ON THE ROBUSTNESS OF A MULTIGRID METHOD FOR ANISOTROPIC REACTION-DIFFUSION PROBLEMS

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Abstract. In this paper we consider a reaction-diffusion boundary value problem in a three-dimensional thin domain. The very different length scales in the geometry result in an anisotropy effect. Our study is motivated by a parabolic heat conduction problem in a thin foil leading to such anisotropic reaction-diffusion problems in each time step of an implicit time integration method [5]. The reaction-diffusion problem contains two important parameters, namely $\varepsilon > 0$ which parameterizes the thickness of the domain and $\mu > 0$ denoting the measure for the size of the reaction term relative to that of the diffusion term. In this paper we analyze the convergence of a multigrid method with a robust (line) smoother. Both, for the W- and the V-cycle method we derive contraction number bounds smaller than one uniform with respect to the mesh size and the parameters ε and μ.

Key words. anisotropic reaction-diffusion problem, robust multigrid method

1. Introduction. In this paper we study a reaction-diffusion boundary value problem on the domain $\Omega_\varepsilon := [0,1]^2 \times [0,\varepsilon]$ with $0 < \varepsilon \leq 1$. We use the notation $(.,.)_0$ and $\|.\|_0$ for the standard scalar product and norm in $L_2(\Omega_\varepsilon)$. The scalar products and corresponding norms in $H^k(\Omega_\varepsilon)$, $k = 1, 2$ are denoted by $(.,.)_k$ and $\|.\|_k$, respectively. Let the Dirichlet and Neumann boundaries of Ω_ε be denoted by

$$\begin{aligned} \Gamma_D &:= \{(x,y,z) \,|\, (x,y) \in [0,1]^2, z \in \{0,\varepsilon\}\}, \\ \Gamma_N &:= \{(x,y,z) \,|\, (x,y) \in \{0,1\}^2, z \in [0,\varepsilon]\}, \end{aligned} \tag{1.1}$$

and define $U_\varepsilon := \{v \in H^1(\Omega_\varepsilon) \,|\, v = 0 \text{ on } \Gamma_D\}$. For $\mu > 0$ we introduce the bilinear form

$$a(u,v) := (\nabla u, \nabla v)_0 + \mu (u,v)_0 \quad \text{for all} \quad u, v \in U_\varepsilon.$$

This bilinear form is continuous and elliptic on U_ε. For given $f \in L_2(\Omega_\varepsilon)$ we consider the following problem: find $u \in U_\varepsilon$ such that

$$a(u,v) = (f,v)_0 \quad \text{for all} \quad v \in U_\varepsilon. \tag{1.2}$$

For the discretization of this problem we use standard linear conforming finite elements on a nested family of uniform tetrahedral grids. To obtain a bound for the discretization error we use the Céa-lemma and a suitable interpolation operator. In a *two*-dimensional domain one can apply the standard Lagrangian interpolation operator even for such anisotropic problems. For the *three*-dimensonal case, however, the latter mentioned operator is not satisfactory. In this paper we use a modified Scott-Zhang interpolation operator which is introduced in [1]. This operator conserves Dirichlet boundary conditions only on the upper and lower faces of the domain Ω_ε explaining why we use a combination of Dirichlet and Neumann boundaries as in (1.1). Based on the modified Scott-Zhang interpolation operator we derive a finite element discretization error bound in which the dependence on the parameters ε and μ is explicit. To solve the discrete problem we consider a multigrid method with a

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symmetric z-line Gauss-Seidel smoother. The main topic of our paper is a convergence analysis of this method. For the multigrid W-cycle we analyze the convergence in the framework of the approximation and smoothing property. We prove robustness of the multigrid W-cycle method in the sense that (for sufficiently many smoothing iterations) its contraction number in the Euclidean norm is bounded by a constant smaller than one independent of all the parameters. On the basis of [9] and [10] we also prove a robustness result for the V-cycle multigrid method. Finally, we present numerical experiments that illustrate these robustness properties.

In the literature the convergence of multigrid methods for anisotropic pure diffusion problems, i.e. a problem as in (1.2) with $\mu = 0$, has been studied in [8, 9, 10] and [11, 12]. In the latter papers the robustness of smoothers is studied, whereas in the former the convergence of W-cycle and V-cycle algorithms is analyzed. These convergence analyses of the multigrid methods are based on the standard Lagrangian interpolation operator and (thus) are restricted to the two-dimensional case. In all these analyses only the case $\mu = 0$ is considered. In the present paper we treat the three-dimensional case and consider the additional parameter $\mu > 0$.

2. Finite element discretization. From the Lax-Milgram lemma it follows that problem (1.2) has a unique solution. Note, that the very different length scales in the (x, y)- and the z-direction (for $\varepsilon \ll 1$) result in an anisotropy effect.

REMARK 1. Instead of (1.2) we could also consider the weak formulation of the following anisotropic reaction-diffusion problem on the unit cube $\Omega := [0, 1]^3$:

$$\begin{cases} -u_{xx} - u_{yy} - \lambda u_{zz} + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N, \end{cases} \tag{2.1}$$

with a parameter $\lambda = 1/\varepsilon^2 \geq 1$. The discrete versions of both formulations (1.2) and (2.1) lead to operators that have very similar anisotropy properties. In this paper we consider (1.2) because our research is motivated by a parabolic heat conduction problem in a foil, which is a domain of the form Ω_ε with $\varepsilon \ll 1$ (cf. [5]). An implicit time integration method applied to this parabolic problem leads to a problem of the form (1.2) in each time step.

For the discretization we apply a standard finite element method based on a *uniform* family of nested triangulations. The uniform subdivision of the domain is based on Kuhn's triangulation, as illustrated in Fig. 2.1. A stable regular (red) refinement

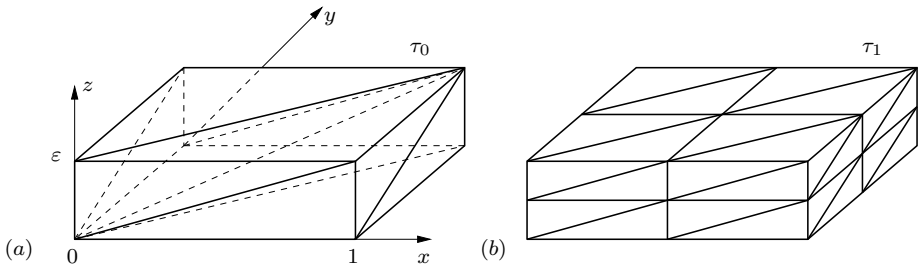

FIGURE 2.1. *(a) Initial Kuhn triangulation $\mathcal{T}_0$ and (b) first refinement $\mathcal{T}_1$.*

strategy results in a family of consistent, nested triangulations (see [2]), which is denoted by $\{\mathcal{T}_k\}_{k\geq 0}$. With $\mathcal{T}_k$ we associate the mesh size parameter $h_k = (\frac{1}{2})^k$. Note

that for all $T \in \mathcal{T}_k$ we have

$$h_T := \operatorname{diam}(T) = \sqrt{2}\, h_k,$$
$$\rho_T := \sup\{\operatorname{diam}(S) \,|\, S \text{ is a ball contained in } T\} \sim \varepsilon h_k.$$

Thus, this family of triangulations is regular in the sense that

$$\sigma := \sup_{k\in\mathbb{N}} \sup_{T\in\mathcal{T}_k} \frac{h_T}{\rho_T} < \infty$$

holds. However, $\sigma \sim \varepsilon^{-1}$ and thus $\sigma \to \infty$ for $\varepsilon \downarrow 0$. In Fig. 2.2 we show one particular octahedron on level k and a typical tetrahedron $T \in \mathcal{T}_k$. Due to the degeneracy of the given domain Ω_ε, inside the elements T arbitrarily small angles appear for $\varepsilon \downarrow 0$. On the other hand the maximum angles that occur are right angles meaning that a *maximum angle condition* is satisfied uniformly w.r.t. k and ε.

