u), 1)^T$$

with $\varepsilon_k := (-1)^k$, $k = 1, 2$. Then the nonlinearity factor χ_k and its derivative ${\chi'}_k$ turn out to be

$$\chi_k(u) = 0.5\,\sigma''(\tau), \quad {\chi'}_k(u) = 0.5\,\sigma'''(u)/\lambda_k(u), \quad k = 1, 2.$$

Note, that both characteristic fields are nonlinear and they degenerate at inflection points of the stress function. The characteristic fields are simply degenerated at this state, because σ'' is supposed to have only simple roots.

3.2 Continuum Mechanics

The state of a fluid is characterized by several macroscopic variables, e.g., mass density ρ, specific internal energy e and particle velocity v. The evolution of the fluid is governed by the balance equations of mass, momentum and total energy. In Lagrangian coordinates these equations can be written in conservative form (1) with $u = (\tau, v, E)^T$ and $f(u) = (-v, p, v\,p)^T$ where $\tau = 1/\rho$ and $E = e + 0.5\,v^2$ denote the specific volume and the specific total energy, respectively. Here we neglect irreversible effects such as viscosity and heat conduction. The system has to be closed by an equation of state (EOS) for the pressure p. According to the first and second law of thermodynamics the pressure can be interpreted as the partial derivative of the internal specific energy, i.e., $p(\tau, s) = -e_\tau(\tau, s)$, which is supposed to be postive. Since we restrict to thermodynamical equilibrium, the specific internal energy e of an equilibrium state is related to the specific entropy s and the specific volume τ. Hence, we can change variables $s = s(\tau, e)$.
The eigenvalues of the Jacobian of f are given by

$$\lambda_1 = -\sqrt{-p_\tau}, \quad , \lambda_2 = 0, \quad \lambda_3 = +\sqrt{-p_\tau}.$$

and the characteristic fields by

$$\chi_{1,2} = \frac{p_{\tau\tau}}{-2p_\tau} = \frac{\mathcal{G}}{\tau}, \quad \chi_2 = 0,$$

where $\mathcal{G}$ denotes the fundamental derivative of gas dynamics. We note that the 2–field is linearly degenerated and the 1(3)–field is nonlinear. In particular the nonlinear fields degenerate at states where the fundamental derivative of gas dynamics $\mathcal{G}$ vanishes. In order to apply Theorem 1 we have to demand $p_{\tau\tau\tau}(\tau, s) \neq 0$ for all inflection points of the isentropes in the p–τ plane, i.e., the nonlinear fields are only simply degenerated. Obviously, the sign of $\mathcal{G}$ is related to the curvature of the isentropes in the p–τ plane. For an ideal gas, the isentropes are convex, i.e., $\mathcal{G}$ is positive. However, close to the vapor–liquid saturation boundary materials are known to exhibit a region where the isentropes are concave, i.e., $\mathcal{G}$ is negative (see Fig. 2). Consequently, there need to be inflection points which are isolated zeros of $\mathcal{G}$ along isentropes. Therefore, we distinguish between *convex EOS* and *non–convex EOS*, e.g., van der Waals, corresponding to the behavior of the isentropes.

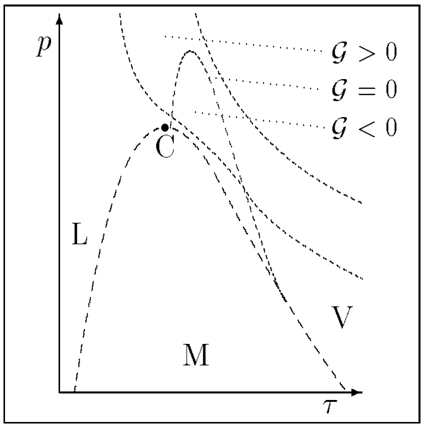

Figure 2: Isentropes in p–τ plane, L=Liquid, V=Vapor, M=Mixture, C=Critical Point

