An hp–Error Estimate for an Unfitted Discontinuous Galerkin Method Applied to Elliptic Interface Problems

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Abstract

In this article an unfitted Discontinuous Galerkin Method is proposed to discretize elliptic interface problems. The method is based on the Symmetric Interior Penalty Discontinuous Galerkin Method and can also be interpreted as a generalization of the method given in [A. Hansbo, P. Hansbo, An unfitted finite element method based on Nitsche's method for elliptic interface problems, Comp. Meth. Appl. Mech. Eng., Vol. 191, (2002), 5537–5552]. We prove the optimal h-convergence of the method for arbitrary p in energy- and in L_2-norm. In fact we present an *hp*-error estimate. The analysis includes grids with hanging nodes and the proposed DG Method is symmetric and inherits the attractive locality property of general Discontinuous Galerkin methods. A variant of the method is proposed which additionally behaves well with respect to the pointwise error in the gradient. The behaviour of the methods in numerical experiments is displayed.

1 Introduction

This article addresses the use of unfitted grids in conjunction with Discontinuous Galerkin (DG) discretizations to obtain hp-convergent methods for the elliptic interface problems given in Problem 1.1.

Problem 1.1 (Elliptic Interface Problem) *Let a polygonal domain $\Omega \subset \mathbb{R}^2$ be given which is decomposed into $\Omega = \Omega^+ \cup \Omega^-$, such that Ω^+ and Ω^- have a Lipschitz-boundary, and an interface $\Gamma \equiv \overline{\Omega^+} \cap \overline{\Omega^-}$. Furthermore, let a positive diffusion coefficient $\kappa \in L_\infty(\Omega)$ be given, with $\kappa^+ \equiv \kappa|_{\Omega^+} \in C(\overline{\Omega^+})$ and $\kappa^- \equiv \kappa|_{\Omega^-} \in C(\overline{\Omega^-})$, as well as $f \in L_2(\Omega)$, $g_D \in H^{1/2}(\partial\Omega)$, $j_D \in H^{1/2}(\Gamma)$, $j_N \in L_2(\Gamma)$. Find $(u^+, u^-) \in H^1(\Omega^+) \times H^1(\Omega^-)$ such that*

$$-div\,(\kappa^+ \nabla u^+) = f \quad \text{in } \Omega^+, \tag{1}$$

$$-div\,(\kappa^- \nabla u^-) = f \quad \text{in } \Omega^-, \tag{2}$$

$$u^+ = g_D \quad \text{on } \partial\Omega \cap \partial\Omega^+, \tag{3}$$

$$u^- = g_D \quad \text{on } \partial\Omega \cap \partial\Omega^-, \tag{4}$$

$$u^+ - u^- = j_D \quad \text{on } \Gamma, \tag{5}$$

$$\kappa^+ \partial_{n^+} u^+ - \kappa^- \partial_{n^+} u^+ = j_N \quad \text{on } \Gamma. \tag{6}$$

Here $\partial_n \cdot$ denotes the directional derivative in direction $n \in \mathbb{R}^2$ and n^+ denotes the outward normal of Ω^+ on Γ.

Problem 1.1 occurs in various applications: The case $j_D = j_N = 0$ models the stationary thermal diffusion with a discontinuous conductivity κ across the (material) interface Γ. In Hele-Shaw flow u is the pressure and j_D is the pressure jump across Γ due to surface tension and $j_N = 0$, see [12]. Furthermore, to approach Problem 1.1 with an unfitted grid method is fundamental to the numerical simulation of multiphase problems. Unfitted grid methods are usually preferred to simulate time dependent multiphase problems with moving phase fronts, and are often combined with the popular level set method [17]. The major reason for choosing an unfitted grid method is that it avoids the expensive remeshing required for fitted grid methods to maintain a good mesh quality. Nethertheless, tackling multiphase problems with an unfitted grid approach requires special techniques to discretize the multiphase balance laws at the phase front, where discontinuities of material properties as well as jump conditions have to be encorporated. In this situtation it is by far more difficult to construct optimally convergent discretizations of the multiphase balance laws than it is for fitted grid methods.

Various unfitted grid methods for Problem 1.1 have been proposed in the literature, falling mainly into the class of Finite Element Methods or Finite Difference Methods. Typically a reference method is adopted to handle the situation at the interface. Here the ultimate goal is to obtain practical methods which do not loose the stability and convergence order of the reference method due to the presence of the interface.

Most unfitted Finite Element Methods to solve Problem 1.1 utilize some form of penalization to impose the jump conditions (5) and (6). A first step in this direction for the case $j_D = j_N = 0$ resulted in a method with suboptimal h-convergence for arbitrary polynomial degree p of the shape functions [3]. This method was again analyzed in [5] and shown to be optimally h-convergent for $p = 1$. The latter work demanded more regularity from the exact solution, but took the approximation of Γ, which is unavoidable in practize, into account. Employing a different penalization, in [11] optimal h-convergence for $p = 1$ was proven in energy norm and in L_2-norm for the case $j_D = 0$. The important thing is that the error estimates obtained are uniform with respect to the relative position of the interface to the grid. Typically this requires the interface to be smooth and that its geometrical features are well resolved by the grid, which is similarly required if a fitted grid method is analyzed, as is the case in [8]. The Finite Element Method given in [10] does not employ a penalization but constructs extensions of the data j_D and g_D.

Finite Difference Methods are very popular unfitted grid methods, as they can rely on the use of cartesian grids, which are very attractive due to their simplicity. One such method is the Immersed Interface Method [14], which is a nonsymmetric method and shows experimentally second order convergence, employing near the interface a modification of the standard 5-point-stencil. The immersed interface method, initially designed for Problem 1.1 has been developed further to treat time dependent mulitphase problems, see [15] and the references therein. A different modification of the standard centered difference is employed in [16], which results in a symmetric method, and shows experimentally first order convergence. It is associated with the Ghost Fluid Method [9], and has been used as a building block for the numerical simulation of multiphase flow problems, see [13]. Various other Finite Difference Methods have been proposed, see for example [7] and the references therein.

To the author's knowledge only h-convergence has been analyzed in the literature concerning unfitted grid methods for Problem 1.1, where all theoretical results achieving optimal h-convergence for multi dimensional problems restrict to the case $p = 1$, i.e. piecewise linear finite element shape functions or second order finite differences are employed.

In the present article, we propose a DG Method based on the Symmetric Interior Penalty Discontinuous Galerkin Method (SIPG) of [2] with a Nitsche-type penalization at the interface similar to [11]. We prove the optimal h-convergence of the method for arbitrary p in energy- and in L_2-norm. In fact we present an hp-error estimate. With respect to p the estimate is slightly worse than the best estimates known for DG Methods for elliptic problems without interface. The analysis includes grids with hanging nodes and the proposed DG Method is symmetric and inherits the attractive locality property of general DG methods. In particular the proposed method employs only changes of the SIPG for grid cells which are cut by the interface and faces belonging to these cells. Actually we introduce two methods. The first method converges in energy- and L_2-norm, but numerical experiments reveal that the pointwise errors in the gradient are not controlled very well. A second method is derived from the first by adding another penalty near the interface in order to control pointwise errors in the gradient. This can be very important when Problem 1.1 is coupled to a transport equation, as is the case in Hele-Shaw flow, where the transport velocity at the interface is determined from the normal derivative of u, compare [12].

