Fictitious Domain - Lagrange Multiplier Approach Smoothness Analysis

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Abstract

We consider the Fictitious domain - Lagrange multiplier approach for second order elliptic boundary value problems with Dirichlet boundary conditions, and study the relation between the smoothness of the solution of the original problem and the solution of the new problem. We present lower bounds for the Besov smoothness for the general n-dimensional case in relation to the choice of the extension for the right-hand side. A numerical experiment illustrates the practical relevance of these estimates.

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1 Introduction

Fictitious domain methods can be conceived as consisting of two parts. The first is a method for solving a certain type of boundary value problem (BVP) on a usually fixed simple domain. The second is some sort of mechanism designed to coerce this method into producing a solution which, when restricted to some inner domain, is the solution to another, given, BVP on that inner domain.

These methods are usually devised for the following reasons. On one hand, it might be the easiest way to get this first solver to deal with more general geometries. Some of the fastest methods for solving boundary value problems are only available for very simple geometries. On the other hand, a fictitious domain method might be more convenient when dealing with a large set of different boundaries [16].

The main pitfalls are, of course, that the solver has to deal with the outer domain too, and this might cancel out any potential gain in efficiency. Another possible pitfall is that the solution on the extended domain might lack the regularity of the solution on the inner domain, thus impoverishing the overall convergence rate of the method.

We are going to investigate here this last aspect for methods derived from the fictitious domain - Lagrange multiplier approach. We briefly sketch this approach in section 2. We are going to analyze the solutions using B-spline wavelet Riesz bases, which we introduce in section 3, and rate the smoothness of the solutions using the results on Sobolev regularity and nonlinear approximation as found in [12]. We include the more important (to our quest) aspects of this body of theory in section 4.

Although we do not leave the wavelet setting in our analysis, one has to remark that these results are also relevant in other contexts. Besov smoothness is known to play a determining role in the convergence rates of adaptive finite element methods. See for example [15].

Section 5 starts with some auxiliary results, and ends with the proof of the central lemma. In section 6, then, we use this lemma to derive theoretical convergence rate estimates applicable for a few variants of these methods. Finally, to illustrate the practical relevance of this material, we include some numerical experiments in section 7.

2 The fictitious domain - Lagrange multiplier approach

Consider the second order elliptic boundary value problem

$$\begin{array}{rcll} Au & = & f_0 & \text{on } \Omega, \\ u & = & g & \text{on } \partial\Omega \end{array} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary, $f_0 \in [H^1(\Omega)]'$, and $g \in H^{\frac{1}{2}}(\partial\Omega)$. We can assume, without loss of generality, that diam $\Omega < 1$, and embed Ω in $\mathbb{T}^d$. Here, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We extend f and A to $\mathbb{T}^d$ in some suitable fashion, and solve this problem by looking for the minimizer u^* in $H^1(\Omega)$ of the functional

$$F(v) = \frac{1}{2} < v, Av > - < v, f >,$$

subject to the additional constraint

$$< Bu - g, q >= 0 \text{ for all } q \in H^{-\frac{1}{2}}(\Gamma),$$

where we take $B : H^1(\mathbb{T}^d) \to H^{\frac{1}{2}}(\Gamma)$ to be the trace or restriction operator $tr_\Gamma u = u_{|\Gamma}$.

The Hilbert space version of the Lagrange multiplier theorem [3] says that there exists $p \in H^{-\frac{1}{2}}(\Gamma)$ such that u^* is a critical point of the functional

$$F(v) = \frac{1}{2} < v, Av > - < v, f > + < Bv, p > - < g, p > .$$

This is equivalent to ask from the pair (u^*, p) that it satisfies

$$\begin{array}{lcccl} < v, Au^* > & + & < Bv, p > & = & < v, f >, \\ < Bu^*, q > & & & = & < g, q >, \end{array} \tag{2}$$

for all $v \in H^1(\mathbb{T}^d)$, $q \in H^{-\frac{1}{2}}(\partial\Omega)$

If the operator A was extended in such a way that $a(\cdot, \cdot) =< \cdot, A\cdot >$ is coercive on the kernel of B, then this system is well posed; the matrix

$$M = \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \tag{3}$$

defines an isomorphism between the product spaces $H^1(\mathbb{T}^d) \times H^{-\frac{1}{2}}(\Gamma)$ and $H^{-1}(\mathbb{T}^d) \times H^{\frac{1}{2}}(\Gamma)$. The restriction to the original domain of the solution u of the problem (2) solves our original problem (1).

For proofs for these facts and more information we refer to [13, 3, 1, 14].

3 Wavelets

The flavor of wavelets we are going to need is a family of Riesz bases for Hilbert spaces. We are going to start by sketching the construction of pairs of biorthogonal wavelet bases for $L^2(\mathbb{R})$ and its extension to the multivariate and the periodic case. Finally we will show how to alter the construction slightly to produce wavelet bases for Sobolev spaces on these domains.

This selection is by no means exhaustive, although it will suffice for our purposes. We have omitted various important constructions, like wavelet bases for L^2 on more general domains (see [9]), or wavelets on manifolds (see [10]).

For a more thorough and complete introduction treatment of the material in this section, see [7].

3.1 Riesz basis

A Riez basis for a Hilbert space H is a collection of elements $F = \{f_\lambda\}$, with λ in some (countable) index set ∇, such that the map $T : \ell^2(\nabla) \to H$ given by $T(\{x_\lambda\}) = \sum_{\lambda\in\nabla} c_\lambda f_\lambda$ is an isomorphism. Through a duality argument we get that there always exists a *dual* Riesz basis $\tilde{F} = \{\tilde{f}_\lambda\}$ in H', with $< f_\lambda, \tilde{f}_\nu >= \delta_{\lambda\nu}$, such that for any $g \in H$, and for any $h \in H'$,

$$g = \sum_{\lambda\in\nabla} < g, \tilde{f}_\lambda > f_\lambda \qquad\qquad h = \sum_{\lambda\in\nabla} < h, f_\lambda > \tilde{f}_\lambda \tag{4}$$

and

$$\|g\|^2 \sim \sum_{\lambda\in\nabla} | < g, \tilde{f}_\lambda > |^2 \qquad\qquad \|h\|^2 \sim \sum_{\lambda\in\nabla} | < h, f_\lambda > |^2. \tag{5}$$

3.2 Multiresolution analysis

A multiresolution analysis (MRA) in $X = L^2(\mathbb{R})$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ satisfying the axioms

I. $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$

II. $\cap_j V_j = \{0\}$

III. $\overline{\cup_j V_j} = L^2(\mathbb{R})$

IV. if $f \in V_j$, then $f(2\,\cdot\,) \in V_{j+1}$

V. if $f \in V_0$, then $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$

VI. there exists $\psi^0 \in V_0$ such that the set $\{\psi^0(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis for V_0. This function is called the *scaling function* of the MRA $\{V_j\}_{j\in\mathbb{Z}}$.

