Convergence of higher order finite volume schemes on irregular grids

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Abstract

We prove convergence to the entropy solution of a general class of higher order finite volume schemes on unstructured, irregular grids for multidimensional scalar conservation laws. Such grids allow for cells to become flat in the limit. We derive a new entropy inequality for higher order schemes built on Godunov's numerical flux. Our result implies convergence of suitably modified versions of MUSCL-type finite volume schemes, ENO schemes and the discontinuous Galerkin finite element method.

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1 Introduction

We shall consider higher order finite volume discretizations for scalar conservation laws in several space dimensions

$$\frac{\partial}{\partial t}u(x,t) + \nabla \cdot f(u(x,t)) = 0, \tag{1.1}$$

with initial values

$$u(x,0) = u_0(x), \tag{1.2}$$

where $x \in \mathbb{R}^d, t \in \mathbb{R}_+, u \in \mathbb{R}, f \in Lip(\mathbb{R}, \mathbb{R}^d)$ and $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. Existence and uniqueness of admissible weak solutions, which satisfy a set of entropy inequalities, have been shown by several authors (see [21, 11] and the references therein).

In [12], DiPerna introduced the notion of measure-valued solutions to (1.1) and gave sufficient conditions for this solution to agree with the admissible weak solution. This opened a new approach for proving convergence of approximate solutions generated by numerical schemes. Before, many convergence results for numerical schemes relied on Helly's compactness theorem and therefore on BV (bounded variation) estimates (see the introductions of [6, 9] for a review of some of that literature). Szepessy [34] and Coquel and LeFloch [8] were the first who applied DiPerna's theory to numerical schemes. Their methods did not use BV estimates. This is important, since according to an observation of Goodman and LeVeque [16] total variation diminishing schemes in several space dimensions can only be first order accurate.

Another interesting technique for proving convergence for finite difference schemes uses Kuznetsov's approximation theory [22]. This approach, which also brings about an L^1-error estimate, was used in [33, 31, 27]. It was applied to first order finite volume schemes in [3, 39]. A first step to apply it to higher order finite volume schemes was accomplished in [4].

In this paper, we prove convergence of first and higher order *finite volume* schemes to the admissible weak solution using DiPerna's theory. These schemes, which include suitably modified versions of MUSCL-type finite volume schemes, ENO schemes and the discontinuous Galerkin finite element method, differ from finite difference schemes by the use of non-cartesian, unstructured grids. First results for *monotone* finite volume schemes on *regular* grids are due to Cockburn, Coquel, LeFloch and Shu [5], Coquel and LeFloch [10] and Kröner and Rokyta [20]. In [19], Kröner, Rokyta and the author derived a convergence result for *higher order* finite volume schemes on regular grids. They applied their result to a modified version of the discontinuous Galerkin finite element method due to Cockburn, Hou and Shu [6]. An independent proof of convergence for this method using finite element techniques was given by Jaffre, Johnson and Szepessy [18].

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The main new aspects of the present paper are the following: First, the use of *irregular* grids, where cells may become flat at a certain rate as the grid-parameter h tends to zero (compare (4.8) and (4.12) below). This is especially useful when refining the grid in a boundary layer or near an essentially planar shock front. In Appendix C, we show that the rate with which our grids may become degenerate is optimal. Since we admit irregular grids, our convergence result generalizes previous results even for first order, monotone finite volume schemes on unstructured grids.

Secondly, we derive a new discrete entropy inequality for higher order finite volume schemes built on Godunov's numerical flux (see Lemmata 3.7, 4.3 and Appendix A). We estimate the rate of entropy dissipation in terms of the local oscillation of the numerical flux function instead of the approximate solution itself. In this way we extend previous results of Kröner, Rokyta and the author [20, 19] obtained for first and higher order schemes built on the Lax-Friedrichs and Engquist-Osher fluxes to general E-fluxes. Our entropy inequality (4.13) is also new for one-dimensional finite difference schemes.

Finally, we generalize a technique due to Tadmor [35] to decompose the multidimensional scheme into its one-dimensional building blocks. The additional flexibility of our decomposition yields a weaker CFL-restriction for the time step than used in previous work (see (4.10) below). This is discussed in Appendix B, where we show that for a linear conservation law in $\mathbb{R}^d$ and rectangular grids, our CFL-condition is arbitrarily close to the von Neumann necessary condition for linear stability.

2 Theoretical Background

In this section we briefly review Kruzkov's admissible weak solutions and DiPerna's theory of measure-valued solutions.

An entropy pair (U,F) for (1.1) consists of a Lipschitz continuous, convex function $U : \mathbb{R} \to \mathbb{R}$ (the **entropy**) and a function $F : \mathbb{R} \to \mathbb{R}^d$ (the **entropy flux**) given by

$$F'(s) = U'(s)f'(s). \tag{2.1}$$

For any $k \in \mathbb{R}$, let $U(s,k) := |s-k|$. $U(\cdot,k)$ is called the Kruzkov entropy, with entropy flux $F(\cdot,k)$. Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ and $u \in L^1 \cap L^\infty(\mathbb{R}^d \times [0,T])$. If for all $k \in \mathbb{R}$

$$\partial_t U(u,k) + \nabla \cdot F(u,k) \leq 0$$

in the sense of distributions, and if for all $R > 0$,

$$\lim_{t \downarrow 0} \int_{|x|<R} |u(x,t) - u_0(x)| dx = 0,$$

then u is called the **admissible** weak solution, or **entropy** solution, of (1.1), (1.2). In [21], Kruzkov shows that given $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, there exists a unique admissible weak solution u of (1.1), (1.2).[1]

Next we review some results due to Tartar [36] and DiPerna [12] which provide sufficient conditions for a sequence u_h to converge to the admissible weak solution u of (1.1), (1.2). Suppose that the sequence u_h is uniformly bounded in $R^d \times [0,T]$. According to a theorem of Tartar [36], there is a subsequence of u_h which converges weak-star to a Young-measure ν (that is a measurable map $\nu : \mathbb{R}^d \times [0,T] \to Prob(\mathbb{R})$, $(x,t) \mapsto \nu_{x,t}$). Moreover, for all continuous $g : \mathbb{R} \to \mathbb{R}$, $g(u_h)$ converges weak-star to $\langle \nu, g\rangle := \int g(\lambda) d\nu(\lambda)$, that is the expected value of g with respect to ν. According to a corollary to Tartar's theorem ([12]), u_h converges strongly to u in $L^p_{loc}(\mathbb{R}^d \times [0,T])$, $1 \leq p < \infty$ if and only if $\nu_{x,t} = \delta_{u(x,t)}$ almost everywhere. Sufficient criteria for the reduction of ν to a Dirac measure centered at the admissible weak solution are derived in [12], the main criterium being a weak entropy inequality for the measure-valued solution ν (see inequality (4.16) below). Summaries of DiPerna's result may be found in [34, 8, 19].

3 Finite volume schemes

In this section we define the class of schemes for which we can prove convergence and prepare the convergence proof by some observations on E-fluxes, in particular Godunov's flux.

[1] Kruzkov only remarks that his result holds for $f \in C^1(\mathbb{R}, \mathbb{R}^d)$. However, one can easily carry over his proof to $f \in Lip(\mathbb{R}, \mathbb{R}^d)$ (private communication, Kruzkov, 1992).