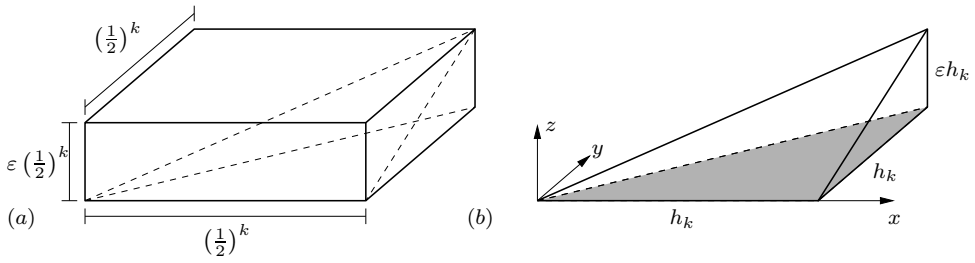

FIGURE 2.2. *(a) Octahedron on level k and (b) typical tetrahedron $T \in \mathcal{T}_k$.*

For the discretization of (1.2) we use conforming finite elements and piecewise linear functions ($\mathcal{P}_1$-elements) with respect to the sequence of nested triangulations $\{\mathcal{T}_k\}_{k\geq 0}$. This results in a hierarchy of nested finite element spaces

$$U_{\varepsilon,0} \subset U_{\varepsilon,1} \subset \ldots \subset U_\varepsilon.$$

The discrete problem on level k is: find $u_k \in U_{\varepsilon,k}$ such that

$$a(u_k, v_k) = (f, v_k)_0 \quad \text{for all} \quad v_k \in U_{\varepsilon,k}. \tag{2.2}$$

Due to the fact that a maximum angle condition is satisfied the spaces $U_{\varepsilon,k}$ are suitable for the spatial discretization of the parabolic problem from which (after implicit time integration) problem (1.2) originates.

3. Interpolation bounds. In our convergence analysis of the multigrid method we need finite element discretization error bounds. If we apply the standard approach based on the Céa-lemma then a key ingredient for obtaining such bounds is a suitable (quasi-)interpolation operator

$$I_k : H^2(\Omega_\varepsilon) \to U_{\varepsilon,k}, \quad k = 0, 1, \ldots$$

This operator I_k should be such that for all $u \in H^2(\Omega_\varepsilon)$ the following error bounds hold

$$\|u - I_k u\|_0 \leq c_1 h_k^2 |u|_2, \tag{3.1}$$
$$\|u - I_k u\|_1 \leq c_2 h_k |u|_2, \tag{3.2}$$

with constants c_1, c_2 *independent of* ε (and, as usual, also of k, u). We refer to [1] for an extensive treatment of interpolation operators for anisotropic finite element spaces. Here we briefly discuss a few issues that are relevant for the analysis in this paper. For the *two*-dimensional case uniform bounds as in (3.1)-(3.2) hold for the standard nodal Lagrangian interpolation operator (cf. corollary 2.1 in [1]). For *three*-dimensional problems the nodal Lagrangian interpolation operator still satisfies the uniform bound (3.1), but a result as in (3.2) is *not* known. However, for this operator the result in (3.2) "almost" holds, in the following sense. For any $p > 2$ an estimate of the form

$$\|u - I_k u\|_{W_p^1(\Omega_\varepsilon)} \leq c_p h_k |u|_{W_p^2(\Omega_\varepsilon)} \quad \text{for all} \quad u \in W_p^2(\Omega_\varepsilon),$$

holds, with c_p independent of ε (cf. corollary 2.1 in [1]). Here $W_p^m(\Omega_\varepsilon)$ denotes the usual Sobolev space of functions whose weak derivatives up to order m belong to $L_p(\Omega_\varepsilon)$. For the Clément and Scott-Zhang interpolation operators the uniform bound (3.1) holds but (3.2) is also not known. Thus, there is a need for other (better) interpolation operators for anisotropic finite element spaces. Such operators are presented in [1]. In particular a modification of the original Scott-Zhang operator is introduced which can be shown to satisfy, for the *three*-dimensional case, both uniform bounds (3.1) and (3.2). This operator is needed in the finite element discretization error analysis in the next section and therefore we describe how this operator is defined. A detailed discussion of this operator and its properties can be found in [1] (section 3.4).

For the description of this modified Scott-Zhang operator we need some additional notation. Let $\{X_i\}_{1\leq i\leq \tilde{n}_k}$ denote the set of vertices of the triangulation $\mathcal{T}_k$ including those on the entire boundary (i.e. in particular on the Dirichlet boundary) and $\{\phi_i\}_{1\leq i\leq \tilde{n}_k}$ the corresponding standard nodal basis which generates the finite element space $V_{\varepsilon,k}$. Note that we consider $\tilde{n}_k$ vertices while n_k denotes the dimension of the original finite element space $U_{\varepsilon,k}$. For an element $T \in \mathcal{T}_k$ we introduce the patch of surrounding elements

$$S_T := \bigcup \{ T' \in \mathcal{T}_k \mid \bar{T}' \cap \bar{T} \neq \emptyset \}.$$

To each node X_i we associate a planar subdomain $\sigma_i \subset \Omega_\varepsilon$ with the following properties:

(P1) σ_i is parallel to the x, y-plane.
(P2) $X_i \in \bar{\sigma}_i$.
(P3) There exists a face E of some element $T \in \mathcal{T}_k$ such that the projection of E on the x, y-plane is identical with the projection of σ_i.
(P4) If the projections of any two points X_i and X_j on the x, y-plane coincide then so do the projections of σ_i and σ_j.

In the triangulation $\mathcal{T}_k$ all the vertices are contained in planes $z = l\,\varepsilon(1/2)^k$, $l = 0, \ldots, 2^k$, which are subdivided into faces (triangles). Such a subdivision and the corresponding degrees of freedom are shown in Fig. 3.1 for the case $k = 1$, $l = 1$.

