MULTIDIMENSIONAL FLUX-VECTOR-SPLITTING AND HIGH-RESOLUTION CHARACTERISTIC SCHEMES

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1. Introduction

Since the work of Godunov, Van Leer, Harten-Lax and Roe, the numerical solution of systems of hyperbolic conservation laws is dominated by Riemann-solver based schemes. These one-dimensional schemes are usually extended to several space-dimensions either by using dimensional-splitting on cartesian grids or by the finite-volume approach on unstructured grids. The first systematic criticism of using one-dimensional Riemann-solvers for multi-dimensional gas-dynamics goes back to Roe himself: the Riemann-solver is applied in the grid- rather than the flow-direction, which may lead to a misinterpretation of the local wave-structure of the solution.

Among the genuinely multi-dimensional alternatives, which were developed since the mid-eighties, let us mention the fluctuation-splitting schemes of Roe, Deconinck, Van Leer et.al. (see (Deconinck et.al., 1993) for references), the Corner-Transport-Upwind (CTU) scheme (Colella, 1990), CLAWPACK (LeVeque, 1997), the Weighted-Average-Flux (WAF) scheme (Billet and Toro, 1997) and the Evolution-Galerkin-Method (Lukacova et.al., 1997).

In this contribution we focus on yet another multi-dimensional approach, Fey's Method of Transport (**MoT**) (Fey, 1998; Fey, 1998), which belongs to the family of flux-vector-splitting schemes. The starting point of Fey's algorithm is a multi-dimensional wave-model, which leads to a reformulation of the system of conservation laws as a finite set of coupled nonlinear advection equations. At the beginning of each timestep, these coupled nonlinear equations are decomposed into a set of linear scalar advection equations with variable coefficients, which are then solved numerically using characteristic schemes.

One-dimensional first-order flux-vector-splitting schemes split the flux-vector in the center of a cell into components which are transported across the left and right boundaries of that cell. At sonic points, where eigenvalues change their sign, this procedure (which we would like to call **C**ell-**C**entered-**E**volution) may lead to inconsistencies (Steger and Warming, 1981). Indeed, we have constructed a linear advection equation with smooth variable coefficients and smooth solutions for which Fey's first-order scalar characteristic scheme diverges at the sonic points.

Motivated by this discovery we have developed a new version of the MoT based on **I**nterface-**C**entered-**E**volution, the **MoT-ICE** (Noelle, 1999). The new method draws ideas from the flux-vector-splitting and the flux-difference-splitting approaches: the multi-dimensional wave-models are inherited from Fey's Method of Transport or other flux-vector-splitting schemes, while a predictor-step which provides auxiliary transport-velocities on the cell-interfaces uses flux-difference-splitting techniques.

For the new method, we have proved uniform first- resp. second-order consistency, including sonic points. Numerical experiments confirm second-order-accuracy for smooth solutions and high-resolution nonoscillatory shock-capturing properties for discontinuous solutions.

The second-order version of the new MoT-ICE is four to five times faster than Fey's second-order scheme and seems to be as fast as standard second-order algorithms. This gain of efficiency is partly due to an improved linearisation and decomposition of the nonlinear system into advection equations, and to our particularly simple characteristic transport algorithm for the resulting linear advection equations.

In ongoing work with C. von Törne, we are developing the MoT-ICE into a distributed-parallel, fully adaptive code, capable of handling general geometries.

2. Multi-dimensional Linearisation

We consider multidimensional systems of n conservation laws in d space dimensions,

$$\partial_t\, U + \nabla \cdot \mathbf{f}\,(\mathbf{U}) = \mathbf{0}$$

which can be written in Fey's advection form

$$\sum_{l=1}^{L} \Big(\partial_t\; S_l(U) + \nabla \cdot (S_l(U)\; \mathbf{a_l}(\mathbf{U})^{\mathbf{T}})\Big) = 0, \tag{1}$$

where the $L \geq n$ waves $S_l : \mathbf{R}^n \to \mathbf{R}^n$ and $\mathbf{a_l} : \mathbf{R^n} \to \mathbf{R^d}$ satisfy the consistency conditions

$$\sum_{l=1}^{L} S_l(U) = U$$

and

$$\sum_{l=1}^{L} S_l(U)\ \mathbf{a_l}(\mathbf{U})^{\mathbf{T}} = \mathbf{f}(\mathbf{U}).$$

Examples of such systems include the wave-equation, the shallow-water-equations, the Euler-equations and the MHD-equations (Fey et.al., 1997).