A *pair of biorthogonal MRAs* $\{V_j\}$, $\{\tilde{V}_j\}$ is a pair of MRAs whose corresponding scaling functions ψ^0, $\tilde{\psi}^0$ satisfy $<\psi^0(\cdot - k), \tilde{\psi}^0(\cdot - l)> = \delta_{kl}$ for all $k, l \in \mathbb{Z}$.

Given a pair of biorthogonal MRAs as above, we set out to construct the sequences of spaces $\{W_j\}$, $\{\tilde{W}_j\}$, with the property that

$$V_{j+1} = V_j \oplus W_j \qquad\qquad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j$$

while

$$V_j \perp \tilde{W}_j \qquad\qquad \tilde{V}_j \perp W_j.$$

Wavelets come from the fact that it is possible to find functions $\psi^1 \in W_0$, $\tilde{\psi}^1 \in \tilde{W}_0$ such that its integer translates form a pair of biorthogonal Riesz basis of W_0 and $\tilde{W}_0$. The usefullness of such functions comes from the fact that they can be constructed in such a fashion that if we write

$$\psi^1_{jk} = \xi_j \psi^1(2^j \cdot - k) \qquad\qquad \tilde{\psi}^1_{jk} = \xi_j \tilde{\psi}^1(2^j \cdot - k) \tag{6}$$

with $\xi_j = 2^{-\frac{j}{2}}$ (this scaling ensures that $\|\xi_j \psi^1_{jk}\| \sim 1$), then the set $\Psi = \{\psi^1_{jk} : j \in \mathbb{Z} \wedge k \in \mathbb{Z}\}$ is a Riesz basis (with dual basis $\tilde{\Psi} = \{\tilde{\psi}^1_{jk} : j \in \mathbb{Z} \wedge k \in \mathbb{Z}\}$) of the whole space $L^2(\mathbb{R})$. We will show here (without proof) how such functions can be found when we are given a pair of biorthogonal MRA as above.

From the axioms IV and VI it follows that

$$\psi^0(x) = \sum_{k\in\mathbb{Z}} a^0_k \psi^0(2 \cdot - k) \qquad\qquad \tilde{\psi}^0(x) = \sum_{k\in\mathbb{Z}} \tilde{a}^0_k \tilde{\psi}^0(2 \cdot - k). \tag{7}$$

The sequences $\{a^0_k\}$, $\{\tilde{a}^0_k\}$ are called the *masks* of ψ^0 and $\tilde{\psi}^0$ respectively, and it is clear that, if these functions are compactly supported, only a finite number of the entries in their masks can be non-zero. Let $\{a^1_k\}$, $\{\tilde{a}^1_k\}$ be sequences whose entries are given by $a_k^1 = (-1)^k \tilde{a}_{1-k}^0$ and $\tilde{a}_k^1 = (-1)^k a_{1-k}^0$. The functions ψ^1, $\tilde{\psi}^1$, that span W_0, $\tilde{W}_0$, respectively, are now given by (see [7])

$$\psi^1 = \sum_{k\in\mathbb{Z}} a_k^1 \psi^0(2\cdot -k), \qquad \tilde{\psi}^1 = \sum_{k\in\mathbb{Z}} \tilde{a}_k^1 \tilde{\psi}^0(2\cdot -k). \tag{8}$$

The functions ψ^1, $\tilde{\psi}^1$ are called the *mother wavelets* of the biorthogonal *wavelet bases* Ψ, $\tilde{\Psi}$.

From the remark made above on the relation of the entries in the mask and the compact support of a scaling function, we conclude that if both the primal and the dual scaling function have compact support, so do the corresponding wavelets.

3.3 Bspline wavelet bases

Given $N \in \mathbb{N}_0$, let us call A the space of functions $f \in C^{N-1}(\mathbb{R}) \cap L^2(\mathbb{R})$ (or just $L^2(\mathbb{R})$ if $N = 0$) such that for every $k \in \mathbb{Z}$, $f_{|[k,k+1)}$ is a polynomial of degree at most N. If we define now the sequence $\{V_j\}_{j\in\mathbb{Z}}$ whose elements are $V_j := \{f \in L^2(\mathbb{R}) : f(2^{-j}\cdot) \in A\}$, we have a sequence which satisfies axioms I to V. We satisfy axiom VI by taking

$$\psi_N^0 = (*)^N \chi_{[0,1)} \tag{9}$$

whose translates are indeed a Riesz basis for V_0.

It is an easy exercise to compute the masks for these functions, and to show that all the elements in the mask are non-negative. This last observation is going to play an important role later on.

Finding the dual MRA is not so simple. Answers can be found in [6], as part of a very exhaustive study of such matters.

3.4 The multivariate and periodic cases

Let $\{V_j\}$, $\{\tilde{V}_j\}$ be a pair of biorthogonal MRAs, and let $d > 1$ be an integer. Write $x = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d$, let $E = \{0,1\}^d$, and consider the functions $\psi^e(x) = \psi^{e_1}(x^1)\psi^{e_2}(x^2)\cdots\psi^{e_d}(x^d)$, $\tilde{\psi}^e(x) = \tilde{\psi}^{e_1}(x^1)\tilde{\psi}^{e_2}(x^2)\cdots\tilde{\psi}^{e_d}(x^d)$ for $e \in E$. We will allways use the superscript 0 to mean the element in E whose coordinates are all zero. This abuse of notation is very usefull, and it seems to never cause confusion.