The key issues of our analysis can be described as follows: Our method is a nonconforming Galerkin discretization and in particular it is nonconforming at the interface, similar to the method of [11]. When deriving error estimates for nonconforming methods one often utilizes inverse estimates. Employing an unfitted grid approach, the interface divides regular grid cells into possibly degenerated subcells. If degenerated subcells occur, standard inverse estimates cannot be applied with uniform constants. The way out of this dilemma is to weight terms in the Galerkin discretization, such that the occurrence of degenerated subcells can be compensated for. To find these weights and inverse estimates and to use them in an *hp*-error analysis are the major steps in the present article. The overall framework of the analysis is similar to [18], where the Local DG Method is applied for an elliptic problem without interface.

The article is organized as follows: In Section 2 we introduce our DG Method, pointing out its relation to the SIPG and the method in [11]. In Section 3.8 we prove the error estimates, and propose a variant of our DG Method, introducing an additional penalty near the interface. In Section 4 we display the behaviour of both methods in numerical experiments. In the Appendices A, B, C we recapitulate some notations concerning norms, some basic facts about shape-regularity of triangles and some results from the literature which are needed in the present work.

2 The Discontinuous Galerkin discretization

Before we propose our method for Problem 1.1 we review the SIPG and introduce the DG notation needed. In particular we consider the SIPG for Problem 2.1 in order to see how Dirichlet and Neumann boundary conditions are imposed.

2.1 The Symmetric Interior Penalty DG Method (SIPG)

Problem 2.1 (Mixed Problem) *Given a polygonal domain* $\Omega \subset \mathbb{R}^2$, *the boundary consisting of two disjoint parts, namely* $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, *and given further* $\kappa \in C(\overline{\Omega})$, $f \in L_2(\Omega)$ *and* $g_D \in H^{1/2}(\partial\Omega_D)$, $g_N \in L_2(\partial\Omega_N)$, *find* $u \in H^1(\Omega)$ *satisfying*

$$\begin{cases} -div\,(\kappa\nabla u) &=& f & \text{in } \Omega, \\ u &=& g_D & \text{on } \partial\Omega_D, \\ \kappa\,\partial_n u &=& g_N & \text{on } \partial\Omega_N. \end{cases} \tag{7}$$

We formulate the SIPG for Problem 2.1: Given a triangulation $\mathcal{T}_h$ of the domain Ω, and for each $T \in \mathcal{T}_h$ a polynomial degree $p_T \in \{1, 2, 3, ...\}$, the DG space is

$$\mathcal{S}^p(\mathcal{T}_h) \;\equiv\; \{v \in L_2(\Omega) : v|_T \in P^{p_T}(T) \;\forall\, T \in \mathcal{T}_h\}\,,$$

where the p in $\mathcal{S}^p$ stands for the array of polynomial degrees p_T. For the faces of the triangulation $\mathcal{T}_h$ we introduce the sets

$$\begin{aligned} \mathcal{E}(\mathcal{T}_h) &\equiv \{e : e \text{ is a face of a triangle } T \in \mathcal{T}_h, \text{ and either } e \subset \partial\Omega \text{ or} \\ &\qquad \text{there exists another triangle } \tilde{T} \in \mathcal{T}_h,\ \tilde{T} \neq T, \text{ such that } e = T \cap \tilde{T}\}, \\ \mathcal{E}_I(\mathcal{T}_h) &\equiv \{e \in \mathcal{E}(\mathcal{T}_h) : e \not\subset \partial\Omega\}, \\ \mathcal{E}_D(\mathcal{T}_h) &\equiv \{e \in \mathcal{E}(\mathcal{T}_h) : e \subset \partial\Omega_D\}, \\ \mathcal{E}_N(\mathcal{T}_h) &\equiv \{e \in \mathcal{E}(\mathcal{T}_h) : e \subset \partial\Omega_N\}. \end{aligned}$$

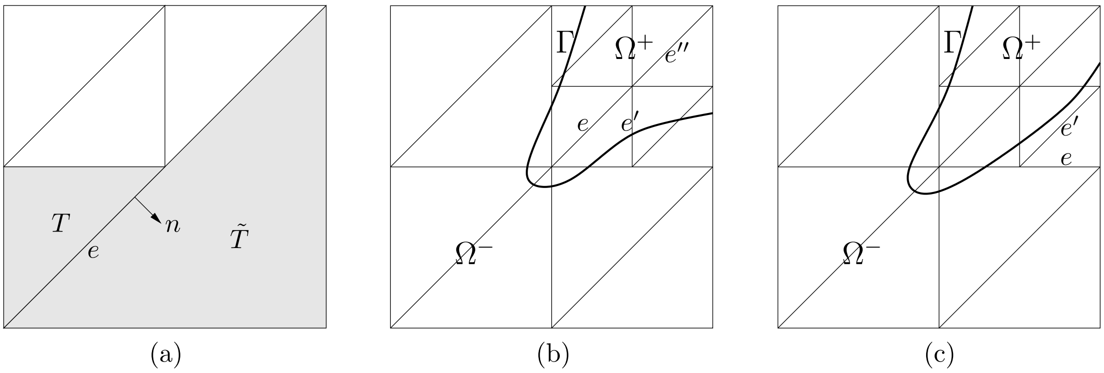


Figure 1: (a): $\omega(e) = \{T, \tilde{T}\}$. (b), (c): Faces e in the vicinity of the interface.

The neighbourhood of an edge $e \in \mathcal{E}(\mathcal{T}_h)$ is

$$\omega(e) \equiv \begin{cases} \{T, \tilde{T} : e = T \cap \tilde{T} \text{ for } T, \tilde{T} \in \mathcal{T}_h\}, & \text{if } e \in \mathcal{E}_I(\mathcal{T}_h), \\ \{T : e = T \cap \partial\Omega \text{ for } T \in \mathcal{T}_h\}, & \text{if } e \in \mathcal{E}_D(\mathcal{T}_h) \cup \mathcal{E}_N(\mathcal{T}_h), \end{cases}$$

which is the shaded region for face e in Fig. 1(a). Finally, for $e \in \mathcal{E}_I(\mathcal{T}_h)$ with $\{T, \tilde{T}\} = \omega(e)$, we introduce mean value, jump and normal: For $T \in \mathcal{T}_h$ let $i(T)$ denote the number of the grid cell T, then if $i(T) < i(\tilde{T})$, we define

$$\overline{v} \equiv \frac{1}{2}(v|_T + v|_{\tilde{T}}) \qquad \text{and} \qquad [v] \equiv v|_T - v|_{\tilde{T}}, \tag{8}$$

the unit normal n on e is the outward normal for T.