4 Appendix

4.1 Alignment of the Equations

In this section we explain how to transform an arbitrary strictly hyperbolic system

$$\tilde{u}_t + (\tilde{f}(\tilde{u}))_x = 0 \tag{10}$$

with $\tilde{u} : \mathbb{R} \times \mathbb{R}_+ \to \tilde{\mathcal{D}} \subset \mathbb{R}^n$, $\tilde{f} : \tilde{\mathcal{D}} \to \mathbb{R}^n$ and $\tilde{f} \in C^3(\tilde{\mathcal{D}})$ into an equivalent system where the Jacobian as well as its eigenvectors and eigenvalues can be represented in the normalized form (8), (9). By $\tilde{\lambda}_k$ we denote the eigenvalues of the Jacobian of the flux $\tilde{f}$ and the corresponding right and left eigenvectors $\tilde{r}_k, \tilde{l}_k : \tilde{\mathcal{D}} \to \mathbb{R}^n$, $\tilde{r}_k \in C^2(\tilde{\mathcal{D}})$ with $\tilde{l}_i^T \tilde{r}_j = \delta_{ij}$.

Let be $k_0 \in \{1, \ldots, n\}$ the index of a nonlinear characteristic field which degenerates at a state $\bar{u}$, i.e.

$$\nabla_{\tilde{u}} \tilde{\lambda}_{k_0}(\bar{u})\, \tilde{r}_{k_0}(\bar{u}) = 0.$$

By means of

$$\bar{\lambda} := \tilde{\lambda}_{k_0}(\bar{u}), \quad \bar{R} := [\tilde{r}_1(\bar{u}), \cdots, \tilde{r}_n(\bar{u})], \quad \bar{L} := [\tilde{l}_1(\bar{u}), \cdots, \tilde{l}_n(\bar{u})]^T$$

we introduce the transformation

$$u = u(\tilde{u}) := \bar{L}(\tilde{u} - \bar{u}), \quad \tilde{u} = \tilde{u}(u) := \bar{R}u + \bar{u},$$

$$f(u) := \bar{L}\tilde{f}(\tilde{u}(u)) - \bar{L}\bar{\lambda}\tilde{u}(u).$$

Then we obtain

$$f_u(u) = \bar{L}\tilde{f}_{\tilde{u}}(\tilde{u}(u))\bar{R} - \bar{\lambda}I, \quad r_k(u) = \bar{L}\tilde{r}_k(\tilde{u}(u)), \quad \lambda_k(u) = \tilde{\lambda}_k(\tilde{u}(u)) - \bar{\lambda}.$$

Note, that the type of the characteristic field is invariant under the transformation. At the new degeneration point 0 we deduce with $\tilde{u}(0) = \bar{u}$

$$f_u(0) = \bar{L}\tilde{f}_{\tilde{u}}(\bar{u})\bar{R} - \bar{\lambda}I = \mathrm{diag}(\tilde{\lambda}_1(\bar{u}) - \bar{\lambda}, \ldots, \underbrace{\tilde{\lambda}_{k_0}(\bar{u}) - \bar{\lambda}}_{=0}, \ldots, \tilde{\lambda}_n(\bar{u}) - \bar{\lambda}),$$

$$r_{k_0}(0) = \bar{L}\tilde{r}_{k_0}(\bar{u}) = (0, \ldots, 0, \underbrace{1}_{k_0}, 0, \ldots, 0)$$

and

$$\lambda_{k_0}(0) = 0.$$

Since we do not use the ordering of the eigenvalues, we choose $k_0 = 1$ without loss of generality.

4.2 The Liapunov–Schmidt Reduction

By the Liapunov–Schmidt reduction an infinite–dimensional problem can be transformed to a finite–dimensional problem, see [GS85]. For our problem it turns out that finding a solution for a system of n equations in $n+1$ unknowns can be reduced to solving a single equation in 2 unknowns. To this end, we consider an arbitrary system of n equations

$$\Phi(u, s) = 0, \quad u \in \mathbb{R}^n,\ s \in \mathbb{R},\ \Phi \in C^1. \tag{11}$$

Let $J := \Phi_u|_{0,0}$ denote the $n \times n$ Jacobian of Φ with respect to u. We assume that J is minimally degenerated, i.e.,

$$\mathrm{Rg}\, J = n - 1. \tag{12}$$

Next we choose vector spaces M and N which are the complements of ker J and range J, i.e.,

$$\mathbb{R}^n = \ker J \oplus M, \quad \mathbb{R}^n = N \oplus \text{range } J. \tag{13}$$

According to (12) we obtain

$$\dim \text{range } J = n-1, \quad \dim \ker J = 1, \quad \dim M = n-1, \quad \dim N = 1.$$