The spaces $V_j^0 = \operatorname{span}\{\psi^0(2^j\cdot -k) : k \in \mathbb{Z}^d\}$ form a MRA, and with the dual spaces $\{\tilde{V}_j^0\}$ (defined analogously) they form a pair of biorthogonal MRAs. The complement spaces W_j^0 such that $V_{j+1}^0 = V_j^0 \oplus W_j^0$ are spanned by the integer translates of the functions ψ^e with $e \in E \setminus (0,0,\ldots,0)$. Using the scaling factor $\xi_j = 2^{\frac{dj}{2}}$, the functions $\{\psi_{jk}^e : j \in \mathbb{Z} \wedge k \in \mathbb{Z}^d \wedge e \in E \setminus (0,0,\ldots,0)\}$, with $\psi_{jk}^e = \xi_j \psi^e(2^j \cdot -k)$, form a Riez basis of the space $L^2(\mathbb{R}^d)$.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\mathcal{Z}_j^d = \mathbb{Z}^d/2^j\mathbb{Z}^d$, and define $\psi_{jk}^{e(\mathbb{T}^d)}(x) = \sum_{z\in\mathbb{Z}} \psi_{jk}^e(x - z)$. The spaces $V_j^{\mathbb{T}} = \operatorname{span}\{\psi_{jk}^{0(\mathbb{T}^d)} : k \in \mathcal{Z}_j^d\}$ satisfy all the axioms for a MRA except for axiom II, because $\cap_j V_j$ yields the space of all constant functions on $\mathbb{T}^d$. Usually, this axiom is just substituted by this assertion and by taking $j \geq 0$. We still have that $\{V_j^{\mathbb{T}}\}_{j\geq 0}$, $\{\tilde{V_j^{\mathbb{T}}}\}_{j\geq 0}$ form a pair of biorthogonal MRAs, for and that the set $\{\psi_{0,0}^{0(\mathbb{T}^d)}\} \cup \{\psi_{jk}^{e(\mathbb{T}^d)} : j \in \mathbb{N} \wedge k \in \mathbb{Z}/2^j\mathbb{Z}\}$ forms a Riesz basis of the space $L^2(\mathbb{T})$. We will drop the $\mathbb{T}^d$ superscript from now on, since it will become clear from the context which set of functions are being used.

3.5 Wavelet bases for Sobolev spaces

Whenever the elements of any of the aforementioned wavelet bases (depending on the underlying set $E \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{T}, \mathbb{T}^d\}$) belong to the space $H^s(E)$, whith $s \in \mathbb{R}$, we get a Riesz basis for that space by using the scaling $\xi_j^s = 2^{-js}\xi_j$ instead of the standard scaling ξ_j. One then has that for every $v \in H^s$, with $s > 0$,

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2(\Omega)} \leq C2^{-2sj}\|v\|_{H^s(\Omega)}. \tag{10}$$

3.6 The fast wavelet transform

Given a sequence $\{x_k\}_{k\in\mathbb{Z}^d}$, and $j \in \mathbb{N}$ we will associate with it the $2^{d(j+1)} \times 2^{dj}$ matrix $M_j^x = (m_{jkl}^x)_{k\in\mathcal{Z}_{j+1}^d, l\in\mathcal{Z}_j^d}$ whose entries are given by

$$m_{jkl}^x = \frac{\xi_j}{\xi_{j+1}} \sum_{z\in\mathbb{Z}} x_{k-2l-2^{j+1}z}. \tag{11}$$

Let $E = \{0,1\}^d$. We will write $\{a_k^e\}_{k\in\mathbb{Z}^d}$ for the sequence whose entries are $a_k^e = a_{k_1}^{e_1} a_{k_2}^{e_2} \cdots a_{k_d}^{e_d}$. For convenience we will also write $\{a_k^0\}_{k\in\mathbb{Z}^d}$ for the sequence $\{a_k^{(0,0,\ldots,0)}\}_{k\in\mathbb{Z}^d}$ (this is just the tensor product of the corresponding masks).

Suppose that $\{V_j\}$, $\{\tilde{V}_j\}$ are a pair of biorthogonal MRA for $L^2(\mathbb{T}^d)$ constructed as above, and let $f \in V_{j+1}$. We have two possible representations immediately available:

$$f = \sum_k (c_0^{j+1})_k \psi_{j+1,k}^0 \tag{12}$$

and

$$f = \sum_{e\in E} \sum_k (c_e^j)_k \psi_{jk}^e \tag{13}$$

Using the tensor product masks, and the matrix mechanism mentioned above, we get

$$c_0^{j+1} = \sum_{e\in E} M_j^{a^e} c_e^j \tag{14}$$

and

$$c_e^j = (M_j^{\tilde{a}^e})^T c_0^{j+1}. \tag{15}$$

The relations (14) and (15) allow us to switch between the possible representations of f. This process is $O(N)$, which has gained it the name *fast wavelet transform.*

We will in the sequel use the notation $M_j^0 = M_j^{a^0}$, $M_j^1 = \sum_{e \in E \setminus 0} M_j^{a^e}$.

4 Best N-term approximation

Let $v \in \ell^2(\mathbb{Z})$. A *sorting* of v is a bijection $\varphi : \mathbb{N} \to \mathbb{Z}$ such that $|v \circ \varphi(n)| \geq |v \circ \varphi(n+1)|$ for each $n \in \mathbb{N}$. Denote by $v_{\{N\}}$ the sequence obtained by setting all elements of v to 0 except those indexed by the set $\varphi(\{1, 2, ..., N\})$. We observe that for $N \in \mathbb{N}$ it holds that

$$\left\| v - v_{\{N\}} \right\| \geq \left\| v - v_{\{N+1\}} \right\|,$$

and in fact that

$$\left\| v - v_{\{N\}} \right\| > \left\| v - v_{\{N+1\}} \right\|,$$

unless $\left\| v - v_{\{N\}} \right\| = 0$.

We call $v_{\{N\}}$ a *best N-term approximation* to v. It is in general not unique, and earns its name from the fact that it is the best way to approximate v with a sequence of only N entries. See [5, 12]

The elements of ℓ^2 can be classified according to how fast $v_{\{N\}}$ converges to v. For $s > 0$, we say that $v \in \mathcal{A}^s$ if $\left\| v - v_{\{N\}} \right\| \leq CN^{-s}$ for some constant C.

It is often easier to prove that sequences belong to the spaces ℓ_τ^w. A sequence $v \in \ell^2$ is in ℓ_w^τ if $\#\{k : |v_k| \geq \epsilon\} \lesssim \epsilon^{-\tau}$. The following theorem settles the relationship between these spaces.

Theorem 4.1. *Let $0 \leq \tau < 2$, and $s = \frac{1}{\tau} - \frac{1}{2}$. Then*

$$\ell_w^\tau = \mathcal{A}^s$$

For the proof of this theorem we refer again to [5, 12]

The following proposition is going to be usefull later.

