On the resolution and stability of central difference schemes

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ABSTRACT. *We present several improvements for the Jiang–Nessyahu–Tadmor second-order, nonoscillatory, central difference scheme on two-dimensional, staggered, Cartesian grids for systems of conservation laws. They concern the flux quadrature to reduce grid orientation effects, a more powerful version of Harten's artificial compression method to improve the resolution of linear discontinuities, some experiments and analysis of instabilities in 2D as well as an adaptive staggered scheme on structured Cartesian grids.*

KEYWORDS: *flux quadrature, artificial compression, stability, adaptive staggered grids*

1. Introduction

Non-oscillatory, central difference schemes [LAX 54, NES 90, ARM 95, JIA 98] approximate the solutions of hyperbolic systems of conservation laws on sequences of staggered one- or multi-dimensional grids. Due to the staggering of the grids these schemes avoid the usual characteristic decompositions or Riemann solvers needed to stabilize upwind schemes. Therefore they yield especially compact and simple computer code and can easily be applied to systems for which no Riemann solver exists. For recent extensions and applications, see [KAR 01, KUR 01, LEV 00, ROM 00, TAI 02].

2. An Improved Quadrature Rule

In the forthcoming paper [LIE 02a], we propose a new staggered scheme which uses a carefully designed integration rule for the flux computations and thereby takes more propagation directions into account than the original cartesian staggered scheme [ARM 95, JIA 98]. This effectively reduces the grid orientation effects and improves the accuracy for smooth solutions.

3. Artificial Compression

In another forthcoming paper [LIE 02b], we study a version of Harten's Artificial Compression Method (ACM) [HAR 83] which has been suggested in [NES 90] to sharpen the resolution of linear discontinuities. For the prototype of a scalar advection equation, we can identify this ACM with a standard second order scheme using a certain limiter function [LIE 02b]. Another member of this class is the well-known limiter of Van Leer. It turns out that the above version of ACM is more dissipative than the second order scheme using Van Leer's limiter, and numerical experiments show that both schemes are not able to keep linear discontinuities sharp. We go on to derive sufficient conditions on the limiters that guarantee the TVD (Total-Variation Diminishing) property. This allows us to derive TVD-limiters which are much more compressive than Van Leer's. Numerical experiments show the effectiveness of schemes based on these limiters when applied to *discontinuous* solutions. In figure 1, we display results of a shock-vortex interaction, all computed on the same grid, but with different limiting strategies. ACM, if applied to the linear fields only, improves the resolution of the density peaks dramatically (middle figure). Applying ACM also to the nonlinear fields creates entropy-violating rarefaction shocks (bottom figure). For further computations, see [LIE 02c].

4. Multidimensional Stability

Our next issue is multi-dimensional stability. In [JIA 98] a local maximum principle is proved for two-dimensional staggered schemes and scalar conservation laws under a rather restrictive CFL-number of about 0.1 (depending on the limiter). In practice, CFL numbers close to 0.5 are often used (this is the largest CFL number under which the staggered scheme may be derived). We give an example of a smooth planar wave diagonal to the grid for which the *local* maximum principle is violated as soon as the theoretically justified CFL condition is not observed. We also show that the numerical solution may become *globally* unstable at CFL numbers around 0.4.

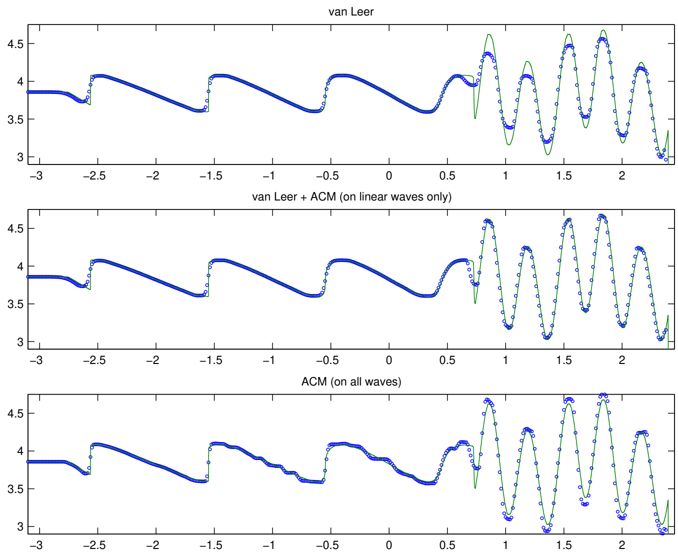

Figure 1. *Comparison of density approximations of the shock-vortex interaction computed on a 800 cell grid by three different limiting strategies.*

In [JIA 98] a numerical convergence study was performed for the scalar transport equation

$$u_t + u_x + u_y = 0, \qquad u(x,y,0) = \sin\bigl(\pi(x+y)\bigr). \tag{1}$$

Using the MM_1 limiter and a CFL number of 0.475, Jiang and Tadmor observed a lack of convergence which they explained by the 'clipping phenomenon'. We here propose another explanation for the observed lack of convergence. Figure 2 shows the solution at time $t = 0.5$ along the line $x = y$ computed with the MM_1 limiter. As is evident from the middle plot (320 grid with CFL number $\nu = 0.475$), the numerical solution contains oscillations. On the 160 grid, the oscillations only appear for CFL numbers close to 0.5, while on the 640 grid they are clearly visible already at $\nu = 0.45$.

