UNIFORM PRECONDITIONERS FOR A PARAMETER DEPENDENT SADDLE POINT PROBLEM WITH APPLICATION TO GENERALIZED STOKES INTERFACE EQUATIONS

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Abstract. We consider an abstract parameter dependent saddle-point problem and present a general framework for analyzing robust Schur complement preconditioners. The abstract analysis is applied to a generalized Stokes problem, which yields robustness of the Cahouet-Chabard preconditioner. Motivated by models for two-phase incompressible flows we consider a generalized Stokes interface problem. Application of the general theory results in a new Schur complement preconditioner for this class of problems. The robustness of this preconditioner with respect to several parameters is treated. Results of numerical experiments are given that illustrate robustness properties of the preconditioner.

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1. Introduction. Let $H_1 \subset H_2$ and M be Hilbert spaces such that $I : H_1 \to H_2$ is a dense embedding. Let there be given continuous elliptic bilinear forms $a : H_1 \times H_1 \to \mathbb{R}$, $c : H_2 \times H_2 \to \mathbb{R}$ and a continuous bilinear form $b : H_1 \times M \to \mathbb{R}$ that satisfies a standard infsup condition. Operators corresponding to these bilinear forms are denoted by $A : H_1 \to H_1'$, $C : H_2 \to H_2'$ and $B : M \to H_1'$, respectively. In this paper we consider the following saddle-point system: Find $(u, p) \in H_1 \times M$ such that

$$\begin{cases} Au + \tau Cu + Bp &=& f \\ B'u &=& 0 \end{cases} \tag{1.1}$$

with $f \in H_1'$ and a parameter $\tau \geq 0$. Similar abstract saddle point problems are thoroughly analyzed in the literature, eg. [6, 11]. Important examples that fit in this general setting are the stationary Stokes equation (then $\tau = 0$) and the so-called generalized Stokes problem, which results from an implicit time integration applied to a nonstationary Stokes equation (then τ is proportional to the inverse of the time step). Another (less standard) example, which motivated the research that led to the results presented in this paper, is the following generalized Stokes *interface* problem. Assume bounded Lipschitz subdomains Ω_1 and Ω_2 of $\Omega \subset \mathbb{R}^d$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. The boundary between the subdomains is denoted by $\Gamma = \overline{\partial \Omega_1} \cap \overline{\partial \Omega_2}$. Consider a problem of the following form: Find $\mathbf{u}$ and p such that

$$\begin{aligned} -\operatorname{div}(\nu(\mathbf{x})\mathrm{D}\mathbf{u}) + \tau\rho(\mathbf{x})\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega_k, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega_k, \quad k = 1, 2 \\ [\mathbf{u}] = 0, [\sigma(\mathbf{u}, p)\mathbf{n}] &= \mathbf{g} \quad \text{on } \Gamma \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

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In this formulation we use standard notations: $\sigma(\mathbf{u}, p) = -p\,I + 2\nu\,\mathrm{D}\mathbf{u}$ is the stress tensor, $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ the rate of deformation tensor, $\mathbf{n}$ is a unit normal vector to Γ, $[a]|_\Gamma = (a|_{\Omega_1} - a|_{\Omega_2})|_\Gamma$. We assume piecewise constant viscosity ($\nu_k > 0$ in Ω_k) and density ($\rho_k > 0$ in Ω_k). An important motivation for considering this type of generalized Stokes interface equations comes from two-phase incompressible flows. Often such two-phase problems can be modeled by time-dependent Navier-Stokes equations with discontinuous density and viscosity coefficients, [4, 9, 12, 21, 22]. A localized force $\mathbf{g}$ on Γ can be used to describe the effect of surface tension. If in such a setting one has highly viscous flows then the nonstationary Stokes equations are a reasonable model problem. After implicit time integration one obtains a problem of the form (1.2). A variational formulation of this problem results in a saddle point problem of the form (1.1).

A Galerkin discretization appoach applied to (1.1) results in a finite dimensional saddle point problem. In the examples mentioned above one applies iterative methods for solving the matrix-vector representation of such discrete problems, cf.[1] for a recent overview. Most of these iterative solvers use block preconditioners [8, 10, 14, 19, 23]. For such methods a good preconditioner of the Schur complement is crucial for the efficiency of the iterative solver. There is an extensive literature on this issue of preconditioning the Schur complement. We mention results that are related to those presented in this paper. For the generalized Stokes problem one is interested in a preconditioner that is robust with respect to variation in both h (mesh size parameter) and τ. Such a preconditioner was introduced by Cahouet and Chabard in [8]. A proof of robustness of this preconditioner (w.r.t. τ and h) in a finite element setting is given in [5]. An analysis of robustness (w.r.t. τ) of this method in a continuous setting can be found in [13]. In a recent paper [15] an analysis is presented which shows how the construction of this Cahouet-Chabard preconditioner is related to certain mapping properties of the gradient operator. This results in a unifying framework in which robust preconditioning of both the continuous and the discrete Schur complement can be analyzed. In [16] it is noted that an important assumption about the regularity of the stationary Stokes problem is hidden in the proof of lemma 1 in [15] and not stated explicitly.

We do not know any literature in which Schur complement preconditioners for the generalized Stokes interface problem (1.2) are treated. A preconditioner for the *stationary* Stokes interface problem, i.e., $\tau = 0$ in (1.2), that is robust with respect to the size of the jump in ν across the interface is introduced and analyzed in [17, 18].

The two main topics of this paper are the following.
Firstly, we extend the analysis that is presented in [15] for the generalized Stokes problem to the general abstract saddle point problem (1.1), resulting in an abstract framework for analyzing the Schur complement $S = B'(A + \tau C)^{-1}B : M \to M'$. In this framework we obtain a natural preconditioner $\tilde{S}$ for this Schur complement. We show that a spectral inequality $S \lesssim \tilde{S}$ that is uniform with respect to τ is easy to derive. For a uniform spectral inequality $\tilde{S} \lesssim S$, however, we need a certain boundedness property for the orthogonal projection $P : H_1' \to B(M) \subset H_1'$ (assumption 1 below). We apply the abstract theory to the continuous and to the discrete generalized Stokes problem. The preconditioner $\tilde{S}$ then coincides with the Cahouet-Chabard preconditioner. To prove the robustness with respect to τ we have to verify the boundedness property, which turns out to be a regularity property for the stationary Stokes problem that is very closely related to the "hidden" assumption in [15] (cf. [16]). The preconditioner $\tilde{S} = \tilde{S}_h$ is of the form $\tilde{S}_h^{-1} = I_h + \tau(B_h'C_h^{-1}B_h)^{-1}$, where I_h is the

identity on the pressure finite element space M_h and B_h, C_h are discrete analogons of the operators B, C in (1.1). We show that if $M_h \subset H^1(\Omega)$ holds, the operator $\tilde{S}_h^{-1}$ is uniformly spectrally equivalent to the simpler operator $\hat{S}_h^{-1} = I_h + \tau N_h^{-1}$, with N_h^{-1} a solution operator of a discrete Neumann problem in the space M_h.

Secondly, we introduce and analyze a Schur complement preconditioner for the generalized Stokes interface problem (1.2). This preconditioner is new and is obtained by applying the general abstract analysis to the variational formulation of the generalized Stokes interface problem. In this interface problem it is interesting (for two-phase flows with large differences in viscosity and density of the two phases) to have a preconditioner that is robust not only with respect to variation in τ but also with respect to the jumps in ν and ρ across the interface. From our general analysis applied to the continuous generalized Stokes interface problem it follows that a spectral inequality $S \lesssim \tilde{S}$ holds uniformly with respect to τ *and* the jumps in ν, ρ. For the spectral inequality $\tilde{S} \lesssim S$ we can only show uniformity with respect to τ. An equality $\tilde{S} \lesssim S$ that is uniform w.r.t. the jumps in ν and ρ, too, would hold if we could verify the boundedness assumption formulated in the abstract theory. It turns out, however, that this requires certain regularity results for the stationary Stokes interface problem that are not known in the literature. This issue of the dependence of the constant in the spectral inequality $\tilde{S} \le c\, S$ on the jumps in ν and ρ will be treated in a forthcoming paper.

The preconditioner for the continuous generalized Stokes interface problem has an obvious discrete analogon. For a standard finite element discretization (P_2-P_1 Hood-Taylor) we present results of numerical experiments that illustrate robustness properties of this preconditoner for the discrete Schur complement.

The remainder of paper is organized as follows. In section 2 we present a general analysis for the abstract problem (1.1). We introduce a preconditioner $\tilde{S}$ for the Schur complement S and derive spectral inequalities $\tilde{S} \lesssim S \lesssim \tilde{S}$. A crucial assumption to obtain the lower spectral inequality uniformly in τ is introduced in section 2.5. In sections 3 we apply the general theory to the continuous generalized Stokes problem and we show that this crucial assumption corresponds to a regularity assumption for the stationary Stokes equations. In section 4 we consider a finite element discretization of the generalized Stokes problem with an LBB stable pair of spaces and show how the general theory can be used to prove robustness of the Cahouet-Chabard preconditioner. In section 5 we apply the abstract analysis to the continuous generalized Stokes interface problem (1.2) and derive a robust preconditioner for the Schur complement. This preconditioner has an obvious discrete analogon. In section 6 results of numerical experiments are presented that illustrate certain robustness properties of this discrete Schur complement preconditioner.

2. General analysis. Consider Hilbert spaces H_1 and M. In section 2.1 we describe a parameter dependent saddle point problem in the pair of spaces $H_1 \times M$. We are interested in a uniform (w.r.t. variation in the parameter) preconditioner for the selfadjoint and positive definite Schur complement operator $S : M \to M'$. In section 2.2 we collect some results that will be used in our analysis. In section 2.3 a Schur complement preconditioner $\tilde{S}$ is introduced. In the sections 2.4 and 2.5 we prove uniform spectral inequalities $S \lesssim \tilde{S}$ and $\tilde{S} \lesssim S$, respectively.

Apart from H_1 we will also use a Hilbert space H_2 such that $H_1 \subset H_2$ and $I : H_1 \to H_2$ is a continuous dense embedding. We use $(\cdot, \cdot)_H$ to denote a scalar product in a Hilbert space H and $\langle \cdot, \cdot \rangle_{H' \times H}$ for the duality pairing. The subscripts are omitted when these are obvious from the context.

2.1. A parameter dependent saddle point problem and its Schur complement. In this section we introduce a parameter dependent saddle point problem. Assume continuous symmetric bilinear forms $a : H_1 \times H_1 \to \mathbb{R}$, $c : H_2 \times H_2 \to \mathbb{R}$ and a continuous bilinear form $b : H_1 \times M$. For the analysis given below it is convenient to introduce corresponding linear mappings

$$A : H_1 \to H_1', \quad \langle A\,u, v \rangle = a(u, v) \quad \text{for all} \ \ u, v \in H_1$$
$$C : H_2 \to H_2', \quad \langle C\,u, v \rangle = c(u, v) \quad \text{for all} \ \ u, v \in H_2$$
$$B : M \to H_1', \quad \langle B\,p, v \rangle = b(v, p) \quad \text{for all} \ \ p \in M, \ v \in H_1$$

For these mappings we assume that there exist positive constants $\gamma_a, \Gamma_a, \gamma_c, \Gamma_c, \gamma_b, \Gamma_b$, such that

$$\gamma_a \|u\|_{H_1} \le \|Au\|_{H_1'} \le \Gamma_a \|u\|_{H_1} \quad \text{for all} \ \ u \in H_1 \tag{2.1}$$
$$\gamma_c \|u\|_{H_2} \le \|Cu\|_{H_2'} \le \Gamma_c \|u\|_{H_2} \quad \text{for all} \ \ u \in H_2 \tag{2.2}$$
$$\gamma_b \|p\|_M \le \|Bp\|_{H_1'} \le \Gamma_b \|p\|_M \quad \text{for all} \ \ p \in M \tag{2.3}$$

Note that the operator $A : H_1 \to H_1'$ is selfadjoint: $\langle Au, v \rangle_{H_1' \times H_1} = \langle Av, u \rangle_{H_1' \times H_1}$ for all $u, v \in H_1$. The operator $C : H_2 \to H_2'$ is selfadjoint, too.