DEFINITION 3.1 *The set $\mathcal{T} := \{T_i \mid i \in I\}$ where $I \subset \mathbb{N}$ and T_i is a convex d-dimensional polygon is called an* **unstructured grid** *of $\mathbb{R}^d$ if $\mathbb{R}^d = \bigcup_{i\in I} T_i$ and if the intersection of two different polygons is a union of common sides and vertices.*

Given $T_i \in \mathcal{T}$, let ρ_i be its inner and h_i its outer diameter. Let $\rho := \inf\{\rho_i | i \in I\}$ and $h := \sup\{h_i | i \in I\}$. Let $S_{ij}, j = 1,\ldots,k_i$ be the sides if T_i, with outward unit normals n_{ij}. Let $J := \{(i,j) \mid \ i \in I, 1 \le j \le k_i\}$. We will denote the center of gravity of T_i by x_i, and by x_{ij} the center of gravity of the j^{th} neighboring polygon of T_i, which we call T_{ij}. Let the polygonal grid $\mathcal{T}$ be given. Discretize time by $0 =: t^0 < t^1 < \ldots < t^N := T$, and let $\Delta t^n := t^{n+1} - t^n$. Let $I(a_1,\ldots,a_n)$ be the closed interval $[\min(a_1,\ldots,a_n), \max(a_1,\ldots,a_n)]$.

Let

$$V_{h,k} := \{v_h \in L^1 \cap L^\infty(\mathbb{R}^d) \mid \ v_h|_{T_i} \in P_k(T_i)\},$$

where $P_k(T_i)$ is the space of polynomials of degree $\le k$ over T_i. Let

$$\begin{aligned} \overline{V}_h &:= V_{h,0} \\ \widetilde{V}_{h,k} &:= \{v_h \in V_{h,k} \mid \int_{T_i} v_h = 0 \text{ for all } i \in I\}. \end{aligned}$$

Then $V_{h,k}$ can be decomposed orthogonally with respect to the L^2-scalar product:

$$\begin{aligned} V_{h,k} &= \overline{V}_h \oplus \widetilde{V}_{h,k}, \\ v_h &= \overline{v}_h + \tilde{v}_h. \end{aligned}$$

We will approximate u by a sequence $u_h : [0,T] \to V_{h,k}$ such that

$$\overline{u}_h(x,t) = u_i^n \text{ for } x \in T_i, t^n \le t < t^{n+1}.$$

Let us begin to define the algorithm for u_i^n. The conservation law (1.1) implies that

$$\int_{T_i} u(x,t^{n+1})dx - \int_{T_i} u(x,t^n)dx + \sum_{j=1}^{k_i} \int_{t^n}^{t^{n+1}} \int_{S_{ij}} f(u(x,t)) \cdot n_{ij} dS dt = 0.$$

If we approximate $\int_{T_i} u(x,t^n)dx$ by $|T_i|u_i^n$ and $\int_{t^n}^{t^{n+1}} \int_{S_{ij}} f(u(x,t)) \cdot n_{ij} dSdt$ by $\Delta t^n |S_{ij}| g_{ij}^n$, we arrive at the discrete approximate equations

$$u_i^0 = \frac{1}{|T_i|} \int_{T_i} u_0(x)dx, \tag{3.1}$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t^n}{|T_i|} \sum_{j=1}^{k_i} |S_{ij}| g_{ij}^n. \tag{3.2}$$

In order to close the algorithm (3.1), (3.2) it remains to define the approximate fluxes g_{ij}^n.

DEFINITION 3.2 *A family $g := \{g_{ij} : \mathbb{R}^2 \to \mathbb{R} \mid (i,j) \in J\}$ of measurable functions is called a* **conservative two point numerical flux consistent with** *$f(u) \cdot n$ (or briefly a* **numerical flux***) if it satisfies*
(i) For all $(i,j) \in J$ and all $u \in \mathbb{R}$,

$$g_{ij}(u,u) = f(u) \cdot n_{ij}.$$

(ii) If $S_{ij} = S_{kl}$ and $i \neq k$, then

$$g_{ij}(u,v) = -g_{kl}(v,u).$$

for all $u, v \in \mathbb{R}$.

A numerical flux g is called an **E-flux** *(see Osher [30]) if it satisfies in addition*
(iii)

$$\frac{g_{ij}(u,v) - f(s) \cdot n_{ij}}{u - v} \ge 0 \tag{3.3}$$

for all $u \neq v \in [-M, M]$ and all $s \in I(u,v)$.

Now we define *first* order finite volume schemes as follows:

DEFINITION 3.3 *Let $k = 0$, and let g be a two-point numerical E-flux. Then a first order finite volume scheme is given by (3.1), (3.2) and*

$$g_{ij}^n := g_{ij}(u_i^n, u_{ij}^n), \tag{3.4}$$

where $u_{ij}^n = u_k^n$ for $T_k = T_{ij}$.

Next we turn to the definition of the general class of *higher* order finite volume schemes considered in this paper. For each $(i,j) \in J$, let $x_{ijl} \in S_{ij}$ and $\omega_l \geq 0,\ l = 1, \ldots, m$ be given such that the quadrature formula

$$\int_{S_{ij}} \varphi(x)dx \approx |S_{ij}| \sum_{l=1}^{m} \omega_l \varphi(x_{ijl})$$

is exact for polynomials of degree $\leq k+1$. Given $\tilde{u}_h(\cdot, t^n) \in \widetilde{V}_{h,k}$, let

$$\begin{aligned} \tilde{u}_{ijl}^n &:= \lim_{x \to x_{ijl}, x \in T_i} \tilde{u}_h(x, t^n) \\ \tilde{v}_{ijl}^n &:= \lim_{x \to x_{ijl}, x \in T_{ij}} \tilde{u}_h(x, t^n). \end{aligned}$$

DEFINITION 3.4 *Let $k \geq 0$, and let g be a two-point numerical E-flux. Then a higher order finite volume scheme is given by (3.1), (3.2) and*

$$g_{ij}^n := \sum_{l=1}^{m} \omega_l g_{ij}(u_i^n + \tilde{u}_{ijl}^n, u_{ij}^n + \tilde{v}_{ijl}^n). \tag{3.5}$$

where $\tilde{u}_h(\cdot, t^n)$ is an arbitrary element of $\widetilde{V}_{h,k}$.

In order to choose a particular algorithm out of this general class, one needs to specify the construction of $\tilde{u}_h(\cdot, t^n)$. Let us give some examples:

EXAMPLE 3.5 *1) If $\tilde{u}_h(\cdot, t^n) \equiv 0$, recipe (3.5) reduces to (3.4), so we recover the first order schemes.*

2) For MUSCL-type finite volume schemes [37, 7] (see also [14] and the references therein) and ENO schemes [17], $\tilde{u}_h(\cdot, t^n)$ is constructed from $\overline{u}_h(\cdot, t^n)$ by interpolation.

3) For the discontinuous Galerkin method [6, 19], $\tilde{u}_h(\cdot, t^n)$ is constructed from $\overline{u}_h(\cdot, t^n)$, $u_h(\cdot, t^{n-1})$ and the differential equation (1.1) by a higher order finite element ansatz.