One possibility to select the subdomains σ_i is to assume a lexicographical numbering of the faces (see Fig. 3.1 (b)) which should be the same in all the planes and to associate to each node X_i the face with maximum number. In this way the properties (P1)–(P3) are obviously satisfied (cf. shaded face in Fig. 2.2 (b) for (P3)). Due to the refinement strategy used, on each fixed level k the corresponding subdivisions into faces are identical for all the planes and thus (P4) is fulfilled, too. Given these subdomains σ_i the modified Scott-Zhang type interpolation operator $\mathcal{L}_k : H^1(\Omega_\varepsilon) \to V_{\varepsilon,k}$

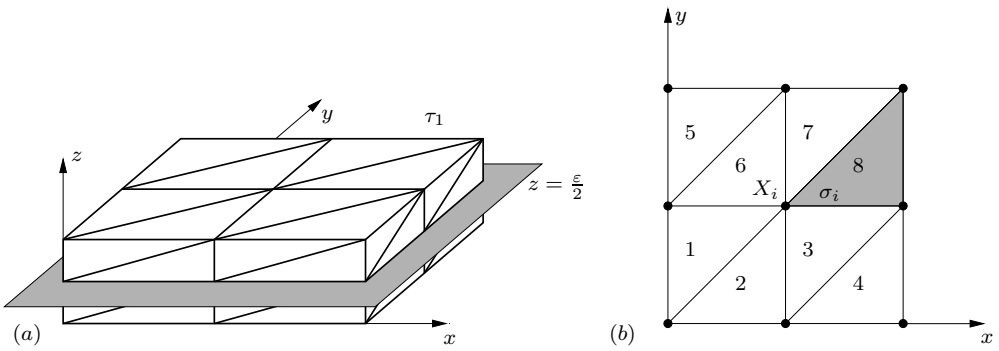

FIGURE 3.1. *(a) $\mathcal{T}_1$ with plane $z = \frac{\varepsilon}{2}$ and (b) corresponding subdivision into faces.*

from [1] is based on the local L_2-orthogonal projections on σ_i:

$$\|u - \Pi_{\sigma_i} u\|_{L_2(\sigma_i)} = \min_{v \in \mathcal{P}_1} \|u - v\|_{L_2(\sigma_i)}, \quad i = 1, \ldots, \tilde{n}_k, \quad \text{for } \; u \in H^1(\Omega_\varepsilon).$$

For $u \in H^1(\Omega_\varepsilon)$ we define

$$\mathcal{L}_k u := \sum_{i=1}^{\tilde{n}_k} a_i \phi_i, \quad \text{with } \; a_i := (\Pi_{\sigma_i} u)(X_i). \tag{3.3}$$

In theorem 3.3 from [1] the following local stability and approximation property of the operator $\mathcal{L}_k$ is given.

THEOREM 1. *The modified Scott-Zhang operator $\mathcal{L}_k$ defined in* (3.3) *satisfies the following estimates for all $T \in \mathcal{T}_k$ and all $u \in W_p^l(S_T)$:*

$$|\mathcal{L}_k u|_{W_q^m(T)} \leq c(\mathrm{meas}_3\, T)^{1/q-1/p} |u|_{W_p^m(S_T)}, \tag{3.4}$$

$$|u - \mathcal{L}_k u|_{W_q^m(T)} \leq c(\mathrm{meas}_3\, T)^{1/q-1/p} \sum_{|\alpha|=l-m} h_k^\alpha |D^\alpha u|_{W_p^m(S_T)}, \tag{3.5}$$

with $0 \leq m \leq l$, $l = 1, 2$. For (3.5) *the numbers $p, q \in [1, \infty]$ must be such that $W_p^l(T) \hookrightarrow W_q^m(T)$. The constant c is independent of k and ε.*

Here $|.|_{W_p^m(T)}$ denotes the seminorm in the Sobolev space $W_p^m(T)$ and for $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{N}$, we use the notation $h_k^\alpha := h_k^{\alpha_1} h_k^{\alpha_2} (\varepsilon h_k)^{\alpha_3} = h_k^{|\alpha|} \varepsilon^{\alpha_3}$ (product of the length scales of the edges of T in the three coordinate directions). Due to the different length scales being exploited in (3.5) estimates of this kind are called *anisotropic estimates*. Theorem 3.3 in [1] is more general than the result formulated in theorem 1. The former gives a similar result for more general (for example, higher order,) polynomial finite elements in d-dimensional spaces, with $d = 2, 3$.

COROLLARY 1. *The modified Scott-Zhang operator $\mathcal{L}_k$ defined in* (3.3) *satisfies the following estimates*

$$\|u - \mathcal{L}_k u\|_0 \leq c_1 h_k^2 |u|_2, \tag{3.6}$$

$$|u - \mathcal{L}_k u|_1 \leq c_2 h_k |u|_2, \tag{3.7}$$

for all $u \in H^2(\Omega_\varepsilon) \cap U_\varepsilon$ and with constants c_1, c_2 independent of k and ε. Moreover, $\mathcal{L}_k : H^2(\Omega_\varepsilon) \cap U_\varepsilon \to U_{\varepsilon,k}$ holds.

Proof. If in (3.5) we take $p = q = 2$, $l = 2$ and $m \in \{0, 1\}$, and sum over all $T \in \mathcal{T}_k$ we obtain (using $\varepsilon \leq 1$) the results in (3.6) and (3.7). Take $u \in H^2(\Omega_\varepsilon) \cap U_\varepsilon$, i.e. $u_{|\Gamma_D} = 0$. From the construction of the modified Scott-Zhang operator $\mathcal{L}_k : H^1(\Omega_\varepsilon) \to V_{\varepsilon,k}$ in (3.3) it follows that $a_i = (\Pi_{\sigma_i} u)(X_i) = 0$ if X_i is a vertex on the Dirichlet boundary Γ_D. Thus $(\mathcal{L}_k u)_{|\Gamma_D} = 0$ holds. Using $U_{\varepsilon,k} = \{v \in V_{\varepsilon,k} \mid v_{|\Gamma_D} = 0\}$ we conclude that $\mathcal{L}_k u \in U_{\varepsilon,k}$. ∎

4. Finite element discretization error bound. In this section, using a standard approach, we derive a finite element discretization error bound that is uniform w.r.t. the parameters μ and ε.

REMARK 2. Due to the special geometry of the considered domain Ω_ε one can use the so-called *Schwarz reflection principle* (see [3] page 143) to show that the solution u of (1.2) lies in $H^2(\Omega_\varepsilon)$ taking into account the mixed boundary conditions. The idea in applying this principle is to get the original problem in a wider domain $\tilde{\Omega}_\varepsilon \supset\supset \Omega_\varepsilon$ with an extended right-hand side $\tilde{f}$ that still belongs to $L_2(\tilde{\Omega}_\varepsilon)$. This can be reached by even extension over the Neumann boundaries and odd extension over the Dirichlet boundaries, respectively. Using the inner regularity of the extended problem solution finally leads to the desired result $u \in H^2(\Omega_\varepsilon)$. Similar results but only concerning pure Neumann or Dirichlet boundary conditions can be found e.g. in [4].

We proceed with an elementary lemma.