REMARK 1. The properties in (2.1)-(2.3) are fulfilled if the following ellipticity, continuity and infsup conditions hold:

$$\gamma_a \|u\|_{H_1}^2 \le a(u, u), \quad a(u, v) \le \Gamma_a \|u\|_{H_1} \|v\|_{H_1} \quad \text{for all} \ \ u, v \in H_1$$
$$\gamma_c \|u\|_{H_2}^2 \le c(u, u), \quad c(u, v) \le \Gamma_c \|u\|_{H_2} \|v\|_{H_2} \quad \text{for all} \ \ u, v \in H_2$$
$$\gamma_b \|p\|_M \le \sup_{v \in H_1} \frac{b(v, p)}{\|v\|_{H_1}}, \quad b(v, p) \le \Gamma_b \|v\|_{H_1} \|p\|_M \quad \text{for all} \ \ v \in H_1, \ p \in M$$

Consider the following general saddle point problem: Given $\tau \ge 0$ and $f \in H_1'$, find $(u, p) \in H_1 \times M$ such that

$$a(u, v) + \tau c(u, v) + b(v, p) + b(u, q) = f(v) \quad \text{for all} \ \ v \in H_1, \ q \in M. \tag{2.4}$$

The problem (2.4) can be rewritten in operator formulation: find $(u, p) \in H_1 \times M$ such that

$$\begin{cases} Au + \tau Cu + Bp & = \ f \\ B'u & = \ 0 \end{cases} \tag{2.5}$$

Standard analyses of saddle point problems (e.g., [6]) yield that this problem has a unique solution. The Schur complement

$$S := B'(A + \tau C)^{-1} B$$

of the system is a selfadjoint positive definite operator $S : M \to M'$. It defines a scalar product (and corresponding norm) on M:

$$\|p\|_S := \langle S\,p, p \rangle^{\frac{1}{2}} = \sup_{v \in H_1} \frac{\langle Bp, v \rangle}{\langle (A + \tau C)v, v \rangle^{\frac{1}{2}}}, \quad p \in M \tag{2.6}$$

EXAMPLE 1. For the nonstationary Stokes system on a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^d$ we take the spaces

$$H_1 := \mathbf{H}_0^1(\Omega), \ H_2 := \mathbf{L}^2(\Omega), \ M = L_0^2(\Omega)$$

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with scalar products

$$(u, v)_{H_1} := (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \quad (u, v)_{H_2} := (\mathbf{u}, \mathbf{v})_{L^2}, \quad (p, q)_M := (p, q)_{L^2}$$

The bilinear forms are

$$a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \quad c(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})_{L^2}, \quad b(\mathbf{v}, p) := -(p, \operatorname{div} \mathbf{v})_{L^2}$$

and the problem is as follows: Find $(\mathbf{u}, p) \in H_1 \times M$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + \tau c(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &=& f(\mathbf{v}) \quad \text{for all} \ \ \mathbf{v} \in H_1 \\ b(\mathbf{u}, q) &=& 0 \quad \text{for all} \ \ q \in M \end{cases} \tag{2.7}$$

Recall the infsup inequality (Nečas inequality):

$$\sup_{\mathbf{v} \in H_1} \frac{(\operatorname{div} \mathbf{v}, p)}{\|\nabla \mathbf{v}\|_{L_2}} \geq \gamma_b \|p\|_{L_2} \quad \text{for all} \ \ p \in M$$

with $\gamma_b > 0$. Using this one easily verifies that conditions in (2.1)–(2.3) are satisfied with $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = \Gamma_b = 1$, $\gamma_b > 0$ the constant from the infsup inequality.

2.2. Preliminaries. In this section we derive some properties of the saddle point problem (2.5) that will be used in the analysis of the Schur complement preconditioner. We use the concept of sums and intersections of vector spaces (cf. [3]). The idea of applying this concept in the analysis of Schur complement preconditioners is introduced in [15].

Let X, Y be compatible normed spaces, i.e., both X and Y are subspaces of some larger topological vector space Z. Then we can form their sum $X + Y$ and intersection $X \cap Y$. The sum $X + Y$ consists of all $z \in Z$ such that we can write $z = x + y$ with $x \in X, y \in Y$. The spaces $X \cap Y$ and $X + Y$ are normed vector spaces with norms

$$\|x\|_{X \cap Y} = \left(\|x\|_X^2 + \|x\|_Y^2 \right)^{\frac{1}{2}} \quad (x \in X \cap Y)$$

$$\|z\|_{X+Y} = \inf_{z = x + y} \left(\|x\|_X^2 + \|y\|_Y^2 \right)^{\frac{1}{2}} \quad (x \in X, \ y \in Y)$$

If X and Y are complete then both $X \cap Y$ and $X + Y$ are complete (Lemma 2.3.1 in [3]). A few properties that we will need further on are given in the following lemma. The space of bounded linear mappings $X \to Y$ is denoted by $\mathcal{L}(X, Y)$.

LEMMA 2.1. *Let X_1, X_2 and Y_1, Y_2 be pairs of compatible normed vector spaces and let T be a linear mapping on $X_1 + X_2$ such that $T \in \mathcal{L}(X_1, Y_1) \cap \mathcal{L}(X_2, Y_2)$. Then $T : X_1 + X_2 \to Y_1 + Y_2$ is bounded and*

$$\|T\|_{X_1 + X_2 \to Y_1 + Y_2} \leq \left(\|T\|_{X_1 \to Y_1}^2 + \|T\|_{X_2 \to Y_2}^2 \right)^{\frac{1}{2}} \tag{2.8}$$

holds. If X_1 and X_2 are Hilbert spaces such that $X_1 \cap X_2$ is dense in both X_1 and X_2, then $(X_1 \cap X_2)' = X_1' + X_2'$ holds and

$$\|g\|_{(X_1 \cap X_2)'} = \|g\|_{X_2' + X_2'} \quad \text{for all} \ \ g \in (X_1 \cap X_2)' \tag{2.9}$$

Proof. Proofs are given in [3]. Since these results are fundamental for our further considerations, we present an elementary proof to make the paper self-contained.

Consider $x \in X$ and an arbitrary decomposition $x = x_1 + x_2$, $x_1 \in X_1$, $x_2 \in X_2$. For $Tx = Tx_1 + Tx_2$ we have

$$\|Tx\|_{Y_1+Y_2} = \inf_{Tx=y_1+y_2} \left(\|y_1\|_{Y_1}^2 + \|y_2\|_{Y_2}^2\right)^{\frac{1}{2}} \leq \left(\|Tx_1\|_{Y_1}^2 + \|Tx_2\|_{Y_2}^2\right)^{\frac{1}{2}}$$

$$\leq \left(\|T\|_{X_1 \to Y_1}^2 + \|T\|_{X_2 \to Y_2}^2\right)^{\frac{1}{2}} \left(\|x_1\|_{X_1}^2 + \|x_2\|_{X_2}^2\right)^{\frac{1}{2}}$$

Since the decomposition $x = x_1 + x_2$ is arbitrary we obtain

$$\|Tx\|_{Y_1+Y_2} \leq \left(\|T\|_{X_1 \to Y_1}^2 + \|T\|_{X_2 \to Y_2}^2\right)^{\frac{1}{2}} \|x\|_{X_1+X_2}.$$

Thus (2.8) is proved.

To prove the second part of the lemma consider arbitrary $g \in X_1' + X_2'$ and take $x \in X_1 \cap X_2$. For an arbitrary decomposition $g = g_1 + g_2$ with $g_1 \in X_1'$, $g_2 \in X_2'$ we have $g(x) = g_1(x) + g_2(x)$ and thus

$$|g(x)| \leq \left(\|g_1\|_{X_1'}^2 + \|g_2\|_{X_2'}^2\right)^{\frac{1}{2}} \left(\|x\|_{X_1}^2 + \|x\|_{X_2}^2\right)^{\frac{1}{2}}$$

This yields

$$|g(x)| \leq \inf_{g=g_1+g_2} \left(\|g_1\|_{X_1'}^2 + \|g_2\|_{X_2'}^2\right)^{\frac{1}{2}} \|x\|_{X_1 \cap X_2} = \|g\|_{X_2'+X_1'} \|x\|_{X_1 \cap X_2}.$$

Therefore $(X_1 \cap X_2)' \supset X_1' + X_2'$ and $\|g\|_{(X_1 \cap X_2)'} \leq \|g\|_{X_2'+X_1'}$.

Now take $g \in (X_1 \cap X_2)'$. Since $X_1 \cap X_2$ with scalar product $(\cdot, \cdot)_{X_1} + (\cdot, \cdot)_{X_2}$ is a Hilbert space, there exists an element $G \in X_1 \cap X_2$ such that $g(x) = (G, x)_{X_1} + (G, x)_{X_2}$ for all $x \in X_1 \cap X_2$ and

$$\|g\|_{(X_1 \cap X_2)'} = \|G\|_{X_1 \cap X_2} = \left(\|G\|_{X_1}^2 + \|G\|_{X_2}^2\right)^{\frac{1}{2}}$$

For $i = 1, 2$ define $\hat{g}_i : x \to (G, x)_{X_i}$ for all $x \in X_i$. Then $\hat{g}_i \in X_i'$, $\|\hat{g}_i\|_{X_i'} = \|G\|_{X_i}$ and $g(x) = \hat{g}_1(x) + \hat{g}_2(x)$ for all $x \in X_1 \cap X_2$. Because $X_1 \cap X_2$ is dense in X_1 and in X_2, both $\hat{g}_1$ and $\hat{g}_2$ are uniquely defined by their values at $x \in X_1 \cap X_2$. Hence we get $g = \hat{g}_1 + \hat{g}_2 \in X_1' + X_2'$ and

$$\|g\|_{X_1'+X_2'} = \inf_{g=g_1+g_2} \left(\|g_1\|_{X_1'}^2 + \|g_2\|_{X_2'}^2\right)^{\frac{1}{2}}$$

$$\leq \left(\|\hat{g}_1\|_{X_1'}^2 + \|\hat{g}_2\|_{X_2'}^2\right)^{\frac{1}{2}} = \left(\|G\|_{X_1}^2 + \|G\|_{X_2}^2\right)^{\frac{1}{2}} = \|g\|_{(X_1 \cap X_2)'}$$

which completes the proof. $\square$

In the remainder we assume $\tau > 0$, unless stated otherwise. By τH_2 we denote the space H_2 with the scaled scalar product $\tau(\cdot, \cdot)_{H_2}$. Using the previous lemma we obtain the following equivalence result for the Schur complement norm in (2.6).

THEOREM 2.2. *For all $p \in M$ we have*

$$\min\{\gamma_a, \gamma_c\}\|p\|_S^2 \leq \|Bp\|_{H_1'+\tau^{-1}H_2'}^2 \leq \max\{\Gamma_a, \Gamma_c\}\|p\|_S^2 \tag{2.10}$$

Proof. For $p \in M$ we have

$$\|Bp\|_{(H_1 \cap \tau H_2)'} = \sup_{v \in H_1} \frac{\langle Bp, v \rangle}{\left(\|v\|_{H_1}^2 + \tau\|v\|_{H_2}^2\right)^{\frac{1}{2}}} \tag{2.11}$$

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Due to the properties of A and C and the definition of $\| \cdot \|_S$ we get

$$\min\{\gamma_a, \gamma_c\}\|p\|_S^2 \leq \|Bp\|_{(H_1 \cap \tau H_2)'}^2 \leq \max\{\Gamma_a, \Gamma_c\}\|p\|_S^2 \quad \text{for all } p \in M$$

Now we apply the result in (2.9) to the case $X_1 = H_1, X_2 = \tau H_2$. Note that $H_1 \cap \tau H_2 = H_1$ (this should be understood as equality of sets) and that the intersection is dense in τH_2. Hence, we get

$$\|Bp\|_{(H_1 \cap \tau H_2)'} = \|Bp\|_{H_1' + \tau^{-1}H_2'}$$

and thus the result is proved. $\quad \square$

REMARK 2. For the nonstationary Stokes problem described in example 1 we obtain

$$(S\,p, p)_{L^2} = \|\nabla p\|_{\mathbf{H}^{-1} + \tau^{-1}\mathbf{L}^2}^2 \qquad \text{for all } p \in L_0^2(\Omega) \ .$$

We introduce a subspace W of M:

$$W = \{\, p \in M \mid Bp \in H_2' \,\} \tag{2.12}$$

and define the following functional on W:

$$\|p\|_W := \sup_{v \in H_2} \frac{\langle Bp, v \rangle}{\langle Cv, v \rangle^{\frac{1}{2}}}. \tag{2.13}$$

The lemma below summarizes several useful properties of W.

LEMMA 2.3. *The following holds*

$$\| \cdot \|_W \ \text{defines a norm and } (W, \| \cdot \|_W) \text{ is a Hilbert space,} \tag{2.14}$$

$$I : W \to M \ \text{is a continuous embedding,} \tag{2.15}$$

$$B(W) \ \text{is a closed subspace of } \ H_2' \,, \tag{2.16}$$

$$\text{If } \dim(H_2) < \infty \ \text{then } W = M \ \text{holds.} \tag{2.17}$$

Proof. First we prove (2.16). From the definition (2.12) we have $B(W) = B(M) \cap H_2'$. Thanks to (2.3) $B(M)$ is closed subspace of H_1'. From the assumption that $H_2' \hookrightarrow H_1'$ is a continuous embedding, it follows that $B(M)$ is closed subspace of H_2'. Thus $B(W)$ is closed in H_2'.

