AN EULERIAN FINITE ELEMENT METHOD FOR ELLIPTIC EQUATIONS ON MOVING SURFACES

MAXIM A. OLSHANSKII*, ARNOLD REUSKEN† , AND JÖRG GRANDE†

Abstract. In this paper a new finite element approach for the discretization of elliptic partial differential equations on surfaces is treated. The main idea is to use finite element spaces that are induced by triangulations of an "outer" domain to discretize the partial differential equation on the surface. The method is particularly suitable for problems in which there is a coupling with a flow problem in an outer domain that contains the surface, for example, two-phase incompressible flow problems. We give an analysis that shows that the method has optimal order of convergence both in the H^1 and in the L^2-norm. Results of numerical experiments are included that confirm this optimality.

Key words. Surface, interface, finite element, level set method, two-phase flow, Marangoni

AMS subject classifications. 58J32, 65N15, 65N30, 76D45, 76T99

1. Introduction.

Moving hypersurfaces and interfaces appear in many physical processes, for example in multiphase flows and flows with free surfaces. Certain mathematical models involve elliptic partial differential equations posed *on* such surfaces. This happens, for example, in multiphase fluids if one takes so-called surface active agents (surfactants) into account. These surfactants induce tangential surface tension forces and thus cause Marangoni phenomena [13, 15]. Numerical simulations play an important role for a better understanding and prediction of processes involving this or other surface phenomena. In mathematical models surface equations are often coupled with other equations that are formulated in a (fixed) domain which contains the surface. In such a setting a common approach is to use a splitting scheme that allows to solve at each time step a sequence of simpler (decoupled) equations. Doing so one has to solve numerically at each time step an elliptic type of equation on a surface. The surface may vary from one time step to another and usually only some discrete approximation of the surface is available. A well-known finite element method for solving elliptic equations on surfaces, initiated by the paper [7], consists of approximating the surface by a piecewise polygonal surface and using a finite element space on a triangulation of this discrete surface, cf. [5, 13]. If the surface is changing in time, then this approach leads to time-dependent triangulations and time-dependent finite element spaces. Implementing this requires substantial data handling and programming effort. Another approach has recently been introduced in [4]. The method in that paper applies to cases in which the surface is given implicitly by some level set function and the key idea is to solve the partial differential equation on a narrow band around the surface. Unfitted finite element spaces on this narrow band are used for discretization.

In this paper we introduce a new technique for the numerical solution of an elliptic equation posed on a hypersurface. The main idea is to use time-*in*dependent finite element spaces that are induced by triangulations of an "outer" domain to

* Department of Mechanics and Mathematics, Moscow State M.V. Lomonosov University, Moscow 119899, Russia; email: Maxim.Olshanskii@mtu-net.ru. This author was partially supported by the Russian Foundation for Basic Research through project 08-01-00415

† Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: reusken@igpm.rwth-aachen.de This work was supported by the German Research Foundation through SFB 540.

discretize the partial differential equation on the surface. Our method is particularly suitable for problems in which the surface is given implicitly by a level set or VOF function and in which there is a coupling with a flow problem in a fixed outer domain. If in such problems one uses finite element techniques for the discetization of the flow equations in the outer domain, this setting immediately results in an easy to implement discretization method for the surface equation. The new approach does not require additional surface elements. If the surface varies in time, one has to recompute the surface stiffness matrix using the same data structures each time. Moreover, quadrature routines that are needed for these computations are often available already, since they are needed in other surface related calculations, for example surface tension forces, cf. section 4. Opposite to the method in [4] we do not use an extension of the surface partial differential equation but instead use a restriction of the outer finite element spaces.

We prove that the method has optimal order of convergence in H^1 and L^2 norms. The analysis requires shape regularity of the outer triangulation, but *does not* require any type of shape regularity for discrete surface elements. The number of unknowns in the resulting algebraic systems is almost the same as in the approach based on the surface finite element spaces. All these properties make the new method very attractive both from the theoretical and the practical (implementation) point of view.

Although our primal objective is to solve efficiently equations on moving and implicitly defined surfaces, the method is also well suited for problems with steady and/or explicitly given surfaces.

The remainder of the paper is organized as follows. In section 2 we present the finite element method for the model example of the Laplace-Beltrami equation. Section 3 contains the main theoretical results of the paper concerning the approximation properties of the finite element spaces and discretization error bounds for the new method. In section 4 we describe a model for two-phase incompressible flows to illustrate a concrete field of application for the method. Finally, in section 5 results of numerical experiments are given, which support the theoretical analysis of the paper.

2. Laplace-Beltrami equation and finite element discretization. In our applications, cf. section 4, the finite element method that is presented in this section is applied to a convection-diffusion equation on a *moving* manifold $\Gamma = \Gamma(t)$. To simplify the presentation and the analysis, we assume a fixed (but unknown) sufficiently smooth connected manifold Γ $(= \Gamma(t_n))$ and instead of a convection-diffusion equation we consider the pure diffusion (i.e., Laplace-Beltrami) equation.

We assume that Ω is an open subset in $\mathbb{R}^3$ and Γ a connected C^2 compact hypersurface contained in Ω. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$ the tangential derivative (along Γ) is defined by

$$\nabla_\Gamma g = \nabla g - \nabla g \cdot \mathbf{n}_\Gamma \, \mathbf{n}_\Gamma. \tag{2.1}$$

The *Laplace-Beltrami operator* on Γ is defined by

$$\Delta_\Gamma g := \nabla_\Gamma \cdot \nabla_\Gamma g.$$

We consider the Laplace-Beltrami problem in weak form: For given $f \in L^2(\Gamma)$ with $\int_\Gamma f \, d\mathbf{s} = 0$, determine $u \in H^1(\Gamma)$ with $\int_\Gamma u \, d\mathbf{s} = 0$ such that

$$\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \, d\mathbf{s} = \int_\Gamma f v \, d\mathbf{s} \qquad \text{for all} \quad v \in H^1(\Gamma). \tag{2.2}$$

The solution u is unique and satisfies $u \in H^2(\Gamma)$ with $\|u\|_{H^2(\Gamma)} \leq c\|f\|_{L^2(\Gamma)}$ and a constant c independent of f, cf. [7].

For the discretization of this problem one needs an approximation Γ_h of Γ. We assume that this approximate manifold is constructed as follows. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of tetrahedral triangulations of a *fixed* domain $\Omega \subset \mathbb{R}^3$ that contains Γ. These triangulations are assumed to be regular, consistent and stable. Take $\mathcal{T}_h \in \{\mathcal{T}_h\}_{h>0}$ and denote the set of tetrahedra that form $\mathcal{T}_h$ by $\{S\}$. We assume that Γ_h is a closed manifold such that

- Γ_h can be decomposed as

$$\Gamma_h = \cup_{T \in \mathcal{F}_h} T, \tag{2.3}$$

  where for each T there is a corresponding tetrahedron $S_T \in \mathcal{T}_h$ with $T = S_T \cap \Gamma_h$ and $\text{meas}_2(T) > 0$. To avoid technical complications we assume that this S_T is unique, i.e., T does not coincide with a face of a tetrahedron in $\mathcal{T}_h$.
- Each T from the decomposition in (2.3) is *planar*, i.e., either a triangle or a quadrilateral.

The main new idea of this paper is that for discretization of the problem (2.2) *we use a finite element space induced by the continuous linear finite elements on* $\mathcal{T}_h$. This is done as follows. We define a subdomain that contains Γ_h:

$$\omega_h := \cup_{T \in \mathcal{F}_h} S_T. \tag{2.4}$$

This subdomain in $\mathbb{R}^3$ is connected and partitioned in tetrahedra that form a subset of $\mathcal{T}_h$. We introduce the finite element space

$$V_h := \{\, v_h \in C(\omega_h) \mid v_{|S_T} \in P_1 \ \text{ for all } \ T \in \mathcal{F}_h \,\}. \tag{2.5}$$

This space induces the following space on Γ_h:

$$V_h^\Gamma := \{\psi_h \in H^1(\Gamma_h) \,|\, \exists\, v_h \in V_h : \, \psi_h = v_h|_{\Gamma_h} \}. \tag{2.6}$$

This space is used for a Galerkin discretization of (2.2): determine $u_h \in V_h^\Gamma$ with $\int_{\Gamma_h} u_h \mathrm{d}\mathbf{s}_h = 0$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h = \int_{\Gamma_h} f_h \psi_h \, \mathrm{d}\mathbf{s}_h \qquad \text{for all } \ \psi_h \in V_h^\Gamma, \tag{2.7}$$

with f_h an extension of f such that $\int_{\Gamma_h} f_h \mathrm{d}\mathbf{s}_h = 0$, cf. section 3.3. Due the Lax-Milgram lemma this problem has a unique solution u_h. In section 3 we present a discretization error analysis of this method that shows that under reasonable assumptions we have optimal error bounds. In section 5 we show results of numerical experiments that confirm the theoretical analysis. As far as we know this method for discretization of a partial differential equation on a surface is new. In the remarks below we give some comments related to this approach.

REMARK 1. The family $\{\mathcal{T}_h\}_{h>0}$ is *shape-regular* but the family $\{\mathcal{F}_h\}_{h>0}$ in general is *not shape-regular*. In our applications, cf. section 4, $\mathcal{F}_h$ contains a significant number of strongly deteriorated triangles that have very small angles. Moreover, neighboring triangles can have very different areas, cf. Fig. 5.1. As we will prove in section 3, optimal discretization bounds hold if $\{\mathcal{T}_h\}_{h>0}$ is shape-regular; for $\{\mathcal{F}_h\}_{h>0}$ shape-regularity is *not* required.