Let E denote the projection of $\mathbb{R}^n$ onto range J and $(I-E)$ the complementary projection. Then we conclude

$$\text{range } E = \text{range } J, \quad \ker E = N, \quad \text{range } (I-E) = N, \quad \ker E = \text{range } J.$$

According to the fact

$$z = 0 \quad \Longleftrightarrow \quad Ez = 0 \quad \text{and} \quad (I-E)z = 0 \quad \text{for all } z \in \mathbb{R}^n$$

the system (11) may be equivalently written as

$$E\Phi(u,s) = 0, \tag{14}$$
$$(I-E)\Phi(u,s) = 0. \tag{15}$$

The basic idea is to apply the Implicit Function Theorem to (14) and show that $n-1$ of the $n+1$ unknowns depend on the two remaining ones. Then we can substitute the $n-1$ variables into (15) and obtain an equation in two unknowns. For this purpose, we first define $F : M \times (\ker J \times \mathbb{R}) \to \text{range } J$ by

$$F(w,v,s) = E\Phi(v+w,s), \quad w \in M, \ v \in \ker J,$$

which is feasible because of the decomposition (13). Differentiation of F with respect to w yields

$$F_w(w,v,s)|_{0,0,0} = E\,\Phi_u|_{0,0} = E\,J = J.$$

If we restrict to M, the map $J : M \to$ range J is invertible. This implies that there is a unique solution of (14) for w near 0. We denote this solution by

$$w = W(v,s), \quad W : \ker J \times \mathbb{R} \to M.$$

In particular, it satisfies

$$E\Phi(v + W(v,s), s) = 0 \quad \text{near } 0 \quad \text{and} \quad W(0,0) = 0.$$

We now introduce the mapping $\phi : \ker J \times \mathbb{R} \to N$ defined by

$$\phi(v,s) = (I-E)\Phi(v + W(v,s), s).$$

Then we note that the roots of Φ and ϕ coincide, i.e.,

$$\phi(v,s) = 0 \quad \Longleftrightarrow \quad \Phi(v + W(v,s), s) = 0.$$

So far ϕ is a mapping with two parameters v and s where v is an element of a onedimensional vector space. However, for our purpose it is more convenient to work with a scalar function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ determined by two scalar parameters. To this end, we finally perform a coordinate transformation by which ϕ is reduced to such a scalar function g. Therefore we choose $v_0 \in \ker J$ and $v_0^* \in (\text{range } J)^\perp$ and define

$$\begin{aligned} g(x,s) &:= \; < v_0^*, \phi(xv_0, s) > = < v_0^*, (I-E)\Phi(xv_0 + W(xv_0,s), s) > \\ &= \; < v_0^*, \Phi(xv_0 + W(xv_0,s), s) > . \end{aligned}$$

Since $\phi \in N$, we conclude

$$g(x,s) = 0 \quad \Longleftrightarrow \quad \phi(xv_0, s) = 0$$

and, moreover, the zeros of g correspond one-to-one to the solutions of (11).

4.3 Derivatives

4.3.1 Derivatives of the Scalar Equation $g(x,s)$

The core ingredients for the proof of Theorem 1 are the derivatives of

$$g(x,s) = < v_0^*, \Phi(\underbrace{xv_0 + W(xv_0,s)}_{u(x,s)}, s) >$$

evaluated at $(0,0)$. For this purpose, we use the derivatives of Φ, u and f and evaluate them at $(0,0)$. Note, that $< v_0^*, J >= 0$, because $v_0^* \in (\text{range } J)^\perp$. In particular, $v_0 = (1,0,\ldots,0) = r(0)$ holds for the reduced system. Moreover, we omit the parameters for sake of simplicity.

$$\begin{aligned} g_x &= < v^*, \Phi_u u_x >=< v^*, J u_x >= 0, \\ g_s &= < v^*, \Phi_u u_s + \Phi_s >= 0, \\ g_{xx} &= < v^*, \Phi_{uu} u_x u_x >=< v^*, f_{uu} rr >=< v^*, f_u r_u r >= 0, \\ g_{xs} &= 0, \\ g_{ss} &= 0, \\ g_{xxx} &= < v^*, \Phi_{uuu} u_x u_x u_x + 3\Phi_{uu} u_x u_{xx} > \\ &= < v^*, 2 f_u r_u r_u r - 3 r \lambda_u r_u r - f_u r_{uu} rr + r\lambda_{ss} + 3 f_{uu} r r_u r > \\ &= < v^*, -3 r \lambda_u r_u r + r \lambda_{ss} + 3(-f_u r_u r_u r + r \lambda_u r_u r) > \\ &= < v^*, r\lambda_{ss} >= \lambda_{ss} = c, \\ g_{xxs} &= 0, \\ g_{xss} &= < v^*, \Phi_{uss} u_x >=< v^*, -\lambda_{ss} >= -c, \\ g_{sss} &= < v^*, \Phi_{sss} >=< v^*, 2\lambda_{ss} >= 2c \end{aligned}$$