LEMMA 1. *For all $u \in H^2(\Omega_\varepsilon) \cap U_\varepsilon$ the identity*

$$|u|_2 = \|\Delta u\|_0 \tag{4.1}$$

holds.

Proof. It is sufficient to prove (4.1) in the dense subset $C^\infty(\Omega_\varepsilon) \cap U_\varepsilon$. For u from this space we have

$$|u|_2^2 = \int_{\Omega_\varepsilon} u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xy}^2 + 2u_{xz}^2 + 2u_{yz}^2 \, dx\, dy\, dz. \tag{4.2}$$

The unit outward pointing normal on $\Gamma_\varepsilon := \partial\Omega_\varepsilon$ is denoted by $\mathbf{n} = (n_x, n_y, n_z)^T$. On the sides of Ω_ε with normal $\mathbf{n} = (0, 0, \pm 1)^T$, i.e. the Dirichlet boundary, we have the identities $u_x = u_{xx} = 0$ and $u_y = u_{yy} = 0$. On the sides with normal $\mathbf{n} = (0, \pm 1, 0)^T$ we have $u_y = 0$ and thus $u_{xy} = 0$. Similar relations hold on the remaining Neumann boundaries. Integration by parts yields

$$\begin{aligned}\int_{\Omega_\varepsilon} u_{xy}^2 \, dx\, dy\, dz &= \int_{\Gamma_\varepsilon} \underbrace{u_x u_{xy} n_y}_{=0} \, d\Gamma_\varepsilon - \int_{\Omega_\varepsilon} u_x u_{xyy} \, dx\, dy\, dz \\ &= -\int_{\Gamma_\varepsilon} \underbrace{u_x u_{yy} n_x}_{=0} \, d\Gamma_\varepsilon + \int_{\Omega_\varepsilon} u_{xx} u_{yy} \, dx\, dy\, dz.\end{aligned}$$

Similar expressions can be derived for the mixed derivatives u_{xz} and u_{yz} in (4.2). This yields

$$\begin{aligned}|u|_2^2 &= \int_{\Omega_\varepsilon} u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xx}u_{yy} + 2u_{xx}u_{zz} + 2u_{yy}u_{zz} \, dx\, dy\, dz \\ &= \int_{\Omega_\varepsilon} (u_{xx} + u_{yy} + u_{zz})^2 \, dx\, dy\, dz = \int_{\Omega_\varepsilon} (\Delta u)^2 \, dx\, dy\, dz = \|\Delta u\|_0^2\end{aligned}$$

and thus the lemma is proved. □

LEMMA 2. *Let u be the solution of the continuous problem* (1.2). *Then the inequalities*

$$\|u\|_0 \leq \frac{1}{\mu}\|f\|_0, \tag{4.3}$$

$$|u|_2 \leq 2\|f\|_0, \tag{4.4}$$

hold.

Proof. Setting $v = u$ in (1.2) gives

$$\mu\|u\|_0^2 \leq a(u,u) = (f,u)_0 \leq \|f\|_0\|u\|_0,$$

and thus (4.3) holds. Since the solution u of (1.2) lies in $H^2(\Omega_\varepsilon)$ (see remark 2) we can write $-\Delta u = f - \mu u$. Using lemma 1 we get

$$|u|_2 = \|\Delta u\|_0 = \|f - \mu u\|_0 \leq \|f\|_0 + \mu\|u\|_0 \leq 2\|f\|_0,$$

which proves the result in (4.4). □

THEOREM 2. *Let u be the solution of the continuous problem* (1.2) *and u_k the solution of the discrete problem* (2.2). *Then*

$$\|u - u_k\|_0 \leq c \min\left\{\frac{1}{\mu}, h_k^2\right\}\|f\|_0 \tag{4.5}$$

holds, with a constant c independent of f, ε, μ and k.

Proof. We use the notation $e_k = u - u_k$. Since $a(e_k, v_k) = 0$ for all $v_k \in U_{\varepsilon,k}$ we get

$$\mu\|e_k\|_0^2 \leq a(e_k,e_k) = a(u,e_k) = (f,e_k)_0 \leq \|f\|_0\|e_k\|_0$$

and thus

$$\|e_k\|_0 \leq \frac{1}{\mu}\|f\|_0. \tag{4.6}$$

Now we apply Nitsche's duality argument and use the interpolation results from corollary 1. Let $w \in U_\varepsilon$ be such that $a(w,v) = (e_k,v)_0$ for all $v \in U_\varepsilon$. From lemma 2 we get

$$|w|_2 \leq 2\|e_k\|_0. \tag{4.7}$$

We also have (due to the Céa-lemma)

$$\begin{aligned} |e_k|_1^2 \leq a(e_k,e_k) &\leq |u - \mathcal{L}_k u|_1^2 + \mu\|u - \mathcal{L}_k u\|_0^2 \\ &\leq (c_2^2 h_k^2 + \mu c_1^2 h_k^4)|u|_2^2 \\ &\leq c(1 + \mu h_k^2) h_k^2 \|f\|_0^2. \end{aligned} \tag{4.8}$$

Thus we get (with a constant c independent of k, ε and μ)

$$\begin{aligned}
\|e_k\|_0^2 = a(w, e_k) &= a(w - \mathcal{L}_k w, e_k) \\
&\leq |w - \mathcal{L}_k w|_1 |e_k|_1 + \mu \|w - \mathcal{L}_k w\|_0 \|e_k\|_0 \\
&\leq (c_2 h_k |e_k|_1 + \mu c_1 h_k^2 \|e_k\|_0) |w|_2 \\
&\overset{(4.6),(4.7)}{\leq} c(h_k |e_k|_1 + h_k^2 \|f\|_0) \|e_k\|_0 \\
&\overset{(4.8)}{\leq} c \left([1 + \mu h_k^2]^{\frac{1}{2}} h_k^2 \|f\|_0 + h_k^2 \|f\|_0 \right) \|e_k\|_0 \\
&= c \left([1 + \mu h_k^2]^{\frac{1}{2}} + 1 \right) h_k^2 \|f\|_0 \|e_k\|_0.
\end{aligned}$$

Hence, for $h_k^2 \leq \frac{1}{\mu}$ we obtain

$$\|e_k\|_0 \leq c\, h_k^2 \|f\|_0.$$

Combining this estimate with the one in (4.6) proves the theorem. □

5. Multigrid convergence analysis. In this section we investigate the convergence behaviour of a multigrid method applied to the discrete problem (2.2). We use the approach introduced by Hackbusch (see [6]) based on the approximation and smoothing property. It is well-known that the anisotropy in the discrete problem (2.2) causes standard pointwise relaxation methods as the damped Jacobi method or the (symmetric) Gauss-Seidel method to smooth the error only in the direction corresponding to the strong couplings. This causes a (strong) deterioration in the rate of convergence of a multigrid method with such smoothers for $\varepsilon \downarrow 0$. One possibility to deal with this problem is to keep pointwise relaxation but to adapt the strategy of grid coarsening e.g. by doubling the mesh size only in the directions in which the error is smooth. An alternative approach, which is used in this paper, is to modify the smoothing procedure from pointwise relaxation to *linewise relaxation* meaning that the unknowns belonging to a line are updated simultaneously. Hackbusch (cf. [6]) introduced the notion of a "robust smoother" for anisotropic problems. Such a smoother should be a fast iterative (or even direct) solver for the discrete problem in the limit case $\varepsilon \downarrow 0$. In the problem that we consider in this paper we do not only have the anisotropy parameter ε but also the parameter μ in front of the reaction term.