Proposition 4.2. *Let $a > 1$. $v \in \ell_w^\tau$ if, and only if*

$$\#\{k : |v_k| \geq a^{-j}\} \lesssim a^{\tau j}$$

Proof. One direction follows from the definition, setting $\epsilon = a^{-j}$. For the other direction, let $j_\epsilon = \lfloor \log_a \epsilon \rfloor$. Then $\#\{k : |v_k| \geq \epsilon\} \leq \#\{k : |v_k| \geq a^{j_\epsilon}\}$, and thus by hypotesis, and noting that $a^{-\tau j_\epsilon} \leq a^\tau \epsilon^{-\tau}$, we get

$$\#\{k : |v_k| \geq \epsilon\} \lesssim \epsilon^{-\tau},$$

as we set out to prove. □

Suppose that $\{\psi_\lambda\}_{\lambda \in \nabla}$ is a wavelet riesz basis for $H^t(\Omega)$, $t \geq 0$, where the wavelets have been chosen to have enough smoothness and vanishing moments (for details see [5, 7]), and let $0 < \tau < 2$. Then the condition $\{c_\lambda\}_{\lambda \in \nabla} \in \ell_\tau$ is equivalent to the requirement that

$$\sum_{\lambda \in \nabla} c_\lambda \psi_\lambda \in B_\tau^{sd+t}(L_\tau(\Omega)) \tag{16}$$

Thus, from $\ell_\tau^w \subset \ell_{\tau+\epsilon}$, with $\epsilon > 0$, it follows that if $\{c_\lambda\}_{\lambda \in \nabla} \in \ell_\tau^w$, then $\sum_{\lambda \in \nabla} c_\lambda \psi_\lambda \in B_{\tau+\epsilon}^{sd+t}(L_{\tau+\epsilon}(\Omega))$ for any $\epsilon > 0$ such that $\tau + \epsilon < 2$.

4.1 Compressible matrices and fast matrix-vector multiplication

The material in this subsection was borrowed from [5] with slight notational modifications.

An infinite matrix B is said to be in the class $\mathcal{B}_s$ if there exist two positive summable sequences $\{\alpha_j\}_{j\in\mathbb{N}}$, $\{\beta_j\}_{j\in\mathbb{N}}$, such that for every $j \geq 0$ there exists a matrix B_j with at most $2^j\alpha_j$ nonzero entries per row and column with the property that

$$\|B - B_j\| \leq 2^{-js}\beta_j. \tag{17}$$

A matrix with this property is called *compressible.*

Proposition 4.3. *Let* $\tau = (s + \frac{1}{2})^{-1}$, *with* $0 < \tau < 2$. *If* $B \in \mathcal{B}_s$, *then* B *maps* ℓ_τ^w *boundedly into itself.*

The important consequence of this proposition is that, if $b \in \ell_\tau^w$, $A : \ell_2 \to \ell_2$ is bounded and invertible, and $A \in \mathcal{B}_s$, then the solution x of the equation $Ax = b$ satisfies $x \in \ell_\tau^w$.

The following theorem shows how to compute an approximation of the action of an operator $B \in \mathcal{B}_s$ on a finitely supported sequence v.

Theorem 4.4. *Let* $v \in \ell_2$ *and* $B \in \mathcal{B}_s$. *Wite* $v_{[0]} = v_{\{1\}}$, *and* $v_{[j]} = v_{\{2^j\}} - v_{\{2^{j-1}\}}$ *if* $j \geq 1$. *Further, let*

$$R_k = \|A\|\|v - v_{\{2^k\}}\| + \sum_{i=0}^{k} a_i\|v_{[k-i]}\|, \tag{18}$$

$$w_k = \sum_{i=0}^{k} A_i v_{[k-i]}. \tag{19}$$

Then

i. $\|Av - w_k\| \leq R_k$

ii. If v *has finite support, the algorithm*

(1) Compute the smallest k *such that* $R_k < \epsilon$

(2) Compute w_k

can be performed at a cost of not more than

$$\epsilon^{-1/s}\|v\|_{\ell_\tau^w}^{1/s} + 2\#(supp\ v) \tag{20}$$

arithmetic operations.

iii. $\#(supp\ w) \leq C\epsilon^{-1/s}\|v\|_{\ell_\tau^w}^{1/s}$

The two last results are the key ingredients of the adaptive wavelet methods devised in [5]. We refer there and also to [4] for further details. An additional spurious logarithmic term in (20) was discovered and removed in [2].

5 A smoothness lemma

This section contains the proof of the main lemma along with all the preparations. The main results are derived in section 6.

5.1 Preliminaries

5.1.1 Canonical projections of local parametrizations

Let $n > 1$ be an integer. For $n \geq m \geq 1$ we call the linear operator $P_m : \mathbb{R}^n \to \mathbb{R}^{n-1}$, given by $P_m(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{m-1}, x_{m+1}, \dots, x_n)$ the *m-th canonical projector.*

Lemma 5.1. *Let $x_0 \in \mathbb{R}^{n-1}$, $I = B(x_0, \varepsilon)$, $\eta : I \to \mathbb{R}$ continuous, and*

$$G = \{x \in \mathbb{R}^n, x = (\overline{x}, x_n) : x_n = \eta(\overline{x}) \wedge \overline{x} \in I\}.$$

Write $z_0 = (x_0, \eta(x_0))$. If $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is orthogonal, then for each $\delta > 0$ there exists $y_0 \in B(z_0, \delta)$, and $\kappa > 0$ such that $P_m Q B(y_0, \kappa) \subset P_m Q G$ whilst $B(y_0, \kappa) \cap G = \emptyset$.

For the proof see apendix A.

5.1.2 Index sets

Let $A \subset \mathbb{R}^d$. We will write $\Lambda_j^\beta(A) = \{k \in \mathbb{Z}^d | 2^{-j}([0,1]^d + k + k') \cap A \neq \emptyset$, where $k' \in \mathbb{Z}$ and $||k'||_\infty \leq \beta\}$

Lemma 5.2. *Suppose that the sequence $a = (a_z)_{z \in \mathbb{Z}^n}$ has support $[\alpha, \beta]^n \cap \mathbb{Z}^n$, with $\alpha, \beta \in \mathbb{Z}$, $\alpha \leq -1$, $1 \leq \beta$. Let I be as before, and let $j \in \mathbb{Z}$, $k_1 \in \Lambda_j^\beta(I) \setminus \Lambda_j^{\beta-1}(I)$. Then*

1. *There exists $k_2 \in \Lambda_j^0(I)$ such that $||k_1 - k_2||_\infty = \beta$*
2. *There exists $k_* \in \mathbb{Z}^d$ such that $k_1, k_2 \in supp\ (a_{z-2k_*})_{z \in \mathbb{Z}^n}$*

Proof. The first assertion follows trivially from the definition of the index sets after observing that $\Lambda_j^\beta(I) \setminus \Lambda_j^{\beta-1}(I) \neq \emptyset$.

