Radially symmetric solutions for a class of hyperbolic systems of conservation laws.

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Dedicated to Professor Rolf Leis on the occasion of his 65th birthday.

Abstract

For a class of two hyperbolic conservation laws in two space dimensions, which includes the Weyl equation of relativistic quantum mechanics and the complex Burgers equation as prototypes, we derive a class of radially symmetric solutions. These solutions are multiple valued, a fact which does have a physical interpretation in case of the Weyl equation.

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We consider conservation laws of the form

$$\overline{U_t} + F(U)_z = 0, \tag{1}$$

where $U = u + iv$, $\overline{U} = u - iv$ and $z = x + iy$ are complex-valued and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. $F = f + ig$ is a holomorphic function of U. Written in real variables, (1) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} b & a \\ a & -b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y = 0,$$

where $a = \frac{1}{2}f_u = \frac{1}{2}g_v$ and $b = -\frac{1}{2}f_v = \frac{1}{2}g_u$. This equation has been proposed by P. Lax [5] as a particularly simple symmetric hyperbolic system of conservation laws in two space variables. The linear system, when $F(U) = 2U$, $a \equiv 1$ and $b \equiv 0$, has a physical interpretation: the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are two of the Pauli-matrices which occur in the Dirac equation and the Weyl equation in relativistic quantum mechanics [4]. The Weyl equation governs the evolution of the waveamplitudes of a particle of spin $\frac{1}{2}$ and mass zero (the neutrino). If these waveamplitudes are real valued, then they depend only on two space dimensions, and their evolution is governed by (1).

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When $F(U)$ is quadratic, (1) is called the complex Burgers equation. In fact, solutions which depend only on one space variable and whose data are related to only one of the two corresponding Riemann-invariants, are governed by the one-dimensional inviscid Burgers equation.

Analytical difficulties arise mainly near points where $F'(U) = 0$. At such points, (1) is not strictly hyperbolic. So far, these difficulties have been studied mostly in the context of the one-dimensional Riemann problem for the complex Burgers equation [8, 3]. Solving this Riemann problems requires the use of non-standard waves, including so-called undercompressive, or crossing shocks.

In this note, we present a class of radially symmetric solutions for the case of a homogeneous flux function F, i.e.

$$\overline{U}_t + \frac{2}{n+1}(U^{n+1})_z = 0, \tag{2}$$

where $n \geq 0$ is fixed. For this purpose, let $z = \rho e^{i\theta}$, so $\rho_z = \overline{z}/2\rho$ and $\theta_z = -i/2z$. We look for solutions of the form

$$U(z,t) = V(\rho,t)\varphi(\theta). \tag{3}$$

Then

$$2U_z = (V_\rho \varphi - \frac{i}{\rho} V \varphi_\theta) e^{-i\theta}.$$

The conservation law (2) now implies that

$$\overline{V}_t \overline{\varphi} + V^n (V_\rho \varphi - \frac{i}{\rho} V \varphi_\theta) \varphi^n e^{-i\theta} = 0.$$

In order to separate variables, we assume that $\varphi_\theta = ik\varphi$, so $\varphi(\theta) = e^{ik\theta}$ and

$$\overline{V}_t + V^n (V_\rho + \frac{k}{\rho} V) e^{i((n+2)k-1)\theta} = 0.$$

If we set $k = \frac{1}{n+2}$, we obtain

$$U(z,t) = V(\rho,t) e^{i\theta/(n+2)}, \tag{4}$$

where V satisfies the one-dimensional equation

$$\overline{V}_t + V^n (V_\rho + \frac{1}{(n+2)\rho} V) = 0. \tag{5}$$

We now rescale variables in order to simplify the one-dimensional problem (5): Let

$$\rho(r) = \left(\frac{2(n+1)}{n+2} r\right)^{\frac{n+2}{2(n+1)}} \tag{6}$$

and

$$W(r,t) = \rho(r)^{1/(n+2)} V(\rho(r),t). \tag{7}$$

Then W satisfies the one-dimensional equation

$$\overline{W}_t + \frac{1}{n+1}(W^{n+1})_r = 0. \tag{8}$$

Remark: (i) The radially symmetric solution given by (4) is $(n+2)$-valued. For the Weyl equation ($n = 0$), the double-valuedness of the solution corresponds to the change of sign of the spin of a particle of spin $\frac{1}{2}$ observed by an observer who travels once around the origin.

(ii) At the origin the solution U blows up with rate $(1/\rho)^{1/(n+2)}$. This is a typical example of the focussing of characteristics (compare Rauch [7]).

(iii) As a subclass of solutions to (8) for $n = 1$, one obtains solutions of the scalar one-dimensional inviscid Burgers equation. Other special solutions, which exhibit undercompressive shocks, were presented in [6]. The radially symmetric solutions corresponding to these one-dimensional solutions can be used to validate numerical algorithms for multi-dimensional systems of conservation laws (see [2] for first numerical experiments).

(iv) It is possible to generalize the calculation given above to show the equivalence of weak space periodic solutions $W \in L^\infty([r_1, r_2] \times [0, T))$ of (8) with weak radially symmetric solutions $U \in L^\infty(\mathcal{R} \times [0, T))$ of (2), where $\mathcal{R} = \{(\rho, \theta) : \rho_1 \leq \rho \leq \rho_2, 0 \leq \theta \leq 2\pi(n+2)\}$. We omit the details.

(v) So far, no general existence theorem for weak solutions of nonstrictly hyperbolic conservation laws has been established even in one space dimension. Recently, however, Chen and Kan [1] announced an existence result for a class of 2×2 systems with quadratic flux functions, which includes the one-dimensional complex Burgers equation, i.e. equation (8) with $n = 1$. It should be possible to obtain weak radially symmetric solutions of the two-dimensional complex Burgers equation as a corollary to their results.

References

[1] Chen, G.-Q. and Kan, P.-T., *Hyperbolic conservation laws with umbilic degeneracy I.* Arch. Rational Mech. Anal. *130*, 231-276 (1995).

[2] Fabian, G., *Numerical experiments for the multi-dimensional upwind method of Colella.* Diploma thesis, Bonn University (1996).

[3] Gomes, M., *Riemann-problems requiring a viscous profile entropy condition.* Adv. Appl. Math. *10*, 285–323 (1989).

[4] Landau, L. and Lifschitz, E., *Relativistic quantum theory.* Course of theoretical physics, Vol. 4.1. Pergamon press (1971).

[5] Lax, P., *Hyperbolic systems of conservation laws in several space variables.* in: Current topics in partial differential equations, edited by Y.Ohya, K. Kasahara & N. Shimakura. Tokyo (1986), p. 327.

[6] Noelle, S., *Development of singularities for the complex Burgers equation.* Nonlinear Analysis, T.M.A., *26*, 1313-1321 (1996).

[7] Rauch, J., BV-estimates fail for most quasilinear hyperbolic systems in dimensions greater than one. Comm. Math. Phys. *106*, 481-484 (1986).

[8] Shearer, M., Schaeffer, D., Marchesin, D. and Paes-Leme, P., *Solutions of the Riemann problem for a prototype* 2×2 *system of nonstrictly hyperbolic conservation laws.* Arch. Rational Mech. Anal. *97*, 299–320 (1987).