In this section, using a fairly standard approach, we derive an approximation property, theorem 3, and a smoothing property for the symmetric z-line Gauss-Seidel method, theorem 4, in which the dependence of the bounds on ε, μ and k is explicit. Combination of these results immediately yields a uniform bound (< 1 for sufficiently many smoothing iterations) for the contraction number of the two-grid method and of the W-cycle iteration.

We introduce the isomorphism

$$P_k : X_k := \mathbb{R}^{n_k} \to U_{\varepsilon,k}, \quad P_k x = \sum_{i=1}^{n_k} x_i \phi_i.$$

In order to establish the norm equivalence

$$c^{-1} \|x\|_\varepsilon \leq \|P_k x\|_0 \leq c \|x\|_\varepsilon \quad \text{for all } x \in X_k, \tag{5.1}$$

with a constant c independent of ε and k we use the scaled Euclidean scalar product

$$\langle x, y\rangle_\varepsilon := \varepsilon h_k^3 \sum_{i=1}^{n_k} x_i y_i, \qquad \|\cdot\|_\varepsilon^2 := \langle \cdot, \cdot\rangle_\varepsilon.$$

Standard arguments yield that for this scaled norm indeed the uniform norm equivalence (5.1) holds. Let the corresponding matrix norm (which is independent of ε) be denoted by $\|\cdot\|$. Note that the adjoint $P_k^* : U_{\varepsilon,k} \to X_k$ satisfies $(P_k x, v)_0 = \langle x, P_k^* v\rangle_\varepsilon$ for all $x \in X_k$, $v \in U_{\varepsilon,k}$. The stiffness matrix A_k on level k is defined by

$$\langle A_k x, y\rangle_\varepsilon = a(P_k x, P_k y) \quad \text{for all } x, y \in X_k. \tag{5.2}$$

In an interior grid point the discrete problem has the stencil given in Fig. 5.1. For

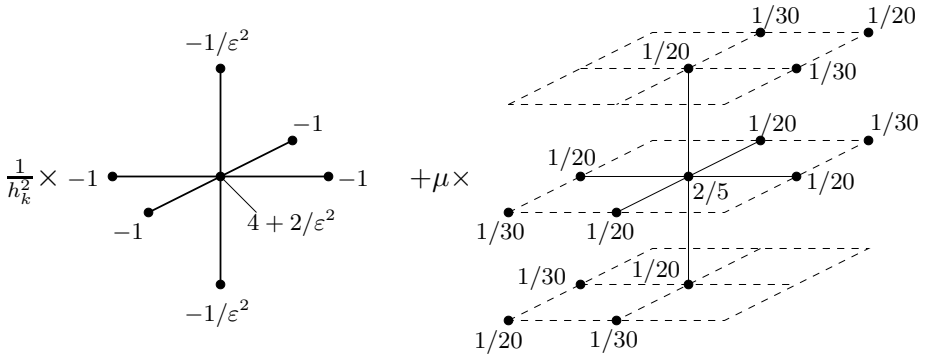

FIGURE 5.1. *3D difference stencil.*

the prolongation and restriction in the multigrid method the canonical choice

$$p_k : X_{k-1} \to X_k, \quad p_k = P_k^{-1} P_{k-1} \tag{5.3}$$

$$r_k : X_k \to X_{k-1}, \quad r_k = P_{k-1}^* (P_k^*)^{-1} = \left(\frac{h_k}{h_{k-1}}\right)^3 p_k^T, \tag{5.4}$$

is used. We use stationary linear iterative methods as smoothers and thus these are of the form

$$x^{new} = x^{old} - W_k^{-1}(A_k x^{old} - b).$$

The corresponding iteration matrix is denoted by $S_k = I - W_k^{-1} A_k$. We consider the damped z-line Jacobi method and the symmetric z-line Gauss-Seidel method as smoothers. For the matrix representation of the discrete operator (see the stencil notation in Fig. 5.1) we assume a z-line ordering of the grid points, i.e. within each line of unknowns in z-direction (the z-lines) the vertices are numbered from bottom to top while the z-lines themselves are ordered lexicographically in the x, y-plane. Let $N_k := 2^k - 1$. The stiffness matrix can be decomposed as

$$A_k = D_k - L_k - L_k^T,$$

with the block-diagonal matrix $D_k = \text{blockdiag}(\hat{D}_k) \in \mathbb{R}^{n_k \times n_k}$ and diagonal blocks

$$\hat{D}_k = \frac{1}{\varepsilon^2 h_k^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} + \frac{4}{h_k^2} I_k + \frac{\mu}{20} \,\text{tridiag}(1, 8, 1) \in \mathbb{R}^{N_k \times N_k}. \tag{5.5}$$

Note that the upper and lower faces of Ω_ε are Dirichlet boundaries. The matrix L_k is strictly lower block-triangular. The choice

$$W_k := \omega^{-1} D_k, \quad \omega \in (0, 1],$$

defines a (damped) line Jacobi smoother, and

$$W_k := (D_k - L_k) D_k^{-1} (D_k - L_k^T) \tag{5.6}$$

yields the symmetric line Gauss-Seidel method. In the convergence analysis and in the numerical experiments below we only consider the symmetric line Gauss-Seidel method. Similar results, however, can be obtained for the damped line Jacobi method if we use a damping factor $\omega = \omega(\varepsilon, \mu, k)$ such that $\omega^{-1} D_k \geq A_k$ holds.

Based on these components a standard multigrid algorithm with ν_1 pre- and ν_2 post-smoothing iterations can be formulated (see [7]) with an iteration matrix that satisfies the recursion

$$\begin{aligned} &M_0(\nu_1, \nu_2) = 0, \\ &M_k(\nu_1, \nu_2) = S_k^{\nu_2} (I - p_k (I - M_{k-1}^\gamma) A_{k-1}^{-1} r_k A_k) S_k^{\nu_1}, \quad k = 1, 2, \ldots \end{aligned} \tag{5.7}$$

The choices $\gamma = 1$ and $\gamma = 2$ correspond to the V- and W-cycle, respectively. Results of numerical experiments with this method are presented in section 6.

We now turn to the convergence analysis of this multigrid method. All constants (denoted by c or c_i) that appear in the analysis are independent of ε, μ and k.

The following lemma gives a result on the scaling of the stiffness matrix.

LEMMA 3. *Let A_k be the stiffness matrix from* (5.2). *Then the inequalities*

$$c_1 \left(\frac{1}{\varepsilon^2 h_k^2} + \mu \right) \leq \|A_k\| \leq c_2 \left(\frac{1}{\varepsilon^2 h_k^2} + \mu \right), \tag{5.8}$$

hold with constants $c_1 > 0$, c_2 independent of ε, μ and k.