We prove the existence of such a k_* componentwise. Write k_*^i, k_0^i, k_1^i for the i-th coordinate of k_*, k_0, k_1, respectively, for $i = 1, ..., n$, and choose $1 \leq i \leq n$.

We are looking for k_*^i such that $2k_*^i + \alpha \leq k_0^i \leq 2k_*^i + \beta$ and $2k_*^i + \alpha \leq k_1^i \leq 2k_*^i + \beta$, which is equivalent to k_*^i satisfying

$$\max\{k_0^i - \beta, k_1^i - \beta\} \leq 2k_*^i \leq \min\{k_0^i - \alpha, k_0^i - \alpha\} \tag{21}$$

Such a k_*^i exists, trivially, if $k_0^i = k_1^i$ and $\beta - \alpha \geq 1$. If $k_0^i > k_1^i$, condition (21) translates to $k_0^i - \beta \leq 2k_*^i \leq k_1^i - \alpha$, which is equivalent to $k_0^i - k_1^i \leq 2k_*^i + \beta - k_1^i \leq \beta - \alpha$. This can be satisfied because $\beta - \alpha > \beta \geq k_0^i - k_1^i$.

The case $k_1^i > k_0^i$ can be handled the same way. □

Suppose that V, W are $\mathbb{R}$-vector spaces on some non-empty subsets of $\mathbb{Z}^d$ denoted I_V, I_W. Let $M : W \to V$ be a linear operator with the property that there exists a number $d_M \in \mathbb{N}$ such that for any $k \in I_W$, $k', k'' \in \operatorname{supp} Me_k$ implies $\|k' - k''\|_\infty < d_M$. Here e_k is the member of the canonical basis corresponding to k.

Proposition 5.3. *Let $A \subset I_V$ be such that for $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $\|a_1 - a_2\| \geq d_M$, and suppose $v \in V$ and $w \in W$ are related by $v = Mw$. If, for some $c_2 > 0$, $|v_a| \geq c_2$ for every $a \in A$, then there exists $c_3 > 0$, and $B \subset I_W$ such that $|w_b| \geq c_3$ for each $b \in B$, and $\#B = \#A$.*

Proof. Choose $a \in A$, and let $C_a = \{k \in I_W : a \in Me_k\}$. Point 2 above implies that $\#C_a$ is uniformly bounded by some constant N. Now, if $|w_c| \leq \frac{c_2}{N\|M\|_\infty}$ for all $c \in C_a$, we reach a contradiction since then

$$|v_a| = \left| \sum_{c \in C} (w_c M e_c)_a \right| \leq c_2.$$

Thus, we take $c_3 = \frac{c_2}{N\|M\|_\infty}$, and choose a b_a in C such that $|v_{b_a}| \geq c_3$.

It remains to prove that if $b_a = b_{a'}$, then $a = a'$. Indeed, if they were not equal, then $a, a' \in \operatorname{supp} Me_{b_a}$, and thus $\|a - a'\| < d_M$, contradicting our hypothesis.

□

5.1.3 Lower bound for single integrals

Lemma 5.4. *Let $n \in \mathbb{N}$, $n > 1$, $x_0 \in \mathbb{R}^{n-1}$, $\varepsilon_1 > \varepsilon_2 > 0$, and $I_i = B(x_0, \varepsilon_i)$, $i = 1, 2$. Further, let*

- *$f : I_1 \to \mathbb{R}^n$ be Lipschitz.*
- *$\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuous, non-negative function such that $[0,1]^n \subset (supp\{\phi\})^\circ$.*

Under these circumstances, there exist $j_0 \in \mathbb{Z}$, and $C > 0$ such that if $j \geq j_0$ and $k \in \Lambda_j^0(f(I_2))$, then

$$\int_I \phi(2^j \cdot - k) \circ f(t) dt \geq C 2^{-j(n-1)}$$

Proof. Since $[0,1]^n \subset (\operatorname{supp}\{\phi\})^\circ$, there exist $\delta, c > 0$ such that $\phi(x) > c$ when $x \in B([0,1]^n, \delta)$.

Let L be such that $||f(x) - f(y)|| \leq L||x - y||$ for all $x, y \in I$. Write $\tau_j = \frac{1}{L} 2^{-j}\delta$, and let j_0 be such that $\varepsilon - \varepsilon_2 > \tau_{j_0}$. Given $j \geq j_0$, $k \in \Lambda_j^0(f(I_2))$, and $\overline{x} \in I_2$ such that $f(\overline{x}) \in 2^{-j}([0,1]^n + k)$, we observe that $f(B(\overline{x}, \tau_j)) \subset 2^{-j}(B([0,1]^n, \delta) + k)$, and thus $\phi_{jk} \circ f(t) \geq c$ when $t \in B(\overline{x}, \tau_j)$. Now

$$\int_I \phi_{jk} \circ f(t) dt \geq c \int_{B(\overline{x}, \tau_j)} dt = C 2^{-j(n-1)}$$

which finishes the proof. □

5.2 The smoothness lemma

Lemma 5.5. *Let $\{\psi_\lambda\}_{\lambda\in\Lambda(\mathbb{T}^d)}$ be a tensor product B-spline wavelet basis for $H^1(\mathbb{T}^d)$, as introduced in subsection 3.3, such that its scaling function ϕ is continuous, compactly supported, and satisfies $[0,1]^d \subset \operatorname{supp}\phi$. Let Ω be a bounded Lipschitz domain, and $I \subset \partial\Omega$ an open set in the relative topology. Further, let $w \in H^{-\frac{1}{2}}(\partial\Omega)$ be a function with $w_{|I} > C_w > 0$. Let*

$$d_\lambda := < B\psi_\lambda, w >$$

where B is the trace operator. If $\{d_\lambda\}_{\lambda\in\Lambda(\mathbb{T}^d)} \in \ell^\tau_w$, then $\tau \geq \frac{2(d-1)}{d}$

Before we start, we review the definition of the space $H^s(\partial\Omega)$. As in the hypothesis, let Ω be a bounded Lipschitz domain. It is the defining property of such a domain that, given $x \in \partial\Omega$, there exists a neigborhood U_x, a function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$, and an affine change of coordinates $T_x : \mathbb{R}^d \to \mathbb{R}^d$ such that $T_x^{-1}(U_x \bigcap \Omega) = \{y \in T_x^{-1}U_x | 0 < y_n < \phi(y_1, \ldots, y_{n-1})\}$.