To prove (2.14) and (2.15) we note that for all $p \in W$ we have

$$\Gamma_c^{-\frac{1}{2}}\|Bp\|_{H_2'} \leq \|p\|_W \leq \gamma_c^{-\frac{1}{2}}\|Bp\|_{H_2'},$$

$$\|p\|_M \leq \gamma_b^{-1}\|Bp\|_{H_1'} \leq c\|Bp\|_{H_2'} \leq c\Gamma_c^{\frac{1}{2}}\|p\|_W \quad \text{with } c \text{ independent of } p$$

Hence, $\| \cdot \|_W$ indeed defines a norm on W and $I : W \to M$ is a coninuous embedding. Moreover, since $B(W)$ is closed in H_2' and $B : W \to B(W)$ is bounded $(W, \| \cdot \|_W)$ is a Banach space. It remains to define a scalar product on W that induces $\| \cdot \|_W$. For this we need the adjoint of $B : W \to H_2'$. Recall that $B' : H_1 \to M'$ is the adjoint of $B : M \to H_1'$, i.e., $\langle B'v, p \rangle_{M' \times M} = \langle Bp, v \rangle_{H_1' \times H_1}$ for all $v \in H_1, p \in M$. To distinguish from this adjoint we use the notation B_2' for the adjoint of $B : W \to H_2'$.

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Hence, $\langle B_2' v, p \rangle_{W' \times W} = \langle Bp, v \rangle_{H_2' \times H_2}$ for all $v \in H_2$, $p \in W$. Using this we define $S_2 : W \to W'$ by

$$S_2 := B_2' C^{-1} B \qquad (2.18)$$

and we get

$$\|p\|_W^2 = \langle S_2 p, p \rangle_{W' \times W} \quad \text{for all } p \in W \qquad (2.19)$$

Thus the scalar product on W that corresponds to $\|\cdot\|_W$ is given by $(p, q)_W = \langle S_2 p, q \rangle_{W' \times W}$.

From $\dim(H_2) < \infty$ and the assumption that the embedding $H_1 \hookrightarrow H_2$ is dense it follows that $H_1 = H_2$ (with possibly different norms) and that $g \in H_1'$ iff $g \in H_2'$. Using that $Bp \in H_1'$ for all $p \in M$ we conclude that $W = M$ holds. $\quad\square$

In our analysis we will need the orthogonal projection on $B(M)$ in H_1'. This projection, which is well-defined since $B(M)$ is a closed subspace of H_1', is denoted by P. The following lemma gives another characterization of this projection P.

LEMMA 2.4. *Let $I_1 : H_1 \to H_1'$ be the Riesz isomorphism, i.e., $\langle I_1 u, v \rangle = (u, v)_{H_1}$ for all $u, v \in H_1$. For $f \in H_1'$ let $(u, p) \in H_1 \times M$ be the unique solution of*

$$I_1 u + Bp = f$$
$$B' u = 0$$

Define the solution operator $S_1 : H_1' \to M$ by $f \to p$. Then $P = B S_1$ holds.

Proof. For arbitrary $f \in H_1'$ we have $B S_1 f = Bp \in B(M)$ and for any $q \in M$:

$$(f - Bp, Bq)_{H_1'} = \langle I_1^{-1}(f - Bp), Bq \rangle_{H_1 \times H_1'} = \langle u, Bq \rangle_{H_1 \times H_1'} = \langle B'u, q \rangle_{M' \times M} = 0$$

and thus the result holds. $\quad\square$

2.3. Schur complement preconditioner. We introduce the norm

$$\|p\|_{M + \tau^{-1} W} = \inf_{q \in W} \left(\|p - q\|_M^2 + \tau^{-1} \|q\|_W^2 \right)^{\frac{1}{2}} \qquad (2.20)$$

From the analysis below (section 2.4 and 2.5) it follows that (under a certain assumption) this norm is uniformly (w.r.t. τ) equivalent to $\|p\|_S = \langle Sp, p \rangle^{\frac{1}{2}}$. It is not obvious how to use $\|p\|_{M + \tau^{-1} W}$ to construct a feasible preconditioner for the Schur complement S. In this section we address this issue.
Let $I_M : M \to M'$ be the Riesz isomorphism. Because $I : W \to M$ is a continuous embedding we have $I_M(W) \subset W'$. The mapping $I_M : W \to W'$ is denoted by I_W (note that in general this is not the Riesz-isomorphism in W).

THEOREM 2.5. *Define $\tilde{S} : M \to M'$ by $\tilde{S} = I_M - I_M(I_W + \tau^{-1} S_2)^{-1} I_M$ with S_2 defined in (2.18). Then $\tilde{S}$ is selfadjoint and positive definite and*

$$\|p\|_{M + \tau^{-1} W}^2 = \langle \tilde{S} p, p \rangle \quad \text{for all } p \in M \qquad (2.21)$$

Proof. By assumption the operator $C^{-1} : H_2' \to H_2'$ is selfadjoint, therefore $\tilde{S}$ is selfadjoint as well.

With the help of elementary variational analysis we see that the infimum on the right handside in (2.20) is attained for $\tilde{q} \in W$ that satisfies

$$(\tilde{q} - p, \xi)_M + \tau^{-1}(\tilde{q}, \xi)_W = 0 \quad \text{for all } \xi \in W.$$

This can be reformulated in operator notation, using the definition of the W-scalar product:

$$\langle I_M(\tilde{q} - p) + \tau^{-1}S_2\tilde{q}, \xi \rangle_{W' \times W} = 0 \quad \text{for all } \xi \in W \tag{2.22}$$

Note that $I_M p \in M' \subset W'$ holds. The solution $\tilde{q} \in W$ of (2.22) is given by

$$(I_W + \tau^{-1}S_2)\tilde{q} = I_M p$$

and thus $\tilde{q} = (I_W + \tau^{-1}S_2)^{-1}I_M p$. A straightforward computation yields

$$\|p\|^2_{M + \tau^{-1}W} = \|p - \tilde{q}\|^2_M + \tau^{-1}\|\tilde{q}\|^2_W = (p - \tilde{q}, p)_M = \langle I_M(p - \tilde{q}), p \rangle.$$

Substituting $\tilde{q} = (I_W + \tau^{-1}S_2)^{-1}I_M p$, we obtain (2.21). From (2.21) it follows that $\tilde{S}$ is positive definite. $\quad\square$

In the setting of preconditioning one is interested in the inverse of the preconditioner. By a straightforward computation one can check that the inverse $\tilde{S}^{-1} : M' \to M$ of $\tilde{S}$ is given by

$$\tilde{S}^{-1} := I_M^{-1} + \tau S_2^{-1}. \tag{2.23}$$

2.4. Uniform spectral bound $S \lesssim \tilde{S}$. The proof of a spectral bound $S \leq c\tilde{S}$ with a constant c independent of τ is very simple.

THEOREM 2.6. *Define* $\Gamma_s = \frac{\Gamma_b^2 + \Gamma_c}{\min\{\gamma_a, \gamma_c\}}$. *For all $p \in M$ we have*

$$\langle Sp, p \rangle \leq \Gamma_s \langle \tilde{S}p, p \rangle$$

Proof. From theorem 2.2 we get

$$\langle Sp, p \rangle \leq \frac{1}{\min\{\gamma_a, \gamma_c\}} \|Bp\|^2_{H_1' + \tau^{-1}H_2'}$$

From (2.2) and the definition of $\| \cdot \|_W$ we have

$$\|B\|_{M \to H_1'} \leq \Gamma_b, \quad \|B\|_{W \to H_2} \leq \Gamma_c^{\frac{1}{2}}$$

and thus from (2.8) we obtain

$$\|Bp\|_{H_1' + \tau^{-1}H_2} \leq (\Gamma_b^2 + \Gamma_c)^{\frac{1}{2}} \|p\|_{M + \tau^{-1}W}$$

Hence, using theorem 2.5, we obtain

$$\|Bp\|^2_{H_1' + \tau^{-1}H_2} \leq (\Gamma_b^2 + \Gamma_c)\langle \tilde{S}p, p \rangle$$

$\square$

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2.5. Uniform spectral bound $\tilde{S} \lesssim S$. The derivation of a spectral inequality $\tilde{S} \leq \hat{c}S$ with a constant $\hat{c} > 0$ independent of τ turns out to be more delicate than the bound $S \leq c\tilde{S}$ that is shown in theorem 2.6. We present an analysis which requires an assumption on the orthogonal projection $P : H_1' \to B(M)$ (cf. section 2.2). This crucial assumption is as follows.

ASSUMPTION 1. *Assume that $P : H_2' \to H_2'$ and that there exist constants $c_P \geq 1,\ d_P \geq 0$ such that*

$$\|Pf\|_{H_2'}^2 \leq c_P^2(\|f\|_{H_2'}^2 + d_P^2\|(I-P)f\|_{H_1'}^2) \quad \text{for all}\ \ f \in H_2' \tag{2.24}$$

LEMMA 2.7. *If assumption 1 holds then we have*

$$B(W) = P(H_2'), \tag{2.25}$$

Proof. Take $p \in W$. Then $Bp \in H_2' \subset H_1'$ and with the solution operator S as in lemma 2.4 we get $S_1 Bp = p$. This yields $P\,Bp = B\,S_1 Bp = Bp$ and thus $Bp \in P(H_2')$, which proves $B(W) \subset P(H_2')$. Take $Pf \in P(H_2')$. Then $Pf \in H_2'$ and $Pf = B\,S_1 f = Bp$ with $p := S_1 f \in M$. Thus $Pf \in B(W)$, i.e., $P(H_2') \subset B(W)$. Hence, the result (2.25) holds. $\quad\square$

Below we use the spaces $(B(M), \|\cdot\|_{H_1'})$ and $(B(W), \|\cdot\|_{H_2'})$.

LEMMA 2.8. *Let assumption 1 hold. Then for all $p \in M$ we have*

$$\|Bp\|_{B(M)+\tau^{-1}B(W)} \leq c_P\|Bp\|_{H_1'+\tau^{-1}H_2'} \quad \text{for all}\ \ \tau \geq d_P^2$$

Proof. We use the notation $f := Bp$. Note that

$$\|f\|_{H_1'+\tau^{-1}H_2'} = \inf_{w\in H_2'}\big(\|f-w\|_{H_1'}^2 + \tau^{-1}\|w\|_{H_2'}^2\big)^{\frac{1}{2}}$$

Take an arbitrary $w \in H_2'$. Using $f \in B(M)$ we get

$$\|f-w\|_{H_1'}^2 = \|P(f-w) + (P-I)w\|_{H_1'}^2 = \|Pf - Pw\|_{H_1'}^2 + \|(P-I)w\|_{H_1'}^2$$
$$= \|f - Pw\|_{H_1'}^2 + \|(I-P)w\|_{H_1'}^2.$$

From $\|Pw\|_{H_2'}^2 \leq c_P^2(\|w\|_{H_2'}^2 + d_P^2\|(I-P)w\|_{H_1'}^2)$ we get $\|w\|_{H_2'}^2 \geq c_P^{-2}\|Pw\|_{H_2'}^2 - d_P^2\|(I-P)w\|_{H_1'}^2$. Hence we obtain, using $\tau \geq d_P^2$ and $c_P \geq 1$,

$$\inf_{w\in H_2'}\big(\|f-w\|_{H_1'}^2 + \tau^{-1}\|w\|_{H_2'}^2\big)^{\frac{1}{2}} \geq \inf_{w\in H_2'}\big(\|f-Pw\|_{H_1'}^2 + \tau^{-1}c_P^{-2}\|Pw\|_{H_2'}^2$$

$$+\quad (1-\tau^{-1}d_P^2)\|(I-P)w\|_{H_1'}^2\big)^{\frac{1}{2}}$$

$$\geq \inf_{w\in H_2'}\big(\|f-Pw\|_{H_1'}^2 + \tau^{-1}c_P^{-2}\|Pw\|_{H_2'}^2\big)^{\frac{1}{2}}$$

$$\geq c_P^{-1}\inf_{w\in H_2'}\big(\|f-Pw\|_{H_1'}^2 + \tau^{-1}\|Pw\|_{H_2'}^2\big)^{\frac{1}{2}}$$

$$\geq c_P^{-1}\inf_{\substack{f = f_1 + f_2 \\ f_1 \in B(M),\ f_2 \in B(W)}}\big(\|f_1\|_{H_1'}^2 + \tau^{-1}\|f_2\|_{H_2'}^2\big)^{\frac{1}{2}}$$

$$= c_P^{-1}\|f\|_{B(M)+\tau^{-1}B(W)}$$

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and thus the result is proved. $\square$

REMARK 3. Consider the *finite dimensional* case $\dim(H_2) < \infty$. We then have $H_1 = H_2$, $H_1' = H_2'$ and $W = M$ (where "=" allows different norms in the spaces). We can apply a symmetry argument involving an alternative to the assumption 1. Let $\hat{P} : H_2' \to B(W)$ be the orthogonal projection on $B(W)$ in H_2'. Assume that $c_P \geq 1$, $\hat{d}_P \geq 0$ are such that

$$\|\hat{P}f\|_{H_1'}^2 \leq c_P^2(\|f\|_{H_1'}^2 + \hat{d}_P^2\|(I - \hat{P})f\|_{H_2'}^2) \quad \text{for all} \ \ f \in H_2' \tag{2.26}$$

Lemma 2.8 then yields, for all $p \in M$:

$$\|Bp\|_{B(W)+\tau^{-1}B(M)} \leq c_P\|Bp\|_{H_2'+\tau^{-1}H_1'} \quad \text{for all} \ \ \tau \geq \hat{d}_P^2$$

which is equivalent to

$$\|Bp\|_{B(M)+\tau^{-1}B(W)} \leq c_P\|Bp\|_{H_1'+\tau^{-1}H_2'} \quad \text{for all} \ \ \tau \leq \hat{d}_P^{-2} \tag{2.27}$$

This will be used in the analysis of the finite element discretization in section 4.