Proof. Let e_i denote the ith basis vector in $\mathbb{R}^{n_k}$ and $S_i := \text{supp}(\phi_i)$ the support of the nodal basis function ϕ_i. Then we have

$$\begin{aligned} (A_k)_{ii} = \frac{\langle A_k e_i, e_i \rangle_\varepsilon}{\langle e_i, e_i \rangle_\varepsilon} &= \varepsilon^{-1} h_k^{-3} a(\phi_i, \phi_i) \\ &= \varepsilon^{-1} h_k^{-3} (|\phi_i|_1^2 + \mu \|\phi_i\|_0^2) \\ &= \varepsilon^{-1} h_k^{-3} \left(\int_{\Omega_\varepsilon} (\nabla \phi_i)^2 d\Omega_\varepsilon + \mu \int_{\Omega_\varepsilon} \phi_i^2 d\Omega_\varepsilon \right) \\ &\sim \varepsilon^{-1} h_k^{-3} \left(\int_{S_i} \left(\frac{1}{\varepsilon h_k} \right)^2 d\Omega_\varepsilon + \mu \int_{S_i} 1 \, d\Omega_\varepsilon \right) \\ &\sim \varepsilon^{-1} h_k^{-3} \left(\varepsilon h_k^3 \frac{1}{\varepsilon^2 h_k^2} + \mu \varepsilon h_k^3 \right) = \frac{1}{\varepsilon^2 h_k^2} + \mu. \end{aligned}$$

Thus, we have $\|A_k\| \geq (A_k)_{ii} \geq c_1\,(1/(\varepsilon^2 h_k^2) + \mu)$ with $c_1 > 0$, yielding the left inequality in (5.8). Using the inverse inequality

$$|P_k x|_1^2 \leq c\,\varepsilon^{-2} h_k^{-2} \|P_k x\|_0^2 \tag{5.9}$$

we get

$$\begin{aligned}\langle A_k x, x\rangle_\varepsilon = a(P_k x, P_k x) &= |P_k x|_1^2 + \mu \|P_k x\|_0^2 \\ &\leq c\left(\frac{1}{\varepsilon^2 h_k^2} + \mu\right)\|P_k x\|_0^2 \leq c_2\left(\frac{1}{\varepsilon^2 h_k^2} + \mu\right)\|x\|_\varepsilon^2.\end{aligned}$$

Finally, we note that all the constants used in this proof are independent of ε, μ and k. ☐

THEOREM 3 (Approximation property). *Let A_k be the stiffness matrix from* (5.2) *and p_k, r_k the prolongation and restriction from* (5.3) *and* (5.4)*, respectively. Then the approximation property*

$$\|A_k^{-1} - p_k A_{k-1}^{-1} r_k\| \leq c\,\frac{1 + \varepsilon^2 \mu h_k^2}{\varepsilon^2} \min\left\{\frac{1}{\mu h_k^2}, 1\right\} \|A_k\|^{-1} \tag{5.10}$$

holds with a constant c independent of ε, μ and k.

Proof. We consider an arbitrary $y_k \in X_k$. Let $w \in U_\varepsilon$, $w_k \in U_{\varepsilon,k}$ and $w_{k-1} \in U_{\varepsilon,k-1}$ be such that

$$\begin{aligned} a(w, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in U_\varepsilon, \\ a(w_k, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in U_{\varepsilon,k}, \qquad (5.11) \\ a(w_{k-1}, v) &= ((P_k^*)^{-1} y_k, v)_0 \quad \text{for all } v \in U_{\varepsilon,k-1}. \qquad (5.12)\end{aligned}$$

Setting $v = P_k y_k \in U_{\varepsilon,k}$ in (5.11) we get the identity

$$\begin{aligned}\langle A_k P_k^{-1} w_k, y_k\rangle_\varepsilon &\overset{(5.2)}{=} a(w_k, P_k y_k) \\ &= ((P_k^*)^{-1} y_k, P_k y_k)_0 = \langle y_k, y_k\rangle_\varepsilon.\end{aligned}$$

Thus we obtain $w_k = P_k A_k^{-1} y_k$. Using the same line of argumentation it follows that $w_{k-1} = P_{k-1} A_{k-1}^{-1} r_k y_k$. We use theorem 2 with $f := (P_k^*)^{-1} y_k \in L_2(\Omega_\varepsilon)$ and obtain

$$\|w - w_l\|_0 \leq c \min\left\{\frac{1}{\mu}, h_l^2\right\} \|(P_k^*)^{-1} y_k\|_0 \quad \text{for } l \in \{k-1, k\}.$$

Using a triangle inequality and $h_{k-1} = 2h_k$ we get

$$\|w_k - w_{k-1}\|_0 \leq c \min\left\{\frac{1}{\mu}, h_k^2\right\} \|(P_k^*)^{-1} y_k\|_0.$$

Due to the norm equivalence (5.1) we have

$$\begin{aligned}\|(A_k^{-1} - p_k A_{k-1}^{-1} r_k) y_k\|_\varepsilon &\leq c\,\|P_k A_k^{-1} y_k - P_k p_k A_{k-1}^{-1} r_k y_k\|_0 \\ &= c\,\|w_k - w_{k-1}\|_0 \\ &\leq c \min\left\{\frac{1}{\mu}, h_k^2\right\} \|(P_k^*)^{-1} y_k\|_0 \\ &\leq c \min\left\{\frac{1}{\mu}, h_k^2\right\} \|y_k\|_\varepsilon,\end{aligned}$$

and thus

$$\|A_k^{-1} - p_k A_{k-1}^{-1} r_k\| \leq c \, \min\left\{\frac{1}{\mu}, h_k^2\right\} \tag{5.13}$$

holds. Using the scaling property from lemma 3 one easily derives the bound in (5.10). □

We now turn to the smoothing property of the symmetric line Gauss-Seidel method for which the matrix W_k in (5.6) is symmetric positive definite. We start with an elementary lemma.

LEMMA 4. *Let $D_k = \text{blockdiag}(\hat{D}_k)$ be the block-diagonal part of A_k. The smallest eigenvalue of D_k is bounded from below by*

$$\lambda_{\min}(D_k) \geq c_1 \left(\frac{1}{\varepsilon^2} + \frac{1}{h_k^2} + \mu\right). \tag{5.14}$$

For the lower block-triangular part L_k of A_k we have

$$\|L_k\| \leq c_2 \left(\frac{1}{h_k^2} + \mu\right). \tag{5.15}$$

The constants $c_1 > 0$ and c_2 are independent of ε, μ and k.