Let $\mathcal{U} = \{U_{x_i}\}_{i\in\{1,2,\ldots,p\}}$ be a finite covering of $\partial\Omega$ by $\{U_x\}_{x\in\partial\Omega}$. Let $B_i = P_n T_{x_i}^{-1} U_{x_i}$, $i = 1, \ldots, p$, and let $\{\theta_i\} \in \mathcal{D}(\mathbb{R})$ be a partition of unity subordinate to the covering $\mathcal{U}$. We will write $\xi_i = T(x_1, \ldots, x_{n-1}, \phi_{x_i}(x_1, \ldots, x_{n-1}))$.

We will say that $u \in H^s(\partial\Omega)$ if $(\theta_i u) \circ \xi_i \in H^s(\mathbb{R}^{d-1})$ for every $i = 1, \ldots, p$, and define the inner product through

$$(u, v)_{H^s(\partial\Omega)} = \sum_{i=1}^{p} ([\theta_i u] \circ \xi_i, [\theta_i v] \circ \xi_i)_{H^s(\mathbb{R}^{d-1})}$$

One can prove that the space obtained in this way is independent of the coice of the covering and the partition of unity selected [17].

Proof of lemma 5.5. In view of this, the way to proceed is to choose a fixed (but arbitrary) set of local parametrizations with a corresponding partition of unity, and prove that the lemma holds. We can simplify a bit further, since we can guarantee that there exist $x_0 \in \mathbb{R}^{d-1}$, $\varepsilon > 0$, and an $i \in \{1, \ldots, p\}$ such that for $f := \xi_{i|B(x_0,\varepsilon)}$ it holds $\theta_i(w \circ f) > c > 0$ on $B(x_0, \varepsilon)$. It will be seen from the argument that we can also safely assume that $\theta_i \equiv 1$.

We write $G = f(B(x_0, \varepsilon))$, and invoke lemma 5.1 to get $y_0 \in \mathbb{R}^d$, $\delta > 0$, and $m \in \{1, \ldots, d\}$ such that

$$B(y_0, \delta) \subset B(f(x_0), \varepsilon/2), \tag{22}$$

$$B(y_0, \delta) \cap G = \emptyset, \tag{23}$$

and (trivially; we shall refine this later)

$$P_m B(y_0, \delta) \subset P_m G. \tag{24}$$

Now choose $k_0 \in \mathbb{Z}^d$, and $j_0 \in \mathbb{N}$ such that $2^{-j_0}([a, b]^d + k_0) \subset B(y_0, \frac{\delta}{2})$, where $[a, b]^d$ is the support of the B-spline scaling function ϕ. Given $j \in \mathbb{N}$, let

$A_j = \{k = k_0 + k' \mid k' \in \mathbb{Z}^d/2^j\mathbb{Z}^d \wedge k'_m = 0\}$. For each $k \in A_j$ we have that there exist $q^0_{jk}, q^1_{jk} \in \mathbb{Z}$ such that

$$2^{-(j+j_0)}([0,1]^d + k + e_m q^0_{jk}) \cap G \neq \emptyset \tag{25}$$

and

$$2^{-(j+j_0)}([a,b]^d + k + e_m q^1_{jk}) \cap G = \emptyset \tag{26}$$

Let us define the sequences $c_j \in \ell(\mathbb{Z}^d)$ by

$$c_{jk} = < w, \phi_{j+j_0,k} >,$$

where the inner product is relative to our selected set of local parametrizations and partition of unity. Now note that given $k \in A_j$, it holds that

$$c_{j,k+e_m q^0_{jk}} \gtrsim \int_{B(x_0,\varepsilon)} (w\phi_{j+j_0,k+e_m q^0_{jk}}) \circ f \tag{27}$$

$$\geq C_1 2^{-j} 2^{\frac{dj}{2}} 2^{-j(d-1)} \tag{28}$$

$$= C_1 2^{-\frac{dj}{2}}, \tag{29}$$

where $C_1 > 0$ is independent of j and k. The powers of two come, in that order, from the H^1 normalization, the L^2 normalization, and lemma 5.4. For a given j, and for every $k \in A_j$, let us collect all q^0_{jk} in the set Q^0_j.

On the other hand,

$$c_{j,k+e_m q^1_{jk}} = 0$$

due to (26).

We use lemma 5.2 to choose q^1_{jk} such that there exists at least one $q^0 \in Q^0_j$ such that $\|q^1_{jk} - q^0\|_\infty = b$. We collect for a given j, and for all $k \in A_j$, all such q^1_{jk} in Q^1_j. Let us denote by $g_j : A_j \to Q^1_j$ the function that maps $k \in A_j$ to its corresponding q^1_{jk}.

Given $q \in Q^1_j$, and given $q' \in Q^0_j$ with $\|q - q'\|_\infty = b$, we use lemma 5.2 to get an index $q^* \in \mathbb{Z}^d$ such that $q, q' \in \mathrm{supp}\,(a_{z-2q^*})$.

This means that $c_{(j-1)q^*} \geq C_2 2^{-\frac{jd}{2}}$, since we can substitute the refinement relation in the integral, and all entries in the mask are positive. We use this last fact again to conclude that $(M^0_{j-1}c_{j-1})_q \geq C_3 2^{-\frac{jd}{2}}$.

Now recall that

$$c_j = M^1_{j-1} d_j + M^0_{j-1} c_{j-1}. \tag{30}$$

Thus our argument implies directly that $(M^0_{j-1}c_{j-1} - c_j)_q \geq 2^{-\frac{jd}{2}}$

Given $j^* < j$, we have that $B^{j^*}_j = \{k_0 + k' | k' \in 2^{j^*}(\mathbb{Z}^d/2^{j-j^*}\mathbb{Z}^d)\}$ is a subset of A_j. We can choose j^* in such a way that if $j > j^*$, and given $k_1, k_2 \in B^{j^*}_j$, then $g_j(k_1), g_j(k_2) \in \mathrm{supp} M^1_j e_k$ implies $k_1 = k_2$. We invoke now proposition (5.3), in conjunction with equation (30), and get that given $j > j*$,

$$\#\{\lambda | j(\lambda) = j + j_0 \wedge |d_\lambda| > C 2^{-\frac{jd}{2}}\} \gtrsim 2^{j(d-1)}.$$

It would thus contradict proposition (4.2) to assume that $\{d_\lambda\} \in \ell^\tau_w$ unless $\tau \geq \frac{2(d-1)}{d}$. □

6 Smoothness analysis of the fictitious domain - Lagrange multiplier approach

Now we are ready for answering the questions on smoothness we asked in the introduction. We begin by rating the compressibility of the trace operator.