Let us discuss how to update (1) from time t_n to $t_{n+1} = t_n + k$. Our first remark is that if one linearises this system by freezing the velocities at the half-timestep $t_{n+1/2} = t_n + k/2$ and solves the resulting coupled linear system, then the linearisation-error during a single timestep is $\mathcal{O}(k^3)$, so the overall linearisation is second-order-accurate in time.

The next step is to decompose the coupled linear system by setting each of the L summands to zero separately. This leads to a first-order decomposition-error. The leading-order term of this error can be removed by modifying the initial data for the decoupled linear equations. The modified initial data are of the form

$$Z_l(t_n) = S_l(U(t_n)) + \frac{k}{2}\ R_l(U(t_n), \nabla\cdot U(t_n))$$

with

$$\begin{aligned} R_l(U, \nabla \cdot U) \quad &:= \quad -S_l'(U)\ \nabla\cdot\mathbf{f}(\mathbf{U}) + \nabla\cdot(\mathbf{S_l}(\mathbf{U})\ \mathbf{a_l}(\mathbf{U})^{\mathbf{T}}) \qquad (2)\\ &= \quad \partial_t\ S_l(U) + \nabla\cdot(S_l(U)\ \mathbf{a_l}(\mathbf{U})^{\mathbf{T}}). \end{aligned}$$

Note that the non-zero term R_l is the residual of the decomposition. In (Noelle, 1999) we prove that the resulting linearisation and decomposition is second-order-accurate in space and time.

For the Euler equations and the shallow-water equations, Morel, Fey and Maurer have also derived a second-order-accurate decomposition into linear advection equations. Instead of evaluating the velocities at the half-timestep they freeze them at the original timestep. As a consequence, they have to add correction terms for a mixed linearisation- and decomposition-error, which are computationally more expensive than (2).

3. Interface Centered Evolution

The linearisation and decomposition given in the previous section lead us, at the beginning of each timestep, to a set of transport equations of the form

$$\partial_t \varphi + \nabla \cdot (\varphi \, \mathbf{a^T}) = \mathbf{0} \tag{3}$$

with initial data φ given at time t_n.

Equation (3) may be solved by introducing the characteristics $\mathbf{z}(\tau; \mathbf{x}, \mathbf{t})$ via

$$\begin{aligned} \mathbf{z} : \mathbf{R}_+ \times \mathbf{R^d} \times \mathbf{R}_+ &\to \mathbf{R}^d \\ \mathbf{z}(\mathbf{t}; \mathbf{x}, \mathbf{t}) &= \mathbf{x} \\ \partial_\tau \mathbf{z}(\tau; \mathbf{x}, \mathbf{t}) &= \mathbf{a}(\mathbf{z}, \tau). \end{aligned}$$

Since the flux-vector $\varphi \mathbf{a^T}$ in (3) is always parallel to the characteristics, we have the following identity for the update: For all $K \subset \mathbf{R}^d$,

$$\int_K \varphi(\mathbf{x}, \mathbf{t_{n+1}}) \, \mathbf{dx} = \int_{\mathbf{z}(\mathbf{t_n}; \mathbf{K}, \mathbf{t_{n+1}})} \varphi(\mathbf{x}, \mathbf{t_n}) \, \mathbf{dx}. \tag{4}$$

We now approximate the characteristic flow as follows: On each edge, or interface, we define an auxiliary transport-velocity in the direction normal to the interface. Using these auxiliary velocities, we move the interfaces backwards in time for a timestep k. For a two-dimensional cartesian grid, this results in a subdivision of each cell into up to nine subcells. These rectangular subcells approximate the subcells which would occur when tracing the interfaces backwards in time using the exact characteristic flow. The update is computed by replacing the exact characteristic flow in (4) by our approximate flow. All we need to do now is to define the auxiliary transport velocities. Up to higher-order terms, there is a unique choice which guarantees second-order consistency in 2D:

$$\hat{a}_{i+\frac{1}{2},j} = \left[a^* + \frac{k}{2}(a^* a_x - b^* a_y) \right]_{i+\frac{1}{2},j} + \mathcal{O}(k^2)$$

on the interfaces in x-direction and

$$\hat{b}_{i,j+\frac{1}{2}} = \left[b^* + \frac{k}{2}(b^* b_y - a^* b_x) \right]_{i,j+\frac{1}{2}} + \mathcal{O}(k^2)$$

on the interfaces in y-direction, where

$$\begin{aligned} a^* &= a + \frac{k}{2} a_t + \mathcal{O}(k^2) \\ b^* &= b + \frac{k}{2} b_t + \mathcal{O}(k^2) \end{aligned}$$

are predicted velocities at the half timestep on the interfaces. In (Noelle, 1999) we prove that this procedure is uniformly second-order-consistent for scalar equations (3) with smooth coefficients $\mathbf{a} = (\mathbf{a}, \mathbf{b})$.