Proof. Consider the diagonal blocks $\hat{D}_k$ of D_k in (5.5). For the third term in this representation we have

$$\frac{\mu}{20}\lambda_{\min}\big(\text{tridiag}(1,8,1)\big) \geq \frac{3}{10}\,\mu.$$

Furthermore, for the smallest eigenvalue of the first summand in (5.5), denoted by $\lambda_{\min}(S_1)$, we have

$$\begin{aligned}\lambda_{\min}(S_1) &= \varepsilon^{-2} h_k^{-2} 4 \sin^2\left(\frac{\pi}{2}(N_k+1)^{-1}\right) \\ &= 4\varepsilon^{-2} h_k^{-2} \sin^2(\pi 2^{-k-1}) \geq c\,\varepsilon^{-2},\end{aligned}$$

with $c > 0$ independent of ε and k. Since all three terms in (5.5) are symmetric positive definite we get the lower bound in (5.14). We now prove (5.15). From Fig. 5.1 we see that the matrix L_k does not contain any entries depending on ε and that $\|L_k\|_p \leq c\,(h_k^{-2} + \mu)$, for $p = 1, \infty$, holds with c indpendent of k and μ. Thus we obtain

$$\|L_k\|^2 \leq \|L_k\|_1 \|L_k\|_\infty \leq c\,(h_k^{-2} + \mu)^2,$$

which completes the proof. □

For the symmetric line Gauss-Seidel method we have

$$W_k = A_k + L_k D_k^{-1} L_k^T \geq A_k. \tag{5.16}$$

The following result can be found in [7].

LEMMA 5. *For all symmetric matrices B with $0 \leq B \leq I$, the inequality*

$$\|B(I-B)^\nu\|_2 \leq \eta_0(\nu) \quad (\nu \geq 0)$$

holds, where the function $\eta_0(\nu)$ is defined by

$$\eta_0(\nu) := \nu^\nu/(\nu+1)^{\nu+1} \le \frac{1}{e\nu+1}.$$

THEOREM 4 (Smoothing property). *For the symmetric z-line Gauss-Seidel method the following property holds*

$$\|A_k S_k^\nu\| \le c\,\frac{\varepsilon^4}{\nu}\,\frac{(1+\mu h_k^2)^2}{\big(h_k^2+\varepsilon^2(1+\mu h_k^2)\big)(1+\varepsilon^2\mu h_k^2)}\,\|A_k\|, \quad \nu = 1,2,\dots \tag{5.17}$$

with a constant c independent of ε, k, μ and ν.

Proof. The symmetric block Gauss-Seidel method corresponds to the splitting $A_k = W_k - R_k$ with $R_k := L_k D_k^{-1} L_k^T$. Note that $R_k = R_k^T$, $W_k = W_k^T > 0$ and $W_k > R_k \ge 0$ holds, and thus $\sigma(W_k^{-1}R_k) \subset [0,1)$. Moreover, $R_k^{\frac{1}{2}}$ is well-defined and

$$0 \le C_k := R_k^{\frac{1}{2}} W_k^{-1} R_k^{\frac{1}{2}} < I.$$

Using the identity (for $\nu \ge 1$)

$$A_k S_k^\nu = (W_k - R_k)(W_k^{-1}R_k)^\nu = R_k^{\frac{1}{2}} C_k^{\nu-1}(I-C_k)R_k^{\frac{1}{2}}$$

and lemma 5 with $B = I - C_k$ we obtain

$$\|A_k S_k^\nu\| \le \|R_k\|\eta_0(\nu-1) \le \frac{c}{\nu}\|R_k\|$$

with c being independent of all the parameters. Using the result in lemma 4 we get

$$\begin{aligned}
\|R_k\| &\le \|L_k\|\|D_k^{-1}\|\|L_k^T\| = \|L_k\|^2 \lambda_{\min}(D_k)^{-1} \\
&\le c\left(\frac{1}{h_k^2}+\mu\right)^2\left(\frac{1}{\varepsilon^2}+\frac{1}{h_k^2}+\mu\right)^{-1} \\
&= c\,\frac{\varepsilon^2(1+\mu h_k^2)^2}{h_k^2(\varepsilon^2+h_k^2+\varepsilon^2\mu h_k^2)}.
\end{aligned} \tag{5.18}$$

In combination with the scaling property of $\|A_k\|$ in lemma 3 we obtain the bound in (5.17). ∎

As a direct consequence of the approximation and smoothing property we obtain the following main result.

THEOREM 5. *For the two-grid iteration matrix with $\nu_1 = \nu$ pre- and $\nu_2 = 0$ post-smoothing iterations with the symmetric z-line Gauss-Seidel method we have*

$$\|(I - p_k A_{k-1}^{-1} r_k A_k) S_k^\nu\| \le \frac{C_T}{\nu}\,\frac{\varepsilon^2(1+\mu h_k^2)}{h_k^2+\varepsilon^2(1+\mu h_k^2)} \le \frac{C_T}{\nu}, \quad \nu = 1,2,\dots \tag{5.19}$$

with C_T independent of ε, μ, k, and ν.

Proof. From theorem 3 and theorem 4 we obtain

$$\begin{aligned}
&\|(I - p_k A_{k-1}^{-1} r_k A_k) S_k^\nu\| \\
&\le \|A_k^{-1} - p_k A_{k-1}^{-1} r_k\|\|A_k S_k^\nu\| \\
&\le c\,\frac{1+\varepsilon^2\mu h_k^2}{\varepsilon^2}\min\left\{\frac{1}{\mu h_k^2}, 1\right\}\frac{\varepsilon^4}{\nu}\,\frac{(1+\mu h_k^2)^2}{\big(h_k^2+\varepsilon^2(1+\mu h_k^2)\big)(1+\varepsilon^2\mu h_k^2)} \\
&= \frac{c}{\nu}\,\frac{\varepsilon^2(1+\mu h_k^2)}{h_k^2+\varepsilon^2(1+\mu h_k^2)}\,(1+\mu h_k^2)\min\left\{\frac{1}{\mu h_k^2}, 1\right\}.
\end{aligned}$$

Finally, note that $(1+x)\min\{\frac{1}{x},1\} \leq 2$ for all $x > 0$. ☐

For the multigrid W-cycle we can apply theorem 10.6.25 from [7] and thus obtain the following result.

THEOREM 6. *For every fixed $\psi \in (0,1)$ there exists $\nu_0 > 0$ independent of ε, μ and k such that for the iteration matrix M_k of the multigrid W-cycle ($\gamma = 2$ in (5.7)) with symmetric z-line Gauss-Seidel smoothing we have*

$$\|M_k(\nu, 0)\| \leq \psi \quad \textit{for all } \nu \geq \nu_0.$$

From the first bound in (5.19) we see that for fixed k and μ the norm of the two-grid iteration matrix tends to zero for $\varepsilon \downarrow 0$. The same holds for the iteration matrix of the multigrid W-cycle. Thus we expect very fast convergence of the multigrid method for $\varepsilon \ll 1$. This is confirmed by numerical experiments in the next section.

We now derive a convergence result for the multigrid V-cycle, based on the analysis given in [10]. We use the energy norm $\|B\|_A := \|A_k^{\frac{1}{2}} B A_k^{-\frac{1}{2}}\|$ for $B \in \mathbb{R}^{n_k \times n_k}$. Note that this norm depends on the parameters k, μ and ε.