THEOREM 2.9. *Let assumption 1 hold. Define* $\gamma_s := \frac{\gamma_b^2\gamma_c}{c_P^2(\gamma_b^2+\gamma_c)\max\{\Gamma_a,\Gamma_c\}}$. *For all* $p \in M$ *we have*

$$\gamma_s\langle \tilde{S}p, p\rangle \leq \langle Sp, p\rangle \quad \text{for all} \ \ \tau \geq d_P^2 \tag{2.28}$$

Proof. From (2.2) and the definition of $\|\cdot\|_W$ we also have

$$\|B^{-1}\|_{B(M)\to M} \leq \gamma_b^{-1}, \quad \|B^{-1}\|_{B(W)\to W} \leq \gamma_c^{-\frac{1}{2}}$$

and thus

$$\|B^{-1}g\|_{M+\tau^{-1}W} \leq (\gamma_b^{-2} + \gamma_c^{-1})^{\frac{1}{2}}\|g\|_{B(M)+\tau^{-1}B(W)} \quad \text{for all} \ \ g \in B(M)$$

Hence,

$$\frac{\gamma_b^2\gamma_c}{\gamma_b^2 + \gamma_c}\|p\|_{M+\tau^{-1}W}^2 \leq \|Bp\|_{B(M)+\tau^{-1}B(W)}^2 \quad \text{for all} \ \ p \in M$$

Using lemma 2.8 we obtain

$$\frac{\gamma_b^2\gamma_c}{c_P^2(\gamma_b^2 + \gamma_c)}\|p\|_{M+\tau^{-1}W}^2 \leq \|Bp\|_{H_1'+\tau^{-1}H_2}^2 \quad \text{for all} \ \ \tau \geq d_P^2$$

and combining this with theorem 2.5 and theorem 2.2 proves the inequality in (2.28). $\square$

REMARK 4. Consider the *finite dimensional* setting as in remark 3 and assume that besides assumption 1 also (2.26) holds. Then (2.27) holds, and a slight modification of the last step in the proof of theorem 2.9 then yields, for all $p \in M$:

$$\gamma_s\langle \tilde{S}p, p\rangle \leq \langle Sp, p\rangle \quad \forall \ \tau \in (0, \hat{d}_P^{-2}] \cup [d_P^2, \infty) \tag{2.29}$$

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The main result of the general analysis is the following.

COROLLARY 2.10. *Suppose assumption 1 holds. The following inequalities hold for any $p \in M$:*

$$\gamma_s \langle \tilde{S}p, p \rangle \leq \langle Sp, p \rangle \leq \Gamma_s \langle \tilde{S}p, p \rangle \quad \text{for all} \;\; \tau \geq d_P^2 \qquad (2.30)$$

Proof. Direct consequence of theorem 2.6 and theorem 2.9. ☐

REMARK 5. Consider the *finite dimensional* case $\dim(H_2) < \infty$. From remark 4 it follows that if the assumptions 1 *and* (2.26) hold, then the equivalence result in (2.30) holds for all $\tau \in (0, \hat{d}_P^{-2}] \cup [d_P^2, \infty)$.

As a final result in this section we give a simple criterion that will be used in the applications in the next sections to show that assumption 1 holds.

LEMMA 2.11. *Let $S_1 : H_1' \to M$ be the solution operator from lemma 2.4.*

Assume that there is a subspace $\tilde{W} \subset M$ with norm $\| \cdot \|_{\tilde{W}}$ such that both $S_1 : H_2' \to \tilde{W}$ and $B : \tilde{W} \to H_2'$ are bounded, i.e.,

$$\|S_1 f\|_{\tilde{W}} \leq c_1 \|f\|_{H_2'} \quad \forall \, f \in H_2' \;, \quad \|Bp\|_{H_2'} \leq c_2 \|p\|_{\tilde{W}} \quad \forall \, p \in \tilde{W} \;,$$

then assumption 1 is fulfilled with $c_P = c_1 c_2$, $d_P = 0$.

Proof. The proof immediately follows from $P = B\,S_1$ and

$$\|Pf\|_{H_2'} = \|B\,S_1 f\|_{H_2'} \leq c_2 \|S_1 f\|_{\tilde{W}} \leq c_2 c_1 \|f\|_{H_2'} \quad \text{for all} \;\; f \in H_2'$$

☐

3. Application to the continuous generalized Stokes problem. In this section we apply the above abstract analysis to the generalized Stokes problem. The spaces and bilinear forms used in the variational problem are as in example 1. It was noted that we have the properties (2.1)-(2.3) with $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = \Gamma_b = 1$, $\gamma_b > 0$ the constant from the infsup inequality . For the operators A, B, B', C corresponding to the bilinear forms we use the (usual) notation

$$A =: -\Delta, \;\; B := \nabla, \;\; B' =: -\mathrm{div}, \;\; C =: \mathrm{I}$$

We now consider assumption 1. We use the criterion given in lemma 2.11. Note that $-\Delta$ is the Riesz isomorphism $\mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega)' =: \mathbf{H}^{-1}$. Thus for $\mathbf{f} \in \mathbf{H}^{-1}$ the solution $p = S_1 \mathbf{f}$, with S_1 from lemma 2.4, satisfies the weak formulation of the *stationary* Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \\ \mathrm{div}\, \mathbf{u} &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0 \end{aligned} \qquad (3.1)$$

In the following lemma it is shown *that H^2-regularity of the Stokes problem implies that assumption 1 holds.*

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LEMMA 3.1. *Assume that the domain Ω is such that the Stokes problem (3.1) is H^2-regular, i.e., there is a constant c_R such that for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the solution $(\mathbf{u}, p)$ is an element of $H^2(\Omega)^d \times H^1(\Omega)$ and satisfies*

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|\nabla p\|_{L^2} \le c_R \|\mathbf{f}\|_{L^2} \tag{3.2}$$

Then assumption 1 is satisfied with $c_P = c_R$ and $d_P = 0$. Furthermore, we have $W = H^1(\Omega) \cap L_0^2(\Omega)$ and $\|p\|_W = \|\nabla p\|_{L^2}$.

Proof. We apply lemma 2.11 with $\tilde{W} := H^1(\Omega) \cap L_0^2(\Omega)$ and norm $\|p\|_{\tilde{W}}^2 = (\nabla p, \nabla p)_{L^2}$. Due to the regularity assumption we have $\|S_1 \mathbf{f}\|_{\tilde{W}} = \|\nabla p\|_{L^2} \le c_R \|\mathbf{f}\|_{L^2}$. Furthermore, for $p \in \tilde{W}$ we have $\|Bp\|_{H_2'} = \|\nabla p\|_{L^2} = \|p\|_{\tilde{W}}$. Thus the assumptions in lemma 2.11 hold with $c_1 = c_R$, $c_2 = 1$. It follows that assumption 1 is fulfilled.

The definition (2.12) of W takes the form $W := \{p \in L_0^2 : \nabla p \in \mathbf{L}^2\}$. Thus $W = H^1(\Omega) \cap L_0^2(\Omega) = \tilde{W}$. Finally by the definition of the W-norm we have for $p \in \tilde{W}$:

$$\|p\|_W := \sup_{v \in H_2} \frac{\langle Bp, \mathbf{v} \rangle}{\langle Cv, v \rangle^{\frac{1}{2}}} = \sup_{\mathbf{v} \in \mathbf{L}^2} \frac{(\nabla p, v)_{L^2}}{\|\mathbf{v}\|_{L^2}} = \|\nabla p\|_{L^2}.$$

□

Now consider the Schur complement of the generalized Stokes problem:

$$S := -\operatorname{div} (\tau I - \Delta)^{-1} \nabla \tag{3.3}$$

We identify $L_0^2(\Omega)$ with its dual. Then $S : L_0^2(\Omega) \to L_0^2(\Omega)$ and $\langle \cdot, \cdot \rangle_{M' \times M} = (\cdot, \cdot)_{L^2}$.

If the stationary Stokes problem is H^2-regular our abstract theory can be applied, with $d_P = 0$ in assumption 1, and we have a uniform equivalence result given in corollary 2.10. This yields the following main result of this section.

THEOREM 3.2. *Assume that the domain $\Omega \subset \mathbb{R}^d$ is such that the Stokes problem (3.1) is H^2-regular. Denote by $-\Delta_N^{-1} : L_0^2(\Omega) \to H^1(\Omega) \cap L_0^2(\Omega)$ the solution operator of the following Neumann pressure problem: Given $f \in L_0^2(\Omega)$ find $p \in H^1(\Omega) \cap L_0^2(\Omega)$ such that*

$$(\nabla p, \nabla q)_{L^2} = (f, q)_{L^2}, \quad \forall \, q \in H^1(\Omega) \cap L_0^2(\Omega).$$

Define $\tilde{S}^{-1} = I - \tau \Delta_N^{-1}$. Then $\tilde{S}^{-1} : L_0^2(\Omega) \to L_0^2(\Omega)$ is selfadjoint and positive definite, and for all $p \in L_0^2(\Omega)$ and all $\tau \ge 0$ the following holds:

$$\gamma_s (\tilde{S}p, p)_{L^2} \le (Sp, p)_{L^2} \le \Gamma_s (\tilde{S}p, p)_{L^2}.$$

with $\gamma_s = \frac{\gamma_b^2}{c_R^2 (\gamma_b^2 + 1)}$, $\Gamma_s = 2$.

Proof. We apply corollary 2.10. In the setting here we have $W = H_0^1(\Omega) \cap L_0^2(\Omega)$, $M = L_0^2(\Omega) = M'$. The mapping $\tilde{S} : M \to M$ is defined by, cf. (2.23), $\tilde{S}^{-1} = I_{L^2}^{-1} + \tau S_2^{-1} = I + \tau S_2^{-1}$ with $S_2 = B_2' C^{-1} B$. For $f \in M$ we have $w = S_2^{-1} f \in W$ iff

$$
\begin{aligned}
& \langle B_2' C^{-1} B w, q \rangle_{W' \times W} = (f, q)_{L^2} \quad \forall q \in W \\
\Leftrightarrow \quad & \langle Bq, C^{-1} B w \rangle_{L^2 \times L^2} = (f, q)_{L^2} \quad \forall q \in W \\
\Leftrightarrow \quad & \langle \nabla q, I_{L^2}^{-1} \nabla w \rangle_{L^2 \times L^2} = (f, q)_{L^2} \quad \forall q \in W \\
\Leftrightarrow \quad & (\nabla w, \nabla q)_{L^2} = (f, q)_{L^2} \quad \forall q \in W
\end{aligned}
$$

and thus S_2^{-1} is equal to the Neumann solution operator $-\Delta_N^{-1}$. Hence $\tilde{S}^{-1} = I + \tau S_2^{-1} = I - \tau \Delta_N^{-1}$. The values for the spectral bounds follow from corollary 2.10 and from $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = \Gamma_b = 1$, and $c_P = c_R$. □

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4. Application to finite element discretization of the generalized Stokes problem. In this section we apply the abstract analysis of section 2 to a finite element discretization of the generalized Stokes problem (2.7).

