The MOTICE: a new multi-dimensional wave-propagation-algorithm based on Fey's Method of Transport with application to the Euler- and MHD-equations

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1 Multidimensional Schemes

Since the work of Godunov, Van Leer, Harten-Lax and Roe, the numerical solution of systems of hyperbolic conservation laws is dominated by Riemann-solver based schemes. These one-dimensional schemes are usually extended to several space-dimensions either by using dimensional-splitting on cartesian grids or by the finite-volume approach on unstructured grids. The first systematic criticism of using one-dimensional Riemann-solvers for multi-dimensional gas-dynamics goes back to Roe himself: the Riemann-solver is applied in the grid- rather than the flow-direction, which may lead to a misinterpretation of the local wave-structure of the solution.

Since the mid-eighties, Roe, Deconinck, Van Leer and many others developed the so-called fluctuation-splitting schemes for the equations of gas-dynamics (see [4] for references). Other multi-dimensional approaches include Colella's Corner-Transport-Upwind (CTU) scheme [3], LeVeque's CLAWPACK [12], the Weighted-Average-Flux (WAF) scheme of Billet and Toro [1] and the Evolution-Galerkin-Method [13].

In this contribution we focus on Fey's Method of Transport (**MoT**) [6, 7], which belongs to the family of flux-vector-splitting schemes. The starting point of Fey's algorithm is a multi-dimensional wave-model, which leads to a reformulation of the system of conservation laws as a finite set of coupled nonlinear advection equations. At the beginning of each timestep, these coupled nonlinear equations are decomposed into a set of linear scalar advection equations, which are then solved numerically using characteristic schemes.

2 Sonic Points

One-dimensional first-order flux-vector-splitting schemes split the flux-vector in the center of a cell into components which are transported across the left and right boundaries of that cell. At sonic points, where eigenvalues change their sign, this procedure (which we would like to call **C**ell-**C**entered-**E**volution) may lead to inconsistencies [19]. Indeed, we have constructed a linear advection equation with smooth variable coefficients and smooth solutions for which Fey's first-order scalar characteristic scheme diverges at the sonic points.

3 The MOTICE

Motivated by this discovery we have developed a new version of the MoT based on **I**nterface-**C**entered-**E**volution, the **MOTICE** [16]. The new method draws ideas from the flux-vector-splitting and the flux-difference-splitting approaches: the multi-dimensional wave-models are inherited from Fey's Method of Transport or other flux-vector-splitting schemes, while a predictor-step which provides auxiliary transport-velocities on the cell-interfaces uses flux-difference-splitting techniques.

For the new method, we have proved uniform first- resp. second-order consistency, including at sonic points. Numerical experiments confirm second-order-accuracy for smooth solutions and high-resolution nonoscillatory shock-capturing properties for discontinuous solutions. For example, the new scheme produces excellent radially symmetric solutions for two-dimensional flows, see Figure 1.

The second-order version of the new MOTICE is several times faster than Fey's second-order scheme and seems to be as fast as standard second-order algorithms. This gain of efficiency is partly due to an improved linearisation and decomposition of the nonlinear system into advection equations, and to our particularly simple characteristic transport algorithm for the resulting linear advection equations.

The method has been implemented in a general framework. Main applications are multi-d Euler- and magnetohydrodynamic (MHD) simulations, but other systems of hyperbolic equations can be implemented as well.

4 Parallelization

Multi-dimensional high-resolution computations are extremely time-consuming. This is especially true for wave models with comparatively many waves (such as MHD) as well as any 3d application. Parallelization of the code becomes essential. We have chosen a block-oriented approach where macro elements (blocks) that contain an equal amount of computational work are distributed to the different processors. Load balancing is done using a space filling curve based on work by Zumbusch [8, 9]. This approach is well-suited for two and three dimensions and adaptive grids.

Results of high resolution computations are shown in figure 2 (see [11], example 3, for the setup).