For the purpose of stabilizing the method, along $e \in \mathcal{E}_I(\mathcal{T}_h) \cup \mathcal{E}_D(\mathcal{T}_h)$, we will make use of the weights

$$\sigma_e \equiv \gamma \cdot \max_{T \in \omega(e)} p_T^2 \,/\, \min_{T \in \omega(e)} h_T. \tag{9}$$

Now, employing the DG space $V_h = \mathcal{S}^p(\mathcal{T}_h)$, the SIPG for Problem 2.1 is to find $u_h \in V_h$ such that for all $\phi \in V_h$ we have (compare [19])

$$\begin{aligned} (f, \phi)_\Omega &= \sum_{T \in \mathcal{T}_h} (\kappa \nabla u_h, \nabla \phi)_T \\ &+ \sum_{e \in \mathcal{E}_I(\mathcal{T}_h)} -(\kappa \overline{\partial_n u_h}, [\phi])_e - (\kappa \overline{\partial_n \phi}, [u_h])_e + \sigma_e \cdot (\kappa [u_h], [\phi])_e \\ &+ \sum_{e \in \mathcal{E}_N(\mathcal{T}_h)} -(g_N, \phi)_e + \sum_{e \in \mathcal{E}_D(\mathcal{T}_h)} -(\kappa \partial_n u_h, \phi)_e - (\kappa \partial_n \phi, u_h - g_D)_e + \sigma_e \cdot (\kappa (u_h - g_D), \phi)_e. \end{aligned} \tag{10}$$

For the case $\partial\Omega = \partial\Omega_D$, a rather elegant error analysis can be found in [18], where the *Local Discontinuous Galerkin Method* is analyzed. This error analysis carries over to the SIPG, which is actually a simple consequence of our results in Theorems 3.14, 3.15, and yields the following error estimates for the case $\partial\Omega = \partial\Omega_D$: Assuming $\gamma > 0$ to be sufficiently large and the exact solution u to satisfy

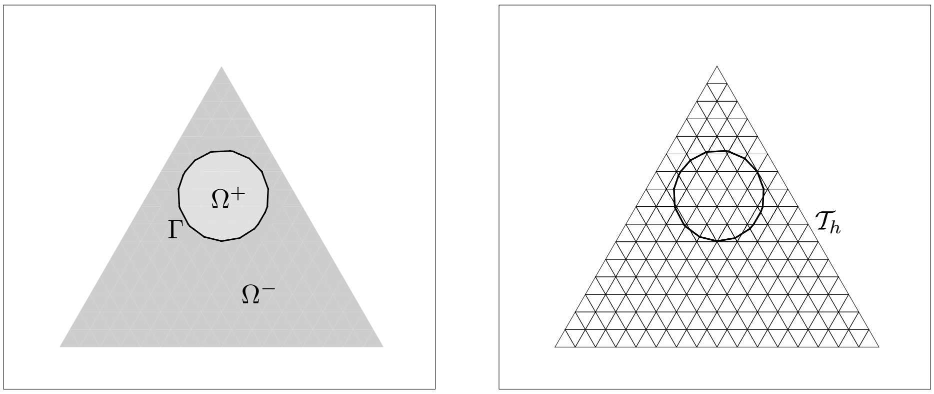


Figure 2: Domains and grid $\mathcal{T}_h$.

$u \in H^s(\Omega)$, and $u|_T \in H^{s_T}(T)$ for each $T \in \mathcal{T}_h$, we have

$$\begin{aligned} ||u - u_h||_h^2 &\leq C \sum_{T \in \mathcal{T}_h} \frac{h_T^{2\min\{p_T+1, s_T\}-2}}{p_T^{2s_T-3}} ||u||_{s_T,T}^2 \,, \\ ||u - u_h||_{0,\Omega} &\leq C \frac{h^{\min\{p+1,s\}}}{p^{s-1}} ||u||_{s,\Omega} \,. \end{aligned} \tag{11}$$

Here $h = \max_{T \in \mathcal{T}_h} h_T$, $p = \min_{T \in \mathcal{T}_h} p_T$ and the grid dependent energy norm $||\cdot||_h$ is given by

$$||u||_h^2 = \sum_{T \in \mathcal{T}_h} (\kappa \nabla u, \nabla u)_T + \sum_{e \in \mathcal{E}_I(\mathcal{T}_h)} \sigma_e \cdot (\kappa[u], [u])_e + \sum_{e \in \mathcal{E}_D(\mathcal{T}_h)} \sigma_e \cdot (\kappa u, u)_e \,. \tag{12}$$

2.2 Discretization of Elliptic Interface Problem

Let us first note that Problem 1.1 can be put in a variational form and shown to have a unique solution. This is done in [10] by lifting the conditions (3), (4) and (5), i.e. by introducing $(\tilde{u}^+, \tilde{u}^-) \in H^1(\Omega^+) \times H^1(\Omega^-)$ which satisfy (3), (4) and (5), and then reducing Problem 1.1 to a variational equation in $H_0^1(\Omega)$.

In order to keep some technicalities, which do not effect the essentials of our result, out of the presentation, we consider Problem 1.1 under the following assumption:

Assumption 2.2 *We assume that* κ^+ *and* κ^- *are constants and that* $g_D = 0$ *in Problem 1.1. Furthermore we assume that* $\Omega^+ \subset\subset \Omega$, *i.e.* $\Gamma \cap \partial\Omega = \emptyset$, $\partial\Omega^+ = \Gamma$, $\partial\Omega^- = \Gamma \cup \partial\Omega$.

We introduce the interface dependent DG spaces V_h^+ and V_h^- and further notation neccessary to define the DG method for Problem 1.1: For $e \in \mathcal{E}(\mathcal{T}_h)$ we use the abbreviation $e^+ \equiv e \cap \Omega^+$, resp. $e^- \equiv e \cap \Omega^-$. Similarly for elements $T \in \mathcal{T}_h$ we set $T^+ \equiv T \cap \Omega^+$, $T^- \equiv T \cap \Omega^-$, $T^0 \equiv T \cap \Gamma$. Then we define

$$\begin{aligned} &\mathcal{T}_h^+ \equiv \{T \in \mathcal{T}_h : |T^+| > 0\}, && \mathcal{T}_h^- \equiv \{T \in \mathcal{T}_h : |T^-| > 0\}, && \mathcal{T}_h^0 \equiv \mathcal{T}_h^+ \cap \mathcal{T}_h^-, \\ &\mathcal{E}_h^+ \equiv \{e \in \mathcal{E}(\mathcal{T}_h) : |e^+| > 0\}, && \mathcal{E}_h^- \equiv \{e \in \mathcal{E}(\mathcal{T}_h) : |e^-| > 0\}, && \mathcal{E}_h^0 \equiv \mathcal{E}_h^+ \cap \mathcal{E}_h^-, \\ &V_h^+ \equiv \{v|_{\Omega^+} : v \in \mathcal{S}^{p^+}(\mathcal{T}_h^+)\}, && V_h^- \equiv \{v|_{\Omega^-} : v \in \mathcal{S}^{p^-}(\mathcal{T}_h^+)\}, && V_h \equiv V_h^+ \times V_h^-, \\ &V^+ \equiv H^1(\Omega^+), && V^- \equiv H^1(\Omega^-), && V \equiv V^+ \times V^-. \end{aligned}$$

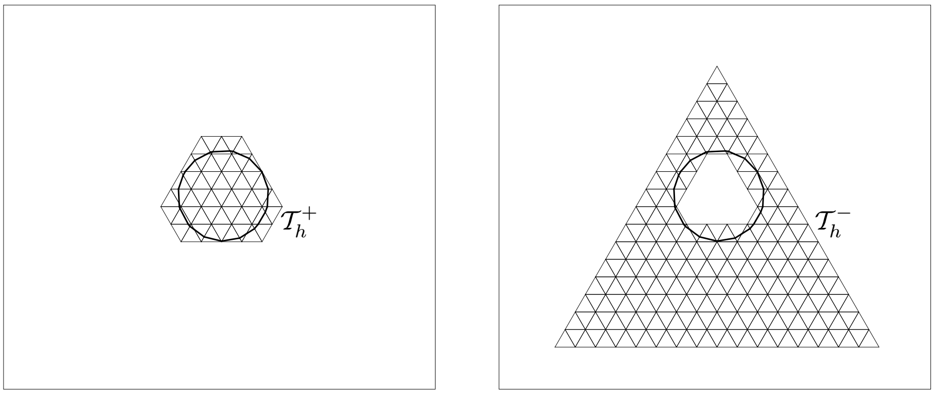


Figure 3: Grids $\mathcal{T}_h^+$ and $\mathcal{T}_h^-$.