In order to prove convergence of the higher order schemes given in Definition 3.4, we will have to impose some further conditions on $\tilde{u}_h(\cdot, t^n)$. This will be discussed in Section 4. We finish this section with some remarks on E-fluxes which will be used in the convergence proofs below.

Any two-point numerical flux which is monotonely nondecreasing (resp. nonincreasing) in the first (resp. second) argument (or briefly **monotone**) and Lipschitz continuous is an E-flux. However, an E-flux needs to be neither monotone nor continuous. Two numerical fluxes are of fundamental importance:

1) *Godunov's flux* g^G (see [15]), which for scalar conservation laws may be written in the form given by Osher [30] as

$$g_{ij}^G(u,v) := \begin{cases} \min\limits_{u \leq s \leq v} f(s) \cdot n_{ij} & \text{if } u \leq v \\ \max\limits_{u \geq s \geq v} f(s) \cdot n_{ij} & \text{if } u > v \end{cases}$$

2) The *Lax-Friedrichs flux* g^{LF} (see [23]) given by

$$g_{ij}^{LF}(u,v) := \frac{f(u) + f(v)}{2} \cdot n_{ij} + \frac{q_{ij}}{2}(u - v),$$

where the q_{ij} are constants satisfying $q_{ij} \geq L_{ij} := \|f' \cdot n_{ij}\|_{L^\infty([-M,M])}$ and $q_{ij} = q_{kl}$ for $S_{ij} = S_{kl}$.

Both of these fluxes are monotone and Lipschitz continuous. Given a numerical flux g define its numerical viscosity Q by $Q_{ij}(u,v) := L_{ij}$ for $u = v$ and

$$Q_{ij}(u,v) := \frac{2}{(u-v)} \left[g_{ij}(u,v) - \frac{f(u) + f(v)}{2} \cdot n_{ij} \right]$$

for $u \neq v$. It was shown by Tadmor [35] that E-fluxes are precisely those numerical fluxes which have no less numerical viscosity than Godunov's flux. Given an E-flux g, let the corresponding Lax-Friedrichs flux be defined by

$$q_{ij} := \sup_{-M \leq u,v \leq M} (Q_{ij}(u,v), L_{ij})$$

Following Tadmor [35], we may now rewrite g as

$$g_{ij}(u,v) = (1-\theta_{ij}(u,v))g_{ij}^G(u,v) + \theta_{ij}(u,v)g_{ij}^{LF}(u,v), \tag{3.6}$$

where $\theta_{ij} : \mathbb{R}^2 \to [0,1]$ are measurable functions satisfying $\theta_{ij} = \theta_{kl}$ for $S_{ij} = S_{kl}$.

REMARK 3.6 *The freedom of choosing discontinuous E-fluxes is not only appealing mathematically, but might very well be significant in applications:*

1) Dubois and LeFloch [13] have shown existence and uniqueness of solutions of a class of boundary value problems in a domain $\Omega \subset \mathbb{R}^d$, where prescribed boundary data are interpreted as Riemann data given at the exterior side of the boundary. If one computes such problems with numerical schemes which are too dissipative, unphysical data will propagate into the interior of the domain. For that reason, we suggest to use Godunov's flux whenever S_{ij} lies on the boundary of the computational domain.

2) In order to resolve discontinuities sharply, one might try to use a flux with very little dissipation at large discontinuities, while using a less expensive flux otherwise. Let, for example, **tol** *be a given tolerance, and choose*

$$Q_{ij}(u,v) := \begin{cases} L_{ij} & \text{if } |u-v| < tol \\ Q_{ij}^G(u,v) & \text{otherwise} \end{cases}$$

Note that the resulting E-flux g_{ij} is discontinuous in (u,v).

The following observation will be essential to derive a discrete entropy inequality for Godunov's flux in Lemma 4.3:

LEMMA 3.7 *There is a family of piecewise continuous functions $w_{ij} : \mathbb{R}^2 \to \mathbb{R}$ such that for all $u,v \in \mathbb{R}$, $w_{ij}(u,v) \in I(u,v)$, $g_{ij}^G(u,v) = f(w_{ij}(u,v)) \cdot n_{ij}$ and*

$$w_{ij}(u,v) = u \quad \text{if } \partial_u g_{ij}^G(u,v) \text{ does not exist or is } \neq 0. \tag{3.7}$$

Proof: Fix $(i,j) \in J$. Since $f \cdot n_{ij}$ is continuous, w_{ij} can be chosen to be piecewise continuous with $w_{ij}(u,v) \in I(u,v)$ (w_{ij} is, of course, a Riemann-solver). If $u = v$, $w_{ij}(u,v) = u$.

Now suppose that $u < v$ and that $\partial_u g_{ij}^G$ does not exist at (u,v) or is not equal to zero. We have to show that $g_{ij}^G(u,v) = f(u) \cdot n_{ij}$. If this is not the case, then there exists a $z \in (u,v]$ with $f(z) \cdot n_{ij} = \min_{u \leq s \leq v} f(s) \cdot n_{ij} < f(u) \cdot n_{ij}$. Since $f \cdot n_{ij}$ is continuous, there exists a $\delta > 0$ such that

$$\min_{u-\delta \leq s \leq u+\delta} f(s) \cdot n_{ij} > f(z) \cdot n_{ij}.$$

Therefore, for all h with $|h| < \delta$,

$$g_{ij}^G(u+h,v) = \min_{u+h \leq s \leq v} f(s) \cdot n_{ij} = f(z) \cdot n_{ij} = g_{ij}^G(u,v).$$

This implies that $\partial_u g_{ij}^G(u,v)$ exists and equals zero, contrary to our assumption. It follows that $g_{ij}^G(u,v) = f(u) \cdot n_{ij}$. The case $u > v$ can be treated analogously. q.e.d.

Let (U,F) be an entropy pair for (1.1). An important property of Godunov's and Lax-Friedrichs' numerical fluxes is that they both posses natural numerical entropy fluxes, given by

$$\begin{aligned} G_{ij}^G(u,v) &:= F(w_{ij}(u,v)) \cdot n_{ij} \\ G_{ij}^{LF}(u,v) &:= \frac{F(u)+F(v)}{2} \cdot n_{ij} + \frac{q_{ij}}{2}(U(u)-U(v)). \end{aligned}$$

These numerical entropy fluxes satisfy partial analogues of the compatibility condition (2.1):

$$\partial_u G_{ij}^G(u,v) = U'(w_{ij}(u,v))\partial_u g_{ij}^G(u,v) \tag{3.8}$$

$$\partial_v G_{ij}^G(u,v) = U'(w_{ij}(u,v))\partial_v g_{ij}^G(u,v) \tag{3.9}$$

$$\partial_u G_{ij}^{LF}(u,v) = U'(u)\partial_u g_{ij}^{LF}(u,v) \tag{3.10}$$

$$\partial_v G_{ij}^{LF}(u,v) = U'(v)\partial_v g_{ij}^{LF}(u,v) \tag{3.11}$$

The compatibility conditions (3.10) and (3.11) where used by Lax [24] and later by many other authors to derive local entropy inequalities for the Lax-Friedrichs scheme. Given an E-flux g, we can now define an associated numerical entropy flux G analogously to (3.6).