5 Ideal Magnetohydrodynamics (MHD)

A second application is the system of ideal MHD equations which describes the motion of conducting fluid in a magnetic field.

One very important issue in this system is the so-called "div $\underline{\mathbf{B}}$ constraint": The set of equations for the magnetic field $\underline{\mathbf{B}}$ implicates that its divergence is a constant in time, such that if initial data yield div $\underline{\mathbf{B}} = 0$, analytically this will remain so for all time.

Unfortunately, this is not necessarily true for the numerical solution. Due to small numerical errors and numerical viscosity, small amounts of divergence will be created. This of course violates the constraint, and a non-vanishing divergence raises the question of the physical relevance of the numerical solution [18].

There are some approaches to overcome this problem: First, one can use a Helmholtz-decomposition of the numerically computed $\underline{\mathbf{B}}$-field, splitting it up in a divergence-free part and a gradient field. This decomposition can be computed by solving a Poisson's problem, with the potential as the unknown and the divergence as the right hand side. A similar problem is known in the incompressible Navier-Stokes equations, the constraint here is on the velocity field $\underline{\mathbf{u}}$.

Second, a source term proportional to div $\underline{\mathbf{B}}$ can be introduced to the MHD equations. This idea is due to Powell [17]. A possible interpretation of this approach is that any divergence created by the numerical scheme is advected away with the speed of the fluid.

Both these approaches can lead to problems. The Helmholtz-decomposition needs some boundary values for the divergence. This problem is not so

severe as it can be overcome [5]. But since the Poisson equation is of elliptic type, information travels with infinite speed – in other words, divergence created at one end of the domain may change the value of the magnetic field at some very distant point of the domain. This is in contrast with the finite characteristic velocities of the hyperbolic system under consideration and so not a desirable feature.

In Powell's idea, a problem arises when there is a stagnation point of the velocity field in the computational domain. Following the interpretation of the "advected divergence", non-vanishing divergence will accumulate at the stagnation point which will lead to physically questionable solutions.

Another interesting approach is by Fey and Limacher [14]. In addition to the wave model, they implement a different explicit solver for the evolution of the magnetic field in such a way that the divergence due to a given discrete operator is not changed. The solver is entirely local, so that no information is passed on at infinite speed.

A recent way of dealing with the divergence problem is by Kemm, Munz et. al. ([10, 15]). The idea is to modify the equation for the magnetic field and to solve an additional equation for a quantity that my be interpreted as a generalized Lagrange multiplier incorporating the constraint on $\underline{\mathbf{B}}$.

6 Numerical results

First results for the 1d MHD Riemann problem by Brio and Wu [2] with different correction techniques for the magnetic field are shown in figure 3. The top left figure shows a comparison of a first and a second order MOTICE without any treatment for the divergence constraint; we can see that while the first order method does not have any obvious problems after the compound wave, the second order method does. However, if we do not update the B_x compoment at all, the undesirable effect disappears (top right). The bottom two graphics show a comparison between second order MOTICE with no B_x update and a second order MOTICE with Powell's source term (left) and with the use of an projection scheme (right). The use of a projection scheme seems to be a bit sharper than Powell's source term, however, Powell's source term is much cheaper in terms of computation time.

7 Future Extensions

In ongoing work we are developing the MOTICE into a fully adaptive, distributed parallel code, capable of handling general geometries.

8 Acknowledgement

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References

[1] Billet, S. and Toro, E., *On WAF-type schemes for multidimensional hyperbolic conservation laws.* J. Comput. Phys. 130 (1997), 1 – 24.

[2] Brio, M. and Wu, C.C. *An Upwind Differencing Scheme for the Equations of Ideal Magnetohydrodynamics.* J. Comp. Phys. 75 (1988), 236–243.

[3] Colella, P., *Multidimensional upwind methods for hyperbolic conservation laws.* J. Comput. Phys. 87 (1990), 171 – 200.

[4] Deconinck, H., Paillère, H. Struijs, R. and Roe, P. *Multidimensional upwind schemes based on fluctuation-splitting for systems of conservation laws.* Comput. Mech. 11 (1993), 323 – 340.