See Fig. 2 and Fig. 3 for an illustration of $\Omega^\pm$, $\mathcal{T}_h$, ${\mathcal{T}_h}^\pm$. To illustrate $\mathcal{E}_h^\pm$ see Fig. 1(b),(c): Although $\omega(e) \subset {\mathcal{T}_h}^0$ for face e of Fig. 1(b), we have $e \in \mathcal{E}_h^+$ but $e \notin \mathcal{E}_h^-$. For face e of Fig. 1(c) we have $e \in \mathcal{E}_h^-$ but $e \notin \mathcal{E}_h^+$.

In Lemmas 3.2, 3.4 we are going to impose conditions on the interface and the grid, which will be needed in the convergence analysis. These conditions ensure the smoothness of the interface and that the geometrical features of the interface are captured by the grid. A first condition which addresses the same issue is given now.

Assumption 2.3 *For the geometrical features of the interface to be well resolved by the grid, we demand that for each $T \in \mathcal{T}_h^0$ the intersection $\partial T \cap \Gamma$ consists of exactly 2 points which lie on different faces of T. Furthermore, in analogy to Assumption 2.2 we demand that $\mathcal{E}(\mathcal{T}_h^0) \cap \mathcal{E}_D(\mathcal{T}_h) = \emptyset$.*

For $e \in \mathcal{E}_I(\mathcal{T}_h)$ with $\{T, \tilde{T}\} = \omega(e)$, respectively for $e \in \mathcal{E}_D(\mathcal{T}_h)$ with $T = \omega(e)$, we redefine (8) for $v^\pm \in V_h^\pm$ by

$$\overline{v^\pm} \equiv \frac{1}{2}\,(v^\pm|_T + v^\pm|_{\tilde{T}}) \qquad \text{on } e \in \mathcal{E}_h^\pm \setminus \mathcal{E}_D(\mathcal{T}_h) \text{ if } T \notin \mathcal{T}_h^0 \text{ and } \tilde{T} \notin \mathcal{T}_h^0, \tag{13}$$

$$\overline{v^\pm} \equiv \lambda_e^\pm\, v^\pm|_T + \tilde{\lambda}_e^\pm\, v^\pm|_{\tilde{T}} \qquad \text{on } e \in \mathcal{E}_h^\pm \setminus \mathcal{E}_D(\mathcal{T}_h) \text{ if } T \in \mathcal{T}_h^0 \text{ or } \tilde{T} \in \mathcal{T}_h^0, \tag{14}$$

$$\overline{v^\pm} \equiv v^\pm|_T \qquad \text{on } e \in \mathcal{E}_h^\pm \cap \mathcal{E}_D(\mathcal{T}_h), \tag{15}$$

$$[v^\pm] \equiv v^\pm|_T - v^\pm|_{\tilde{T}} \qquad \text{on } e \in \mathcal{E}_h^\pm \setminus \mathcal{E}_D(\mathcal{T}_h) \text{ if } i(T) < i(\tilde{T}),$$

$$[v^\pm] \equiv v^\pm|_T \qquad \text{on } e \in \mathcal{E}_h^\pm \cap \mathcal{E}_D(\mathcal{T}_h),$$

where we have replaced the mean value by convex combinations in the vicinity of the interface, i.e. the λ should satisfy

$$0 \le \lambda_e^\pm, \tilde{\lambda}_e^\pm \le 1, \quad \lambda_e^\pm + \tilde{\lambda}_e^\pm = 1, \quad \text{for } e \in \mathcal{E}_h^\pm \setminus \mathcal{E}_D(\mathcal{T}_h) \text{ if } T \in \mathcal{T}_h^0 \text{ or } \tilde{T} \in \mathcal{T}_h^0.$$

Note that for the faces indicated in Fig. 1(b) the following definitions for $\overline{v^\pm}$ apply: Along $e \in \mathcal{E}_h^+ \setminus \mathcal{E}_h^-$, (14) applies for $\overline{v^+}$ and along $e' \in \mathcal{E}_h^0$, (14) applies for $\overline{v^\pm}$. Similarly for the faces in Fig. 1(c) we have: $e, e' \in \mathcal{E}_h^- \setminus \mathcal{E}_h^+$ and (14) applies for $\overline{v^-}$. The only face, that does not ly on the boundary of the grid sections shown in Fig. 1(b),(c) and for which (13) applies, is the face e'' of Fig. 1(b), which satisfies $e'' \in \mathcal{E}_h^+ \setminus \mathcal{E}_h^-$.

Let us also introduce weights along interface sections T^0, which will be needed very soon,

$$0 \le \lambda_T^+, \lambda_T^- \le 1, \quad \lambda_T^+ + \lambda_T^- = 1, \quad \text{for } T \in \mathcal{T}_h^0. \tag{16}$$

The exact choice of the λ will be given in Lemmas 3.2, 3.4. In order to give an idea about this choice, for the moment we provide a slightly different choice,

$$\lambda_e^\pm \equiv \frac{|T^\pm|}{|T^\pm| + |\tilde{T}^\pm|}, \quad \tilde{\lambda}_e^\pm \equiv \frac{|\tilde{T}^\pm|}{|T^\pm| + |\tilde{T}^\pm|} \quad \text{on } e \in \mathcal{E}_h^\pm \setminus \mathcal{E}_D(\mathcal{T}_h) \text{ if } T^\pm \in \mathcal{T}_h^0 \text{ or } \tilde{T}^\pm \in \mathcal{T}_h^0, \tag{17}$$

$$\lambda_T^+ \equiv |T^+|/|T|, \qquad \lambda_T^- \equiv |T^-|/|T| \qquad \text{for } T \in \mathcal{T}_h^0. \tag{18}$$

Finally, in analogy to (9), we introduce

$$\begin{aligned}
p_e^\pm &\equiv \max_{T\in\omega(e)} p_T^\pm && \text{for } e \in \mathcal{E}_h^\pm,\\
\sigma_e^\pm &\equiv \gamma \cdot (p_e^\pm)^2 \cdot |e^\pm|^{-1} && \text{for } e \in \mathcal{E}_h^\pm,\\
p_T^0 &\equiv \max\{p_T^+, p_T^-\} && \text{for } T \in \mathcal{T}_h^0,\\
\kappa^0 &\equiv \max\{\kappa^+, \kappa^-\},\\
\sigma_T^0 &\equiv \gamma \cdot (p_T^0)^2 \cdot (h_T)^{-1} && \text{for } T \in \mathcal{T}_h^0.
\end{aligned}$$

Now we propose the DG Method to approximate Problem 1.1 with Assumptions 2.2, 2.3:

Find $u_h^+ \in V_h^+$, $u_h^- \in V_h^-$, such that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h^+} (\kappa^+\nabla u_h^+, \nabla\phi)_{T^+} + \sum_{e\in\mathcal{E}_h^+} -(\kappa^+\overline{\partial_n u_h^+}, [\phi])_{e^+} - (\kappa^+\overline{\partial_n\phi}, [u_h^+])_{e^+} + \sigma_e^+ \cdot (\kappa^+[u_h^+], [\phi])_{e^+}\\
&+ \sum_{T\in\mathcal{T}_h^0} -\lambda_T^-(j_N + \kappa^-\partial_{n^+}u_h^-, \phi)_{T^0} - \lambda_T^+(\kappa^+\partial_{n^+}u_h^+, \phi)_{T^0} - \lambda_T^+(\kappa^+\partial_{n^+}\phi, u_h^+ - u_h^- - j_D)_{T^0}\\
&\qquad + \sigma_T^0 \cdot (\kappa^0(u_h^+ - u_h^- - j_D), \phi)_{T^0}\\
&\qquad = (f, \phi)_{\Omega^+}
\end{aligned}$$

$$\text{for all } \phi \in V_h^+ \tag{19}$$

and

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h^-} (\kappa^-\nabla u_h^-, \nabla\phi)_{T^-} + \sum_{e\in\mathcal{E}_h^-} -(\kappa^-\overline{\partial_n u_h^-}, [\phi])_{e^-} - (\kappa^-\overline{\partial_n\phi}, [u_h^-])_{e^-} + \sigma_e^- \cdot (\kappa^-[u_h^-], [\phi])_{e^-}\\
&+ \sum_{T\in\mathcal{T}_h^0} -\lambda_T^+(j_N + \kappa^+\partial_{n^-}u_h^+, \phi)_{T^0} - \lambda_T^-(\kappa^-\partial_{n^-}u_h^-, \phi)_{T^0} - \lambda_T^-(\kappa^-\partial_{n^-}\phi, u_h^- - u_h^+ + j_D)_{T^0}\\
&\qquad + \sigma_T^0 \cdot (\kappa^0(u_h^- - u_h^+ + j_D), \phi)_{T^0}\\
&\qquad = (f, \phi)_{\Omega^-}
\end{aligned}$$

$$\text{for all } \phi \in V_h^- \tag{20}$$

Equation (19) can be interpreted as derived from (10) by imposing both a Dirichlet condition and a Neumann condition on T^0 with

$$\begin{aligned}
g_D &= j_D + u_h^-,\\
g_N &= j_N + \kappa^-\partial_{n^+}u_h^-,
\end{aligned}$$

and in order to ensure consistency, we use a convex combination of the terms from (10):

$$-\lambda_T^-(g_N,\phi)_{T^0} \;-\; \lambda_T^+(\kappa\partial_{n^+}u_h^+,\phi)_{T^0} \;-\; \lambda_T^+(\kappa\partial_{n^+}\phi,u_h^+-g_D)_{T^0}$$

An analogous interpretation can be given for (20). This treatment together with the penalty terms $\sigma_T^0\cdot(\kappa^0(u_h^+-u_h^--j_D),\phi)_{T^0}$ is in fact a discretization along the interface already employed in [11]. That this is so, can be read off more clearly from the formulation of the DG Method given below in (22). Our DG Method employs the *unfitted* spaces $V_h^\pm\equiv\mathcal{S}^{p^\pm}(\mathcal{T}_h^\pm)$ to approximate Problem 1.1. If $(u_h^+,u_h^-)\in V_h$ is the solution of (19) & (20), then we are actually only interested in $u_h^+|_{\Omega^+}$ and $u_h^-|_{\Omega^-}$.

For the purpose of writing the system (19) & (20) compactly as one variational equation, for $v\equiv(v^+,v^-)$, $w\equiv(w^+,w^-)\in V_h$ we set

$$\begin{aligned}
[v] &\equiv v^+-v^- && \text{on } T^0 \text{ for } T\in\mathcal{T}_h^0\\
\overline{v} &\equiv \lambda_T^+v^+ + \lambda_T^-v^- && \text{on } T^0 \text{ for } T\in\mathcal{T}_h^0\\
\langle v,w\rangle_T &\equiv (v^+,w^+)_{T^+} + (v^-,w^-)_{T^-} &&\\
\langle v,w\rangle_e &\equiv (v^+,w^+)_{e^+} + (v^-,w^-)_{e^-}\,. &&
\end{aligned} \tag{21}$$

Note that these formulaes are to be understood with priority given to the operations $\overline{\cdot}$, $[\cdot]$, $\langle\cdot,\cdot\rangle$, e.g. $\overline{\partial_n v} = \lambda_T^+\,\partial_n v^+ + \lambda_T^-\,\partial_n v^-$. Summing (19) and (20) yields our DG Method in the following form:

Find $u\equiv(u_h^+,u_h^-)\in V_h$, such that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\langle\kappa\nabla u,\nabla\phi\rangle_T \;+\; \sum_{e\in\mathcal{E}(\mathcal{T}_h)} -\langle\kappa\overline{\partial_n u},[\phi]\rangle_e \;-\; \langle\kappa\overline{\partial_n\phi},[u]\rangle_e \;+\; \langle\sigma_e\,\kappa\,[u],[\phi]\rangle_e\\
&\qquad + \sum_{T\in\mathcal{T}_h^0} -(\overline{\kappa\partial_{n^+}u},[\phi])_{T^0} \;-\; (\overline{\kappa\partial_{n^+}\phi},[u])_{T^0} \;+\; \sigma_T^0\cdot(\kappa^0[u],[\phi])_{T^0}\\
&= \sum_{T\in\mathcal{T}_h}\langle f,\phi\rangle_T \;+\; \sum_{T\in\mathcal{T}_h^0}\lambda_T^-(j_N,\phi^+)_{T^0} + \lambda_T^+(j_N,\phi^-)_{T^0} \;-\; (\overline{\kappa\partial_{n^+}\phi},j_D)_{T^0} \;+\; \sigma_T^0\cdot(\kappa^0 j_D,[\phi])_{T^0}\\
&\qquad\qquad \text{for all } \phi\equiv(\phi^+,\phi^-)\in V_h,
\end{aligned} \tag{22}$$

where we have used $f^\pm(x)\equiv f(x)$ for $x\in\Omega^\pm$. The l.h.s. of (22) defines the bilinear form $\mathcal{B}_h(u,\phi)$: $V_h\times V_h\to\mathbb{R}$, whereas the r.h.s defines the linear functional $\mathcal{F}_h(\phi):V_h\to\mathbb{R}$. We easily see that $\mathcal{B}_h$ is symmetric. In fact with the choice (18) the method reduces to the method of [11], if we replace V_h^+ by $V_h^+\cap V^+$ and V_h^- by $\{v\in V_h^-\cap V^-: v=0 \text{ on } \partial\Omega\}$ and use $p_T^\pm=1$ for all $T\in\mathcal{T}_h$. Taking $j_D=0$ as in [11], this reads

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\langle\kappa\nabla u,\nabla\phi\rangle_T \;+\; \sum_{T\in\mathcal{T}_h^0} -(\overline{\kappa\partial_{n^+}u},[\phi])_{T^0} \;-\; (\overline{\kappa\partial_{n^+}\phi},[u])_{T^0} \;+\; \sigma_T^0\cdot(\kappa^0[u],[\phi])_{T^0}\\
&\qquad = \sum_{T\in\mathcal{T}_h}\langle f,\phi\rangle_T \;+\; \sum_{T\in\mathcal{T}_h^0}\lambda_T^-(j_N,\phi^+)_{T^0} + \lambda_T^+(j_N,\phi^-)_{T^0}\,.
\end{aligned}$$

In [11] the use of (18) was crucial for proving coercivity of the bilinear form.