Let $\mathbf{V}_h \times M_h \subset \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be a pair of conforming finite element spaces. We assume the LBB stability condition:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, p_h)_{L^2}}{\|\nabla \mathbf{v}_h\|_{L_2}} \geq \gamma_b \|p_h\|_{L_2} \quad \text{for all } p_h \in M_h$$

with a constant $\gamma_b > 0$ independent of h. We also assume a global inverse inequality and an approximation property:

$$\|\nabla \mathbf{v}_h\|_{L^2} \leq c_{\text{inv}} h^{-1} \|\mathbf{v}_h\|_{L^2} \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$
$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2} + \inf_{q_h \in M_h} \|p - q_h\|_{L^2} \leq Ch(\|\mathbf{u}\|_{H^2} + \|p\|_{H^1})$$

for all $\mathbf{u} \in H^2(\Omega)^d \cap \mathbf{H}_0^1(\Omega)$, $p \in H^1(\Omega)$. In the setting of the general analysis we take the spaces

$$H_1 = (\mathbf{V}_h, (\nabla \cdot, \nabla \cdot)_{L^2}), \quad H_2 = (\mathbf{V}_h, (\cdot, \cdot)_{L^2}) \quad M = (M_h, (\cdot, \cdot)_{L^2})$$

In this finite dimensional case we have $W = M$ (note, however, that in general $\|\cdot\|_W \neq \|\cdot\|_M$). The bilinear forms are the same as in section 3. The operators corresponding to these bilinear forms are denoted by A_h, C_h, B_h. As in the continuous case *we identify M_h with its dual M_h'.* Thus we have $\langle \cdot, \cdot \rangle_{M' \times M} = \langle \cdot, \cdot \rangle_{W' \times W} = (\cdot, \cdot)_{L^2}$ and

$$
\begin{aligned}
A_h &: \mathbf{V}_h \to \mathbf{V}_h', & \langle A_h \mathbf{u}_h, \mathbf{v}_h \rangle &= a(\mathbf{u}_h, \mathbf{v}_h) & \text{for all } \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h \\
C_h &: \mathbf{V}_h \to \mathbf{V}_h', & \langle C_h \mathbf{u}_h, \mathbf{v}_h \rangle &= c(\mathbf{u}_h, \mathbf{v}_h) & \text{for all } \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h \\
B_h &: M_h \to \mathbf{V}_h', & \langle B_h p_h, \mathbf{v}_h \rangle &= b(\mathbf{v}_h, p_h) & \text{for all } p_h \in M_h, \ \mathbf{v}_h \in \mathbf{V}_h \\
B_h' &: \mathbf{V}_h \to M_h, & (B_h' \mathbf{v}_h, p_h)_{L^2} &= b(\mathbf{v}_h, p_h) & \text{for all } p_h \in M_h, \ \mathbf{v}_h \in \mathbf{V}_h
\end{aligned}
$$

The discrete generalized Stokes problem is as follows: given $\mathbf{f}_h \in \mathbf{V}_h'$ find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ such that

$$
\begin{aligned}
A_h \mathbf{u}_h + \tau C_h \mathbf{u}_h + B_h p_h &= \mathbf{f}_h \\
B_h' \mathbf{u}_h &= 0
\end{aligned}
\tag{4.1}
$$

The corresponding Schur complement is given by $S_h = B_h'(A_h + \tau C_h)^{-1} B_h : M_h \to M_h$. Application of the general analysis yields the following main result of this section.

THEOREM 4.1. *Assume that $\Omega \subset \mathbb{R}^d$ is such that the continuous stationary Stokes problem (3.1) is H^2-regular. With I_h the identity operator on M_h define $\tilde{S}_h^{-1} = I_h + \tau(B_h' C_h^{-1} B_h)^{-1}$. Then the following inequalities hold for any $p_h \in M_h$ with $c_d > 0$ independent of τ and h:*

$$c_d(\tilde{S}_h p_h, p_h)_{L^2} \leq (S_h p_h, p_h)_{L^2} \leq 2(\tilde{S}_h p_h, p_h)_{L^2}$$

Proof. The properties (2.1)-(2.3) hold with $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = \Gamma_b = 1$, and $\gamma_b > 0$ the constant from the LBB condition.

We now treat assumption 1. Let $P_h : H_1' \to B_h(M_h)$ be the orthogonal projection on $B_h(M_h)$ in H_1'. From lemma 2.4 we have $P_h = B_h S_{1,h}$, where (for $\mathbf{f}_h \in H_2'$)

$S_{1,h}\mathbf{f}_h = p_h$ is the solution operator corresponding to the discrete stationary Stokes problem

$$A_h\mathbf{u}_h + B_h p_h = \mathbf{f}_h$$
$$B_h'\mathbf{u}_h = 0 \tag{4.2}$$

The functional $\mathbf{f}_h$ can be extended to $\mathbf{f} \in \mathbf{L}^2(\Omega)'$ with $\mathbf{f}(\mathbf{v}_h) = \mathbf{f}_h(\mathbf{v}_h)$ for all $\mathbf{v}_h \in \mathbf{V}_h$ and $\|\mathbf{f}\|_{L^2(\Omega)'} = \|\mathbf{f}_h\|_{H_2'}$ (recall, $H_2 = (\mathbf{V}_h, \|\cdot\|_{L^2})$). Consider the continuous stationary Stokes problem with right handside $\mathbf{f}$:

$$A\mathbf{u} + Bp = \mathbf{f}$$
$$B'\mathbf{u} = 0 \tag{4.3}$$

Comparison of (4.2) and (4.3) yields

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) = 0 \quad \text{for all} \quad \mathbf{v}_h \in \mathbf{V}_h$$

Using the H^2-regularity, the inverse inequality and the approximation property of the finite element spaces we obtain

$$\|P_h\mathbf{f}_h\|_{H_2'} = \|B_h S_{1,h}\mathbf{f}_h\|_{H_2'} = \|B_h p_h\|_{H_2'} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{L^2}}$$

$$\leq \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p)}{\|\mathbf{v}_h\|_{L^2}} + \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{L^2}}$$

$$\leq \|\nabla p\|_{L^2} + \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\|\nabla \mathbf{v}_h\|_{L^2}}{\|\mathbf{v}_h\|_{L^2}}$$

$$\leq \|\mathbf{f}_h\|_{H_2'} + c\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2} h^{-1}$$

$$\leq \|\mathbf{f}_h\|_{H_2'} + c\|\mathbf{u}\|_{H^2(\Omega)} \leq c\|\mathbf{f}_h\|_{H_2'}$$

Hence, assumption 1 holds with $d_P = 0$ and a constant c_P independent of h. Application of corollary 2.10 proves the result. $\quad\square$

Both in the analysis of the continuous generalized Stokes problem (theorem 3.2) and of its finite element discretization(theorem 4.1) we need a H^2-regularity assumption. We now show, that for a certain range of τ values a regularity assumption can be avoided.

THEOREM 4.2. *Let $\tilde{S}_h^{-1} = I_h + \tau(B_h' C_h^{-1} B_h)^{-1}$ be as in theorem 4.1. There exist positive constants c_1, c_2, independent of h and τ, such for all $p_h \in M_h$ the following holds:*

$$\gamma_s(\tilde{S}_h p_h, p_h)_{L^2} \leq (S_h p_h, p_h)_{L^2} \leq \Gamma_s(\tilde{S}_h p_h, p_h)_{L^2} \quad \text{for all} \quad \tau \in [0, c_1] \cup [c_2 h^{-2}, \infty),$$

with $\gamma_s = \frac{\gamma_b^2}{2(\gamma_b^2+1)}$, $\Gamma_s = 2$.

Proof. We use the result given in remark 5. The properties (2.1)-(2.3) hold with $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = \Gamma_b = 1$, and γ_b the constant from the LBB condition.
Let $P_h : H_1' \to B_h(M_h)$ be the orthogonal projection on $B_h(M_h)$ in H_1'. Using the

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inverse inequality we get, for $\mathbf{f}_h \in H_2'$:

$$\|P_h\mathbf{f}_h\|_{H_2'}^2 \le 2\|\mathbf{f}_h\|_{H_2'}^2 + 2\|(I-P_h)\mathbf{f}_h\|_{H_2'}^2 = 2\|\mathbf{f}_h\|_{H_2'}^2 + 2\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{((I-P_h)\mathbf{f}_h, \mathbf{v}_h)_{L^2}^2}{\|\mathbf{v}_h\|_{L^2}^2}$$

$$\le 2\|\mathbf{f}_h\|_{H_2'}^2 + 2c_{\text{inv}}^2 h^{-2}\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{((I-P_h)\mathbf{f}_h, \mathbf{v}_h)_{L^2}^2}{\|\nabla\mathbf{v}_h\|_{L^2}^2}$$

$$= 2\big(\|\mathbf{f}_h\|_{H_2'}^2 + c_{\text{inv}}^2 h^{-2}\|(I-P_h)\mathbf{f}_h\|_{H_1'}^2\big)$$

Thus assumption 1 holds with $c_P = \sqrt{2}$ and $d_P = c_{\text{inv}}h^{-1}$.

Let $\hat{P}_h : H_2' \to B_h(M_h)$ be the orthogonal projection on $B_h(M_h)$ in H_2'. Using the Friedrichs inequality, $\|\mathbf{v}_h\|_{L^2} \le c_F\|\nabla\mathbf{v}_h\|_{L^2}$, we obtain, for $\mathbf{f}_h \in H_1'$:

$$\|\hat{P}\mathbf{f}_h\|_{H_1'}^2 \le 2\|\mathbf{f}_h\|_{H_1'}^2 + 2\|(I-\hat{P}_h)\mathbf{f}_h\|_{H_1'}^2 \le 2\big(\|\mathbf{f}_h\|_{H_1'}^2 + c_F^2\|(I-\hat{P}_h)\mathbf{f}_h\|_{H_2'}^2\big)$$

Thus (2.26) holds with $c_P = \sqrt{2}$, $d_P = c_F$. Using the result in remark 5 we obtain the equivalence result with spectral constants $\gamma_s = \frac{\gamma_b^2}{2(\gamma_b^2+1)}$, $\Gamma_s = 2$. $\quad\square$

REMARK 6. The equivalence result for the Schur complement operator $S_h : M_h \to M_h$ has an obvious analogon if we use matrix representations. Assume that we have chosen (nodal) bases in $\mathbf{V}_h$ and M_h. The coefficient vectors of $\mathbf{u}_h$ and p_h in these bases are denoted by $\bar{\mathbf{u}}_h$, $\bar{p}_h$, respectively. The Euclidean scalar product in $\mathbb{R}^n$ is denoted by $\langle\cdot,\cdot\rangle_2$. Let $\mathbf{Q}_h$ be the mass matrix in M_h. The matrix representations of A_h, C_h, B_h are defined by

$$\langle\mathbf{A}_h\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h\rangle_2 = (\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)_{L^2} \quad\text{for all}\quad \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$$

$$\langle\mathbf{C}_h\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h\rangle_2 = (\mathbf{u}_h, \mathbf{v}_h)_{L^2} \quad\text{for all}\quad \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$$

$$\langle\mathbf{B}_h\bar{p}_h, \bar{\mathbf{v}}_h\rangle_2 = (\text{div}\,\mathbf{v}_h, p_h)_{L^2} \quad\text{for all}\quad p_h \in M_h,\ \mathbf{v}_h \in \mathbf{V}_h$$

The discrete generalized Stokes problem has a matrix-vector formulation with matrix

$$\begin{pmatrix} \mathbf{A}_h + \tau\mathbf{C}_h & \mathbf{B}_h \\ \mathbf{B}_h^T & 0 \end{pmatrix}$$

and thus the Schur complement matrix is $\mathbf{S}_h = \mathbf{B}_h^T(\mathbf{A}_h + \tau\mathbf{C}_h)^{-1}\mathbf{B}_h$. Using

$$(B_h'C_hB_h p_h, p_h)_{L^2} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)^2}{\|\mathbf{v}_h\|_{L^2}^2} = \sup_{\bar{\mathbf{v}}_h \in \mathbb{R}^n} \frac{\langle\mathbf{B}_h\bar{p}_h, \bar{\mathbf{v}}_h\rangle_2^2}{\langle\mathbf{C}_h\bar{\mathbf{v}}_h, \bar{\mathbf{v}}_h\rangle_2}$$

$$= \langle\mathbf{B}_h^T\mathbf{C}_h^{-1}\mathbf{B}_h\bar{p}_h, \bar{p}_h\rangle_2 \quad\text{for all}\quad p_h \in M_h\ ,$$

it follows that the Schur complement preconditioner $\tilde{S}_h$ given in in theorem 4.1 and in theorem 4.2 has the matrix representation $\tilde{\mathbf{S}}^{-1} = \mathbf{Q}_h^{-1} + \tau(\mathbf{B}_h^T\mathbf{C}_h^{-1}\mathbf{B}_h)^{-1}$.

The operator $B_h'C_h^{-1}B_h$ in the definition of $\tilde{S}_h$ corresponds to a mixed discretization of the saddle point formulation of a Neumann problem: find $\mathbf{u} \in \mathbf{H}_0(\text{div}\,)$, $p \in L_0^2(\Omega)$ such that

$$\mathbf{u} + \nabla p = 0$$

$$\text{div}\,\mathbf{u} = g$$

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

This mixed discretization is convenient when the discrete pressure is not continuous, i.e. $M_h \not\subset H^1(\Omega)$. On the other hand, if $M_h \subset H^1(\Omega)$, then one may wish to use a conforming finite element discretization of the Neumann problem and thus obtain a discrete analogon of the preconditioner given for the continuous case in theorem 3.2. This is treated in section 4.1 below.