[5] DeSterck, H., private communication, 1998.

[6] Fey, M. *Multidimensional upwinding. I. The method of transport for solving the Euler equations.* J. Comput. Phys. 143 (1998), 159 – 180.

[7] Fey, M. *Multidimensional upwinding. II. Decomposition of the Euler equations into advection equations.* J. Comput. Phys. 143 (1998), 181 – 199.

[8] Griebel, M., Zumbusch, G. *Hash-Storage Techniques for Adaptive Multilevel Solvers and their Domain Decomposition Parallelization.* Proceedings of Domain Decomposition Methods 10, editor: J. Mandel, C. Farhat, X.-C. Cai, Contemporary Mathematics 218, AMS, 279–286 (1998)

[9] Griebel, M., Zumbusch, G. *Parallel multigrid in an adaptive PDE solver based on hashing.* Proceedings of ParCo '97, editors: E. D'Hollander, G.R. Joubert, F.J. Peters, U. Trottenberg, Elsevier, 589–599, 1998.

[10] Kemm, F.; Munz, C.-D.; Dedner, A.; Krner, D.; Wesenberg, M. *Hyperbolic Divergence Cleaning for MHD Equations.* Preprint (in preparation), March 2000.

[11] Lax, P.D. and Liu, X.D. *Solution of Two-Dimensional Riemann Problems of Gas Dynamics by Positive Schemes.* SIAM J. Sc. Comp. 19 (2), 319–340, 1998.

[12] LeVeque, R.J., *Wave Propagation Algorithms for Multidimensional Hyperbolic Systems.* J. Comput. Phys. 131 (1997), 327 – 353.

[13] Lukacova-Medvidova, M., Morton, K. and Warnecke, G. *Evolution Galerkin methods for hyperbolic systems in two space dimensions.* Report 97-44, Univ. Magdeburg, Germany (1997).

[14] Limacher, R. *Simulation der MHD-Gleichungen mit der Transportmethode.* Diplom thesis, ETH Zurich, 2000 (unpublished).

[15] Munz, C.-D., Schneider, R., Sonnendrcker, E., Voss, U. *Maxwell's Equations When the Charge Conservation is Not Satisfied.* to appear in C. R. Acad. Sci. Paris, 1999.

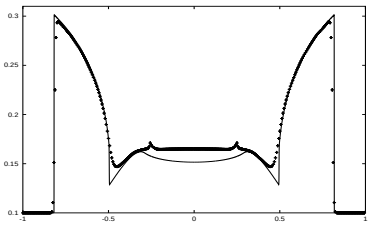
[16] Noelle, S., *The MOTICE: a new high-resolution wave-propagation algorithm based on Fey's Method of Transport.* Invited plenary lecture. "Proceedings of the Second International Symposium on Finite Volumes for Complex Applications – Problems and Perspectives", Duisburg, Germany, July 1999, 95 – 115.

[17] Powell, K.G. *An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension).* ICASE Report No. 94-24, Langley, VA, 1994.

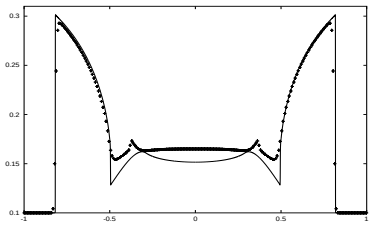
[18] Quirk, J.J. *A contribution to the great Riemann solver debate.* Intern. J. Numer. Meth. Fluids 18 (6), 555–574 (1994)

[19] Steger, J. and Warming, R., *Flux vector splitting of the inviscid gas-dynamic equations with applications to finite difference methods.* J. Comput. Phys. 40 (1981), 263 – 293.