For nonlinear systems of conservation laws, where the velocity-field (a, b) is not known a-priori, it remains to define $a^*, b^*, a_x, a_y, b_x, b_y$ on the interfaces. We predict the values at the half-timestep by a one-dimensional flux-difference-splitting step, for example simply by a Lax-Friedrichs-step. Based on these predicted values a^*, b^*, we choose an upwind-weighted mean of the derivatives on both sides of an interface for a_x, a_y, b_x, b_y. These algorithmic details are crucial for stability in the presence of shocks. See (Noelle, 1999) for details.

4. Numerical Experiments

In Figure 1 we display 2D-computations of a radially-symmetric explosion for the shallow-water equations. One can see glitches at the sonic points for Fey's first-order scheme, which uses Cell-Centered-Evolution (MoT-CCE-P0) common to other flux-vector-splitting schemes. These kinks do not occur for the first-order version of our new scheme (MoT-ICE-P0), and our second-order version (MoT-ICE-P1) resolves this problem very well.

Let us give a first comparison of efficiencies. Morel (Morel, 1997) reports that the MoT-CCE-P0, Van Leer's flux-vector-splitting and CLAWPACK $T^{1,0}$ (using the first-order Roe-solver without transverse wave-propagation) all use the same amount of cpu-time (say one time unit). CLAWPACK $T^{1,1}$ takes 1.5 units, and a first-order fix at sonic points proposed by Morel 2.9 units. Our preliminary experience with the MoT-ICE is the following: the MoT-ICE-P0 takes 0.9 to 1.0 units and, hence, it is as fast as standard first-order schemes. The MoT-ICE-P1 takes 2.0 - 2.2 units, which is the same as the second-order CLAWPACK $T^{2,2}$. This compares favorably with Fey's MoT-CCE-P1 (Fey, 1998), which is consistent at sonic points, but needs 10.5 units of cpu-time.

Acknowledgement: This work was supported by DFG-SPP ANumE and SFB256 at Bonn University.

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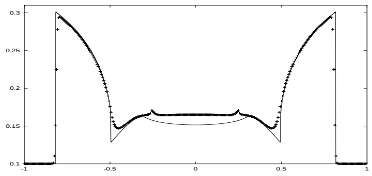
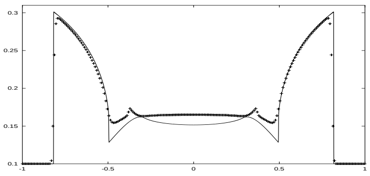
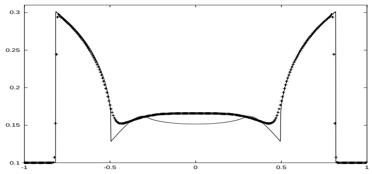


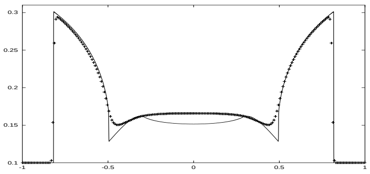
MoT-CCE-P0, x-axis



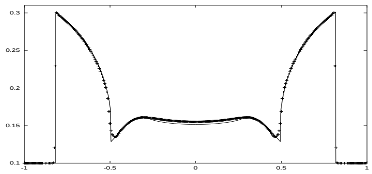
MoT-CCE-P0, diagonal



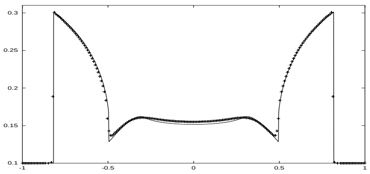
MoT-ICE-P0, x-axis



MoT-ICE-P0, diagonal



MoT-ICE-P1, x-axis



MoT-ICE-P1, diagonal

Figure 1. Cross-sections of 2D-computations of a radially-symmetric explosion for the shallow-water-equations. Plot of water-height for grids of 320×320 points. Fey's first-order MoT-CCE-P0 (top), our new first-order MoT-ICE-P0 (middle) and second-order MoT-ICE-P1 (bottom). Left column: cuts along the x-axis. Right column: cuts along the diagonal. Solid line: exact solution (one-dimensional computation with 3200 points). Note the kinks at the sonic points for the MoT-CCE-P0.

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