REMARK 2. Let $(\xi_i)_{1\leq i\leq m}$ be the collection of all vertices of all tetrahedra in ω_h and ϕ_i the nodal linear finite element basis function corresponding to ξ_i. Then V_h^Γ is spanned by the functions $\phi_i|_{\Gamma_h}$, $1 \leq i \leq m$. These functions, however, are *not* necessarily independent. In computations we use this generating system $\phi_i|_{\Gamma_h}$, $1 \leq i \leq m$, for solving the discrete problem (2.7). Properties that are of interest for the numerical solution of the resulting linear system, such as conditioning of the mass and stiffness matrix are analyzed in a forthcoming paper.

REMARK 3. In the implementation of this method one has to compute integrals of the form

$$\int_T \nabla_{\Gamma_h}\phi_j \nabla_{\Gamma_h}\phi_i \,\mathrm{d}\mathbf{s}, \quad \int_T f_h\phi_i \,\mathrm{d}\mathbf{s} \quad \text{for} \quad T \in \mathcal{F}_h.$$

The domain T is either a triangle or a quadrilateral. The first integral can be computed exactly. For the second one standard quadrature rules can be applied.

REMARK 4. Each quadrilateral in $\mathcal{F}_h$ can be subdivided into two triangles. Let $\tilde{\mathcal{F}}_h$ be the induced set consisting of *only* triangles and such that $\cup_{T\in\tilde{\mathcal{F}}_h} T = \Gamma_h$. Define

$$W_h^\Gamma := \{\, \psi_h \in C(\Gamma_h) \mid \psi_h|_T \in P_1 \quad \text{for all} \quad T \in \tilde{\mathcal{F}}_h \,\}. \tag{2.8}$$

The space W_h^Γ is the space of continuous functions that are piecewise linear on the triangles of Γ_h. Clearly $V_h^\Gamma \subset W_h^\Gamma$ holds. There are, however, situations in which $V_h^\Gamma \neq W_h^\Gamma$. A 2D illustration of this is given in Fig. 2.1.

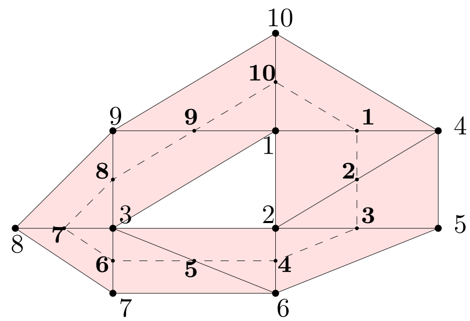

FIG. 2.1. *Example*

In this example ω_h consists of 10 triangles (shaded). The nodal basis functions correponding to these basis functions are denoted by $\{\phi_i\}_{1\leq i\leq 10}$. The line segments of the interface Γ_h (denoted by - -) intersect midpoints of edges of the triangles. The space W_h^Γ consists of piecewise linears on Γ_h and is spanned by the 1D nodal basis functions at the intersection points labeled by boldface $\mathbf{1},\ldots,\mathbf{10}$. Clearly $\dim(W_h^\Gamma) = 10$. In this example we have $\dim(V_h^\Gamma) = 9$. For the piecewise linear function $v = \sum_{i=1}^{10}\alpha_i\phi_i$ with $\alpha_i = -1$ for $i = 1,2,3$ and $\alpha_i = 1$ for $i = 4,\ldots,10$ we have $v_{|\Gamma_h} = 0$.

The example in remark 4 shows that the finite element space V_h^Γ can be smaller then W_h^Γ, and therefore approximation properties of V_h^Γ do not follow directly from those of W_h^Γ. Moreover, the triangulations $\{\tilde{\mathcal{F}}_h\}_{h>0}$ of Γ_h are not shape regular, cf. remark 1 and Fig. 5.1. Thus it is not clear how (optimal) approximation error bounds for the standard linear finite element space W_h^Γ in (2.8) can be derived.

3. Discretization error analysis.

In this section we derive discretization error bounds, both in the H^1- and the L^2-norm on Γ_h. We first collect some preliminaries in section 3.1, then derive approximation error bounds in section 3.2 and finally present discretization error bounds in section 3.3.

3.1. Preliminaries.

We will need a Poincare type inequality that is given in the following lemma.

LEMMA 3.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and a subdomain $S \subset \Omega$. Assume that Ω is such that the Neumann-Poincare inequality is valid:*

$$\|f\|_{L^2(\Omega)} \leq C_P\,|\Omega|^{\frac{1}{n}}\|\nabla f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in H^1(\Omega) \quad \text{with} \quad \int_\Omega f\,d\mathbf{x} = 0. \tag{3.1}$$

Then for any $f \in H^1(\Omega)$ the following estimate holds:

$$\|f\|^2_{L^2(\Omega)} \leq \frac{|\Omega|}{|S|}\left(2\|f\|^2_{L^2(S)} + 3C_P^2|\Omega|^{\frac{2}{n}}\|\nabla f\|^2_{L^2(\Omega)}\right). \tag{3.2}$$

Proof. The proof uses a technique developed by Sobolev ([22], Ch.I) for building equivalent norms on $W^l_q(\Omega)$ (Sobolev spaces). We consider the simple case with $q = 2$, $l = 1$, i.e. $H^1(\Omega)$. Introduce the projectors $\Pi_k : H^1(\Omega) \to \mathbb{R}$, $k = 1, 2$:

$$\Pi_1 f := |\Omega|^{-1}\int_\Omega f\,d\mathbf{x}, \qquad \Pi_2 f := |S|^{-1}\int_S f\,d\mathbf{x}.$$

Since $\|(I - \Pi_1)f\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - |\Omega||\Pi_1 f|^2$, the Neumann-Poincare inequality (3.1) can be rewritten in the equivalent form:

$$\|f\|^2_{L^2(\Omega)} \leq |\Omega||\Pi_1 f|^2 + C_P^2|\Omega|^{\frac{2}{n}}\|\nabla f\|^2_{L^2(\Omega)} \quad \text{for all} \quad f \in H^1(\Omega). \tag{3.3}$$

For any $f \in H^1(\Omega)$ with $\Pi_1 f = 0$ the Cauchy and Neumann-Poincare inequality implies

$$\begin{aligned} |\Pi_2 f| = |S|^{-1}\left|\int_S f\,d\mathbf{x}\right| &\leq |S|^{-\frac{1}{2}}\|f\|_{L^2(S)} \\ &\leq |S|^{-\frac{1}{2}}\|f\|_{L^2(\Omega)} \leq C_p|\Omega|^{\frac{1}{n}}|S|^{-\frac{1}{2}}\|\nabla f\|_{L^2(\Omega)}. \end{aligned} \tag{3.4}$$

Define $M := C_P|\Omega|^{\frac{1}{n}}|S|^{-\frac{1}{2}}$. Note that for $f \in H^1(\Omega)$ we have $\Pi_1(I - \Pi_1)f = 0$ and thus from (3.4) we obtain:

$$|(\Pi_2 - \Pi_1)f| = |\Pi_2(I - \Pi_1)f| \leq M\|\nabla(I - \Pi_1)f\|_{L^2(\Omega)} = M\|\nabla f\|_{L^2(\Omega)}.$$

Hence, for any $f \in H^1(\Omega)$ we have

$$\begin{aligned} |\Pi_1 f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)} &\leq 2|\Pi_2 f|^2 + 2|(\Pi_2 - \Pi_1)f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)} \\ &\leq 2|\Pi_2 f|^2 + 3M^2\|\nabla f\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.5}$$

Estimates (3.3) and (3.5) imply:

$$\begin{aligned}
\|f\|_{L^2(\Omega)}^2 &\leq \max\{|\Omega|, C_P^2|\Omega|^{\frac{2}{n}}M^{-2}\}\left(|\Pi_1 f|^2 + M^2\|\nabla f\|_{L^2(\Omega)}^2\right)\\
&= |\Omega|\left(|\Pi_1 f|^2 + M^2\|\nabla f\|_{L^2(\Omega)}^2\right)\\
&\leq |\Omega|\left(2|\Pi_2 f|^2 + 3M^2\|\nabla f\|_{L^2(\Omega)}^2\right)\\
&\leq |\Omega|\left(2|S|^{-1}\|f\|_{L^2(S)}^2 + 3M^2\|\nabla f\|_{L^2(\Omega)}^2\right)\\
&= |\Omega||S|^{-1}\left(2\|f\|_{L^2(S)}^2 + 3C_P^2|\Omega|^{\frac{2}{n}}\|\nabla f\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

which proves the inequality in (3.2). ☐

Remark 5. It is well known that the constant C_F from the Friedrichs-Poincare inequality

$$\|f\|_{L^2(\Omega)} \leq C_F|\Omega|^{\frac{1}{n}}\|\nabla f\|_{L^2(\Omega)} \quad \text{for} \quad f \in H_0^1(\Omega),$$

depends only on the space dimension n. For the constant C_P in (3.1) this is not true. Bounds for C_P that are available in the literature depend on the geometry of the domain Ω. Results from [3] for domains with piecewise smooth boundaries ensure that in the cases that we consider in this paper C_P can be uniformly bounded.