4 Convergence

In this section we state and prove our main convergence result. The strategy of the proof is as follows:

In order to prove convergence of u_h using DiPerna's theory, one needs to establish two main estimates: an L^∞ bound, which assures that the sequence of approximate solutions contains a subsequence which converges to a Young measure, and an entropy inequality, which guarantees that this Young measure coincides with the admissible weak solution almost everywhere.

The L^∞ bound was already derived by Cockburn, Hou and Shu [6], who introduced the notion of a B-triangulation, and later by Liu [26] and Geiben [14] for general triangulations. Therefore, we will assume such a bound in the statement of our main theorem and focus on the derivation of the entropy inequality. The desired entropy inequality will be derived in three steps:

In a first step, we decompose the update formulae (3.2), (3.5) of the multidimensional finite volume scheme into its one-dimensional building blocks (see (4.1) – (4.3) below). Here we further refine an idea of Tadmor [35], which was already used and refined in [9, 5]. In some cases, this additional flexibility will enable us to weaken the CFL-condition for the timestep.

In a second step, we derive local discrete entropy inequalities for the one-dimensional Godunov and Lax-Friedrichs schemes, which are the building blocks of the multidimensional scheme (compare (4.2) – (4.3) and (4.13) – (4.14) below). The entropy inequality for the higher order Lax-Friedrichs scheme is taken from [19], while the one for the higher order Godunov scheme is new. It was shown by Osher and Tadmor [32] that in general higher order finite difference (and hence finite volume) schemes cannot satisfy a discrete entropy inequality as derived in Lemma 4.3 below. We therefore resort to a modification of such schemes called the antidiffusive flux approach. This approach requires that the higher order corrections $\tilde{u}^n_{ijl}$ of u^n_i be of order $O(h^\alpha)$, $\frac{1}{2} < \alpha \leq 1$. Even with such an assumption, the derivation of a discrete entropy inequality remains nontrivial. A more detailed discussion of the antidiffusive flux approach is given below.

In the third step, we combine the *local* discrete entropy inequalities derived for the one-dimensional building blocks into a *weak* entropy inequality for the approximate solution $\overline{u}_h$ (see (4.15) below). In order to conclude the convergence proof, one has to pass to the limit $h \to 0$ in this weak entropy inequality. Here one has to control a remainder, which is essentially of order $O(h^{2\alpha}/\Delta t)$. In the statement of our main theorem, we *assume* that $\lim_{h\to 0} h^{2\alpha}/\Delta t = 0$, which is a geometrical condition on the space-time grid. Since the timestep is coupled to the inner diameter of the cells via the CFL-condition (4.10), this geometrical condition may be translated into another one on the spatial grid alone (see (4.12) below). It turns out that condition (4.12) is weaker than the standard regularity condition (4.11) used by other authors and admits *irregular* families of grids, where cells may become flat in the limit $h \to 0$.

After finishing the proof of the main theorem, we state an alternative convergence theorem for first order finite volume schemes, where we may weaken some of the assumptions made for the higher order schemes. In particular, this allows taking a larger, and in some cases optimal, timestep.

We begin our analysis with the decomposition of the multidimensional update formulae (3.2), (3.5) into contributions from each node x_{ijl}. Let $f^n_{ij} := f(u^n_i) \cdot n_{ij}$ and note that $\sum_{j=1}^{k_i} |S_{ij}| f^n_{ij} = 0$. Let $\Delta x^n_{ij} \geq 0$ such that

$$\sum_{j=1}^{k_i} |S_{ij}| \Delta x^n_{ij} = |T_i|$$

and $\Delta x^n_{ij} \neq 0$ if $g^n_{ij} \neq f^n_{ij}$. Define

$$\begin{aligned} \lambda^n_{ij} &:= \begin{cases} \Delta t^n/\Delta x^n_{ij} & \text{if } \Delta x^n_{ij} \neq 0 \\ \Delta t^n/h_i & \text{if } \Delta x^n_{ij} = 0 \end{cases} \\ \sigma_{ij} &:= |S_{ij}| \Delta x^n_{ij}/|T_i|. \end{aligned}$$

Note that $\sum_{j=1}^{k_i} \sigma_{ij} = 1$. Let

$$g^n_{ijl} := g_{ij}(u^n_i + \tilde{u}^n_{ijl}, u^n_{ij} + \tilde{v}^n_{ijl})$$

$$\theta_{ijl}^n \;:=\; \theta_{ij}(u_i^n + \tilde{u}_{ijl}^n, u_{ij}^n + \tilde{v}_{ijl}^n)$$

and rewrite (3.2) as

$$u_i^{n+1} = \sum_{j=1}^{k_i} \sum_{l=1}^{m} \sigma_{ij}\omega_l \left[(1-\theta_{ijl}^n) u_i^{jl,n+1,G} + \theta_{ijl}^n u_i^{jl,n+1,LF}\right] \tag{4.1}$$

where $u_i^{jl,n+1,G}$ resp. $u_i^{jl,n+1,LF}$ are obtained via a one-dimensional Godunov resp. Lax-Friedrichs step:

$$u_i^{jl,n+1,G} \;:=\; u_i^n - \lambda_{ij}^n (g_{ijl}^{n,G} - f_{ij}^n) \tag{4.2}$$

$$u_i^{jl,n+1,LF} \;:=\; u_i^n - \lambda_{ij}^n (g_{ijl}^{n,LF} - f_{ij}^n). \tag{4.3}$$

It follows from the definition of ω_l and θ that u_i^{n+1} is a convex combination of $u_i^{jl,n+1,G}$ and $u_i^{jl,n+1,LF}$, $j = 1, \ldots, k_i$, $l = 1, \ldots, m$. Note that the parameters Δx_{ij}^n never have to be computed in practice, where one uses the update formulae (3.2) and (3.5) instead of (4.1) – (4.3). They are merely introduced in order to simplify the following analysis without restricting the timestep unnecessarily. An intuitive interpretation of this procedure will be given in Appendices A and B.

Next, we state our main convergence theorem. For simplicity of exposition, we restrict ourselves to initial data $u_0 \in L^\infty(\mathbb{R}^d)$ with compact support in $B_R \subset \mathbb{R}^d$. Our result may be extended to the case of data in $L^1 \cap L^\infty(\mathbb{R}^d)$ via techniques developed in [19].

THEOREM 4.1 *(main convergence theorem) Let $u_0 \in L^\infty(\mathbb{R}^d)$ with compact support in a ball of radius R. Let u_h be the family of approximate solutions of (1.1), (1.2) constructed with the (first or higher order) finite volume E-scheme (3.1), (3.2) and (3.5). Suppose that*

(i) There is a constant $M < \infty$ such that for all $h > 0$

$$\|u_h\|_{L^\infty(\mathbb{R}^d \times [0,T])} \leq M. \tag{4.4}$$

(ii) There are constants $C_0, C_1, C_2 > 0$ and $\alpha \in (\frac{1}{2}, 1]$ such that for all i, j, l and n

$$\tilde{u}_{ijl}^n = 0 \text{ for } |x_i| > R + C_0 t^n, \tag{4.5}$$

$$|\tilde{u}_{ijl}^n| \leq C_1 h^\alpha, \tag{4.6}$$

$$(u_i^n - u_{ij}^n)\tilde{u}_{ijl}^n \leq C_2 h^{2\alpha}. \tag{4.7}$$

(iii) For the same α, and for $\Delta t := \min_{0 \leq n < N} \Delta t^n$,

$$\lim_{h \to 0} h^{2\alpha}/\Delta t = 0. \tag{4.8}$$

(iv) There is a constant $\varepsilon > 0$ independent of h such that for $\delta := \varepsilon h/\sqrt{\Delta t}$ and for all i, j and n,

$$\lambda_{ij}^n q_{ij} \leq 1 \tag{4.9}$$

and

$$(1+\delta)\lambda_{ij}^n L_{ij} \leq 1. \tag{4.10}$$

Then for $1 \leq p < \infty$, u_h converges strongly in $L_{loc}^p(\mathbb{R}^d \times [0,T])$ to the entropy solution of (1.1), (1.2).