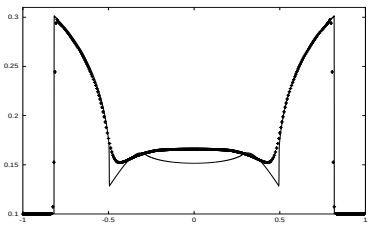
Figure 1: Explosion problem for the shallow water-equations. Plot of water-height for grids of 320 $\times$ 320 points. Fey's first-order MoT-CCE-P0 (top), our new first-order MOTICE-P0 (middle) and second-order MOTICE-P1 (bottom). Left column: cuts along the x-axis. Right column: cuts along the diagonal. Solid line: one-dimensional solution with 3200 points. Note the kinks at the sonic points for the MoT-CCE-P0.



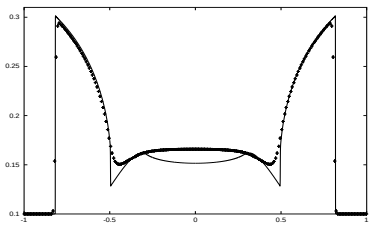
MoT-CCE-P0, x-axis



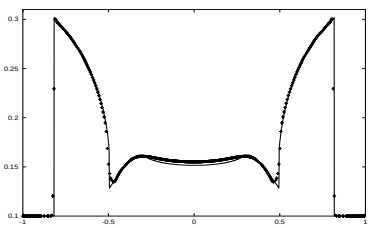
MoT-CCE-P0, diagonal



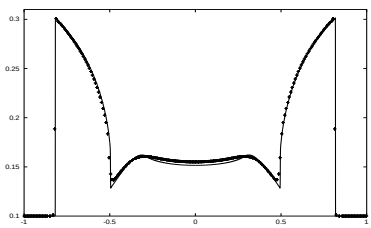