Let $\{\psi_\lambda\}_{\lambda\in\nabla^{\mathbb{T}^d}}$, $\{\psi^\Gamma_\lambda\}_{\lambda\in\nabla^\Gamma}$ be B-spline wavelet Riesz basis for $H^1(\mathbb{T}^d)$, $H^{1/2}(\Gamma)$ as constructed in section 3, capable of reproducing cuadratic polynomials (note that the approximation order of the wavelets only weights in because of the support it imposes on the scaling functions), and with corresponding duals not shown here. Further, let $B = (b_{\lambda,\mu})_{\lambda\in\nabla^\Gamma,\mu\in\nabla^{\mathbb{T}^d}}$ given by $b_{\lambda,\mu} =< B\psi_\mu, \psi^\Gamma_\lambda >$.

Theorem 6.1. *The matrtix B is not in $\mathcal{B}_s$ for any $s > \frac{1}{2(d-1)}$.*

Proof. Let $\lambda_0 \in \nabla^\Gamma$. The sequence $\{c_\mu\}$ with entries $c_\mu = \delta_{\lambda_0,\mu}$ is in ℓ^w_τ for any $0 \leq \tau < 2$, but $B^T(\{c_\mu\})$ is not in any ℓ^w_τ for $\tau \leq \frac{2(d-1)}{d}$. To see this, just observe that $B^T(\{c_\mu\}) = \{b_{\lambda_0,\mu}\}$ and use lemma 5.5. We conclude that, for this range of τ, B^T cannot map ℓ^w_τ into itself, and thus $B^T \notin \mathcal{B}_s$ for $s \geq \frac{1}{2(d-1)}$.

On the other hand, if it would hold that $B \in \mathcal{B}_s$ for $s \geq \frac{1}{2(d-1)}$, then from the definition we get that $B^T \in \mathcal{B}_s$. This contradiction proves the claim. □

This sets a lower bound on the approximabilty of the action of the trace operator and its transpose. See Theorem 4.4. Another consequence of lemma 5.5 is an estimate on the smoothness that can be achieved.

Theorem 6.2. *Let f be the extension in $H^{-1}(\mathbb{T}^d)$ of f_0 in (1), and write $f = \sum_{\lambda\in\nabla} f_\lambda\tilde{\psi}_\lambda$. If $\{f_\lambda\}_{\lambda\in\nabla} \in \ell^w_\tau$ for some $\tau < \frac{2(d-1)}{d}$, the pair (u,p) is the solution of the system (2), the basis $\{\psi\}_{\lambda\in\nabla}$ was chosen in such a way that $A \in \mathcal{B}_s$ for some $s > \frac{1}{2(d-1)}$, and $p > c > 0$ on some open set $U \subset \Gamma$, then the following (equivalent) assertions hold.*

- *Writing $u = \sum_{\lambda\in\nabla} u_\lambda\psi_\lambda$, we have that $\{u_\lambda\}_{\lambda\in\nabla} \notin \ell^w_\tau$ for any $\tau \leq \frac{2(d-1)}{d}$.*
- *For every $\epsilon > 0$, $u \in B^{s_\epsilon+1}_{\tau_\epsilon}(L_{\tau_\epsilon}(\Omega))$ with $s_\epsilon = \frac{d+\epsilon}{2(d-1)}$ and $\tau_\epsilon = \frac{2(d-1)}{d+\epsilon}$.*

Proof. Writing $f - Au = B^T p$. If $u \in \ell^w_\tau$ for some $\tau < \frac{2(d-1)}{d}$, then so would $f - Au$, and thus $B^T p$. But this would contradict lemma 5.5. □

This result limits the convergence rate of the best $N-$term approximations of u to be of at most $O(N^{-\frac{1}{2(d-1)}})$, for wavelets with approximation order at least 3, whenever we find this type of function as a Lagrange multiplier. The following theorem puts this result into perspective, showing us also the way to better convergence rates.

Theorem 6.3. *Let u_0 be the solution of (1), and suppose that $u_0 \in B_\tau^{sd+t}(L_\tau(\Omega))$ for some $\tau < \frac{2(d-1)}{d}$ and where $s = \frac{1}{\tau} - \frac{1}{2}$. Then there exists an extension f of f_0 to $\mathbb{T}^d$ such that the solution u of (2) is in $B_\tau^{sd+t}(L_\tau(\mathbb{T}^d))$*

Proof. By a result from DeVore and Sharpley [11] we know that we can find an extension $u \in B_\tau^{sd+t}(L_\tau(\mathbb{T}^d))$ of u_0 to $\mathbb{T}^d$. We obtain f by applying the differential operator to u. □

We conclude from all the above that the role of the extension cannot be ignored if a good convergence rate is desired. Note that it is not enough to choose a smooth extension to the right hand side. It has to be smooth *and* produce a zero Lagrange multiplier.

7 Numerical experiment

In this section we show some numerical results that illustrate the effect a non-zero Lagrange multiplier has on the smoothness of the solution on the extended domain. Consider the problem

$$\begin{aligned} -\triangle u &= 1 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{31}$$

with $\Omega = \{x \in \mathbb{R}^2 : \|x - (\frac{1}{2}, \frac{1}{2})\| < \frac{1}{4}$. Extending the data and imbedding the domain into $\mathbb{T}^2$ is trivial; we obtain

$$\begin{aligned} -\triangle u &= 1 \quad \text{on } \mathbb{T}^2, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{32}$$