Let us discuss the assumptions of the theorem in some more detail:

REMARK 4.2 *1) Note that conditions (4.4) – (4.7) are trivially satisfied by the first order schemes, where the E-flux property (3.3) together with the CFL-condition (4.10) give the L^∞ bound, and for which all $\tilde{u}_{ijl}^n$ equal zero.*

2) Condition (4.6) has sometimes been called the antidiffusive flux approach. This terminology goes back to Boris and Book [1]. Similar conditions to the one used here have been used by several authors to enforce an L^∞-bound [6] or convergence to the entropy solution [28, 25, 38, 2, 9, 4]. Conditions (4.6) - (4.7) do not reduce the accuracy of the scheme in regions where the solution is sufficiently smooth.

Near discontinuities, however, the schemes are modified and degenerate to first order. It remains an open problem whether or not the unmodified MUSCL-type finite volume schemes and ENO schemes are convergent, even for a strictly convex, one-dimensional scalar conservation law. First steps in that direction are [29, 32]. As discussed in [6, 19, 14], conditions (4.6) – (4.7) may be implemented numerically via a flux-limiting projection.

3) Even though the previous results [5, 20, 10, 19] on finite volume schemes used unstructured *polygonal grids, they all assumed a* **regularity** *condition,*

$$h \le C\rho, \tag{4.11}$$

which is often used in the theory of finite elements. In Appendix B we show that the timestep is of the order of magnitude of ρ. Therefore, we only have to require the weaker geometrical condition

$$\lim_{h\to 0} h^{2\alpha}/\rho = 0 \tag{4.12}$$

in order to enforce convergence, so our convergence result is new even for first order, monotone finite volume schemes. Condition (4.12) allows the family of grids to become **irregular**, *i.e. cells may become flat in the limit. This may be important in applications, especially when a grid is refined adaptively near an essentially planar shock front or in a boundary layer. In Appendix C, we show that (4.8) and (4.12) are in fact necessary for convergence, and hence optimal.*

4) From (4.8), $\lim_{h\to 0}\delta = 0$. In Appendix B we show how the local CFL-conditions (4.10) may in some cases be translated into one global CFL-number of 1 in the limit $h \to 0$.

Besides the treatment of irregular grids, the main achievement of this paper is the derivation of a discrete entropy inequality for Godunov's flux (see (4.13) below), which is new even for one-dimensional finite difference schemes. As it was done by Kröner, Noelle and Rokyta [20, 19] for the first and higher order Lax-Friedrichs and Enquist-Osher schemes, we write the entropy dissipation of the higher order Godunov scheme in terms of the local oscillation of the numerical flux function instead of estimating it more roughly in terms of the local oscillation of the approximate solution itself, as in [9]. For the higher order schemes, there is a remainder in the entropy inequalities whose sign cannot be controlled (compare Osher and Tadmor [32], Theorem 7.3). We now use condition (4.6) to expand this remainder in powers of h^α. As for the monotone schemes, terms of order $O(1)$ have the correct sign. Terms of order $O(h^{2\alpha})$ can be controlled via (4.8). It remains to control the terms of order $O(h^\alpha)$. For the Lax-Friedrichs flux, these terms have the correct sign due to (4.7). For Godunov's flux, they vanish identically due to Lemma 3.7.

LEMMA 4.3 *(local entropy inequality) Let (U,F) be a convex entropy pair with $U \in C^2$, and let $\sigma := \min_{-M\le u\le M} U''(u)$. Then there is a constant C_3 depending only on C_0, C_1 and C_2 such that at each node x_{ijl}, the following entropy inequalities hold:*

$$\begin{aligned} & U(u_i^{jl,n+1,G}) - U(u_i^n) + \lambda_{ij}^n \left(G_{ijl}^{n,G} - F(u_i^n)\cdot n_{ij}\right) + \delta\frac{\sigma}{2}(\lambda_{ij}^n)^2\left(g_{ijl}^{n,G} - f(u_i^n)\cdot n_{ij}\right)^2 \\ \le\ & C_3 h^{2\alpha}\|U''\|_{L^\infty([-M,M])}\chi_i^n, \end{aligned} \tag{4.13}$$

$$\begin{aligned} & U(u_i^{jl,n+1,LF}) - U(u_i^n) + \lambda_{ij}^n \left(G_{ijl}^{n,LF} - F(u_i^n)\cdot n_{ij}\right) + \frac{\delta}{2+\delta}\frac{\sigma}{2}(\lambda_{ij}^n)^2\left(g_{ijl}^{n,LF} - f(u_i^n)\cdot n_{ij}\right)^2 \\ \le\ & C_3 h^{2\alpha}\|U''\|_{L^\infty([-M,M])}\chi_i^n, \end{aligned} \tag{4.14}$$

where $\chi_i^n = 1$ if $|x_i| \le R + C_0 t^n$ and $\chi_i^n = 0$ otherwise.

Proof: Let us first prove (4.13). Let i, j, l, n be fixed. We will use the notation $u := u_i$, $v := u_{ij}$, $\tilde u := \tilde u_{ijl}^n$, $\tilde v := \tilde v_{ijl}^n$, $\lambda := \lambda_{ij}^n$, $L := L_{ij}(M)$. For $0 \le \tau \le 1$ let $\gamma(\tau)$ (respectively $\gamma_1(\tau)$, $\gamma_2(\tau)$, $\Gamma(\tau)$, $\Gamma_1(\tau)$, $\Gamma_2(\tau)$, $\omega(\tau)$) be the function g_{ij}^G (respectively $\partial_u g_{ij}^G$, $\partial_v g_{ij}^G$, G_{ij}^G, $\partial_u G_{ij}^G$, $\partial_v G_{ij}^G$, w_{ij}) evaluated at $(u + \tau\tilde u, u + \tau(v - u + \tilde v))$. Let $\gamma_0 := \gamma(0), \Gamma_0 := \Gamma(0)$ and

$$b(\tau) := u - \lambda(\gamma(\tau) - \gamma_0).$$

Note that $b(0) = u$ and $b(1) = u_i^{jl,n+1,G}$. Denote the LHS of (4.13) by E. E is the entropy produced at node x_{ijl} during time $t^n \le t \le t^{n+1}$. We can calculate

$$E = \int_0^1 \left[U'(b(\tau))b'(\tau) + \lambda\Gamma'(\tau) + \delta\sigma\lambda^2(\gamma(\tau) - \gamma_0)\gamma'(\tau)\right] d\tau.$$