Taking the particular situation of this example into account, we have proved a discrete local maximum-principle as in Theorem 1 of [JIA 98],

$$\min\{\bar{w}^n_{jk}, \bar{w}^n_{j+1,k}, \bar{w}^n_{j,k+1}, \bar{w}^n_{j+1,k+1}\} \le \bar{w}^n_{j+\frac{1}{2},k+\frac{1}{2}} \le \max\{\bar{w}^n_{jk}, \bar{w}^n_{j+1,k}, \bar{w}^n_{j,k+1}, \bar{w}^n_{j+1,k+1}\} \tag{2}$$

under the sharper CFL condition

$$\nu \le C_\theta := \frac{1}{2\sqrt{2}\theta}\left(\sqrt{2} - \sqrt{1 + (\theta - 1)^2}\right) \tag{3}$$

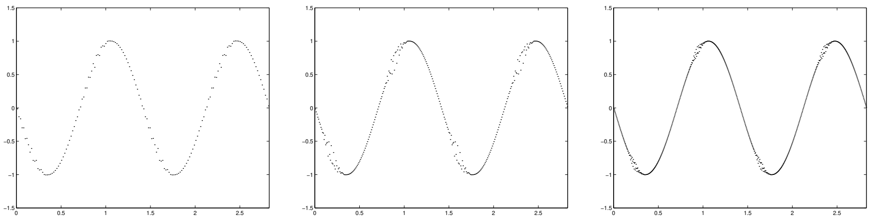

Figure 2. *Solution at time* $t = 0.5$ *along the line* $x = y$. *Computations with CFL number* ν *and* MM_1 *limiter on a* $N \times N$ *grid;* $N = 160$, $\nu = 0.495$ *(left),* $N = 320$, $\nu = 0.475$ *(middle), and* $N = 640$, $\nu = 0.45$ *(right).*

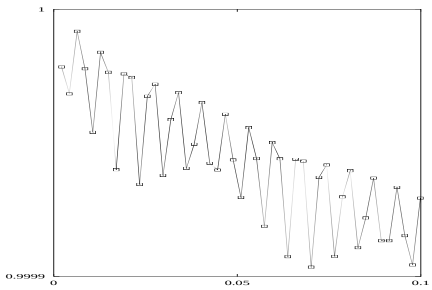

Figure 3. *Violation of the* local *maximum-principle for the* MM_1 *limiter; plot of* $\max u$ *versus time with 160 points* $\nu = 0.34$.

Here the parameter $\theta \in [0, 2]$ determines the limiter MM_θ. Compare [LIE 02a] for the notation.

Let us first discuss a number of experiments for the MM_1 limiter. Note that $C_1 = (\sqrt{2} - 1)/2\sqrt{2} \approx 0.146$. Running the example with $\nu = 0.15$ and 160 points, the local maximum-principle [2] is already violated for some time steps, which shows the sharpness of condition [3]. In most time steps, however, the maximum decreases, and the global maximum-principle

$$\min_{x,y} u_0(x, y) \leq \bar{w}^n_{jk} \leq \max_{x,y} u_0(x, y) \text{ for all } j, k \qquad [4]$$

is still satisfied. At this stage there is a balance of the dissipative nature of the scheme and its tendency to become oscillatory. This is illustrated in Figure 3 which shows the results of a single calculation with $\nu = 0.34$ as time progresses. Note that if [2] were satisfied, the points would form a monotonely decreasing curve. For values of ν larger than 0.36 the maximum begins to *grow* with time, so [4] is violated. Note that the variation of the solution increases with the extrema of the solution, so the TVD-principle is violated as well. Increasing the CFL number even further will lead to the creation of new

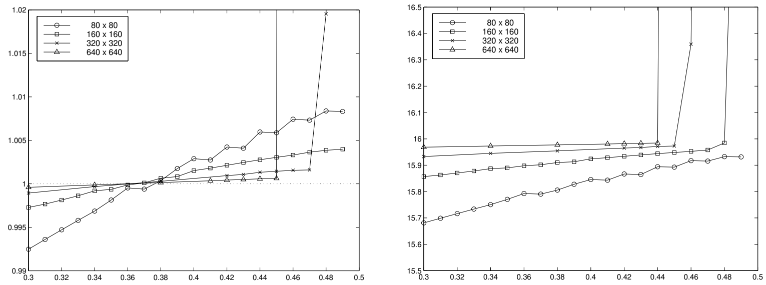

Figure 4. *Violation of the* global *maximum-principle and the TVD-principle for the MM_1 limiter; (left) plot of* $\max|u|$ *at time* $t = 0.5$ *versus CFL number, (right) plot of TV-norm at time* $t = 0.5$ *versus CFL number.*