We define a neighborhood of Γ:

$$U = \{\, \mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma) < c\,\},$$

with c sufficiently small and assume that $\Gamma_h \subset U$. Let $d : U \to \mathbb{R}$ be the signed distance function, $|d(x)| := \mathrm{dist}(\mathbf{x}, \Gamma)$ for all $\mathbf{x} \in U$. Thus Γ is the zero level set of d. We assume $d < 0$ on the interior of Γ and $d > 0$ on the exterior. Note that $\mathbf{n}_\Gamma = \nabla d$ on Γ. We define $\mathbf{n}(\mathbf{x}) := \nabla d(\mathbf{x})$ for all $\mathbf{x} \in U$. Thus $\mathbf{n} = \mathbf{n}_\Gamma$ on Γ and $\|\mathbf{n}(\mathbf{x})\| = 1$ for all $\mathbf{x} \in U$. Here and in the remainder $\|\cdot\|$ denotes the Euclidean norm. The Hessian of d is denoted by $\mathbf{H}$:

$$\mathbf{H}(\mathbf{x}) = D^2 d(\mathbf{x}) \in \mathbb{R}^{3\times 3} \quad \text{for all} \quad \mathbf{x} \in U. \tag{3.6}$$

The eigenvalues of $\mathbf{H}(\mathbf{x})$ are denoted by $\kappa_1(\mathbf{x}), \kappa_2(\mathbf{x})$ and 0. For $\mathbf{x} \in \Gamma$ the eigenvalues $\kappa_i(\mathbf{x})$, $i = 1, 2$, are the principal curvatures.

We will need the orthogonal projection

$$\mathbf{P}(\mathbf{x}) = \mathbf{I} - \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x})^T \quad \text{for} \quad \mathbf{x} \in U.$$

Note that the tangential derivative can be written as $\nabla_\Gamma g(\mathbf{x}) = \mathbf{P}\nabla g(\mathbf{x})$ for $\mathbf{x} \in \Gamma$. We introduce a locally orthogonal coordinate system by using the projection $\mathbf{p} : U \to \Gamma$:

$$\mathbf{p}(\mathbf{x}) = \mathbf{x} - d(\mathbf{x})\mathbf{n}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in U.$$

We assume that the decomposition $\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\mathbf{n}(\mathbf{x})$ is unique for all $\mathbf{x} \in U$. Note that

$$\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{p}(\mathbf{x})) \quad \text{for all} \quad \mathbf{x} \in U.$$

We use an extension operator defined as follows. For a function v on Γ we define

$$v^e(\mathbf{x}) := v(\mathbf{x} - d(\mathbf{x})\mathbf{n}(\mathbf{x})) = v(\mathbf{p}(\mathbf{x})) \quad \text{for all} \;\; \mathbf{x} \in U,$$

i.e., v is extended along normals on Γ. We define a discrete analogon of the orthogonal projection $\mathbf{P}$:

$$\mathbf{P}_h(\mathbf{x}) := \mathbf{I} - \mathbf{n}_h(\mathbf{x})\mathbf{n}_h(\mathbf{x})^T \quad \text{for} \;\; \mathbf{x} \in \Gamma_h, \; \mathbf{x} \text{ not on an edge.}$$

Here $\mathbf{n}_h(\mathbf{x})$ denotes the (outward pointing) normal at $\mathbf{x} \in \Gamma_h$ ($\mathbf{x}$ not on an edge). The tangential derivative along Γ_h can be written as $\nabla_{\Gamma_h} g(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})\nabla g(\mathbf{x})$ for $\mathbf{x} \in \Gamma_h$ (not on an edge).

In the analysis we use techniques from [5, 7]. For example, the formula

$$\nabla u^e(\mathbf{x}) = (\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on} \;\; U \tag{3.7}$$

(cf. section 2.3 in [5]), which implies,

$$\nabla_{\Gamma_h} v^e(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})\big(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x})\big)\nabla_\Gamma v(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on} \;\; \Gamma_h. \tag{3.8}$$

Furthermore, for u sufficiently smooth and $|\mu| = 2$, the inequality

$$|D^\mu u^e(\mathbf{x})| \leq c\big(\sum_{|\mu|=2} |\mathrm{D}_\Gamma^\mu u(\mathbf{p}(\mathbf{x}))| + \|\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))\|\big) \quad \text{a.e. on} \;\; U \tag{3.9}$$

holds, cf. lemma 3 in [7]. We define an h-neighborhood of Γ:

$$U_h = \{\, \mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma) < c_1 h \,\}$$

and assume that h is sufficiently small, such that $\omega_h \subset U_h \subset U$ and

$$4c_1 h < \big(\max_{i=1,2} \|\kappa_i\|_{L^\infty(\Gamma)}\big)^{-1}. \tag{3.10}$$

From (2.5) in [5] we have the following formula for the principal curvatures κ_i:

$$\kappa_i(\mathbf{x}) = \frac{\kappa_i(\mathbf{p}(\mathbf{x}))}{1 + d(\mathbf{x})\kappa_i(\mathbf{p}(\mathbf{x}))}, \quad \text{for } \mathbf{x} \in U. \tag{3.11}$$

Hence, from (3.10) and (3.11) it follows that

$$\|d\|_{L^\infty(U_h)} \max_{i=1,2} \|\kappa_i\|_{L^\infty(U_h)} \leq \frac{1}{4} \tag{3.12}$$

holds. In the remainder we assume that

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h} |d(\mathbf{x})| \leq c_0 h^2, \tag{3.13}$$

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h} \|\mathbf{n}(\mathbf{x}) - \mathbf{n}_h(\mathbf{x})\| \leq \tilde{c}_0 h, \tag{3.14}$$

holds. .

LEMMA 3.2. *There are constants $c_1 > 0$ and c_2 independent of h such that for all $u \in H^2(\Gamma)$ the following inequalities hold:*

$$c_1\|u^e\|_{L^2(U_h)} \leq \sqrt{h}\|u\|_{L^2(\Gamma)} \leq c_2\|u^e\|_{L^2(U_h)}, \tag{3.15}$$

$$c_1\|\nabla u^e\|_{L^2(U_h)} \leq \sqrt{h}\|\nabla_\Gamma u\|_{L^2(\Gamma)} \leq c_2\|\nabla u^e\|_{L^2(U_h)}, \tag{3.16}$$

$$\|D^\mu u^e\|_{L^2(U_h)} \leq c_2\sqrt{h}\|u\|_{H^2(\Gamma)}, \quad |\mu| = 2. \tag{3.17}$$

Proof. Note that $u \in H^2(\Gamma)$ is continuous and thus u^e is well-defined. Define

$$\mu(\mathbf{x}) := \big(1 - d(\mathbf{x})\kappa_1(\mathbf{x})\big)\big(1 - d(\mathbf{x})\kappa_2(\mathbf{x})\big), \quad \mathbf{x} \in U_h.$$

From (2.20), (2.23) in [5] we have

$$\mu(\mathbf{x})\mathrm{d}\mathbf{x} = \mathrm{d}r\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in U,$$

where $\mathrm{d}\mathbf{x}$ is the measure in U_h, $\mathrm{d}\mathbf{s}$ the surface measure on Γ and r the local coordinate at $\mathbf{x} \in \Gamma$ in the direction $\mathbf{n}(\mathbf{p}(\mathbf{x})) = \mathbf{n}(\mathbf{x})$. Using (3.12) we get

$$\frac{9}{16} \le \mu(\mathbf{x}) \le \frac{25}{16} \quad \text{for all } \; \mathbf{x} \in U_h. \tag{3.18}$$

Using the local coordinate representation $\mathbf{x} = (\mathbf{p}(\mathbf{x}), r)$, for $\mathbf{x} \in U$, we have

$$\begin{aligned}\int_{U_h} u^e(\mathbf{x})^2 \mu(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{-c_1 h}^{c_1 h} \int_\Gamma \left[u^e(\mathbf{p}(\mathbf{x}), r)\right]^2 \,\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))\mathrm{d}r \\ &= \int_{-c_1 h}^{c_1 h} \int_\Gamma \left[u(\mathbf{p}(\mathbf{x}), 0)\right]^2 \,\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})) = 2c_1 h \|u\|^2_{L^2(\Gamma)}.\end{aligned}$$

Combining this with (3.18) yields the result in (3.15).

From (3.7) we have that $u^e \in H^1(U_h)$. Note that

$$\int_{U_h} \left[\nabla u^e(\mathbf{x})\right]^2 \mu(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{-c_1 h}^{c_1 h} \int_\Gamma \left[(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))\right]^2 \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))\,\mathrm{d}r.$$

Using this in combination with $\|d(\mathbf{x})\mathbf{H}(\mathbf{x})\| \le \frac{1}{4}$ for all $\mathbf{x} \in U_h$ (cf. (3.12)) and the bounds in (3.18) we obtain the result in (3.16). Finally, using similar arguments and the bound in (3.9) one can derive the bound in (3.17). ∎

3.2. Approximation error bounds. Let $I_h : C(\overline{\omega_h}) \to V_h$ be the nodal interpolation operator. We use the approximation property of the linear finite element space V_h: For $v \in H^2(\omega_h)$

$$\|v - I_h v\|_{H^k(\omega_h)} \le C\, h^{2-k} \|v\|_{H^2(\omega_h)}, \quad k = 0, 1. \tag{3.19}$$

A consequence of this approximation result is given in the following lemma.