From the compatibility conditions (3.8), (3.9),

$$\Gamma'(\tau) = \Gamma_1(\tau)\tilde{u} + \Gamma_2(\tau)(v - u + \tilde{v}) = U'(\omega(\tau))[\gamma_1(\tau)\tilde{u} + \gamma_2(\tau)(v - u + \tilde{v})] = U'(\omega(\tau))\gamma'(\tau).$$

Since $U \in C^2$, it is possible to choose $\xi(\tau) \in I(b(\tau), \omega(\tau))$ such that

$$U'(\omega(\tau)) - U'(b(\tau)) = U''(\xi(\tau))(\omega(\tau) - b(\tau)).$$

Therefore, E can be rewritten as

$$E = \lambda \int\limits_{[0,1]\cap\{\omega(\tau)\neq u\}} \left[U''(\xi(\tau)) + [U''(\xi(\tau)) + \delta\sigma]\lambda \frac{\gamma(\tau) - \gamma_0}{\omega(\tau) - u}\right] (\omega(\tau) - u)\gamma'(\tau)d\tau$$

Note that from (4.10),

$$0 \leq U''(\xi(\tau)) + [U''(\xi(\tau)) + \delta\sigma]\lambda \frac{\gamma(\tau) - \gamma_0}{\omega(\tau) - u} \leq 2\|U''\|_{L^\infty([-M,M])}.$$

Next let us analyze $(\omega(\tau) - u)\gamma'(\tau)$.
a) If $|x_i| > R + C_0 t^n$, then from (4.5) $\tilde{u} = \tilde{v} = 0$. Since $\gamma_2(\tau) \leq 0$ and $\omega(\tau) \in I(u + \tau\tilde{u}, u + \tau(v - u + \tilde{v})) \subset I(u, v)$, we obtain

$$(\omega(\tau) - u)\gamma'(\tau) = (\omega(\tau) - u)(v - u)\gamma_2(\tau) \leq 0,$$

and $E \leq 0$.
b) If $|x_i| \leq R + C_0 t^n$ and $|v - u| \leq C_1 h^\alpha$, then from (4.6)

$$|(\omega(\tau) - u)\gamma'(\tau)| \leq |\omega(\tau) - u| \left[|\gamma_1(\tau)||\tilde{u}| + |\gamma_2(\tau)||v - u + \tilde{v}|\right] \leq 3C_1^2 L h^{2\alpha},$$

so

$$E \leq 6C_1^2 \lambda L h^{2\alpha} \|U''\|_{L^\infty([-M,M])} \leq 6C_1^2 h^{2\alpha} \|U''\|_{L^\infty([-M,M])}.$$

c) If $|x_i| \leq R + C_0 t^n$ and $|v - u| > C_1 h^\alpha$, then we use (3.7) to assert that if $\gamma_1(\tau) \neq 0$, then $\omega(\tau) = u + \tau\tilde{u}$. Therefore, $(\omega(\tau) - u)\gamma_1(\tau)\tilde{u} = \tau\gamma_1(\tau)\tilde{u}^2$. Moreover, $\mathrm{sign}(v - u + \tilde{v}) = \mathrm{sign}(v - u)$, so $\gamma_2(\tau)(v - u + \tilde{v})(\omega(\tau) - u) \leq 0$ as in a). From this and (4.6)

$$E \leq 2C_1^2 \lambda L h^{2\alpha} \|U''\|_{L^\infty([-M,M])} \leq 2C_1^2 h^{2\alpha} \|U''\|_{L^\infty([-M,M])}.$$

This completes the derivation of (4.13). The estimate (4.14) for the Lax-Friedrichs scheme can be derived similarly, and using (4.7) (see also [19]). q.e.d.

Using the decomposition (4.1) – (4.3) of the multidimensional finite volume scheme, we now combine the *algebraic* entropy inequalities derived above for the one-dimensional building blocks into a weak discrete entropy inequality for the multidimensional scheme. The remainder which occurs in this entropy inequality can then be controlled using the *geometrical* constraints (4.8) respectively (4.12) for the space-time grid.

Proof of Theorem 4.1: With the help of Lemma 4.3, the proof can be finished using techniques similar to those already established in [34, 8, 5, 20, 19]. Here we restrict ourselves to showing how the regularity condition (4.11) used in those papers is replaced by (4.8).

As discussed in Section 2, $\overline{u}_h$ converges weak-star to a Young measure ν. We now derive a weak entropy inequality for ν. Let $\phi \in C_0^\infty(\mathbb{R}^d \times [0, T))$, $\phi \geq 0$,

$$\phi_i^n := \frac{1}{|T_i|} \int_{T_i} \phi(x, t^n) dx,$$

and

$$\phi_{ij}^n := \frac{1}{\Delta t^n |S_{ij}|} \int\limits_{t^n}^{t^{n+1}} \int_{S_{ij}} \phi(x, t) dS dt.$$

Using the decomposition (4.1) – (4.3), Lemma 4.3 and Jensen's inequality, one can show that

$$\int_0^T \int_{\mathbb{R}^d} [U(\overline{u}_h(x,t))\partial_t\phi(x,t) + F(\overline{u}_h(x,t)) \cdot \nabla\phi(x,t)] dx dt + \int_{\mathbb{R}^d} U(\overline{u}_h(x,0))\phi(x,0) dx \geq R_h + S_h \quad (4.15)$$

where

$$R_h := -\sum_{n=0}^{N-1} \Delta t^n \sum_{i\in I} |T_i| \sum_{j=1}^{k_i} \sigma_{ij} (U(u_i^{j,n+1}) - U(u_i^n))(\phi_i^{n+1} - \phi_{ij}^n),$$

$$u_i^{j,n+1} := u_i^n - \lambda_{ij}^n (g_{ij}^n - f_{ij}^n)$$

and

$$S_h := -C(\phi) \|U''\|_{L^\infty([-M,M])} h^{2\alpha} \sum_{n=0}^{N-1} \sum_{i\in I} |T_i| \chi_i^n.$$

From (4.8), $\lim_{h\to 0} S_h = 0$. We now show that $\lim_{h\to 0} R_h = 0$. Using Hölder's inequality we obtain

$$\begin{aligned}
|R_h| &\leq \|U'\|_{L^\infty([-M,M])} \sum_{n=0}^{N-1} \sum_{i\in I} |T_i| \sum_{j=1}^{k_i} \sigma_{ij} |\phi_i^{n+1} - \phi_{ij}^n| \, \lambda_{ij}^n \, |g_{ij}^n - f_{ij}^n| \\
&\leq \|U'\|_{L^\infty([-M,M])} \left(\sum_{n=0}^{N-1} \sum_{i\in I} |T_i| \sum_{j=1}^{k_i} \sigma_{ij} |\phi_i^{n+1} - \phi_{ij}^n|^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{n=0}^{N-1} \sum_{i\in I} |T_i| \sum_{j=1}^{k_i} \sigma_{ij} (\lambda_{ij}^n)^2 (g_{ij}^n - f_{ij}^n)^2 \right)^{1/2}.
\end{aligned}$$

The first factor is $O(h^2/\Delta t)$. Using the entropy inequalities (4.13) and (4.14) together with Jensen's inequality, one can show that the second factor is $O((1 + h^{2\alpha}/\Delta t)/\delta) = O(\delta^{-1})$. Therefore,

$$|R_h| \leq C_4 \left(\frac{h^2}{\delta \Delta t} \right)^{\frac{1}{2}} \leq C_5 \left(\frac{h^2}{\Delta t} \right)^{\frac{1}{4}}$$

which tends to zero from (4.8). We have thus shown that

$$\int_0^T \int_{\mathbb{R}^d} [\langle \nu, U\rangle \phi_t + \langle \nu, F\rangle \cdot \nabla \phi] \geq \int_{\mathbb{R}^d} U(u_0)\phi(\cdot, 0). \tag{4.16}$$

This is the main estimate needed for DiPerna's convergence theory. q.e.d.