where $c = \lambda_{ss} = \chi'(\overline{u}) \neq 0$.

4.3.2 Derivatives of the Equation for the Composite Curve $\Phi(u,s)$ and the Flux $f(u)$

First of all, we have to calculate the derivatives of

$$\Phi(u,s) = f(u) - f(u^*(s)) - \lambda(u^*(s))(u - u^*(s)).$$

By a straightforward computation we obtain

$$\begin{aligned}
\Phi_u &= f_u(u) - \lambda(u^*),\\
\Phi_s &= -\lambda_s(u^*)(u-u^*),\\
\Phi_{uu} &= f_{uu}(u),\\
\Phi_{us} &= -\lambda_s(u^*),\\
\Phi_{ss} &= -\lambda_{ss}(u^*)(u-u^*) + \lambda_s(u^*)r(u^*),\\
\Phi_{uuu} &= f_{uuu}(u),\\
\Phi_{uus} &= 0,\\
\Phi_{uss} &= -\lambda_{ss}(u^*),\\
\Phi_{sss} &= -\lambda_{sss}(u^*)(u-u^*) + 2\lambda_{ss}(u^*)r(u^*) + \lambda_s(u^*)r_u(u^*)r(u^*).
\end{aligned}$$

Evaluating these derivatives at $(0,0)$ yields

$$\begin{aligned}
\Phi_u &= f_u(0) = J, \quad \Phi_s = 0,\\
\Phi_{uu} &= f_{uu}(0) = 0, \quad \Phi_{us} = 0, \quad \Phi_{ss} = 0,\\
\Phi_{uuu} &= f_{uuu}(0), \quad \Phi_{uus} = 0, \quad \Phi_{uss} = -\lambda_{ss}(0), \quad \Phi_{sss} = 2\lambda_{ss}(0)r(0).
\end{aligned}$$

For the flux we obtain at $(0,0)$

$$\begin{aligned}
&f_{uu}rr = -f_u r_u r,\\
&f_{uu}rr_u r = -f_u r_u r_u + r\lambda_u r_u r,\\
&f_{uuu}rrr = 2f_u r_u r_u r - 3r\lambda_u r_u r - f_u r_{uu} rr + r\lambda_{ss}.
\end{aligned}$$

4.3.3 Derivatives of the Reduced Equation $\Phi(u(x,s),s)$

We need the derivatives of the mapping $\Psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^n$ defined by

$$\Psi(x,s) = \Phi(\underbrace{xv_0 + W(xv_0,s)}_{=u(x,s)}, s).$$

with respect to x and s up to order three. For convenience, we consider the implicitly given mapping W as a mapping $W(x,s) := W(xv_0,s)$. Hence, the derivative with respect to x is well–defined.

$$\begin{aligned}
\Psi_x &= \Phi_u u_x,\\
\Psi_s &= \Phi_u u_s + \Phi_s,\\
\Psi_{xx} &= \Phi_{uu}u_x u_x + \Phi_u u_{xx}\\
\Psi_{xs} &= \Phi_{uu}u_s u_x + \Phi_{us}u_x + \Phi_u u_{us}\\
\Psi_{ss} &= \Phi_{uu}u_s u_s + 2\Phi_{us}u_s + \Phi_u u_{ss} + \Phi_{ss}\\
\Psi_{xxx} &= \Phi_{uuu}u_x u_x u_x + 3\Phi_{uu}u_x u_{xx} + \Phi_u u_{xxx}\\
\Psi_{xxs} &= \Phi_{uuu}u_s u_x u_x + \Phi_{uus}u_x u_x + 2\Phi_{uu}u_{xs}u_x + \Phi_{uu}u_s u_{xx} + \Phi_{us}u_{xx}\\
&\quad +\Phi_u u_{xxs}\\
\Psi_{xss} &= \Phi_{uuu}u_x u_s u_s + 2\Phi_{uus}u_x u_s + 2\Phi_{uu}u_{xs}u_s + \Phi_{uu}u_{ss}u_x + 2\Phi_{us}u_{us}\\
&\quad +\Phi_{uss}u_x + \Phi_u u_{xss}\\
\Psi_{sss} &= \Phi_{uuu}u_s u_s u_s + 3\Phi_{uus}u_s u_s + 3\Phi_{uss}u_s + 3\Phi_{uu}u_{ss}u_s + 3\Phi_{us}u_{ss} +\\
&\quad \Phi_u u_{sss} + \Phi_{sss}.
\end{aligned}$$