Lemma 3.3. *For $u \in H^2(\Gamma)$ and $k = 0, 1$ we have*

$$\|u^e - I_h u^e\|_{H^k(\omega_h)} \le C\, h^{\frac{5}{2}-k} \|u\|_{H^2(\Gamma)}. \tag{3.20}$$

Proof. From (3.19) and (3.16) we obtain

$$\|u^e - I_h u^e\|_{H^k(\omega_h)} \le C\, h^{2-k} \|u^e\|_{H^2(\omega_h)} \le C\, h^{2-k} \|u^e\|_{H^2(U_h)} \le C\, h^{\frac{5}{2}-k} \|u\|_{H^2(\Gamma)},$$

which proves the result. ∎

The following two lemmas play a crucial role in the analysis. In both lemmas we use a “pull back” strategy based on lemma 3.1. For this we introduce a special local coordinate system as follows. For a subdomain $\omega \subset \mathbb{R}^n$ let $\rho(\omega)$ be the diameter of the largest ball that is contained in ω. Take an arbitrary planar segment T of Γ_h, i.e., $T \in \mathcal{F}_h$. Let $S_T \in \mathcal{T}_h$ be the tetrahedron such that $\Gamma_h \cap S_T = T$. There exists a planar extension T^e of T such that $T^e \subset U$, $\mathbf{p}(T^e) = \mathbf{p}(S_T)$ and

$$\operatorname{diam}(T^e) \simeq \rho(T^e) \simeq h, \tag{3.21}$$

cf. remark 6. This extension T^e is used to define a coordinate system in the neighborhood $N_T := \{\, \mathbf{x} \in U \mid \mathbf{p}(\mathbf{x}) \in \mathbf{p}(S_T) = \mathbf{p}(T^e) \,\}$. Note that $S_T \subset N_T$. Every $\mathbf{x} \in N_T$ has a unique decomposition of the form

$$\mathbf{x} = \mathbf{s} + \tilde{d}(\mathbf{x})\mathbf{n}(\mathbf{x}), \quad \text{with } \mathbf{s} \in T^e,\ \tilde{d}(\mathbf{x}) := \pm\|\mathbf{s} - \mathbf{x}\|. \tag{3.22}$$

On which side of the plane T^e the point $\mathbf{x}$ lies determines the sign of $\tilde{d}(\mathbf{x})$. Note that $\tilde{d}$ is a signed distance, along the normal $\mathbf{n}(\mathbf{x})$, to the planar segment T^e. The representation in this coordinate system is denoted by Φ, i.e., $\Phi(\mathbf{x}) = (\mathbf{s}(\mathbf{x}), \tilde{d}(\mathbf{x}))$. This coordinate system is illustrated, for the 2D case, in Fig. 3.1.

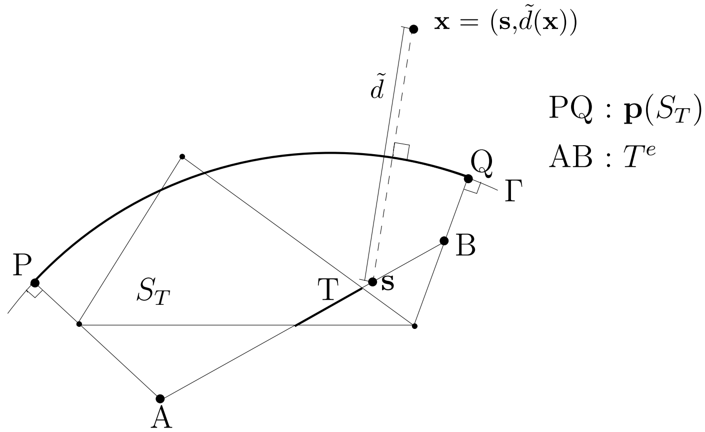

FIG. 3.1. *2D Illustration of coordinate system*

For $\mathbf{x} \in T^e$ we thus have $\Phi(\mathbf{x}) = (\mathbf{s}(\mathbf{x}), 0)$. Due to the shape-regularity of $\mathcal{T}_h$ there exists, in the Φ-coordinate system, a cylinder B_T that has the following properties:

$$B_T = T_b^e \times [d_0, d_1] \subset S_T, \quad T_b^e \subset T^e, \ \ |T_b^e| \simeq h^2, \ \ d_1 - d_0 \simeq h. \tag{3.23}$$

This coordinate system and the cylinder $B_T \subset S_T$ are used in the analysis below.

REMARK 6. The following shows that an extension T^e of T with the properties described above exists. Take a fixed $\mathbf{x}_0 \in T$. Let W_Γ be the tangent plane at $\mathbf{p}(\mathbf{x}_0)$. The normal vector of W_Γ is $\mathbf{n}(\mathbf{x}_0)$. There is a subdomain w_Γ of this plane such that $\mathbf{p}(w_\Gamma) = \mathbf{p}(S_T)$. Due to shape regularity of $\mathcal{T}_h$ this subdomain is such that $\operatorname{diam}(w_\Gamma) \simeq \rho(w_\Gamma) \simeq h$ holds. Let $w_{\mathbf{x}_0}$ be a planar subdomain that is parallel to w_Γ, contains $\mathbf{x}_0$ and such that $\mathbf{p}(w_\Gamma) = \mathbf{p}(w_{\mathbf{x}_0})$. Using assumption (3.13) it follows that $\operatorname{diam}(w_{\mathbf{x}_0}) \simeq \rho(w_{\mathbf{x}_0}) \simeq h$ holds. The point $\mathbf{x}_0$ belongs to the planar subdomains $w_{\mathbf{x}_0}$ and T, which have normals $\mathbf{n}(\mathbf{x}_0)$ and $\mathbf{n}_h(\mathbf{x}_0)$, respectively. Due to assumption (3.14) the angle between these normals is bounded by ch and thus there exists a planar extension T^e of T such that $T^e \subset U$ and $\mathbf{p}(T^e) = \mathbf{p}(w_{\mathbf{x}_0})$. This T^e has the property (3.21).

LEMMA 3.4. *Let v_h be a linear function on N_T and $u \in H^2(\Gamma)$. There exists a constant c independent of v_h, u and T such that the following inequality holds:*

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T^e)} \le ch^{-\frac{1}{2}}\|\nabla(u^e - v_h)\|_{L^2(S_T)} + h\|u\|_{H^2(\mathbf{p}(S_T))}. \tag{3.24}$$

Here ∇_{Γ_h} denotes the projection of the gradient on T^e.

Proof. Using lemma 3.1 and (3.9) we obtain

$$\begin{aligned}\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T^e)} &\le c\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T_b^e)} + ch^2\|\nabla^2_{\Gamma_h}u^e\|^2_{L^2(T^e)} \\ &\le c\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T_b^e)} + ch^2\|u\|^2_{H^2(\mathbf{p}(S_T))}. \end{aligned} \tag{3.25}$$

We consider the first term in (3.25). We write $\nabla v_h =: c_T$ and use the notation $\mathbf{x} = (\mathbf{s}(\mathbf{x}), \tilde{d}(\mathbf{x})) =: (\mathbf{s}, y)$ in the Φ-coordinate system. From (3.7) we have

$$\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = \nabla u^e(\mathbf{s}, y) + d(\mathbf{x})\mathbf{H}(\mathbf{x})\nabla_\Gamma u(\mathbf{p}(\mathbf{x})).$$

Using this and (3.8) we obtain

$$\begin{aligned}\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T_b^e)} &= \|\nabla_{\Gamma_h}u^e - \mathbf{P}_h c_T\|^2_{L^2(T_b^e)} \\ &\le 2\|\mathbf{P}_h(\nabla_\Gamma u)\circ\mathbf{p} - \mathbf{P}_h c_T\|^2_{L^2(T_b^e)} + 2\|d\mathbf{H}(\nabla_\Gamma u)\circ\mathbf{p}\|^2_{L^2(T_b^e)} \\ &\le c\|(\nabla_\Gamma u)\circ\mathbf{p} - c_T\|^2_{L^2(T_b^e)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\ &= c\int_{T_b^e}\|\nabla_\Gamma u(\mathbf{p}(\mathbf{s},0)) - c_T\|^2\,\mathrm{d}\mathbf{s} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\ &\le ch^{-1}\int_{d_0}^{d_1}\int_{T_b^e}\|\nabla_\Gamma u(\mathbf{p}(\mathbf{s},0)) - c_T\|^2\,\mathrm{d}\mathbf{s}\mathrm{d}y + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\ &\le ch^{-1}\int_{d_0}^{d_1}\int_{T_b^e}\|\nabla u^e(\mathbf{p}(\mathbf{s},y)) - c_T\|^2\,\mathrm{d}\mathbf{s}\mathrm{d}y + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\ &\le ch^{-1}\|\nabla(u^e - v_h)\|^2_{L^2(B_T)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\ &\le ch^{-1}\|\nabla(u^e - v_h)\|^2_{L^2(S_T)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))}.\end{aligned}$$

Combination of this result with the one in (3.25) completes the proof. ∎

LEMMA 3.5. *There are constants c_i independent of h such that for all $u \in H^2(\Gamma)$ and all $v_h \in V_h$ the following inequality holds:*

$$\|u^e - v_h\|_{L^2(\Gamma_h)} \le c_1 h^{-\frac{1}{2}}\|u^e - v_h\|_{L^2(\omega_h)} + c_2 h^{\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)} + c_3 h^2\|u\|_{H^2(\Gamma)}. \tag{3.26}$$

Proof. We consider an arbitrary element $T \in \Gamma_h$. Let T^e be its extension as defined above. Take $v_h \in V_h$. The extension of v_h to a linear function on T^e is denoted by v_h, too. Using lemma 3.1 we get:

$$\begin{aligned}\|u^e - v_h\|^2_{L^2(T)} \le \|u^e - v_h\|^2_{L^2(T^e)} &= \int_{T^e}(u^e(\mathbf{s},0) - v_h(\mathbf{s},0))^2\,\mathrm{d}\mathbf{s} \\ &\le c\int_{T_b^e}(u^e(\mathbf{s},0) - v_h(\mathbf{s},0))^2\,\mathrm{d}\mathbf{s} \\ &+ ch^2\int_{T^e}\|\nabla_{\Gamma_h}\big(u^e(\mathbf{s},0) - v_h(\mathbf{s},0)\big)\|^2\,\mathrm{d}\mathbf{s}.\end{aligned} \tag{3.27}$$