As noted above, Theorem 4.1 includes convergence of first order finite volume schemes. In this case, however, we may replace the CFL-condition (4.10) by an even weaker condition: Now in the derivation of the L^∞ bound and the entropy inequality only the derivative of the numerical flux function with respect to the second argument comes into play. The proof of Theorem 4.1 thus includes the following result:

THEOREM 4.4 *Let $u_h : [0,T] \to V_{h,0}$ be defined by (3.1), (3.2) and (3.4), and let*

$$L_{ij}^n := \|\partial_v g_{ij}(u_i^n, \cdot)\|_{L^\infty(I(u_i^n, u_{ij}^n))}.$$

Then u_h converges to the entropy solution provided that

(i)

$$\lim_{h\to 0} h^2/\Delta t = 0. \tag{4.17}$$

(ii) There is a constant $\varepsilon > 0$ independent of h such that for $\delta := \varepsilon h/\sqrt{\Delta t}$ and for all i, j and n,

$$\lambda_{ij}^n q_{ij} \leq 1 \tag{4.18}$$

and

$$(1+\delta)\lambda_{ij}^n L_{ij}^n \leq 1. \tag{4.19}$$

In the Appendices we give a geometrical interpretation of the algebraic entropy inequality (4.13), discuss the CFL-conditions (4.10) and (4.19) and the irregularity conditions (4.8) and (4.12) for the space-time grid.

A Geometrical interpretation of the entropy production of the first order Godunov scheme

In this appendix we interpret the entropy dissipation of Godunov's scheme in terms of the waves entering a cell during one timestep.

Let u be the entropy solution of the one-dimensional Riemann-problem

$$u_t + f(u)_x = 0 \tag{A.1}$$

$$u(x,0) = \begin{cases} u_L & \text{for } x \leq 0 \\ u_R & \text{for } x > 0. \end{cases} \tag{A.2}$$

Let $\lambda := \frac{\Delta t}{\Delta x}$, $u_M := u(0, \Delta t)$, and

$$\begin{aligned} \overline{u} &:= \frac{1}{\Delta x}\int_0^{\Delta x} u(x,\Delta t)dx, \\ \overline{U} &:= \frac{1}{\Delta x}\int_0^{\Delta x} U(u(x,\Delta t))dx. \end{aligned}$$

Let g be Godunov's flux consistent with $-f$. Then $g(u_R, u_L) = -f(u_M)$, and

$$\overline{u} = u_R - \lambda(f(u_R) - f(u_M)).$$

Note that this is the building block of the multidimensional finite volume scheme (4.1) – (4.3). In Lemma 4.3, it was shown that if

$$(1+\delta)\lambda L \leq 1, \tag{A.3}$$

then

$$E := U(\overline{u}) - U(u_R) + \lambda(F(u_R) - F(u_M)) + \delta\frac{\sigma}{2}\lambda^2(f(u_R) - f(u_M))^2 \leq 0. \tag{A.4}$$

We will now give a simple alternative proof. Let

$$\begin{aligned} E_{ex} &:= \overline{U} - U(u_R) + \lambda(F(u_R) - F(u_M)) \\ E_{av} &:= U(\overline{u}) - \overline{U} \\ E^* &:= \delta\frac{\sigma}{2}\lambda^2(f(u_R) - f(u_M))^2 \end{aligned}$$

Then $E = E_{ex} + E_{av} + E^*$. Since u is the entropy solution of the Riemann problem (A.1) (A.2), $E_{ex} \leq 0$, where equality holds if $u(\cdot, \Delta t)$ is continuous. We thus have to show that

$$E_{av} + E^* \leq 0.$$

Let

$$s := \frac{f(u_R) - f(u_M)}{u_R - u_M},$$

i.e. if the wave entering a cell is a single shock, it will travel with speed s. From (3.3), $s \geq 0$. Now

$$\overline{u} = \lambda s\, u_M + (1 - \lambda s)u_R.$$

From (A.3), $u(x, \Delta t) = u_R$ for $x \geq \frac{\Delta x}{1+\delta}$, i.e. even the fastest waves do not reach the opposite boundary of the computational cell during one timestep. If $s > 0$, Jensen's inequality will therefore be a strict inequality. To quantify this, let

$$z := \frac{1+\delta}{\Delta x}\int_0^{\frac{\Delta x}{1+\delta}} u(x,\Delta t)dx.$$

Then

$$\overline{u} = \frac{1}{1+\delta}z + \frac{\delta}{1+\delta}u_R.$$

Let

$$V(v) := \frac{v - u_R}{z - u_R}U(z) + \frac{v - z}{u_R - z}U(u_R) + \frac{\sigma}{2}(v - z)(v - u_R)$$

i.e. $V(v)$ is the parabola through $(z, U(z))$, $(u_R, U(u_R))$ with $V''(v) = \sigma$. Since $\sigma = \min_{B_R} U''$, we have that for all $v \in I(z, u_R)$, $U(v) \leq V(v)$. In particular,

$$U(\overline{u}) \leq V(\overline{u}) = \frac{1}{1+\delta} U(z) + \frac{\delta}{1+\delta} U(u_R) - \delta \frac{\sigma}{2} \lambda^2 s^2 (u_R - u_M)^2.$$

On the other hand, by Jensen's inequality,

$$\overline{U} \geq \frac{1}{1+\delta} U(z) + \frac{\delta}{1+\delta} U(u_R).$$

Combining the last two inequalities we obtain

$$-E_{av} = \overline{U} - U(\overline{u}) \geq \delta \frac{\sigma}{2} \lambda^2 s^2 (u_R - u_M)^2 = E_*,$$

which implies (A.4).

B Discussion of the CFL-condition

In this section we give some sufficient CFL-conditions which may be useful in practice. For linear and monotone conservation laws, we argue that our CFL-condition is weaker than those used in previous convergence proofs (see, for example, [35, 5, 20, 19]) and compare it to the von Neumann necessary condition for linear stability.

Let

$$\Delta x_{ij}^n = \frac{|T_i| L_{ij}}{\sum_{l=1}^{k_i} |S_{il}| L_{il}},$$

and for first order schemes define Δx_{ij}^n analogously, but with L_{ij} replaced by L_{ij}^n. If we define $L_i := (\sum_{j=1}^{k_i} |S_{ij}| L_{ij}) / (\sum_{j=1}^{k_i} |S_{ij}|)$, $L_i^n := (\sum_{j=1}^{k_i} |S_{ij}| L_{ij}^n) / (\sum_{j=1}^{k_i} |S_{ij}|)$, $\Delta x_i := (|T_i|) / (\sum_{j=1}^{k_i} |S_{ij}|)$, and $\lambda_i^n := (\Delta t^n)/(dxi)$, then (4.10) becomes

$$\sup_{i,n,h} \lambda_i^n L_i < 1, \tag{B.1}$$

respectively

$$\sup_{i,n,h} \lambda_i^n L_i^n < 1 \tag{B.2}$$

for first order schemes. We have thus chosen Δx_{ij}^n in such a way as to allow the maximal timestep over each cell.