3 Convergence analysis

3.1 Assumptions on grid and interface

For the convergence analysis to work, additionally to Assumption 2.3 we will require some more conditions regarding the smoothness of the interface Γ, the regularity of the grid $\mathcal{T}_h$, and we will

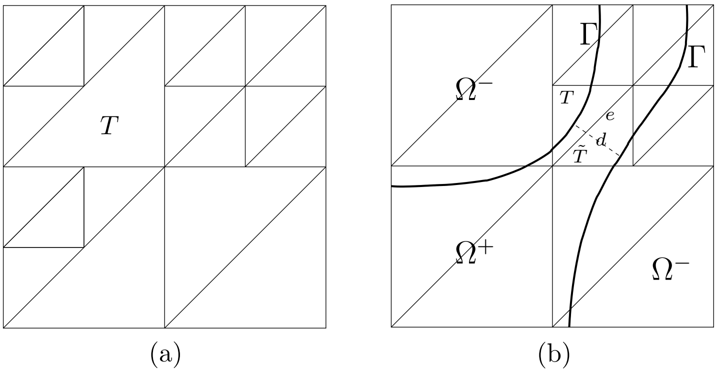


Figure 4: (a): $T \in \omega(e)$ for 6 different $e \in \mathcal{E}(\mathcal{T}_h)$. (b): Ω^+ locally looks like a tube containing face e.

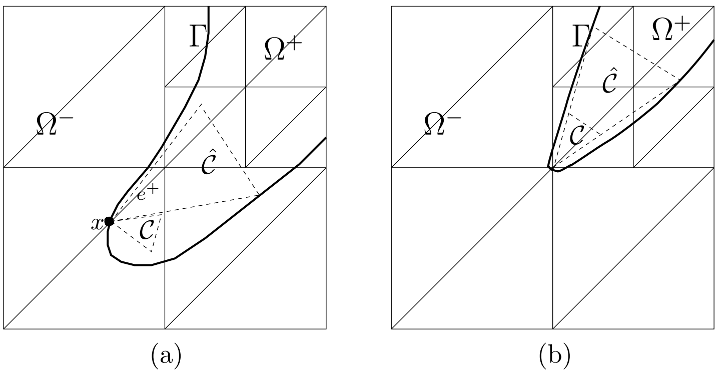


Figure 5: Cone condition for Ω^+.

demand that the geometrical features of the interface are resolved by the grid. These conditions are made precise in Assumption 3.1 below and in Assumption 3.3 later on.

Assumption 3.1 *For the smoothness of* Γ *we require that*

$$\Omega^+ \text{ and } \Omega^- \text{ both satisfy the uniform } C^m\text{–regularity property, in the sense of [1], page 84.} \quad (23)$$

For the regularity of the grid we assume a constant $\varrho \geq 2$ *to be given such that*

$$\varrho\, r_T \;\geq\; h_T \quad \forall\, T \in \mathcal{T}_h \quad \text{and each interior angle of each } T \in \mathcal{T}_h \text{ is } \leq \pi/2. \quad (24)$$

For the grading of the polynomial degrees $p_T^{\pm}$ *and the grading of the grid we require the existence of positive constants* $G_1,\ G_2,\ G_3$ *such that*

$$\frac{1}{G_1} \;\leq\; \frac{p_T^{\pm}}{p_{\tilde{T}}^{\pm}} \;\leq\; G_1, \qquad \text{if } \{T,\tilde{T}\} = \omega(e),\ \ e \in \mathcal{E}_h^{\pm}, \quad (25)$$

$$\frac{1}{G_1} \;\leq\; \frac{p_T^{+}}{p_T^{-}} \;\leq\; G_1, \qquad \text{if } T \in \mathcal{T}_h^0, \quad (26)$$

$$\#\{e \in \mathcal{E}(\mathcal{T}_h) : T \in \omega(e)\} \;\leq\; G_2, \qquad \text{if } T \in \mathcal{T}_h, \quad (27)$$

$$h_T \;\leq\; G_3|e|, \qquad \text{if } T \in \omega(e),\ \ e \in \mathcal{E}(\mathcal{T}_h). \quad (28)$$

For the geometrical features of the interface to be well resolved by the grid, we require that a constant $M_1 > 0$ *is given, such that whenever* (14) *applies for* $\overline{v^{\pm}}$, *i.e. when* $e \in \mathcal{E}_h^{\pm}$ *with* $\{T,\tilde{T}\} = \omega(e)$ *and* $T \in \mathcal{T}_h^0$ *or* $\tilde{T} \in \mathcal{T}_h^0$, *then triangles* $S^{\pm} \subset T^{\pm}$, $\tilde{S}^{\pm} \subset \tilde{T}^{\pm}$ *exist, such that* $e^{\pm}$ *is a face of both* $S^{\pm}$ *and* $\tilde{S}^{\pm}$, *and*

$$|S^{\pm}| + |\tilde{S}^{\pm}| \;\;\geq\;\; |e^{\pm}|^2/M_1\,. \quad (29)$$

Here $S^{\pm}$, $\tilde{S}^{\pm}$ *are allowed to degenerate, i.e.* $|S^{\pm}| = 0$ *or* $|\tilde{S}^{\pm}| = 0$ *is allowed.*

Condition (23) ensures that we can find smooth extensions of both parts of the solution, namely of u^+ and u^-, compare Theorem C.5. The shape regularity in (24) is well-known, the additional condition that the interior angles are not greater than $\pi/2$ is typically met by triangular grids used in practize, and the condition only serves to avoid the consideration of too many cases in the proofs of Lemma 3.4, 3.5, 3.6. Condition (25) is well-known for DG Methods, (26) is an obvious analogue at the interface. Conditions (27) and (28) are typically met in practize. Let us shortly find the constants in the common case that initially we have a coarse triangulation of Ω without hanging nodes and that $\mathcal{T}_h$ is obtained by refinements of the coarse triangulation, where a triangle refinement is performed by dividing a triangle into four congruent subtriangles. We assume that on each face of $T \in \mathcal{T}_h$ not more than one hanging node can occur, see Fig. 4(a). Then we have $G_2 = 6$, and making use of (24) we obtain $h_T \leq \varrho\, r_T \leq \varrho\,|e_T|/2 \leq \varrho\,|e|$, where e_T is the face of T with $e \subset e_T$, so that $G_3 = \varrho$.

Finally we explain why (29) is a plausible assumption, if (23), (24) hold and the geometrical features of the interface are resolved by the grid: Assume first that Γ intersects e, $x = \Gamma \cap e$. Condition (23) yields the cone condition, see [1], page 84, valid from both sides of Γ. Let $\mathcal{C}$ be the corresponding cone in Ω^+ with its tip at x and denote by $\hat{\mathcal{C}}$ the largest cone contained in Ω^+ which has tip and tip angle in common with $\mathcal{C}$, compare Fig. 5(a). Let $h_{\hat{\mathcal{C}}}$ be the diameter of $\hat{\mathcal{C}}$. Then we can say that the geometrical features of the interface are resolved by the grid, if $|e| \leq h_{\hat{\mathcal{C}}}$, i.e. $|e^+| \leq h_{\hat{\mathcal{C}}}$. Obviously, $|\hat{\mathcal{C}}| \geq h_{\hat{\mathcal{C}}}^2 M_1'$, where M_1' depends on the tip angle. Similarly, $|\hat{\mathcal{C}} \cap (T^+ \cup \tilde{T}^+)| = |\hat{\mathcal{C}} \cap T^+| + |\hat{\mathcal{C}} \cap \tilde{T}^+| \geq |e^+|^2/M_1$ for a constant M_1 depending on the tip angle and ϱ. This yields (29) since we can choose S^+ and $\tilde{S}^+$ such that $S^+ \supset \hat{\mathcal{C}} \cap T^+$ and $\tilde{S}^+ \supset \hat{\mathcal{C}} \cap \tilde{T}^+$. In the case that Γ does not intersect e, as in Fig. 4(b) or Fig. 5(b), we can define $d \equiv dist(T^0, \tilde{T}^0)$. Now either we need $|e|$ to be on the scale of d as in Fig. 4(b) and so again (29) follows. Or $d \ll |e|$

and Γ gets very close to an end point of e as in Fig. 5(b), and $|e|$ is on the scale of $\hat{\mathcal{C}}$. Here $\mathcal{C}$ and $\hat{\mathcal{C}}$ are as above but with their tip in the end point of e. This implies (29) again.