We consider the first term on the right handside of (3.27). For this we need an elementary result from calculus. Let f be a scalar linear function with $f(0) = 1$. An elementary computation yields that for $\delta < 1$ the inequalities

$$\frac{1}{1-\delta}\int_\delta^1 f(t)^2\,dt \ge \frac{1}{6}\big(f(\delta)^2 + f(1)^2\big) \ge \frac{1}{12}(1-\delta)^2$$

hold. Thus for an arbitrary scalar linear function g and $\delta < 1$ we have

$$g(0)^2 \leq 12(1-\delta)^{-3} \int_\delta^1 g(t)^2 \, dt.$$

Using a variable transformation we obtain that for δ_0, δ_1 with $0 \leq \delta_0 < \delta_1$ or $\delta_0 < \delta_1 \leq 0$ and for an arbitrary scalar linear function g we have

$$g(0)^2 \leq 12\Big(\frac{\max\{|\delta_0|, |\delta_1|\}}{\delta_1 - \delta_0}\Big)^2 \frac{1}{\delta_1 - \delta_0} \int_{\delta_0}^{\delta_1} g(t)^2 \, dt. \tag{3.28}$$

Without loss of generality we can assume that d_0, d_1 from (3.23) satisfy $d_0 < d_1 \leq 0$ or $0 \leq d_0 < d_1$. Furthermore, we have $\frac{|d_i|}{d_1 - d_0} \leq c$ for $i = 1, 2$, with c independent of h. Using this and the result in (3.28) applied to the linear function $y \to c + v_h(\mathbf{s}, y)$ we obtain

$$\begin{aligned}
&\int_{T_b^e} (u^e(\mathbf{s}, 0) - v_h(\mathbf{s}, 0))^2 \, \mathrm{d}\mathbf{s} \leq ch^{-1} \int_{T_b^e} \int_{d_0}^{d_1} (u^e(\mathbf{s}, 0) - v_h(\mathbf{s}, y))^2 \, \mathrm{d}y\mathrm{d}\mathbf{s} \\
&= ch^{-1} \int_{T_b^e} \int_{d_0}^{d_1} (u^e(\mathbf{s}, y) - v_h(\mathbf{s}, y))^2 \, \mathrm{d}y \, \mathrm{d}\mathbf{s} = ch^{-1} \|u^e - v_h\|^2_{L^2(B_T)} \\
&\leq ch^{-1} \|u^e - v_h\|^2_{L^2(S_T)}.
\end{aligned} \tag{3.29}$$

For the second term on right handside of (3.27) we can apply lemma 3.4 and thus we get

$$\|u^e - v_h\|^2_{L^2(T)} \leq ch^{-1}\|u^e - v_h\|^2_{L^2(S_T)} + ch\|\nabla(u^e - v_h)\|^2_{L^2(S_T)} + ch^4\|u\|^2_{H^2(\mathbf{p}(S_T))}.$$

Summation over all triangles in $T \in \mathcal{F}_h$ gives (3.26). ◻

Lemma 3.6. *There are constants c_1, c_2 independent of h such that for all $u \in H^2(\Gamma)$ and all $v_h \in V_h$ the following inequality holds:*

$$\|u^e - v_h\|_{H^1(\Gamma_h)} \leq c_1 h^{-\frac{1}{2}} \|u^e - v_h\|_{H^1(\omega_h)} + c_2 h\|u\|_{H^2(\Gamma)}. \tag{3.30}$$

Proof. Take $u \in H^2(\Gamma)$ and $v_h \in V_h$. By definition of the H^1-norm on Γ_h we get

$$\|u^e - v_h\|^2_{H^1(\Gamma_h)} = \|u^e - v_h\|^2_{L^2(\Gamma_h)} + \|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(\Gamma_h)}.$$

For the first term on the right handside we can apply lemma 3.5 and use

$$h^{-\frac{1}{2}}\|u^e - v_h\|_{L^2(\omega_h)} + c_2 h^{\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)} \leq ch^{-\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)}.$$

We now consider the second term

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(\Gamma_h)} = \sum_{T \in \mathcal{F}_h} \|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T)}.$$

Take a $T \in \mathcal{F}_h$ and extend v_h linearly outside T. This extension is denoted by v_h, too. Using lemma 3.4 we get

$$\begin{aligned}
\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T)} &\leq \|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T^e)} \\
&\leq ch^{-1}\|\nabla(u^e - v_h)\|^2_{L^2(S_T)} + h^2\|u\|^2_{H^2(\mathbf{p}(S_T))}.
\end{aligned}$$

Summation over $T \in \mathcal{F}_h$ yields

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(\Gamma_h)} \le c\, h^{-1}\|u^e - v_h\|^2_{H^1(\omega_h)} + ch^2\|u\|^2_{H^2(\Gamma)}$$

and thus the proof is completed. ▯

As a direct consequence of the previous two lemmas we obtain the following main theorem.

THEOREM 3.7. *For each $u \in H^2(\Gamma)$ the following holds*

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{L^2(\Gamma_h)} \le \|u^e - (I_h u^e)_{|\Gamma_h}\|_{L^2(\Gamma_h)} \le C\, h^2 \|u\|_{H^2(\Gamma)}, \tag{3.31}$$

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{H^1(\Gamma_h)} \le \|u^e - (I_h u^e)_{|\Gamma_h}\|_{H^1(\Gamma_h)} \le C\, h \|u\|_{H^2(\Gamma)}, \tag{3.32}$$

with a constant C independent of u and h.

Proof. Combine the results in the lemmas 3.5 and 3.6 with the result in lemma 3.3. ▯

3.3. Finite element error bounds. . In this section we prove optimal discretization error bounds both in the $H^1(\Gamma_h)$ and the $L^2(\Gamma_h)$ norm. The arguments are very close to those in [7]. A difference is that in [7] the convergence results are derived in the $H^1(\Gamma)$ and the $L^2(\Gamma)$ norms by lifting the discrete solutions from Γ_h on Γ, whereas we consider the error between the finite element solution $u_h \in V_h^\Gamma$ and the extension u^e of the continuous solution to the discrete interface.

In the analysis we need a few results from [5]. For $\mathbf{x} \in \Gamma_h$ define $\tilde{\mathbf{P}}_h(\mathbf{x}) = \mathbf{I} - \mathbf{n}_h(\mathbf{x})\mathbf{n}(\mathbf{x})^T/(\mathbf{n}_h(\mathbf{x})^T\mathbf{n}(\mathbf{x}))$. In (2.19) in [5] the following representation of the surface gradient of $u \in H^1(\Gamma)$ in terms of $\nabla_{\Gamma_h} u^e$ is given:

$$\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = \big(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x})\big)^{-1}\tilde{\mathbf{P}}_h(\mathbf{x})\nabla_{\Gamma_h} u^e(\mathbf{x}) \quad \text{a.e. on } \ \Gamma_h. \tag{3.33}$$

For $\mathbf{x} \in \Gamma_h$ define

$$\mu_h(\mathbf{x}) = (1 - d(\mathbf{x})\kappa_1(\mathbf{x}))(1 - d(\mathbf{x})\kappa_1(\mathbf{x}))\mathbf{n}(\mathbf{x})^T\mathbf{n}_h(\mathbf{x}).$$

The integral transformation formula

$$\mu_h(\mathbf{x})\mathrm{d}\mathbf{s}_h(\mathbf{x}) = \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in \Gamma_h, \tag{3.34}$$

holds, where $\mathrm{d}\mathbf{s}_h(\mathbf{x})$ and $\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))$ are the surface measures on Γ_h and Γ, respectively (cf. (2.20) in [5]). From

$$\|\mathbf{n}(\mathbf{x}) - \mathbf{n}_h(\mathbf{x})\|^2 = 2\big(1 - \mathbf{n}(\mathbf{x})^T\mathbf{n}_h(\mathbf{x})\big),$$

assumption (3.14) and $|d(\mathbf{x})| \le ch^2$, $|\kappa_i(\mathbf{x})| \le c$ we obtain

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h} |1 - \mu_h(\mathbf{x})| \le ch^2, \tag{3.35}$$

with a constant c independent of h.

THEOREM 3.8. *Let $u \in H^2(\Gamma)$ be the solution of (2.2) and $u_h \in V_h^\Gamma$ the solution of (2.7) with $f_h = f^e - c_f$, where c_f is such that $\int_{\Gamma_h} f_h \,\mathrm{d}\mathbf{s} = 0$. The following discretization error bound holds*

$$\|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)} \leq c\, h\, \|f\|_{L^2(\Gamma)} \tag{3.36}$$

with a constant c independent of f and h.