local extrema in the numerical solution. Thus another stability-principle, the "Number-of-Extrema-Decreasing" (NED) principle [LIU 98] will be violated. For example, this happens for $\nu = 0.42$ and 160 points at time $t \approx 4.5$, with 320 points at $t \approx 3.8$, with 640 points at $t \approx 2.2$ and with 1280 points at $t \approx 1.2$ (see also Figure 2). From here on, the variation begins to grow exponentially and convergence breaks down. Thus increasing the CFL number from the theoretically justified value C_θ towards the intuitively reasonable value of 0.5, the following principles are violated one after another: the local maximum-principle and the local (in time) TVD-principle, the global maximum-principle and global TVD-principle, and finally the NED-principle. This behavior is illustrated in Figure 4, which shows a plot of $\max|u|$ and TV-norm versus CFL number ν for a series of refined grids at $t = 0.5$.

Let us briefly discuss the two other limiters used in [JIA 98]: For both limiters, the local maximum-principle is already violated for $\nu = 0.01$, which confirms the sharpness of the bound $C_2 = 0$, i.e. [3] for $\theta = 2$. For the MM_2 limiter, the global maximum-principle is violated at $\nu = 0.4$, and the NED-principle at $\nu = 0.44$ (640 points, $t \approx 4$). For the UNO limiter, the global maximum-principle holds up to $\nu = 0.43$ and is violated at $\nu = 0.44$. The NED-principle is violated at $\nu = 0.46$ (640 points, $t \approx 4.8$).

5. An adaptive staggered grid scheme

In ongoing joint work with M. Rumpf, we develop new adaptive central schemes on structured cartesian grids. This is a challenging task, which is particularly important when computing unsteady flows. Since the grids are now staggered, new techniques need to be developed. First results in 2-d may be found in [ROS 00] and [NOE 01].

Given an adaptively refined rectangular *primal grid* there is no unique choice of a corresponding staggered *dual grid*. Our approach relies on the following natural design principles:

– the *corners* of the staggered cells should lie in the *interior* of the primal cells, and vice versa (this is our definition of staggered grids)
– the local resolution of the dual grid should reflect that of the primal grid
– the edges of the dual cells should be parallel to the axis, but we do *not* require the dual cells to be rectangular.

As our primal grid we use an adaptive rectangular grid with at most 1-level transitions over common edges organized in a quadtree. We construct the staggered dual grid

– locally on each cell of the primal grid
– using only direct neighbour information
– finally composing local pattern

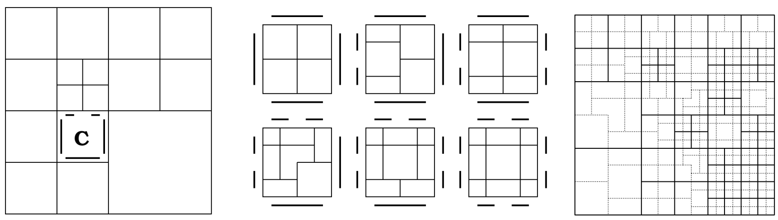

Figure 5. *local situation, dual pattern and resulting corresponding grids*

For all four edges of a primal cell we distinguish between two cases:

1) there exists a neighbour cell on a finer level
2) the neighbour cell is not on a finer level or there is no neighbour

By rotation symmetry we get six different local situations (note that there are 144 cases in 3-d). Composing scaled and rotated versions of these pattern we get the complete dual grid consisting of almost exclusively rectangular dual cells. Figure 5 shows an example of two corresponding grids, the primal one with thick and the dual grid with thin lines.

As the staggered grid corresponding to the dual grid we use the primal grid again. We refine and coarsen only the primal and not the dual grid.

In [ROS 00] we have applied our scheme to various one- and two-dimensional test problems of scalar advection, the gas dynamics and the ideal magnetohydrodynamics. Figure 6 shows results combining the adaptive scheme with the techniques of artificial compression described in [LIE 02b]. Without the added compression, we were not able to resolve the instability of the shear layer emanating from the triple point.

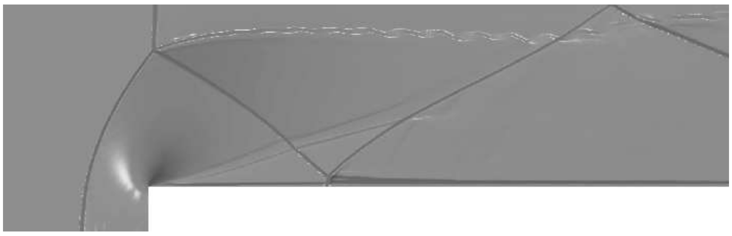

Figure 6. *slight vortices in the contact discontinuity*

6. Ongoing Projects

Currently, Kurganov, Lie and Noelle are designing new high resolution central schemes which combine the smoothness indicators of [KAR 01] with the compressive limiters of [LIE 02b].

In another joint project, Noelle, Rosenbaum and Rumpf are extending the adaptiv staggered scheme to structured grids in 3-d.

Combining these two efforts, we hope to design very high resolution, Riemann-solver free algorithms in 3-d.

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