These comments on Assumption 3.1 reveal two important observations: First (23)–(29) allow Γ to pass through the grid by getting arbitrarily close to grid vertices and by cutting faces at arbitrarily small angles. In this sense the constructed method is stable with respect to the relative position of the interface to the grid. We require only that Γ is smooth and that its geometrical features are resolved by the grid. Secondly, if an interface fitted grid was used, good quality meshes also require that we resolve the geometrical features of Γ. If the fitted mesh is on the scale of our unfitted meshes as shown in Fig. 4(b) and Fig. 5, then the cone angles will also enter the convergence analysis of the fitted method.

3.2 Inverse estimate at the interface

In the following lemma we redefine (17) by replacing $T^{\pm}$, $\tilde{T}^{\pm}$ by $S^{\pm}$, $\tilde{S}^{\pm}$ from Assumption 3.1. In Section 4 we employ a simple approximation of the sets $S^{\pm}$, $\tilde{S}^{\pm}$. How to find more sophisticated approximations will be detailed elsewhere.

Lemma 3.2 (Inverse estimates at the interface) *For $e \in \mathcal{E}_h^{\pm}$ let $\{T, \tilde{T}\} = \omega(e)$ and let corresponding $S^{\pm}$, $\tilde{S}^{\pm}$ exist as in Assumption 3.1 and such that* (29) *holds. Then, setting*

$$\lambda_e^{\pm} \quad = \quad \frac{|S^{\pm}|}{|S^{\pm}| + |\tilde{S}^{\pm}|}\,,$$

we have

$$\lambda_e^{\pm}\, ||v||_{0,e^{\pm}}^2 \quad \leq \quad C_{(3.2)}\, \frac{p^2}{|e^{\pm}|} ||v||_{0,T^{\pm}}^2 \qquad \forall\, v \in P^p(T),$$

where $C_{(3.2)} = M_1 C_{(66)}$.

Proof:

$$\lambda_e^{\pm}\, ||v||_{0,e^{\pm}}^2 \;\leq\; \frac{M_1 |S^{\pm}|}{|e^{\pm}|^2}\, ||v||_{0,e^{\pm}}^2 \;\leq\; \frac{M_1 C_{(66)}\, p^2}{|e^{\pm}|}\, ||v||_{0,S^{\pm}}^2 \;\leq\; \frac{M_1 C_{(66)}\, p^2}{|e^{\pm}|}\, ||v||_{0,T^{\pm}}^2 \;\square$$

Next we explore how a similar estimate can be derived for $\lambda_T^{\pm}\, ||v||_{0,T^0}^2$. For this purpose, we choose local coordinates in $T^{\pm}$. In the following steps we point out, how to choose these coordinates, based on a few decisions concerning the relative position of T^0 within T. We refer to Fig. 6–8 for an illustration.

- Let T_1^0 denote the straight interface approximation of T^0, which is obtained by connecting the two intersection points of Γ and ∂T by a straight line, see Fig. 6.
- T_1^0 divides T into a triangle, denoted by $T_1^{\triangle}$, and a quadrilateral, denoted by $T_1^{\square}$. Similarly, T^0 divides T into a triangle with a curved side, denoted by $T^{\triangle}$, and a quadrilateral with a curved side, denoted by $T^{\square}$. In each case, the curved side is T^0, see Fig. 7 and Fig. 8(a). We have $T^{\pm} = T^{\triangle} \Leftrightarrow T^{\mp} = T^{\square}$. In Fig. 6–8 we display points lying on Γ by a circular symbol, whereas points which do not belong to Γ, but do belong to $T^{\triangle}$, respectively $T^{\square}$, are displayed with a triangular symbol, respectively a square symbol.
- *Angle condition*: Find the face that is closest to being parallel to T_1^0, i.e. find the face that creates the smallest angle with T_1^0; the smallest angle is α in Fig. 6. Denote this face by e^*. Denote by N^* the node opposite to e^*.

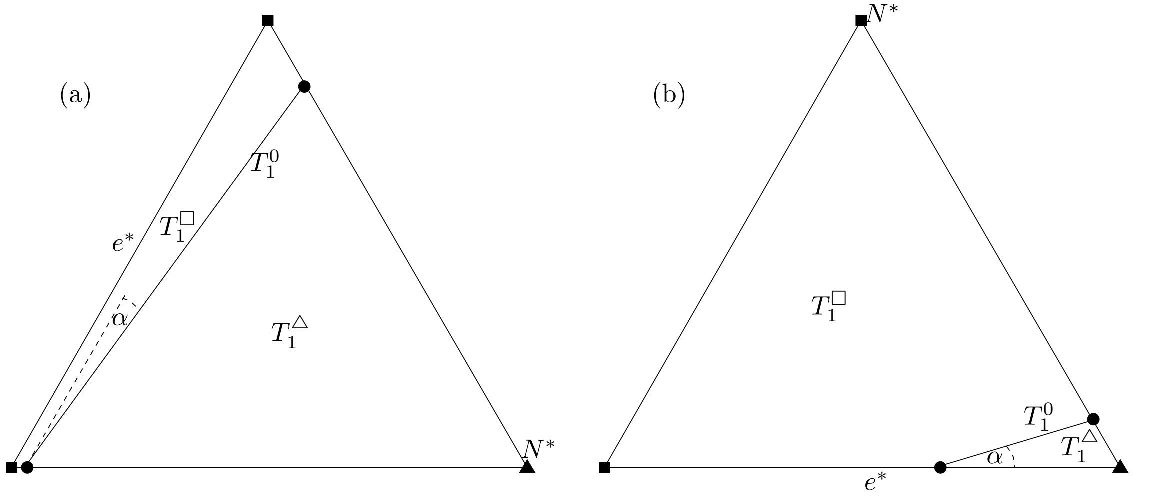


Figure 6: Straight interface approximation T_1^0 and angle condition.

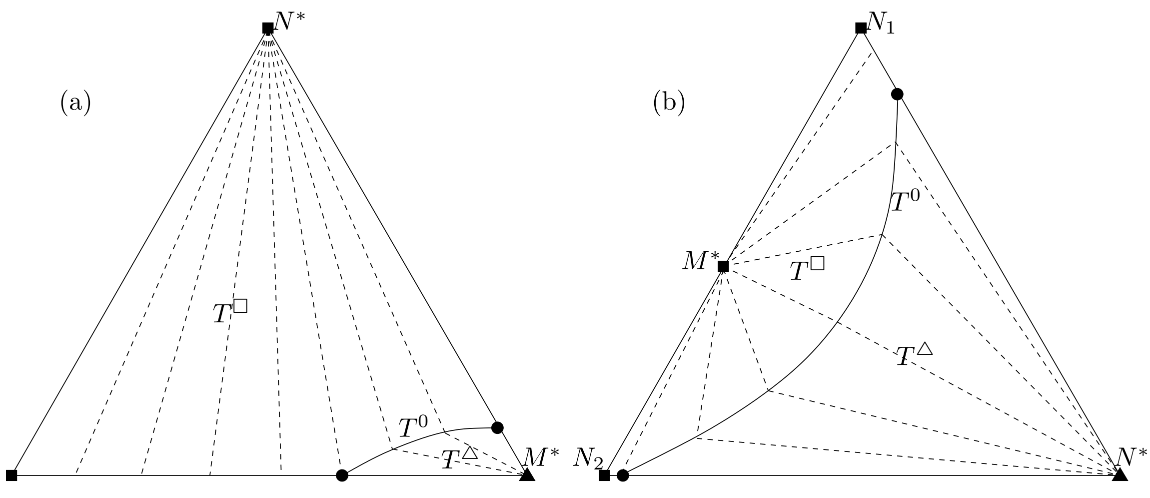


Figure 7: (a): Case (A). (b): Case (B)-(C).