MOTICE-P0, x-axis



MOTICE-P0, diagonal



MOTICE-P1, x-axis



MOTICE-P1, diagonal

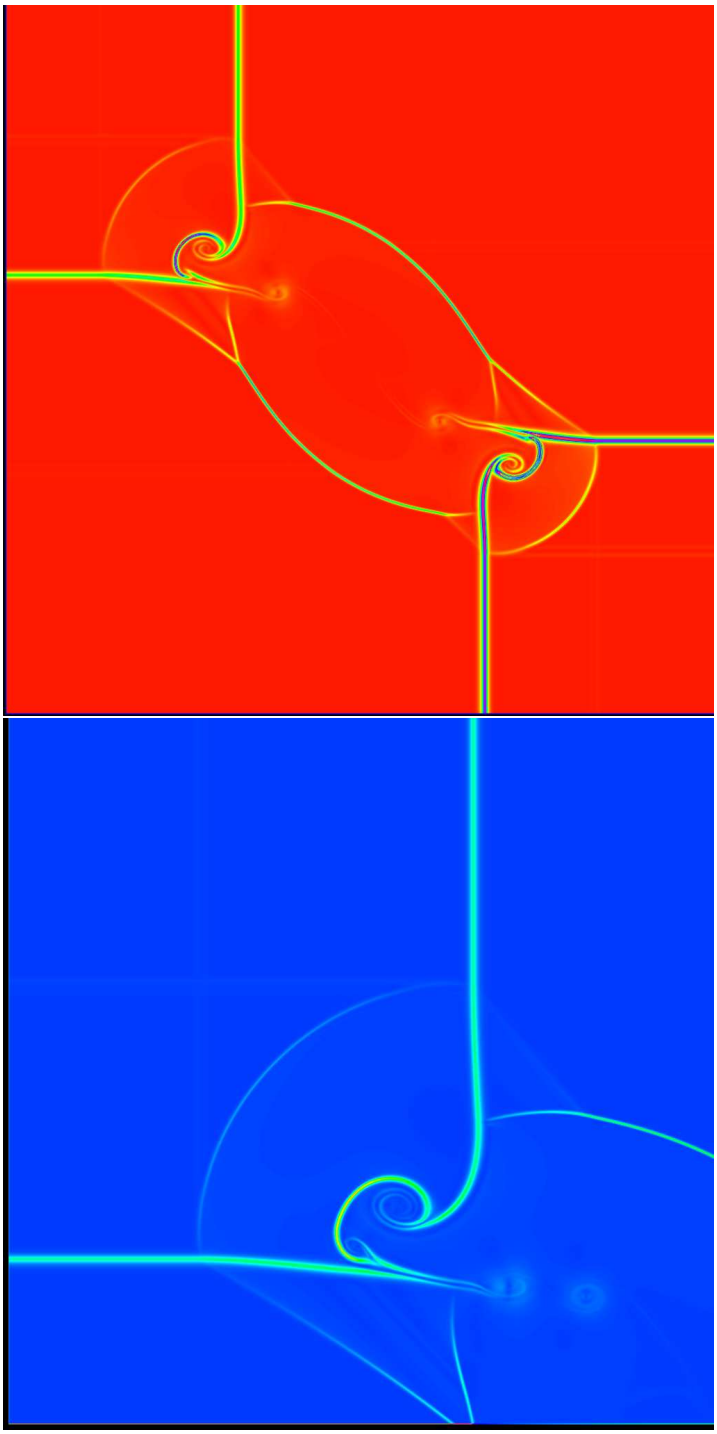


Figure 2: 2d Riemann problem for the Euler equations, schlieren-type plots. $\Delta x = 1/1024$ (top) and $\Delta x = 1/2048$ (bottom). The lower picture is a detail of the NW region of the top picutre.

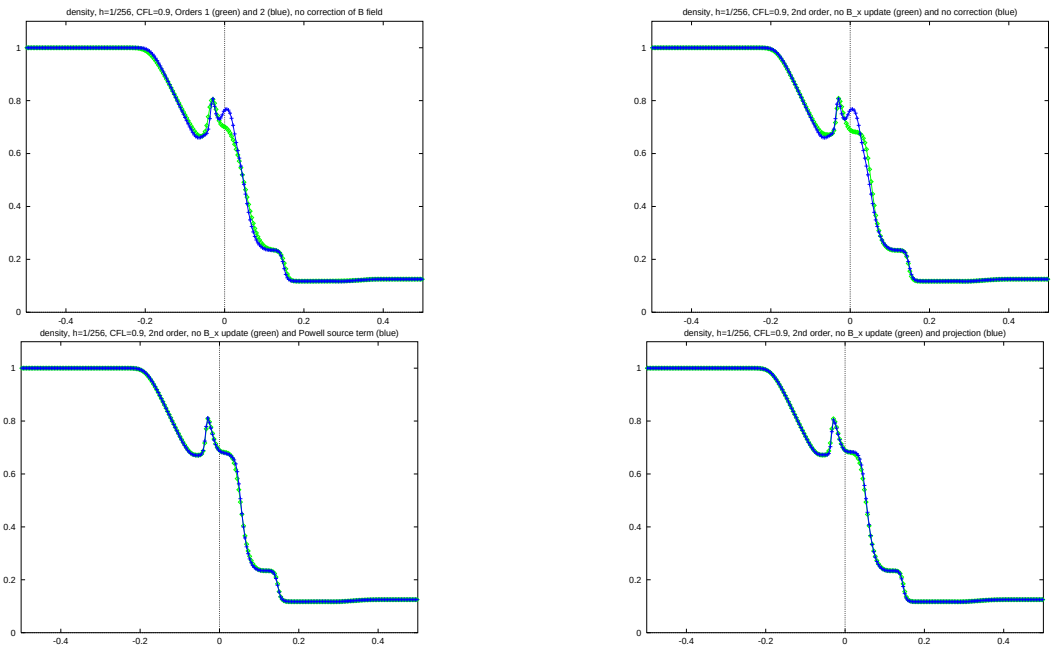


Figure 3: Riemann Problem of Brio and Wu. No correction of magnetic field, 1st and 2nd order (top left), no update of B_x and no correction (top right), Powell source term and no B_x update (bottom left), and projection and no B_x update (bottom right)