Proof. Using (3.35) we obtain $|1 - \frac{1}{\mu_h(\mathbf{x})}| \leq ch^2$ on Γ_h. Define

$$c_f := \int_{\Gamma_h} f^e \,\mathrm{d}\mathbf{s}_h, \quad \delta_f := (1-\mu_h) f^e - c_f.$$

Note that $f_h = f^e - c_f$ and due to $\int_\Gamma f \,\mathrm{d}\mathbf{s} = 0$ we get

$$|c_f| = \Big| \int_{\Gamma_h} f^e \,\mathrm{d}\mathbf{s}_h \Big| = \Big| \int_\Gamma f (\frac{1}{\mu_h} - 1) \,\mathrm{d}\mathbf{s} \Big| \leq ch^2 \|f\|_{L^2(\Gamma)}.$$

Furthermore,

$$\|\delta_f\|_{L^2(\Gamma_h)} \leq \operatorname{ess}\sup_{\mathbf{x}\in\Gamma_h} |1 - \mu_h(\mathbf{x})| \|f^e\|_{L^2(\Gamma_h)} + |\Gamma_h|^{\frac{1}{2}} |c_f| \leq ch^2 \|f\|_{L^2(\Gamma)}. \tag{3.37}$$

Using relation (3.33) and (3.34) we obtain

$$\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \,\mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v^e \,\mathrm{d}\mathbf{s}_h \quad \text{for all} \quad v \in H^1(\Gamma), \tag{3.38}$$

with $\mathbf{A}_h(\mathbf{x}) = \mu_h(\mathbf{x}) \tilde{\mathbf{P}}_h(\mathbf{x})(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))^{-2} \tilde{\mathbf{P}}_h(\mathbf{x})$. Any $\psi_h \in H^1(\Gamma_h)$ can be lifted on Γ by defining $\psi_h^l(\mathbf{p}(\mathbf{x})) := \psi_h(\mathbf{x})$. Then $\psi_h^l \in H^1(\Gamma)$ holds. From the definition of the discrete solution u_h in (2.7) we get, for arbitrary $\psi_h \in V_h^\Gamma$:

$$\begin{aligned}
\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h &= \int_{\Gamma_h} f_h \psi_h \,\mathrm{d}\mathbf{s}_h = \int_\Gamma (f - c_f) \mu_h(\mathbf{x})^{-1} \psi_h^l \,\mathrm{d}\mathbf{s} \\
&= \int_\Gamma f \psi_h^l \,\mathrm{d}\mathbf{s} + \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h \\
&= \int_\Gamma \nabla_\Gamma u \nabla_\Gamma \psi_h^l \,\mathrm{d}\mathbf{s} + \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h \\
&= \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h + \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h.
\end{aligned}$$

Using this we obtain, for arbitrary $\psi_h \in V_h^\Gamma$,

$$\begin{aligned}
\int_{\Gamma_h} \nabla_{\Gamma_h}(u^e - u_h) \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h &= \int_{\Gamma_h} (\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h \\
&= \int_{\Gamma_h} \mathbf{P}_h (\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h.
\end{aligned} \tag{3.39}$$

Therefore we get

$$\begin{aligned}
\|\nabla_{\Gamma_h}(u^e - u_h)\|^2_{L^2(\Gamma_h)} &= \int_{\Gamma_h} \nabla_{\Gamma_h}(u^e - u_h) \nabla_{\Gamma_h}(u^e - \psi_h) \,\mathrm{d}\mathbf{s}_h \\
&\quad + \int_{\Gamma_h} \mathbf{P}_h (\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h}(\psi_h - u_h) \,\mathrm{d}\mathbf{s}_h \\
&\quad - \int_{\Gamma_h} \delta_f (\psi_h - u_h) \,\mathrm{d}\mathbf{s}_h.
\end{aligned}$$

From $\|\tilde{\mathbf{P}}_h - \mathbf{A}_h\| \leq ch^2$ a.e. on Γ_h and $\mathbf{P}_h\tilde{\mathbf{P}}_h = \mathbf{P}_h$ we obtain, for $\mathbf{x} \in \Gamma_h$,

$$\|\mathbf{P}_h(\mathbf{x})(\mathbf{I} - \mathbf{A}_h(\mathbf{x}))\| = \|\mathbf{P}_h(\mathbf{x})(\tilde{\mathbf{P}}_h(\mathbf{x}) - \mathbf{A}_h(\mathbf{x}))\| \leq ch^2. \tag{3.40}$$

Furthermore, using (3.8) we get

$$\|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)} \leq \text{ess} \sup_{\mathbf{x}\in\Gamma_h} \|\mathbf{P}_h(\mathbf{x})(\mathbf{I} - d\mathbf{H}(\mathbf{x}))\| \|\nabla_\Gamma u\|_{L^2(\Gamma)} \leq c\|f\|_{L^2(\Gamma)}. \tag{3.41}$$

Introduce the notation $E_h := \|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)}$. The Cauchy inequality and $\|\nabla_{\Gamma_h}(u_h - \psi_h)\|_{L^2(\Gamma_h)} \leq E_h + \|\nabla_{\Gamma_h}(u^e - \psi_h)\|_{L^2(\Gamma_h)}$, in combination with the approximation result (3.32) leads to

$$\begin{aligned} E_h^2 &\leq E_h ch\|f\|_{L^2(\Gamma)} + ch^2\|f\|_{L^2(\Gamma)}\big(E_h + ch\|f\|_{L^2(\Gamma)}\big) \\ &\leq \frac{1}{2}E_h^2 + ch^2\|f\|^2_{L^2(\Gamma)}. \end{aligned}$$

This yields the bound in (3.36). ∎

We now apply a duality argument to obtain an $L^2(\Gamma_h)$-error bound.

Theorem 3.9. *Let u and u_h be as in theorem 3.8. The following error bound holds*

$$\|u^e - u_h\|_{L^2(\Gamma_h)} \leq c\, h^2\, \|f\|_{L^2(\Gamma)} \tag{3.42}$$

with a constant c independent of f and h.

Proof. Denote $e_h := (u^e - u_h)|_{\Gamma_h}$ and let e_h^l be the lift of e_h on Γ and $c_e := \int_\Gamma e_h^l \,\mathrm{d}\mathbf{s}$. Consider the problem: Find $w \in H^1(\Gamma)$ with $\int_\Gamma w \,\mathrm{d}\mathbf{s} = 0$ such that

$$\int_\Gamma \nabla_\Gamma w \nabla_\Gamma v \,\mathrm{d}\sigma = \int_\Gamma (e_h^l - c_e) v \,\mathrm{d}\mathbf{s} \qquad \text{for all} \quad v \in H^1(\Gamma). \tag{3.43}$$

The solution w satisfies $w \in H^2(\Gamma)$ and $\|w\|_{H^2(\Gamma)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$ with $\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}} := \|e_h^l - c_e\|_{L^2(\Gamma)}$. Furthermore, $\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$ and $\|w^e\|_{L^2(\Gamma_h)} \leq c\|w\|_{L^2(\Gamma)} \leq c\|\nabla_\Gamma w\|_{L^2(\Gamma)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$. Due to (3.43) and (3.39) we have, for any $\psi_h \in V_h^\Gamma$,

$$\begin{aligned} \|e_h^l\|^2_{L^2(\Gamma)/\mathbb{R}} &= \int_\Gamma \nabla_\Gamma w \nabla_\Gamma (e_h^l - c_e) \,\mathrm{d}\mathbf{s} = \int_\Gamma \nabla_\Gamma w \nabla_\Gamma e_h^l \,\mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} e_h \nabla_{\Gamma_h} w^e \,\mathrm{d}\mathbf{s}_h \\ &= \int_{\Gamma_h} \nabla_{\Gamma_h} e_h \nabla_{\Gamma_h}(w^e - \psi_h) \,\mathrm{d}\mathbf{s}_h + \int_{\Gamma_h} \mathbf{P}_h(\mathbf{A}_h - \mathbf{I}) \nabla_{\Gamma_h} e_h \nabla_{\Gamma_h} w^e \,\mathrm{d}\mathbf{s}_h \\ &\quad + \int_{\Gamma_h} \mathbf{P}_h(\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \,\mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h. \end{aligned}$$

Introduce $E_h := \|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$. Thanks to the approximation property (3.32) one can choose ψ_h such that $\|\nabla_{\Gamma_h}(w^e - \psi_h)\|_{L^2(\Gamma_h)} \leq ch\|w\|_{H^2(\Gamma)} \leq chE_h$. Using Cauchy-Schwarz and triangle inequalities and the bounds in (3.37), (3.40) we get

$$\begin{aligned} E_h^2 &\leq \|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)} chE_h + ch^2\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} \\ &\quad + ch^2\|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)}\big(\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} + chE_h\big) + ch^2\|f\|_{L^2(\Gamma)}\big(\|w^e\|_{L^2(\Gamma_h)} + chE_h\big) \\ &\leq ch^2\|f\|_{L^2(\Gamma)}E_h + ch^2\|f\|_{L^2(\Gamma)}\big(E_h + chE_h\big). \end{aligned}$$

Hence, $E_h \le ch^2\|f\|_{L^2(\Gamma)}$ holds. We have

$$|c_e| = \Big|\int_\Gamma u - u_h^e \,\mathrm{d}\mathbf{s}\Big| = \Big|\int_\Gamma u_h^e \,\mathrm{d}\mathbf{s}\Big| = \Big|\int_{\Gamma_h} (\mu_h - 1)u_h^e \,\mathrm{d}\mathbf{s}_h\Big| \le ch^2\|f\|_{L^2(\Gamma)},$$

and thus

$$\|e_h\|_{L^2(\Gamma_h)} \le c\|\mu_h^{-\frac{1}{2}} e_h\|^2_{L^2(\Gamma_h)} = c\|e_h^l\|_{L^2(\Gamma)} \le c(E_h + |c_e|) \le ch^2\|f\|_{L^2(\Gamma)},$$

which completes the proof. ▯

4. Application to two-phase incompressible flows with surfactants. We consider a standard Navier-Stokes model for incompressible two-phase flows in which a localized force at the interface describes the effect of surface tension. Surfactant concentration *at the interface* is modeled by a scalar convection-diffusion equation at the interface. In section 4.1 we desribe a model for this two-phase flow problem and in section 4.2 a few discretization issues are discussed.