The following sharp estimate relates ρ_i and Δx_i:

LEMMA B.1 *Let $T_i \in \mathcal{T}$ be a d-dimensional convex polygon with inner diameter ρ_i. Then*

$$\frac{\rho_i}{2d} \leq \Delta x_i \leq \frac{\rho_i}{2}. \tag{B.3}$$

Proof: The proof relies on the observation that

$$|T_i| = \frac{1}{d} \sum_{j=1}^{k_i} h_i^j |S_{ij}| \geq \frac{\rho_i}{2d} \sum_{j=1}^{k_i} |S_{ij}|,$$

where h_i^j is the distance of the center of a maximal ball contained in T_i to the side S_{ij}. This gives the first inequality in (B.3). The second inequality is trivial. q.e.d.

We can thus see that Δt is coupled to the inner diameter of the cells, $\Delta t \leq C\rho$. Together with Theorem 4.1 the lemma implies that for unstructured, irregular meshes the finite volume schemes will converge provided that

$$\frac{\Delta t^n L_*}{\rho} < \frac{1}{2d}, \tag{B.4}$$

where $L_* := \|f'\|_{L^\infty([-M,M])}$. If the cells are made of d-dimensional rectangles, we can do better: let u_* and n_* be such that $|n_*| = 1$ and $L_* = f'(u_*) \cdot n_*$. Note that $k_i = 2d$. Now

$$\begin{aligned}\sum_{j=1}^{k_i} |S_{ij}| L_{ij} &\leq \frac{|T_i|}{\rho_i} \sum_{j=1}^{2d} |f'(u_*) \cdot n_{ij}| \\ &\leq 2\sqrt{d}\, \frac{|T_i|}{\rho_i} |f'(u_*)|,\end{aligned}$$

which gives the sufficient CFL-condition

$$\frac{\Delta t^n L_*}{\rho} < \frac{1}{2\sqrt{d}}. \tag{B.5}$$

However, conditions (B.4) and (B.5) are much stronger than (B.1) and (B.2). In fact, for linear conservation laws and rectangular meshes, (B.2) is arbitrarily close to the von Neumann necessary condition for linear stability, which can easily be computed to be

$$\sup_{i,n} \lambda_i^n L_i^n \leq 1 + O(\Delta t).$$

More generally, if the grid is rectangular and the fluxes $f \cdot n_{ij}$ are monotone, then the CFL-condition

$$\frac{\Delta t^n L_*}{\rho} < \frac{1}{\sqrt{d}}$$

guarantees convergence of the first order Godunov scheme to the entropy solution. We have thus overcome the usual CFL-restriction of $\frac{1}{2}$ (compare the discussion in [35]). For the one-dimensional Godunov scheme, this improvement may be interpreted in terms of the waves entering a cell of length h: a CFL condition of 0.5 guarantees that incoming waves cannot interact during a timestep Δt and thus allows to decompose the scheme into building blocks of the form (4.2). This corresponds to choosing all $\Delta x_{ij}^n = \frac{h}{2}$. If, however, waves enter the cell only from one side, as is the case for monotone conservation laws, we set $\Delta x_{ij}^n = h$ on that side and $\Delta x_{ij}^n = 0$ on the opposite side. Now we can compute with a CFL number of 1. Carrying out the analogous argument for a monotone conservation law and the Lax-Friedrichs scheme yields a CFL-condition of 2/3 instead of 0.5.

C Sharp lower bounds on Δt and ρ.

Consider the two-dimensional scalar conservation law

$$\frac{\partial}{\partial_t} u(x,y,t) + \frac{\partial}{\partial_x} f(u(x,y,t)) + \frac{\partial}{\partial_y} g(u(x,y,t)) = 0, \tag{C.1}$$

where $f, g \in C^2(\mathbb{R})$ with $L_f := \|f'\|_{L^\infty([-M,M]} \neq 0$ and $L_g := \|g'\|_{L^\infty([-M,M]} \neq 0$. Consider a cartesian grid with mesh sizes Δx and Δy, so that $h^2 = \Delta x^2 + \Delta y^2$ and $\rho = \min(\Delta x, \Delta y)$. Let T_0 be a cell with center (x_0, y_0) and let $u^0 := \overline{u}_h(x_0, y_0, t^{n+1})$, $u_0 := \overline{u}_h(x_0, y_0, t^n)$, $u_1 := \overline{u}_h(x_0 + \Delta x, y_0, t^n)$, $u_2 := \overline{u}_h(x_0, y_0 + \Delta y, t^n)$, $u_3 := \overline{u}_h(x_0 - \Delta x, y_0, t^n)$, $u_4 := \overline{u}_h(x_0, y_0 - \Delta y, t^n)$. Let $\sigma_{0j}^n := \frac{1}{4}$ and $q_{0j}^n := \frac{1}{\lambda_{0j}^n}$, $j = 1, ..., 4$. Then the first order Lax-Friedrichs scheme may be written as

$$\frac{u^0 - u_0}{\Delta t} + \frac{f(u_1) - f(u_3)}{2\Delta x} + \frac{g(u_2) - g(u_4)}{2\Delta y} = \frac{\Delta x^2}{8\Delta t} \frac{u_1 - 2u_0 + u_3}{\Delta x^2} + \frac{\Delta y^2}{8\Delta t} \frac{u_2 - 2u_0 + u_4}{\Delta y^2}. \tag{C.2}$$

Suppose now that (4.10) holds, i.e. there is a constant $\delta > 0$ such that

$$4(1+\delta) L_f \Delta t / \Delta x \leq 1, \;\; 4(1+\delta) L_g \Delta t / \Delta y \leq 1. \tag{C.3}$$

Then Theorem 4.1 shows that the scheme (C.2) converges to the entropy solution u of (C.1) if

$$\lim_{h \to 0} \frac{h^2}{\Delta t} = 0. \tag{C.4}$$

We will now show that (C.4) is also necessary for convergence:

The finite difference equation (C.2) is consistent with the differential equation

$$\frac{\partial}{\partial_t}u(x,y,t) + \frac{\partial}{\partial_x}f(u(x,y,t)) + \frac{\partial}{\partial_y}g(u(x,y,t)) = \mu\frac{\partial^2}{\partial_x^2}u(x,y,t) + \nu\frac{\partial^2}{\partial_y^2}u(x,y,t),$$

where $\mu := \lim_{h\to 0}\Delta x^2/(8\Delta t)$, $\nu := \lim_{h\to 0}\Delta y^2/(8\Delta t)$. Therefore, (C.2) is consistent with (C.1) only if $\mu+\nu = 0$, which is equivalent to (C.4).

Finally, let us consider irregular cartesian meshes, w.l.o.g. $\lim_{h\to 0}\Delta y/\Delta x = 0$. Then (C.3) implies that $\Delta t \le C\Delta y = C\rho$, so (C.4) implies that

$$\lim_{h\to 0}\frac{h^2}{\rho} = 0,$$

which gives the optimal rate with which cells may become flat in the limit.

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