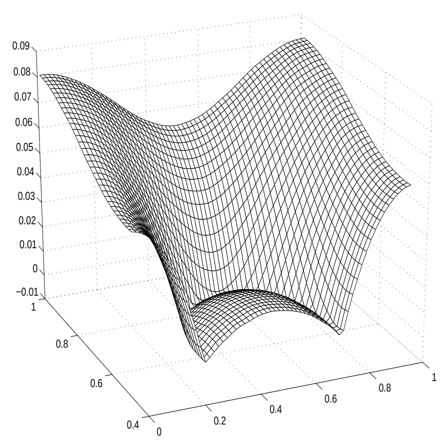
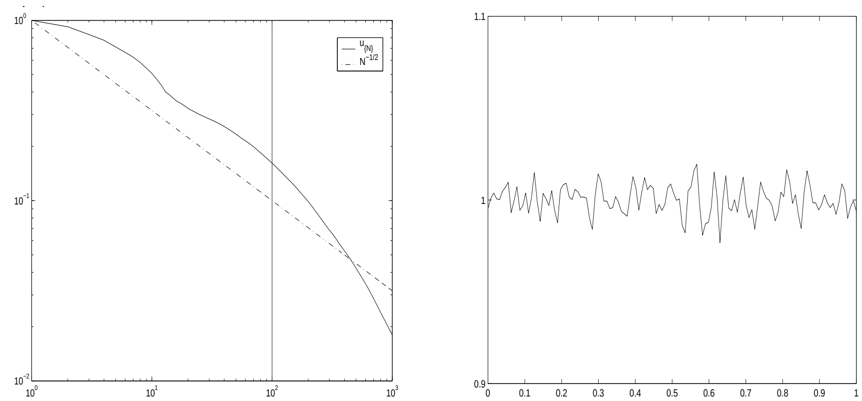


Figure 1: Cross section of the solution to (32)

It is evident that the solution u_0 of (31) is C^∞. Thus, we expect fast convergence of the best N-term approximations if we use basis with high order of polynomial approximation. Using (10) and proposition 6.5 in [8], one derives that, if we use B-spline wavelets with polynomial approximation of order $d = 4$ on the primal side, $d' = 8$ on the dual side, then the coefficient sequence of the solution should be at least in $\ell^w_{\tau_0}$ with $\tau_0 = \frac{2}{3}$, and thus the convergence rate of its best N-term approximation should be of order $O(N^{-1})$. Nevertheless, we obtain the rate predicted in 6.2 of $O(N^{-\frac{1}{2}})$.



Convergeence of $u_{\{N\}}$ **Lagrange multiplier**

Figure 2: Convergence of best N-term approximation and Lagrange multiplier

On the left of figure 2 we show the best N-term convergence history of the solution to (32) in the H^1 norm, along with a reference convergence path $N^{-\frac{1}{2}}$. The solution has been normalized to have norm 1, and the vertical bar indicates where we think that the compactly supported nature of the data available starts to accelerate the convergence. On the right we show the lagrange multiplier as produced by the numerical solver.

The dataset used was the solution of (32) using the wavelet bases mentioned above (and the standard Galerkin approach, see [14]). The resolution corresponds to V_9 in the associated MRA.

APPENDIX

A Proof of lemma 5.1

Given 2 points $x_1, x_2 \in \mathbb{R}^n$, we denote by $\gamma = \gamma[x_1 x_2] : [0,1] \to \mathbb{R}^n$ the function $\gamma(t) = (1-t)x_1 + tx_2$. Given $\gamma, \eta : [0,1] \to \mathbb{R}^n$, and if $\gamma(1) = \eta(0)$ we write $\iota = \gamma * \eta$ for the function $\iota : [0,1] \to \mathbb{R}^n$ given by

$$\iota(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t < \frac{1}{2} \\ \eta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

We also write $\gamma[x_1x_2\cdots x_m] = \gamma[x_1x_2] * \gamma[x_2x_3] * \cdots * \gamma[x_{m-1}x_m]$.

Lemma A.1. *Let $x_0 \in \mathbb{R}^{n-1}$, $I = B(x_0, \varepsilon)$, $\eta : I \to \mathbb{R}$ continuous, and*

$$G = \{x \in \mathbb{R}^n, x = (\overline{x}, x_n) : x_n = \eta(\overline{x}) \wedge \overline{x} \in I\}.$$

For each orthogonal operator $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ there exist $1 \leq m \leq n$ such that

$$(P_m AG)^\circ \neq \emptyset$$

Proof. Let $\alpha = \sup\{r > 0 : B_\infty(w, r) \subset B_2(w, 1)\}$ and $\delta \leq \frac{\varepsilon}{2}$. Set $W = B(w, \delta) \setminus G$, and note that this set is open and disconected. There is, in particular, no path between the points $w_- = w - \alpha\frac{\delta}{2}e_n$ and $w_+ = w + \alpha\frac{\delta}{2}e_n$. If $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is orthogonal, then QW is also disconected, and we cannot find a path between Qw_+ and Qw_-in QW.

Now suppose that the lemma is false for a certain orthogonal $Q_0 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

Set $W' = B_\infty(Q_0w, \alpha\delta) \setminus Q_0G$, $w'_+ = Q_0w_+$, and $w'_- = Q_0w_-$. By the way we constructed the parameters, we have that $W' \subset Q_0W$, that W' is open, and that $w'_+, w'_- \in W'$, and that there is no path between w'_+ and w'_- in W'. Let δ_0 be such that $B_\infty(w'_+, \delta_0) \subset W'$ and $B(w'_-, \delta_0) \subset W'$.

Now, let $\xi_1 = w'_-$, and choose $\zeta_1 \in B_\infty(\xi_1, \frac{\delta_0}{n})$ such that $P_1\zeta_1 \notin P_1W'$. This is possible because we assumed our lemma false. We next choose $\lambda_1 \in \mathbb{R}$ such that the first coordinates of $\xi_2 = \zeta_1 + \lambda_1 e_1$ and w'_+ are equal. Note that the path $\gamma[\xi_1\zeta_1\xi_2]$ lies fully in W'.

Next choose $\zeta_2 \in B_\infty(\xi_2, \frac{\delta_0}{n})$ such that $P_2\zeta_2 \notin P_2W'$ (which again has to exist), and choose λ_2 such that the second coordinates of $\xi_3 = \zeta_2 + \lambda_2 e_2$ and w'_+ are equal. Note that, for the same reasons as above, the path $\gamma[\xi_1\zeta_1\xi_2\zeta_2\xi_2]$ lies fully in W'.

We proceed in this fashion until we have constructed ξ_n, and note inmediately that $\xi_n \in B_\infty(w'_+, \delta_0)$, since each coordinate of ξ_n is at most at a distance of $\frac{(n-1)\delta_0}{n}$ of the corresponding one in w'_+. But we took care to never leave W', which implies that the path $\gamma[w'_-\zeta_1\xi_2\zeta_2\cdots\zeta_{n-1}w'_+]$ lies in W'. This contradiction finishes the proof. □

Proof of lemma 5.1. Without loss of generality, we can assume that $\delta < \frac{\varepsilon}{2}$, and apply the same proof as before. The point we are looking for is the last ξ_i obtained before the process cannot be continued, and κ can be chosen as $\kappa = \frac{\delta_0}{n}$. □

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