THEOREM 7. *Let $M_k = M_k(\nu, \nu)$ be the iteration matrix of the multigrid V-cycle ($\gamma = 1$ in (5.7)) with symmetric z-line Gauss-Seidel smoothing. There exists a constant c independent of ε, μ and k such that*

$$\|M_k\|_A \leq \frac{c}{c+\nu} \quad \textit{for all} \quad \nu \geq 1.$$

Proof. From theorem 2.1 in [10], with $\alpha = 1$, $\gamma = 1$, it follows that

$$\begin{aligned} \|M_k\|_A &\leq \frac{\delta}{1+\delta} \quad \text{with} \\ \delta &:= \rho(A_k^{-1} - p_k A_{k-1}^{-1} r_k)\, \rho\big(A_k[(I - S_k^{2\nu})^{-1} - I]\big). \end{aligned} \tag{5.20}$$

Theorem 2.3 in [10] yields

$$\rho\big(A_k[(I - S_k^{2\nu})^{-1} - I]\big) \leq \frac{\rho(R_k)}{2\nu}.$$

Using the bounds in (5.13) and (5.18) we obtain

$$\begin{aligned} \delta &\leq \frac{1}{2\nu} \|A_k^{-1} - p_k A_{k-1}^{-1} r_k\|\, \|R_k\| \\ &\leq \frac{c}{2\nu} \min\left\{\frac{1}{\mu}, h_k^2\right\} \frac{\varepsilon^2(1+\mu h_k^2)^2}{h_k^2(\varepsilon^2 + h_k^2 + \varepsilon^2 \mu h_k^2)} \\ &\leq \frac{c}{2\nu} \min\left\{\frac{1}{\mu h_k^2}, 1\right\} \big(1 + \mu h_k^2\big) \leq \frac{c}{2\nu} \max_{x>0} \left[\min\left\{\frac{1}{x}, 1\right\}(1+x)\right] = \frac{c}{\nu}. \end{aligned}$$

Substitution of this result in (5.20) yields

$$\|M_k\|_A \leq \frac{c/\nu}{1 + c/\nu} = \frac{c}{c+\nu},$$

and thus the theorem is proved. ☐

A very similar result holds for the V-cycle method with ν_1 pre- and ν_2 post-smoothing iterations (with the symmetric z-line Gauss-Seidel method), if $\nu_1 + \nu_2 > 0$ but not necessarily $\nu_1 = \nu_2$ (cf. [10]).

6. Numerical experiments. In this section we present results of some numerical experiments using the multigrid method with a symmetric z-line Gauss-Seidel smoother. The results confirm the robustness of the W- and the V-cycle multigrid algorithms w.r.t. variation in the discretization parameter h_k and the problem parameters ε and μ.

We consider a linear system with stiffness matrix A_k as in (5.2) and a random right-hand side vector. The zero vector is used as starting vector. For the stopping criterion we take a reduction of the relative residual by a factor 10^6. The block problems arising within the smoother are solved with a tridiagonal LU decomposition.

First we study the smoother without coarse grid correction. In the tables 6.1 and 6.2 we observe expected properties of the symmetric z-line Gauss-Seidel method. In these and all other tables the numbers between the brackets give the average residual reduction per iteration. For fixed parameters ε and μ (not too large) the rate of convergence decreases with increasing refinement level. For fixed values of μ and k the rate of convergence increases if ε decreases.

	h_k			
ε	1/8	1/16	1/32	1/64
1	**82**(0.85)	**279**(0.95)	**988**(0.99)	-
1e-1	**3**(0.11e-1)	**6**(0.1)	**14**(0.43)	**42**(0.77)
1e-3	**1**(0.5e-10)	**1**(0.27e-9)	**1**(0.15e-8)	**1**(0.85e-8)

TABLE 6.1
Number of Gauss-Seidel iterations and average reduction factor for $\mu = 1$.

	h_k			
μ	1/8	1/16	1/32	1/64
1e-3	**3**(0.11e-1)	**6**(0.1)	**14**(0.43)	**42**(0.77)
1e+3	**3**(0.52e-2)	**3**(0.11e-1)	**8**(0.2)	**23**(0.61)
1e+6	**8**(0.18)	**8**(0.16)	**7**(0.14)	**7**(0.12)

TABLE 6.2
Number of Gauss-Seidel iterations and average reduction factor for $\varepsilon = 0.1$.

We now turn to the W-cycle multigrid algorithm with $\nu_1 = 2$ pre- and $\nu_2 = 0$ post-smoothing iterations. Table 6.3 shows very fast convergence for $\varepsilon \ll 1$ which is consistent with the first bound in (5.19). Furthermore, we clearly observe a uniform upper bound < 1 for the reduction number w.r.t. variation in all three parameters.

	h_k			
ε	1/8	1/16	1/32	1/64
1	**7**(0.13)	**7**(0.11)	**6**(0.11)	**6**(0.98e-1)
1e-1	**2**(0.54e-4)	**3**(0.21e-2)	**4**(0.17e-1)	**4**(0.27e-1)
1e-3	**1**(0.36e-15)	**1**(0.52e-15)	**1**(0.86e-15)	**1**(0.11e-13)

TABLE 6.3
Number of W-cycle iterations and average reduction factor for $\mu = 1$.

	h_k			
μ	1/8	1/16	1/32	1/64
1e-3	**2**(0.55e-4)	**3**(0.21e-2)	**4**(0.17e-1)	**4**(0.27e-1)
1e+3	**2**(0.22e-4)	**2**(0.72e-4)	**3**(0.45e-2)	**4**(0.21e-1)
1e+6	**4**(0.21e-1)	**4**(0.19e-1)	**3**(0.12e-1)	**3**(0.86e-2)

TABLE 6.4

Number of W-cycle iterations and average reduction factor for $\varepsilon = 0.1$.

Finally, in the tables 6.5 and 6.6 we show results for the V-cycle multigrid algorithm with $\nu_1 = 2$ pre- and $\nu_2 = 0$ post-smoothing iterations. These results show no significant differences compared to those for the W-cycle algorithm.

	h_k			
ε	1/8	1/16	1/32	1/64
1	**8**(0.15)	**8**(0.15)	**8**(0.15)	**8**(0.15)
1e-1	**2**(0.54e-4)	**3**(0.21e-2)	**4**(0.17e-1)	**4**(0.3e-1)
1e-3	**1**(0.36e-15)	**1**(0.52e-15)	**1**(0.86e-15)	**1**(0.11e-13)

TABLE 6.5

Number of V-cycle iterations and average reduction factor for $\mu = 1$.

	h_k			
μ	1/8	1/16	1/32	1/64
1e-3	**2**(0.55e-4)	**3**(0.21e-2)	**4**(0.17e-1)	**4**(0.3e-1)
1e+3	**2**(0.22e-4)	**2**(0.72e-4)	**3**(0.45e-2)	**4**(0.2e-1)
1e+6	**4**(0.22e-1)	**4**(0.19e-1)	**3**(0.12e-1)	**3**(0.86e-2)

TABLE 6.6

Number of V-cycle iterations and average reduction factor for $\varepsilon = 0.1$.

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