4.1. Schur complement preconditioner for the case $M_h \subset H^1(\Omega)$. Assume $M_h \subset H^1(\Omega)$. Let $N_h^{-1} : M_h \to M_h$, $N_h^{-1} g_h = p_h$ be the solution operator of the discrete Neumann problem in M_h:

$$(\nabla p_h, \nabla q_h)_{L^2} = (g_h, q_h)_{L^2} \quad \text{for all} \quad q_h \in M_h.$$

Note that $(N_h p_h, p_h) = \|\nabla p_h\|_{L^2}^2$ for all $p_h \in M_h$. We define $\tilde{S}_{h,N} : M_h \to M_h$ by

$$\tilde{S}_{h,N}^{-1} := I_h^{-1} + \tau N_h^{-1}$$

This preconditioner has been proposed in [8] and analyzed in [5]. The preconditioner $\tilde{S}_{h,N}$ is uniformly (w.r.t. h and τ) spectrally equivalent to $\tilde{S}_h$ from theorem 4.1 iff N_h is, uniformly in h, spectrally equivalent to $B_h' C_h^{-1} B_h$. Note that, for all $p_h \in M_h$,

$$(B_h' C_h^{-1} B_h \, p_h, p_h)_{L^2} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(p_h, \operatorname{div} \mathbf{v}_h)_{L^2}^2}{\|\mathbf{v}_h\|_{L^2}^2} \le \|\nabla p_h\|_{L^2}^2 = (N_h p_h, p_h)_{L^2}$$

Hence, $\tilde{S}_{h,N}$ is uniformly spectrally equivalent to $\tilde{S}_h$ iff

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(p_h, \operatorname{div} \mathbf{v}_h)_{L^2}}{\|\mathbf{v}_h\|_{L^2}} \ge \gamma_w \|\nabla p_h\|_{L^2} \quad \text{for all} \quad p_h \in M_h \tag{4.4}$$

holds with $\gamma_w > 0$, independent of h. This modified stability condition (also called weak inf-sup condition) can be found at several places in the literature, e.g., [2, 7, 24]. In [2] a proof of this result for $P_1 \mathrm{iso} P_2 - P_1$ and for the Hood-Taylor P_2-P_1 pair is given for the two-dimensional case. Below we give a simplified analysis for P_2-P_1 Hood-Taylor finite elements in a d-dimensional domain, with $d = 2, 3$. We assume that the family of triangulations $\{\mathcal{T}_h\}$ is regular but not necessarily quasi-uniform.

LEMMA 4.3. *For $\Omega \subset \mathbb{R}^d$, let $\{\mathcal{T}_h\}$ be a regular family of triangulations consisting of d-simplices. Assume that every simplex has at least one vertex which is not on $\partial\Omega$. Then the Hood-Taylor P_2-P_1 pair of finite element spaces satisfies (4.4).*

Proof. The Hood-Taylor P_2-P_1 pair is denoted by $(\mathbf{V}_h, M_h)$. Take $q_h \in M_h$, $q_h \neq 0$. The constants used below are independent of $\mathcal{T}_h \in \{\mathcal{T}_h\}$ and of q_h. The set of edges in $\mathcal{T}_h$ is denoted by $\mathcal{E}$. This set is partitioned in edges which are in the interior of Ω and edges which are part of $\partial\Omega$: $\mathcal{E} = \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{bnd}}$. For every $E \in \mathcal{E}$, m_E denotes the midpoint of the edge E. Every $E \in \mathcal{E}_{\mathrm{int}}$ with endpoints $a_1, a_2 \in \mathbb{R}^d$ is assigned a vector $\mathbf{t}_E := a_1 - a_2$. For $E \in \mathcal{E}_{\mathrm{bnd}}$ we define $\mathbf{t}_E := 0$. Since q_h is continuous piecewise linear the function $x \to \mathbf{t}_E \cdot \nabla q_h(x)$ is continuous across E, for $E \in \mathcal{E}_{\mathrm{int}}$. We define

$$\hat{\mathbf{t}}_E := \|\mathbf{t}_E\|_2^{-1} \mathbf{t}_E \quad (\hat{\mathbf{t}}_E := 0 \ \text{if} \ E \in \mathcal{E}_{\mathrm{bnd}})$$
$$\mathbf{w}_E := \big(\hat{\mathbf{t}}_E \cdot \nabla q_h(m_E)\big) \hat{\mathbf{t}}_E, \quad \text{for} \ E \in \mathcal{E}$$

A unique $\mathbf{w}_h \in \mathbf{V}_h$ is defined by

$$\mathbf{w}_h(x_i) = \begin{cases} 0 & \text{if} \ x_i \ \text{is a vertex of} \ T \in \mathcal{T}_h \\ \mathbf{w}_E & \text{if} \ x_i = m_E \ \text{for} \ E \in \mathcal{E} \end{cases}$$

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The set of edges of $T \in \mathcal{T}_h$ is denoted by E_T. By using quadrature we see that for any $p \in P_2$ which is zero at the vertices of T we have

$$\int_T p(x)\, dx = \frac{|T|}{2d - 1} \sum_{E \in E_T} p(m_E)$$

We obtain:

$$
\begin{aligned}
-\int_\Omega q_h \mathrm{div}\, \mathbf{w}_h\, dx &= \int_\Omega \nabla q_h \cdot \mathbf{w}_h\, dx = \sum_{T \in \mathcal{T}_h} (\nabla q_h)_{|T} \cdot \int_T \mathbf{w}_h\, dx \\
&= \sum_{T \in \mathcal{T}_h} \frac{|T|}{2d - 1} (\nabla q_h)_{|T} \cdot \sum_{E \in E_T} \mathbf{w}_h(m_E) \\
&= \sum_{T \in \mathcal{T}_h} \frac{|T|}{2d - 1} \sum_{E \in E_T} \left(\hat{\mathbf{t}}_E \cdot \nabla q_h(m_E) \right)^2
\end{aligned}
\tag{4.5}
$$

Using the fact that $(\nabla q_h)_{|T}$ is constant and for each T at least two independent nonzero vectors $\hat{\mathbf{t}}_E$ exist, one easily checks that

$$c \|\nabla q_h\|^2_{L^2(T)} \leq |T| \sum_{E \in E_T} \left(\hat{\mathbf{t}}_E \cdot \nabla q_h(m_E) \right)^2 \leq \tilde{c} \|\nabla q_h\|^2_{L^2(T)}, \quad c > 0$$

Combining this with (4.5) we get

$$-\int_\Omega q_h \mathrm{div}\, \mathbf{w}_h\, dx \geq C \sum_{T \in \mathcal{T}_h} \|\nabla q_h\|^2_{L^2(T)} = C \|\nabla q_h\|^2_{L^2} \tag{4.6}$$

Let $E_{\hat{T}}$ be the set of edges of the unit d-simplex. In the space $\{ \hat{v} \in P_2 \mid \hat{v} \text{ is zero at the vertices of } \hat{T} \}$ the norms $\|\hat{v}\|_{L^2(\hat{T})}$ and $\left(\sum_{E \in E_{\hat{T}}} \hat{v}(m_E)^2 \right)^{\frac{1}{2}}$ are equivalent. Using this componentwise for the vector-function $\hat{\mathbf{w}}_h := \mathbf{w}_h \circ F$, with F the affine mapping such that $F(\hat{T}) = T$, we get:

$$
\begin{aligned}
\|\mathbf{w}_h\|^2_{L^2(T)} &\leq C |T| \, \|\hat{\mathbf{w}}_h\|^2_{L^2(\hat{T})} \\
&\leq C |T| \sum_{E \in E_{\hat{T}}} \|\hat{\mathbf{w}}_h(m_E)\|^2_2 = C |T| \sum_{E \in E_T} \|\mathbf{w}_E\|^2_2
\end{aligned}
$$

Summation over all simplices T yields

$$
\begin{aligned}
\|\mathbf{w}_h\|^2_{L^2} &\leq C \sum_{T \in \mathcal{T}_h} |T| \sum_{E \in E_T} \|\mathbf{w}_E\|^2_2 = C \sum_{T \in \mathcal{T}_h} |T| \sum_{E \in E_T} \left(\hat{\mathbf{t}}_E \cdot \nabla q_h(m_E) \right)^2 \\
&\leq C \sum_{T \in \mathcal{T}_h} \|\nabla q_h\|^2_{L^2(T)} = C \|\nabla q_h\|^2_{L^2}
\end{aligned}
\tag{4.7}
$$

¿From (4.6) and (4.7) we obtain

$$\frac{(q_h, \mathrm{div}\, (-\mathbf{w}_h))_{L^2}}{\|\mathbf{w}_h\|_{L^2}} \geq C \|\nabla q_h\|_{L^2}$$

with a constant $C > 0$ independent of q_h and of $\mathcal{T}_h \in \{\mathcal{T}_h\}$. $\quad \square$

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5. A Stokes interface problem. In this section we consider a generalized Stokes interface problem. Assume bounded Lipschitz subdomains Ω_1 and Ω_2 of Ω such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. The interface between the subdomains is denoted by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The problem we consider is as follows: Find $\mathbf{u}$ and p such that

$$-\operatorname{div}(\nu(\mathbf{x})\mathrm{D}\mathbf{u}) + \tau\rho(\mathbf{x})\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_k, \tag{5.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_k, \quad k = 1, 2 \tag{5.2}$$

$$[\mathbf{u}] = 0, [\sigma(\mathbf{u}, p)\mathbf{n}] = \mathbf{g} \quad \text{on } \Gamma \tag{5.3}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{5.4}$$

In this formulation we use standard notations: $\sigma(\mathbf{u}, p) = -p\,I + 2\nu\,\mathrm{D}\mathbf{u}$ is the stress tensor, $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ the rate of deformation tensor, $\mathbf{n}$ is a unit normal vector to Γ, $[a]|_\Gamma = (a|_{\Omega_1} - a|_{\Omega_2})|_\Gamma$.

We assume piecewise constant viscosity and density. A localized force term g occurs, for example, in models that take surface tension effects into account, cf. [12, 22]. Suitable scaling can be used to ensure that viscosity and density are equal to one in Ω_1. Hence, we assume

$$\nu = \begin{cases} 1 & \text{in } \Omega_1 \\ \nu_2 > 0 & \text{in } \Omega_2 \end{cases} \qquad \rho = \begin{cases} 1 & \text{in } \Omega_1 \\ \rho_2 > 0 & \text{in } \Omega_2 \end{cases} \tag{5.5}$$

The weak formulation leads to a saddle point problem as in (2.4), (2.5): find $\mathbf{u}, p \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$a_\nu(\mathbf{u}, \mathbf{v}) + \tau c_\rho(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) = f(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \ q \in L_0^2(\Omega), \tag{5.6}$$

with

$$a_\nu(\mathbf{u}, \mathbf{v}) := \int_\Omega \nu\operatorname{tr}(\mathrm{D}\mathbf{u}\mathrm{D}\mathbf{v})\,dx, \quad c_\rho(\mathbf{u}, \mathbf{v}) := (\rho\mathbf{u}, \mathbf{v})_{L^2}, \quad b(\mathbf{v}, p) := -(p, \operatorname{div}\mathbf{v})_{L^2},$$

$$f(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_{L^2} + \int_\Gamma \mathbf{g} \cdot \mathbf{v}\,ds$$

In view of the general analysis it is natural to introduce the following Hilbert spaces:

$$H_1 = \{\ \mathbf{H}_0^1(\Omega), \text{ with } \|\mathbf{v}\|_{H_1}^2 := \int_\Omega \nu\operatorname{tr}\big((\mathrm{D}\mathbf{v})^2\big)\,dx\ \}$$

$$H_2 = \{\ \mathbf{L}^2(\Omega), \text{ with } \|\mathbf{v}\|_{H_2} := \|\rho^{\frac{1}{2}}\mathbf{v}\|_{L^2}\ \}$$

Due to Korn's inequality $\|\cdot\|_{H_1}$ defines a norm on $\mathbf{H}_0^1(\Omega)$. Related to this norm we need a *uniform* equivalence result that is proved in [17], lemma 6.1. Assume that one of the following conditions is satisfied:

$$\operatorname{meas}(\partial\Omega_k \cap \partial\Omega) > 0 \quad \text{for } k = 1, 2 \tag{5.7}$$

$$\operatorname{meas}(\partial\Omega_1 \cap \partial\Omega) > 0 \quad \text{and } \nu_2 \leq C \tag{5.8}$$

Then there exists a constant $\tilde{c} > 0$ independent of ν such that

$$\tilde{c}\|\nu^{\frac{1}{2}}\nabla\mathbf{v}\|_{L^2} \leq \|\mathbf{v}\|_{H_1} \leq \|\nu^{\frac{1}{2}}\nabla\mathbf{v}\|_{L^2} \quad \text{for all } \mathbf{v} \in H_1 \tag{5.9}$$