4.1. A model for two-phase incompressible flow with surfactants. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain containing two different immiscible incompressible phases. The time dependent subdomains containing the two phases are denoted by $\Omega_1(t)$ and $\Omega_2(t)$ with $\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. We assume that Ω_1 and Ω_2 are connected and $\partial\Omega_1 \cap \partial\Omega = \emptyset$ (i. e., Ω_1 is completely contained in Ω). The interface is denoted by $\Gamma(t) = \bar\Omega_1(t) \cap \bar\Omega_2(t)$. A typical example is a rising air bubble or liquid droplet in a surrounding fluid. The standard model for describing incompressible two-phase flows consists of the Navier-Stokes equations in the subdomains with the coupling condition

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = \tau\mathcal{K}\mathbf{n} \tag{4.1}$$

at the interface, i. e., the surface tension balances the jump of the normal stress at the interface. The surface tension coefficient τ is assumed to be constant. We use the notation $[v]_\Gamma$ for the jump of v across Γ, $\mathbf{n} = \mathbf{n}_\Gamma$ is the unit normal at the interface Γ (pointing from Ω_1 into Ω_2), $\mathcal{K}$ the curvature of Γ and $\boldsymbol{\sigma}$ the stress tensor defined by $\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{D}(\mathbf{u})$ with $\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T$. Furthermore $p = p(\mathbf{x}, t)$ denotes the pressure, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ the velocity and μ the viscosity. We assume continuity of $\mathbf{u}$ across the interface. Based on the conservation laws for mass and momentum the fluid dynamics is modeled by the Navier-Stokes equations in the two subdomains combined with $[\mathbf{u}]_\Gamma = 0$ and the coupling condition in (4.1), cf. for example [17, 18, 24, 23]. A level set method can be used for capturing the unknown interface, cf. [16, 20, 21]. The level set function, denoted by $\phi = \phi(\mathbf{x}, t)$ is a scalar function with $\phi(\mathbf{x}, 0) < 0$ for $\mathbf{x} \in \Omega_1(0)$, $\phi(\mathbf{x}, 0) > 0$ for $\mathbf{x} \in \Omega_2(0)$, $\phi(\mathbf{x}, 0) = 0$ for $\mathbf{x} \in \Gamma(0)$. It is desirable to have the level set function at $t = 0$ as an approximate signed distance function.

The evolution of the interface is given by the linear hyperbolic partial differential equation $\phi_t + \mathbf{u} \cdot \nabla\phi = 0$ for $t \ge 0$ and $\mathbf{x} \in \Omega$.

The jumps in the coefficients ρ and μ can be described using the level set function (which has its zero level set precisely at the interface Γ) in combination with the Heaviside function H. We define

$$\begin{aligned} \rho(\phi) &:= \rho_1 + (\rho_2 - \rho_1)H(\phi), \\ \mu(\phi) &:= \mu_1 + (\mu_2 - \mu_1)H(\phi). \end{aligned} \tag{4.2}$$

The effect of the surface tension can be expressed in terms of a localized force at the interface, cf. the so-called continuum surface force (CSF) model [2, 16]. Combination of the CSF approach with the level set method leads to the following model for the two-phase problem in $\Omega \times [0,T]$

$$\rho(\phi)\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) = -\nabla p + \rho(\phi)\mathbf{g} + \operatorname{div}(\mu(\phi)\mathbf{D}(\mathbf{u})) + \tau\mathcal{K}\delta_\Gamma \mathbf{n}_\Gamma \tag{4.3}$$

$$\operatorname{div}\mathbf{u} = 0 \tag{4.4}$$

$$\phi_t + \mathbf{u}\cdot\nabla\phi = 0 \tag{4.5}$$

together with suitable initial and boundary conditions for $\mathbf{u}$ and ϕ. This is the continuous problem that we use to model our two-phase flow problem. It is also used in, for example, [16, 12, 14, 16, 24, 23, 26].

We treat this problem using finite element techniques. Thus we need an appropriate weak formulation. We do not discuss this subject here, but refer to the literature, for example, [8, 25, 26, 27, 28]. We only briefly address the weak formulation of the localized surface tension force. The surface tension term in (4.3) results in the functional

$$f_\Gamma(\mathbf{v}) := \tau \int_\Gamma \mathcal{K}\mathbf{n}_\Gamma \cdot \mathbf{v}\,\mathrm{d}\mathbf{s}, \quad \mathbf{v} \in \mathbf{V} := H_0^1(\Omega)^3. \tag{4.6}$$

For Γ sufficiently smooth we have $\sup_{\mathbf{x}\in\Gamma} |\mathcal{K}(\mathbf{x})| \le c < \infty$ and

$$|f_\Gamma(\mathbf{v})| \le c\,\tau \int_\Gamma |\mathbf{n}_\Gamma \cdot \mathbf{v}|\,\mathrm{d}\mathbf{s} \le c\,\|\mathbf{v}\|_{L^2(\Gamma)} \le c\|\mathbf{v}\|_{H_0^1(\Omega)} \quad \text{for all } \ \mathbf{v} \in \mathbf{V}. \tag{4.7}$$

Thus $f_\Gamma \in \mathbf{V}'$ holds.

In many two-phase flow systems surface active agents (surfactants) are present as impurities or added to the bulk fluid. To describe the effect of such surfactants a convection-diffusion equation *at* the interface is added to the fluid dynamics model (4.3)-(4.5). Let the velocity field $\mathbf{u}$ be decomposed in a tangential and normal component: $\mathbf{u} = \mathbf{u}_\Gamma + u_\perp \mathbf{n}$. Let $D_\Gamma > 0$ be a given diffusion coefficient of Γ. The following type of transport equation for the surfactant concentration $c = c_\Gamma$ can be found in the literature, cf. [1, 13]:

$$\partial_{t,n} c - D_\Gamma \Delta_\Gamma c + \nabla_\Gamma \cdot (c\mathbf{u}_\Gamma) - \mathcal{K} u_\perp c = 0, \tag{4.8}$$

where $\partial_{t,n} c$ denotes the derivative of c along a purely normal path. In case of a soluble surfactant a source term is added that describes the process of ad- and desorption of the surfactant. The flow field $\mathbf{u}$ results from the fluid dynamics model (4.3)-(4.5). In most two-phase flow models that take transport of surfactants into account there also is a dependence of the flow equations (4.3)-(4.5) on the concentration c, namely a dependence of the surface tension coefficient τ on c, i.e., $\tau = \tau(c)$, cf. [1, 13].

4.2. Discretization of the two-phase flow problem. We outline the main ideas of the discretization methods used for the two-phase flow problem (4.3)-(4.5) (in weak formulation). For further information we refer to the literature [9, 11, 10]. The finite element spaces are based on a hierarchy of multilevel tetrahedral meshes $\{\mathcal{T}\}_{h>0}$ that is constructed using red/green refinement strategies. These triangulations are regular, consistent and stable. In our applications the triangulations are locally refined close to the interface. For discretization of the Navier-Stokes equations we use either the Hood-Taylor P_2-P_1 pair, or a pair consisting of continuous piecewise quadratics for velocity (i.e., P_2) and an *extended* linear finite element space for the pressure. The

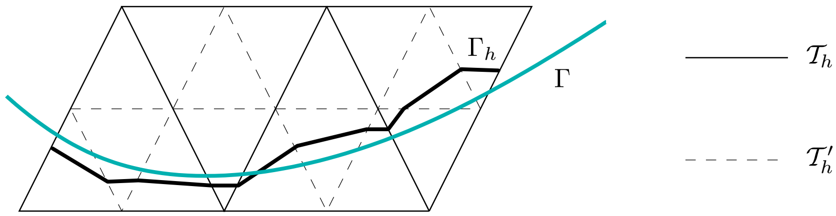

FIG. 4.1. *Construction of approximate interface for 2D case.*

latter space is much better suited for the approximation of the discontinuous pressure function than the standard P_1 finite element space, cf. [10, 19]. The linear hyperbolic level set equation is approximated using *quadratic* finite element combined with a standard streamline diffusion stablilization. For time discretization we use a variant of the θ-scheme.

The polyhedral approximation Γ_h of Γ is constructed as follows.

The level set equation for ϕ (signed distance function) is discretized with continuous piecewise quadratic finite elements on the tetrahedral triangulation $\mathcal{T}_h$. The piecewise *quadratic* finite element approximation of ϕ on $\mathcal{T}_h$ is denoted by ϕ_h. We now introduce one further regular refinement of $\mathcal{T}_h$, resulting in $\mathcal{T}_h' = \mathcal{T}_{\frac{h}{2}}$. Let $I(\phi_h)$ be the continuous piecewise *linear* function on $\mathcal{T}_h'$ which interpolates ϕ_h at all vertices of all tetrahedra in $\mathcal{T}_h'$. The approximation of the interface Γ is defined by

$$\Gamma_h := \{\, \mathbf{x} \in \Omega \mid I(\phi_h)(\mathbf{x}) = 0 \,\}. \tag{4.9}$$

and consists of piecewise planar segments. The mesh size parameter h is the maximal diameter of these segments. This maximal diameter is approximately the maximal diameter of the tetrahedra in $\mathcal{T}_h'$ that contain the discrete interface, i.e., $h = h_\Gamma$ is approximately the maximal diameter of the tetrahedra in $\mathcal{T}_h'$ that are close to the interface. In Figure 4.1 we illustrate this construction for the two-dimensional case.

Each of the planar segments of Γ_h is either a triangle or a quadrilateral. Note that *this construction of* Γ_h *satisfies the assumptions made in section* 2.