4.4 Derivatives of the implicitly given mapping $W(x, s)$

According to the Liapunov–Schmidt reduction there is a mapping

$$W : \ker J \times \mathbb{R} \to M$$

which satisfies

$$E\Phi(\underbrace{xv_0 + W(xv_0, s)}_{=u(x,s)}, s) = 0 \quad \text{near } 0 \quad \text{and} \quad W(0,0) = 0.$$

With our results from above, the following relations hold at $(0, 0)$

$$u_x = v_0 + W_x, \quad u_{x^j s^k} = W_{x^j s^k} \quad j \neq 1, k > 0.$$

Since $EJ = J$, we obtain for W at $(0, 0)$

$$\frac{d}{dx} E\Phi \quad = \quad E\Phi_u u_x = EJ(\underbrace{v_0}_{\in \ker J} + W_x) = JW_x = 0.$$

Then we conclude $W_x = 0, u_x = v_0 = r(0)$, because J is invertible on M. Similarly, we derive $u_s = u_{ss} = u_{xs} = 0$ and $u_{xx} = r_u r$ at $(0, 0)$.

4.5 The Implicit Function Theorem

For our investigations, we apply the Implicit Function Theorem in the following representation, see [Nir74], p. 59.

Implicit Function Theorem: Let X,Y and Z be Banach spaces and Φ a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that Φ has a Fréchet derivative with respect to x, $\Phi_x(x, y)$, which is continuous in U. Let $(x_0, y_0) \in U$ and $\Phi(x_0, y_0) = 0$. If $A = \Phi_x(x_0, y_0)$ is an isomorphism of X onto Z then

i. there is a ball $\{y \ : \ \|y - y_0\| < r\} = B_r(y_0)$ and a unique continuous map $u : B_r(y_0) \to X$ such that $u(y_0) = x_0$ and $\Phi(u(y), y) \equiv 0$.

ii. If Φ is of class C^1 then $u(y)$ is of class C^1 and

$$u_y(y) = -[\Phi_x(u(y), y)]^{-1} \circ \Phi_y(u(y), y).$$

iii. $u_y(y)$ belongs to C^p if Φ is in C^p, $p > 1$.

Acknowledgement

We gratefully acknowledge the discussions with V. LeBlank and M. Hender on bifurcation theory as well as Heinrich Freistühler and Christian Fries on Liu's construction of the composite curve.

References

[Gel59] I. Gelfand. Some problems in the theory of quasilinear equations. *Usp. Math. Nauk*, 14:87 ff, 1959. Am. Math. Soc. Transl., Ser. 2, 29 (1963), 295 ff.

[GS85] M. Golubitsky and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, volume I. Springer Verlag, 1985.

[Lax57] P.D. Lax. Hyperbolic systems of conservation laws, II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[Liu74] T.-P. Liu. The Riemann problem for general systems 2×2 conservation laws. *Am. Math. Soc.*, 199:89–112, 1974.

[Liu75] T.-P. Liu. The Riemann problem for general systems of conservation laws. *J. Diff. Eqns.*, 18:218–234, 1975.

[MP89] R. Menikoff and B.J. Plohr. The Riemann problem for fluid flow of real materials. *Rev. Mod. Physics*, 61:75–130, 1989.

[Nir74] L. Nirenberg. *Topics on Nonlinear Functional Analysis.* Courant Institute of Mathematical Sciences, New York University, 1974. Lecture Notes.

[Ole59] O. Oleinik. Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi–linear equation. *Usp. Mat. Nauk.*, 14:165 ff, 1959. Am. Math. Soc. Transl., Ser. 2, 33 (1964), 285 ff.