Before we introduce the (pressure) space M we recall a result from [18] that we need in the analysis below. Let $\tilde{p}$ be the piecewise constant function

$$\tilde{p} = \begin{cases} |\Omega_1|^{-1} & \text{on } \Omega_1 \\ -|\Omega_2|^{-1} & \text{on } \Omega_2. \end{cases} \tag{5.10}$$

Since $(\tilde{p}, 1)_{L^2} = 0$, we have $\tilde{p} \in L_0^2(\Omega)$. Consider the one-dimensional subspace $P_0 := \text{span}\{\tilde{p}\}$ of $L_0^2(\Omega)$ and an L^2-orthogonal decomposition $L_0^2(\Omega) = P_0 \oplus P_0^\perp$. For $p \in L_0^2(\Omega)$ we use the notation

$$p = p_0 + p_0^\perp, \quad p_0 \in P_0, \ p_0^\perp \in P_0^\perp \tag{5.11}$$

One easily checks that

$$P_0^\perp = \{\, p \in L_0^2(\Omega) \mid (p,1)_{L^2(\Omega_1)} = (p,1)_{L^2(\Omega_2)} = 0 \,\}, \tag{5.12}$$

Using this splitting we can define an appropriate norm on $L_0^2(\Omega)$:

$$M = \{\, L_0^2(\Omega), \text{ with } \|p\|_M := \big(\|p_0\|_{L^2}^2 + \|\nu^{-\frac{1}{2}} p_0^\perp\|_{L^2}^2\big)^{\frac{1}{2}} \,\}.$$

The scalar product corresponding to $\|\cdot\|_M$ is denoted by $(\cdot, \cdot)_M$. In [18], lemma 1 and theorem 1, the following results are proved. There exist constants $\hat{C}$ and $\hat{c} > 0$ independent of ν such that

$$|(\text{div } \mathbf{v}, p)_{L^2}| \leq \hat{C}\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\|_{L^2}\|p\|_M \quad \text{for all } \mathbf{v} \in H_1,\ p \in M \tag{5.13}$$

$$\sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{(\text{div } \mathbf{v}, p)_{L^2}}{\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\|_{L_2}} \geq \hat{c}\|p\|_M \quad \text{for all } p \in M \tag{5.14}$$

We identify $L^2(\Omega)$ with its dual and then have

$$H_1' = \{\, \mathbf{H}^{-1}, \text{ with } \|\mathbf{f}\|_{H_1'} = \sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{H_1}} \,\};$$

$$H_2' = \{\, \mathbf{L}^2(\Omega), \text{ with } \|\mathbf{v}\|_{H_2'} := \|\rho^{-\frac{1}{2}} \mathbf{v}\|_{L^2} \,\}$$

$$M' = \{\, L_0^2(\Omega), \text{ with } \|p\|_{M'} = \big(\|p_0\|_{L^2}^2 + \|\nu^{\frac{1}{2}} p_0^\perp\|_{L^2}^2\big)^{\frac{1}{2}} \,\}.$$

With these spaces H_1 and M the properties (2.1)-(2.3) can be shown to hold.

LEMMA 5.1. *Assume that (5.7) or (5.8) is satisfied. The properties (2.1)-(2.3) hold with $\gamma_a = \Gamma_a = \gamma_c = \Gamma_c = 1$ and constants $\Gamma_b, \gamma_b > 0$ independent of ν and ρ.*

Proof. From $a_\nu(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_{H_1}^2$ we get property (2.1) with $\gamma_a = \Gamma_a = 1$. Due to $c_\rho(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_{H_2}^2$ property (2.2) holds. Using (5.9) and (5.13) we get

$$(\text{div } \mathbf{v}, p)_{L^2} \leq \hat{C}\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\|_{L^2}\|p\|_M \leq \hat{C}\,\tilde{c}^{-1}\|\mathbf{v}\|_{H_1}\|p\|_M$$

and thus the upper bound in (2.3) with $\Gamma_b = \hat{C}\,\tilde{c}^{-1}$. Using (5.9) and (5.14) we obtain

$$\sup_{\mathbf{v} \in H_1} \frac{(\text{div } \mathbf{v}, p)_{L^2}}{\|\mathbf{v}\|_{H_1}} \geq \sup_{\mathbf{v} \in H_1} \frac{(\text{div } \mathbf{v}, p)_{L^2}}{\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\|_{L_2}} \geq \hat{c}\|p\|_M$$

and thus $\|Bp\|_{H_1'} \geq \hat{c}\|p\|_M =: \gamma_b\|p\|_M$, i.e., the lower bound in (2.3). Because ρ is not used in the definitions of H_1 and M the constants Γ_b and γ_b are independent of ρ. $\quad\square$

The norm $\| \cdot \|_{H_2}$ is equivalent to the standard L^2-norm. Hence, the space $W = \{ p \in M \mid Bp \in H_2' \}$ is the same as the one for the generalized Stokes problem in section 3:

$$W = H^1(\Omega) \cap L_0^2(\Omega), \quad \text{with norm} \ \ \|p\|_W = \sup_{\mathbf{v} \in H_2} \frac{\langle Bp, \mathbf{v} \rangle}{\|\mathbf{v}\|_{H_2}} = \|\rho^{-\frac{1}{2}} \nabla p\|_{L^2} \quad (5.15)$$

The Schur complement $S_{\nu,\rho} : L_0^2(\Omega) \to L_0^2(\Omega)$ corresponding to this Stokes interface problem is characterized by

$$(S_{\nu,\rho} p, p)_{L^2}^{\frac{1}{2}} = \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})_{L^2}}{\left(a_\nu(\mathbf{v}, \mathbf{v}) + \tau c_\rho(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}}} \quad (5.16)$$

We take the preconditioner from theorem 2.5:

$$(\tilde{S}_{\nu,\rho} p, p)_{L^2}^{\frac{1}{2}} = \|p\|_{M + \tau^{-1} W} = \inf_{q \in W} \left(\|p - q\|_M + \tau^{-1} \|\rho^{-\frac{1}{2}} \nabla q\|_{L^2}^2 \right)^{\frac{1}{2}} \quad (5.17)$$

This preconditioner can be characterized using a Neumann solution operator by applying a similar approach as in theorem 3.2. We can apply the general analysis of section 2.4 to derive a uniform spectral bound $S \lesssim \tilde{S}$. This is summarized in the following theorem.

THEOREM 5.2. *Assume that one of the conditions (5.7) or (5.8) is satisfied. Denote by $-\Delta_\rho^{-1} : L_0^2(\Omega) \to H^1(\Omega) \cap L_0^2(\Omega)$ the solution operator of the following Neumann interface problem: Given $f \in L_0^2(\Omega)$ find $p \in H^1(\Omega) \cap L_0^2(\Omega)$ such that*

$$(\rho^{-1} \nabla p, \nabla q)_{L^2} = (f, q)_{L^2}, \quad \forall \, q \in H^1(\Omega) \cap L_0^2(\Omega).$$

$I_\nu : L_0^2(\Omega) \to L_0^2(\Omega)$ is defined by $(I_\nu p, q)_{L^2} = (p, q)_M$ for all $p, q \in L_0^2(\Omega)$. Then

$$\tilde{S}_{\nu,\rho}^{-1} = I_\nu^{-1} - \tau \Delta_\rho^{-1}$$

holds and or all $p \in L_0^2(\Omega)$

$$(S_{\nu,\rho} p, p)_{L^2} \leq c(\tilde{S}_{\nu,\rho} p, p)_{L^2} \,,$$

with a constant c independent of τ, ν and ρ.

Proof. For W as in (5.15) define $\tilde{S}$ as in (5.17). Using the same arguments as in the proof of theorem 3.2 it can be shown that $\tilde{S}^{-1} = I_\nu^{-1} - \tau \Delta_\rho^{-1}$ holds. The spectral inequality $S \leq c\tilde{S}$ follows from theorem 2.6. The constant c is uniform not only in τ but also w.r.t. ν and ρ, due to the fact that $\Gamma_a, \Gamma_c, \gamma_a$ and γ_c are independent of ν and ρ. $\square$

REMARK 7. We comment on a discrete version of the preconditioner $\tilde{S}_{\nu,\rho}$ in a pair of finite element spaces $\mathbf{V}_h \times M_h$ as considered in section 4.1. In the operator I_ν the scalar product $(\cdot, \cdot)_M$ and thus the orthogonal projection on the *one* dimensional subspace $P_0 = \operatorname{span}(\tilde{p})$ is used. This projection is avoided in the following operator $\hat{I}_\nu : L_0^2(\Omega) \to L_0^2(\Omega)$

$$(\hat{I}_\nu p, q)_{L^2} = (\nu^{-1} p, q)_{L^2} \quad \text{for all} \ \ q \in L_0^2(\Omega)$$

Note that $\hat{I}_\nu p = I_\nu p$ for all $p \in L_0^2(\Omega) \cap P_0^\perp$. In general a poor behaviour of a preconditioner on a *one* dimensional subspace is harmless if the preconditioner is combined with a CG method. Therfore we base our discrete preconditioner on the simpler operator $\hat{I}_\nu$ instead of on I_ν. Let $\hat{I}_{\nu,h}^{-1} : M_h \to M_h$, $\hat{I}_{\nu,h}^{-1} g_h = p_h$ be such that:

$$(\nu^{-1} p_h, q_h)_{L^2} = (g_h, q_h)_{L^2} \quad \text{for all} \quad q_h \in M_h.$$

Let $\hat{N}_{\rho,h}^{-1} : M_h \to M_h$, $N_{\nu,h}^{-1} g_h = p_h$ be the solution operator of the following discrete Neumann problem:

$$(\rho^{-1} \nabla p_h, \nabla q_h)_{L^2} = (g_h, q_h)_{L^2} \quad \text{for all} \quad q_h \in M_h.$$

We define $\tilde{S}_{\nu,\rho,h} : M_h \to M_h$ by

$$\tilde{S}_{\nu,\rho,h}^{-1} := \hat{I}_{\nu,h}^{-1} + \tau N_{\rho,h}^{-1}$$

This preconditioner is used in our numerical experiments. To evaluate $\tilde{S}_{\nu,\rho,h}^{-1} g_h$ one has to solve a system with a pressure mass matrix (w.r.t. scalar product $(\nu^{-1} \cdot, \cdot)_{L^2}$ and a discrete Neumann interface problem.

Note that in theorem 5.2 we have a spectral inequality $S_{\nu,\rho} \lesssim \tilde{S}_{\nu,\rho}$ that is uniform with respect to both the parameter τ and the jumps in the coefficients ν, ρ *without using any regularity assumptions*. To derive a spectral inequality $\tilde{S}_{\nu,\rho} \lesssim S_{\nu,\rho}$ we need (at least in our analysis) regularity results for a stationary Stokes interface problem of the form

$$-\text{div}\,(\nu(\mathbf{x})\mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega_k, \tag{5.18}$$

$$\text{div}\,\mathbf{u} = 0 \quad \text{in } \Omega_k, \quad k = 1, 2 \tag{5.19}$$

$$[\mathbf{u}] = 0, [\sigma(\mathbf{u}, p) \cdot \mathbf{n}] = \mathbf{g} \quad \text{on } \Gamma \tag{5.20}$$

$$\mathbf{u} = 0 \quad \text{on } \Omega. \tag{5.21}$$

Similarly to the Stokes case in section 3, verifying assumption 1 is based on regularity properties of the solution of this problem. This important issue is largely unsolved. The following result is found in the literature (see, [20]): If the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is sufficiently smooth and has no common points with $\partial\Omega$ and $\mathbf{f} \in \mathbf{L}^2$ then a solution $\mathbf{u}, p$ of (5.18)–(5.21) belongs to $H^2(\Omega_k)^d \times H^1(\Omega_k)$, $k = 1, 2$. However, in these results and in other analyses known in the literature the dependence of constants in a priori estimates on ν is not known. Due to this we are not able to prove a result $\tilde{S}_{\nu,\rho} \lesssim S_{\nu,\rho}$ that is uniform both with respect to τ and the jumps in ν, ρ. Below we present an analysis where the spectral inequality is *uniform with respect to τ only*.

THEOREM 5.3. *Assume that one of the conditions* (5.7) *or* (5.8) *is satisfied and that the domain $\Omega \subset \mathbb{R}^d$ is such that the Stokes problem* (3.1) *is H^2-regular. Let $\tilde{S}_{\nu,\rho}$ be the preconditioner from* (5.17). *There exists a constant $c > 0$ independent of τ such that for all $p \in L_0^2(\Omega)$*

$$c(\tilde{S}_{\nu,\rho} p, p)_{L^2} \leq (S_{\nu,\rho} p, p)_{L^2}$$

holds.