If we assume $|I(\phi_h)(\mathbf{x}) - \phi(\mathbf{x})| \leq c\, h_\Gamma^2$ for all $\mathbf{x}$ in a neighbourhood of Γ, which is reasonable for a smooth ϕ and piecewise quadratic ϕ_h, then we have

$$\max_{\mathbf{x}\in\Gamma_h} |\phi(\mathbf{x}) - \phi_h(\mathbf{x})| = \max_{\mathbf{x}\in\Gamma_h} |\phi(\mathbf{x})| = \max_{\mathbf{x}\in\Gamma_h} |\phi(\mathbf{x}) - I(\phi_h)(\mathbf{x})| \leq c\, h_\Gamma^2, \tag{4.10}$$

cf. assumption (3.13).

The approximation of the localized surface tension force in (4.6) is based on the following Laplace-Beltrami characterization of the curvature. Let $\mathrm{id}_\Gamma : \Gamma \to \mathbb{R}^3$ be the identity on Γ and $\mathcal{K} = \kappa_1 + \kappa_2$ the sum of the principal curvatures. For all sufficiently smooth vector functions $\mathbf{v}$ on Γ the following holds, cf. (4.6):

$$f_\Gamma(\mathbf{v}) = \int_\Gamma \mathcal{K}\, \mathbf{n}_\Gamma \cdot \mathbf{v}\, \mathrm{d}\mathbf{s} = -\int_\Gamma (\Delta_\Gamma\, \mathrm{id}_\Gamma) \cdot \mathbf{v}\, \mathrm{d}\mathbf{s} = \int_\Gamma \nabla_\Gamma\, \mathrm{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v}\, \mathrm{d}\mathbf{s}. \tag{4.11}$$

In [11] we introduced and analyzed the following discretization method for the surface tension force. Define

$$\tilde{\mathbf{n}}_h(\mathbf{x}) := \frac{\nabla\phi_h(\mathbf{x})}{\|\nabla\phi_h(\mathbf{x})\|}, \qquad \tilde{\mathbf{P}}_h(\mathbf{x}) := \mathbf{I} - \tilde{\mathbf{n}}_h(\mathbf{x})\tilde{\mathbf{n}}_h(\mathbf{x})^T, \quad \mathbf{x} \in \Gamma_h,\ \mathbf{x} \text{ not on an edge.}$$

The discrete surface tension force is given by

$$f_{\Gamma_h}(\mathbf{v}_h) = \tau \sum_{i=1}^{3} \int_{\Gamma_h} \tilde{\mathbf{P}}_h(\mathbf{x})\mathbf{e}_i \cdot \nabla_{\Gamma_h}(\mathbf{v}_h)_i \, \mathrm{d}\mathbf{s}, \tag{4.12}$$

with $\mathbf{e}_i$ the i-th basis vector in $\mathbb{R}^3$ and $(\mathbf{v}_h)_i$ the i-th component of $\mathbf{v}_h$.

REMARK 7. The implementation of this functional requires the numerical integration over the triangulated surface $\Gamma_h = \cup_{T\in\mathcal{F}_h} T$ of functions that are smooth on the planar segments T this triangulation. *If methods for this are implemented then these can also be used for the realization of the finite element discretization method introduced in section* 2, cf. remark 2.

We now turn to the discretization of a transport equation as in (4.8) on a moving sufficiently smooth surface $\Gamma = \Gamma(t)$. The flow problem (4.3)-(4.5) coupled with the transport equation (4.8) is typically handled using an implicit time integration method and an iterative decoupling procedure to solve, in each time step, the coupled system. In such an approach for a fixed $t = t_n$ and *given* finite element velocity field $\mathbf{u}_h(\mathbf{x})$, pressure finite element function $p_h(\mathbf{x})$ and level set finite element function $\phi_h(\mathbf{x})$ (which induces an approximate interface $\Gamma_h(t_n)$) one has to solve a stationary convection-diffusion equation of the form

$$-D_\Gamma \Delta_\Gamma c + \nabla_\Gamma \cdot (c\mathbf{u}_\Gamma) - Ec = 0$$

for the surfactant concentration c at the approximate interface $\Gamma_h(t_n)$. For this the method described in section 2 can be used. For the outer finite element space we then use the continuous piecewise linears on the triangulation $\mathcal{T}_{\frac{h}{2}}$. The data structure for this space is already available in the solver for the two-phase flow problem (4.3)-(4.5). Furthermore, for the treatment of surface tension, routines for numerical integration over Γ_h are available that can be used for the finite element discretization at the interface as well, cf. remark 7. Thus *in this setting we need only very little additional implementation effort to extend the solver for the two-phase flow problem* (4.3)-(4.5) *such that it can deal with a surfactant convection-diffusion equation at the interface.*

5. Numerical experiments. In this section we present results of a numerical experiment. As a test problem we consider the Laplace-Beltrami equation

$$-\Delta_\Gamma u + u = f \quad \text{on } \Gamma,$$

with $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|_2 = 1\}$ and $\Omega = (-2, 2)^3 + \mathbf{b}$ with $\mathbf{b} = (29^{-1}, 31^{-1}, 37^{-1})^T$. This example is taken from [4]. The shift over $\mathbf{b}$ is introduced for the following technical reason. The grids we use are obtained by regular (local) refinement as explained below. For the case $\mathbf{b} = 0$ there are grid points of the outer triangulation that lie *exactly* on Γ. It turns out that for this case it can happen that the iterative method (in our case CG) for solving the discrete problem does not converge.

The zero order term is added to guarantee a unique solution. The source term f is taken such that the solution is given by

$$u(\mathbf{x}) = a\frac{\|\mathbf{x}\|^2}{12 + \|\mathbf{x}\|^2}\left(3x_1^2x_2 - x_2^3\right), \quad \mathbf{x} = (x_1, x_2, x_3) \in \Omega,$$

with $a = -\frac{13}{8}\sqrt{\frac{35}{\pi}}$. A family $\{\mathcal{T}_l\}_{l\geq 0}$ of tetrahedral triangulations of Ω is constructed as follows. We triangulate Ω by starting with a uniform subdivision into 48 tetrahedra

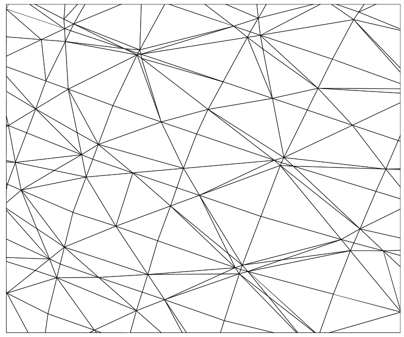

FIG. 5.1. *Detail of the induced triangulation of* Γ_h.

with mesh size $h_0 = \sqrt{3}$. Then we apply an adaptive red-green refinement-algorithm (implemented in the software package DROPS [6]) in which in each refinement step the tetrahedra that contain Γ are refined such that on level $l = 1, 2, \dots$ we have

$$h_T \leq \sqrt{3}\, 2^{-l} \quad \text{for all } T \in \mathcal{T}_l \quad \text{with} \quad T \cap \Gamma \neq \emptyset.$$

The family $\{\mathcal{T}_l\}_{l \geq 0}$ is consistent and shape-regular. The interface Γ is the zero-level of $\varphi(\mathbf{x}) := \|\mathbf{x}\|^2 - 1$. Let $\varphi_h := I(\varphi)$ where I is the standard nodal interpolation operator on $\mathcal{T}_l$. The discrete interface is given by $\Gamma_{h_l} := \{\, \mathbf{x} \in \Omega \mid I(\phi_h)(\mathbf{x}) = 0 \,\}$, cf. (4.9). Let $\{\phi_i\}_{1 \leq i \leq m}$ be the nodal basis functions corresponding to the vertices of the tetrahedra in ω_h, as explained in remark 1. The entries $\int_{\Gamma_h} \nabla_{\Gamma_h} \phi_i \cdot \nabla_{\Gamma_h} \phi_j + \phi_i \phi_j \mathrm{d}\mathbf{s}$ of the stiffness matrix are computed within machine accuracy. For the right-handside we use a quadrature-rule that is exact up to order five. The discrete problem is solved using a standard CG method with symmetric Gauss-Seidel preconditioner to a relative tolerance of 10^{-6}. The number of iterations needed on level $l = 1, 2, \dots, 7$, is 14, 26, 53, 104, 201, 435, 849, respectively.
The discretization errors in the $L^2(\Gamma_h)$-norm are given in table 5.1.

level l	$\|u - u_h\|_{L^2(\Gamma_h)}$	factor
1	0.1124	–
2	0.03244	3.47
3	0.008843	3.67
4	0.002186	4.05
5	0.0005483	3.99
6	0.0001365	4.02
7	3.411e-05	4.00

TABLE 5.1
Discretization errors and error reduction.

These results clearly show the h^2 behaviour as predicted by our theoretical analysis. To illustrate the fact that in this approach the triangulation of the approximate manifold Γ_h is strongly shape-irregular we show a part of this triangulation in Figure 5.1. The discrete solution is visualized in Fig. 5.2.

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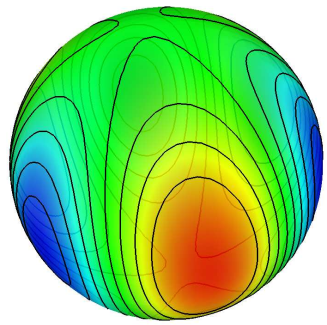

Fig. 5.2. *Level lines of the discrete solution* u_h

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