Proof. Let $S : L_0^2(\Omega) \to L_0^2(\Omega)$ as in (3.3) be the Schur complement for the generalized Stokes problem and Let $\tilde{S}$ the preconditioner from theorem 3.2. For this preconditioner we have

$$(\tilde{S}p, p)_{L^2}^{\frac{1}{2}} = \inf_{q \in H^1(\Omega) \cap L_0^2(\Omega)} \left(\|p - q\|_{L^2}^2 + \tau^{-1}\|\nabla q\|_{L^2}^2 \right)^{\frac{1}{2}}$$

Using (5.17) and the norm equivalences $\|\cdot\|_M \sim \|\cdot\|_{L^2}$ (on $L_0^2(\Omega)$), $\|\rho^{-\frac{1}{2}}\nabla\cdot\|_{L^2} \sim \|\nabla\cdot\|_{L^2}$ (on $H^1(\Omega) \cap L_0^2(\Omega)$) it follows that there exists a constant c independent of τ such that

$$(\tilde{S}_{\nu,\rho}p, p)_{L^2} \leq c\,(\tilde{S}p, p)_{L^2} \quad \text{for all } p \in L_0^2(\Omega) \tag{5.22}$$

From theorem 3.2 it follows that

$$(\tilde{S}p, p)_{L^2} \leq c\,(Sp, p)_{L^2} \quad \text{for all } p \in L_0^2(\Omega) \tag{5.23}$$

holds with a constant c independent of τ. From

$$(Sp, p)_{L^2} = \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div}\mathbf{v})_{L^2}}{\left(\|\nabla\mathbf{v}\|_{L^2}^2 + \tau\|\mathbf{v}\|_{L^2}^2\right)^{\frac{1}{2}}}\,,$$

the representation for $S_{\nu,\rho}$ in (5.16) and the equivalences $a_\nu(\cdot,\cdot) \sim \|\nabla\cdot\|_{L^2}^2$, $c_\rho(\cdot,\cdot) \sim \|\mathbf{v}\|_{L^2}^2$ (on $\mathbf{H}_0^1(\Omega)$) it follows that

$$(Sp, p)_{L^2} \leq c\,(S_{\nu,\rho}p, p)_{L^2} \quad \text{for all } p \in L_0^2(\Omega) \tag{5.24}$$

Combination of the results in (5.22), (5.23) and (5.24) completes the proof. $\quad\square$

6. Numerical experiments. We present results of a model generalized Stokes interface problem. Numerical results for a *stationary* ($\tau = 0$) Stokes interface problem can be found in [17].

We take $\Omega = (0,1)^3$ with subdomains $\Omega_2 = (0,\frac{1}{2})^3$, $\Omega_1 = \Omega \setminus \overline{\Omega}_2$. The model problem reads: Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\hat{a}_\nu(\mathbf{u}, \mathbf{v}) + \tau c_\rho(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = 0 \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$
$$b(\mathbf{u}, q) = 0 \quad \text{for all } q \in L_0^2(\Omega).$$

The bilinear forms $b(\cdot,\cdot)$ and $c_\rho(\cdot,\cdot)$ are as in section 5, $\hat{a}_\nu(\mathbf{u}, \mathbf{v}) := (\nu\,\nabla\mathbf{u}, \nabla\mathbf{v})_{L^2}$ with ν and ρ piecewise constants as in (5.5).

For the discretization we start with a uniform tetrahedral grid with $h = \frac{1}{2}$ and apply regular (red) refinement to this triangulation. The resulting grids $\mathcal{T}_h$ satisfy a conformity condition:

$$\exists\,\mathcal{T}_h^{(i)} \subset \mathcal{T}_h : \cup\{T \mid T \in \mathcal{T}_h^{(i)}\} = \overline{\Omega}_i, \quad i = 1, 2$$

We use the LBB-stable P_2-P_1 Hood-Taylor finite element pair, denoted by $\mathbf{V}_h \times M_h$, and perform computations for $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and various values of ν_2, ρ_2 and τ. For $h = \frac{1}{32}$ we have approximately $7.5 \cdot 10^5$ velocity unknowns and $3.3 \cdot 10^4$ pressure unknowns.

The following matrices are introduced (notation as in remark 6):

$$\langle \mathbf{A}_h \bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h \rangle_2 = (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)_{L^2} \quad \text{for all } \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h,$$
$$\langle \mathbf{C}_h \bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h \rangle_2 = (\rho\mathbf{u}_h, \mathbf{v}_h)_{L^2} \quad \text{for all } \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h,$$
$$\langle \mathbf{B}_h \bar{p}_h, \bar{\mathbf{v}}_h \rangle_2 = -(\operatorname{div}\mathbf{v}_h, p_h)_{L^2} \quad \text{for all } p_h \in M_h, \; \mathbf{v}_h \in \mathbf{V}_h,$$
$$\langle \mathbf{A}_h^{\mathrm{N}} \bar{p}_h, \bar{q}_h \rangle_2 = (\rho^{-1}\nabla p_h, \nabla q_h)_{L^2} \quad \text{for all } p_h, q_h \in M_h,$$
$$\langle \mathbf{Q}_h \bar{p}_h, \bar{q}_h \rangle_2 = (\nu^{-1}p_h, q_h)_{L^2} \quad \text{for all } p_h, q_h \in M_h$$

The discrete model problem has the following matrix-vector formulation: Find $\bar{\mathbf{u}}_h \in \mathbb{R}^n$, $\bar{p}_h \in \mathbb{R}^m$, such that

$$\begin{pmatrix} \mathbf{A}_h + \tau\mathbf{C}_h & \mathbf{B}_h \\ \mathbf{B}_h^T & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}}_h \\ \bar{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ 0 \end{pmatrix}.$$

In the experiments we use $\mathbf{f}_h = 0$ and a fixed starting vector $(\bar{\mathbf{u}}_h^{(0)}, \bar{p}_h^{(0)}) \neq (0,0)$. The Schur complement is

$$\mathbf{S}_h = \mathbf{B}_h^T(\mathbf{A}_h + \tau\mathbf{C}_h)^{-1}\mathbf{B}_h$$

The linear system of equations is solved with a Uzawa method:

$$
\begin{aligned}
&(1) \quad \text{Solve } (\mathbf{A}_h + \tau\mathbf{C}_h)\bar{\mathbf{z}} = \mathbf{f}_h \\
&(2) \quad \text{Solve } \mathbf{S}_h\bar{p}_h = \mathbf{B}_h^T\bar{\mathbf{z}}. \\
&(3) \quad \text{Solve } (\mathbf{A}_h + \tau\mathbf{C}_h)\bar{\mathbf{u}}_h = \mathbf{f}_h - \mathbf{B}_h\bar{p}_h.
\end{aligned}
\tag{6.1}
$$

In step (1), (2) and (3) the equations of the form $(\mathbf{A}_h + \tau\mathbf{C}_h)\mathbf{x} = \mathbf{r}$ are all solved with a standard multigrid V-cycle with one pre- and one post-smoothing iteration with a symmetric Gauss-Seidel method. The iteration is stopped as soon as the relative scaled residual satisfies

$$\frac{\|\mathbf{D}^{-1}\big((\mathbf{A}_h + \tau\mathbf{C}_h)\mathbf{x}^{(k)} - \mathbf{r}\big)\|}{\|\mathbf{D}^{-1}\big((\mathbf{A} + \tau\mathbf{C}_h)\bar{\mathbf{u}}_h^{(0)} - \mathbf{r}\big)\|} \leq 10^{-10}, \quad \mathbf{D} := \mathrm{diag}(\mathbf{A}_h + \tau\mathbf{C}_h). \tag{6.2}$$

Here $\|\cdot\|$ denotes the Euclidean norm. The system in step (2) is solved with a preconditioned conjugate gradient method. The iteration is stopped as soon as the relative preconditioned residual satisfies

$$\frac{\|\mathbf{Q}_S^{-1}(\mathbf{S}_h\bar{p}_h^k - \mathbf{B}_h^T\bar{\mathbf{z}})\|}{\|\mathbf{Q}_S^{-1}(\mathbf{S}_h\bar{p}_h^{(0)} - \mathbf{B}_h^T\bar{\mathbf{z}})\|} \leq 10^{-6}. \tag{6.3}$$

The preconditioner $\mathbf{Q}_S$ is derived from our theoretical analysis as explained in remark 7. We compute $\mathbf{Q}_S^{-1}\mathbf{r}$ as follows

$$
\begin{aligned}
&\text{(a)} \quad \text{solve } \mathbf{A}_h^N\mathbf{a} = \mathbf{r}, \\
&\text{(b)} \quad \text{solve } \mathbf{Q}_h\mathbf{b} = \mathbf{r}, \\
&\text{(c)} \quad \text{compute } \mathbf{Q}_S^{-1}\mathbf{r} := \tau\mathbf{a} + \mathbf{b}.
\end{aligned}
\tag{6.4}
$$

The linear systems in (a) and (b) are solved up to machine accuracy (using an SSOR-preconditioned conjugate gradient method).

Of course, in practice the Uzawa method is not very attractive because one has to solve the systems with $\mathbf{A}_h + \tau\mathbf{C}_h$ accurately. In this paper we take the Uzawa method to illustrate the robustness of the multigrid solver and of the preconditioner $\mathbf{Q}_S$ for the Schur complement. In practice one would use a block-preconditioner combined with a MINRES method or an inexact version of the Uzawa method, cf. [19]. The efficiency of such methods is mainly determined by the efficiency of the preconditioners. The quality of the MG preconditioner (for the $\mathbf{A}_h + \tau\mathbf{C}_h$ systems) and of the $\mathbf{Q}_S$ preconditioner for the Schur complement is illustrated in the numerical experiments below.

In the first experiment (table 6.1) we take $h = \frac{1}{16}$, $\tau = h^{-1}$ and vary $\nu_2 = \nu_{|\Omega_2}$ and $\rho_2 = \rho_{|\Omega_2}$ (recall that $\nu_{|\Omega_1} = \rho_{|\Omega_1} = 1$). We present the average iteration numbers of the solvers in the Uzawa method (6.1). The first row (#-MG) gives the average number of V-cycle steps for solving the systems with $\mathbf{A}_h + \tau \mathbf{C}_h$. In the second row (#-PCG) we give the average iteration number of the preconditioned conjugate gradient solver in step (2) of (6.1).

TABLE 6.1
Iteration counts for MG and PCG in Uzawa method, with $h = 1/16$, $\tau = h^{-1}$

$\nu_2 = \rho_2$	1e4	1e2	1e0	1e-2	1e-4
#-MG	13	13	13	13	13
#-PCG	23	23	20	24	22
$\nu_2 = \frac{1}{10}\rho_2,\ \rho_2$	1e4	1e2	1e0	1e-2	1e-4
#-MG	13	13	13	13	13
#-PCG	22	22	21	22	22
$\nu_2 = 10\rho_2,\ \rho_2$	1e4	1e2	1e0	1e-2	1e-4
#-MG	13	13	13	13	13
#-PCG	23	24	23	22	23

We repeat the experiment from table 6.1 for the case $\nu_2 = 10\rho_2$, but now with $h = \frac{1}{32}$. The results are given in table 6.2.

TABLE 6.2
Iteration counts for MG and PCG in Uzawa method, with $h = 1/32$, $\tau = h^{-1}$

$\nu_2 = 10\rho_2,\ \rho_2$	1e4	1e2	1e0	1e-2	1e-4
#-MG	13	13	13	13	13
#-PCG	24	24	23	21	22

In table 6.3 we present results for different τ values, with $h = \frac{1}{16}$, $\nu_2 = \frac{1}{10}\rho_2$

TABLE 6.3
Iteration counts for MG and PCG in Uzawa method, with $h = 1/16$, $\nu_2 = \frac{1}{10}\rho_2$

$\rho_2 = 10,\ \tau$	1e2	1e0	1e-2
#-MG	12	13	13
#-PCG	20	20	20
$\rho_2 = \frac{1}{10},\ \tau$	1e2	1e0	1e-2
#-MG	12	13	13
#-PCG	23	24	24

In all these experiments we observe a clear robustness both of the MG and the PCG method in large parameter ranges. We observe robustness of the PCG method with respect to the jumps in ν and ρ across the interface, too. Note, however, that the analysis in section 5 does not yield such a robustness result. We observed in numerical experiments, that if we take very large jumps in opposite directions in ν and ρ (which

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is not likely to occur in realistic two-phase problems) then the Schur complement preconditioner turns out to become (much) less efficient, whereas the muligrid method remains robust. Results of one such an experiment are given in table 6.4.

TABLE 6.4
Iteration counts for MG and PCG in Uzawa method, with $\nu_2 = 1e\text{-}6$, $\rho_2 = 1e\text{+}4$, $\tau = 10$

h	1/8	1/16	1/32
#-MG	14	14	14
#-PCG	125	211	324

These results motivate a further theoretical analysis of the Schur complement preconditioner with respect to jumps in the coefficients.

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