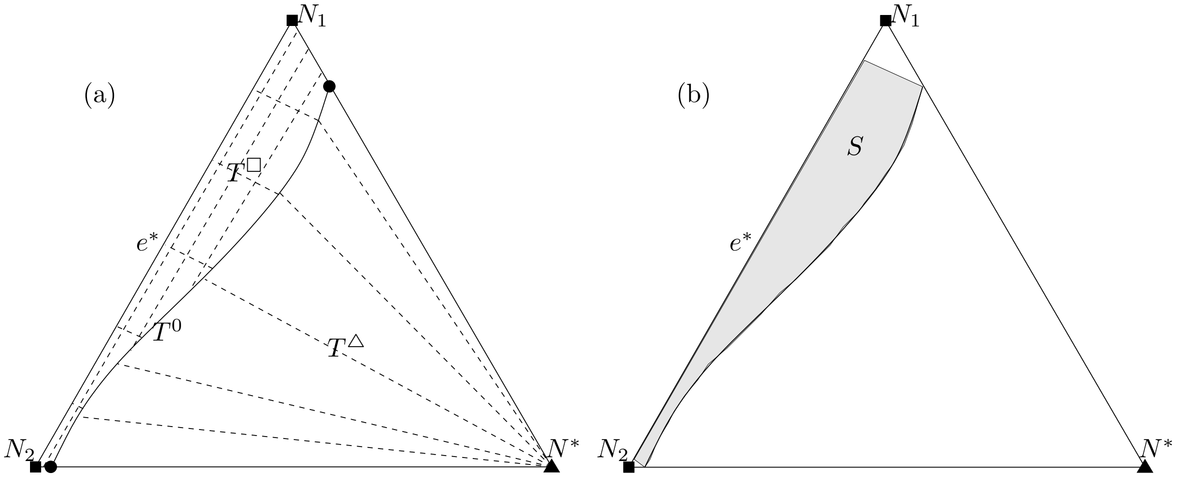


Figure 8: (a): Case (B)-(D). (b): The domain $S \subset T^\square$ defined in (D).

- Choice of local coordinates:
  - (A) If $N^* \in T_1^\square$, denote by M^* the node of T, such that $M^* \in T_1^\triangle$. Use polar coordinates in $T^\square$ with center N^*, and polar coordinates in $T^\triangle$ with center M^*, see Fig. 7(a).
  - (B) Else, i.e. if $N^* \in T_1^\triangle$, denote by M^* the midpoint of e^*. Use polar coordinates in $T^\triangle$ with center N^*, see Fig. 7(b) and 8(a).
    - (C) If $(T^0 \setminus T_1^0) \cap T_1^\square = \emptyset$, use polar coordinates in $T^\square$ with center M^*, see Fig. 7(b).
    - (D) Else, use e^* and the inward normal on e^* as coordinate system in $T^\square$, see Fig. 8(a).

Note, that the steps above uniquely determine the local coordinate system and that exactly one of the three cases (A), (B)-(C), (B)-(D) occurs. In (A), (B), (C) polar coordinates are used, thereas in (D) cartesian coordinates are used. We consider the two different types of coordinate systems separately:

Polar coordinates (A), (B), (C): Let S be $T^\triangle$ or $T^\square$ and assume that S is star-shaped with respect to Z, where $Z = N^*$ if $N^* \in S$ and $Z = M^*$ if $M^* \in S$.

We define polar coordinates,

$$\psi(\theta, r) \;=\; Z + r \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \qquad 0 \le r \le R(\theta),\; \theta \in [\theta_1, \theta_2], \tag{30}$$

where $R(\theta) \in C^1[\theta_1, \theta_2]$ is such that

$$\begin{aligned} T^0 &\subset \{\psi(\theta, R(\theta)) : \theta \in [\theta_1, \theta_2]\}, \\ S &= \{\psi(\theta, r) : 0 \le r \le R(\theta),\; \theta \in [\theta_1, \theta_2]\}. \end{aligned}$$

Introducing the intervals $I_\theta := (0, R(\theta))$ and using Theorem C.2, we find for arbitrary $v \in P^p(T)$

that

$$\begin{aligned} R(\theta)^2 v(\psi(\theta,R(\theta)))^2 &= \int_{I_\theta} \frac{\partial}{\partial r}\left(r^2 v(\psi(\theta,r))^2\right) dr \\ &\leq 2\left\| \frac{\partial}{\partial r}(rv)\right\|_{0,I_\theta} \|rv\|_{0,I_\theta} \\ &\leq 2\,C_{(64)}\frac{(p+1)^2}{R(\theta)}\|rv\|^2_{0,I_\theta} \\ &\leq 8\,C_{(64)}\,p^2 \int_{I_\theta} r\,v(\psi(\theta,r))^2\,dr\,. \end{aligned}$$

Here we have used that, for constant θ, $r\,v(\psi(\theta,r))$ is a polynomial in r, which is of degree $\leq p+1$. Introducing R_{min} and assuming constants $M_2, M_3 > 0$ to exist, such that

$$R_{min} := \min_{\theta\in[\theta_1,\theta_2]} R(\theta) \tag{31}$$

$$|R'(\theta)| \leq M_2 \cdot h_T \tag{32}$$

$$R_{min} \geq h_T/M_3, \tag{33}$$

we obtain

$$\begin{aligned} \|v\|^2_{0,T^0} &\leq \int_{\theta_1}^{\theta_2} v(\psi(\theta,R(\theta)))^2 \left\|\frac{\partial}{\partial\theta}\psi(\theta,R(\theta))\right\|_2 d\theta \\ &= \int_{\theta_1}^{\theta_2} v(\psi(\theta,R(\theta)))^2 \sqrt{R'(\theta)^2+R(\theta)^2}\,d\theta \\ &\leq \sqrt{M_2^2+1}\int_{\theta_1}^{\theta_2} h_T\,v(\psi(\theta,R(\theta)))^2 d\theta \qquad (34) \\ &\leq 8\,C_{(64)}\sqrt{M_2^2+1}\,\frac{p^2 h_T}{R^2_{min}}\int_{\theta_1}^{\theta_2}\int_{I_\theta} r\,v(\psi(\theta,r))^2 dr\,d\theta\,. \end{aligned}$$

Thus our final result for (A), (B), (C) is that if (32) and (33) are satisfied, then

$$\|v\|^2_{0,T^0} \leq C_{(35)}\frac{p^2}{h_T}\|v\|^2_{0,S} \qquad \forall\, v\in P^p(T), \tag{35}$$

where $C_{(35)} = 8\,C_{(64)}M_3^2\sqrt{M_2^2+1}$.

Cartesian coordinates (D): Let N_1, N_2 be nodes of T, such that $e^* = \overline{N_1N_2}$ and let n be the inward pointing unit normal on the face e^* of T. We define the local coordinates (s,r) by

$$\psi(s,r) = N_1 + s\,(N_2-N_1) + r\,n,$$

and we assume that $R\in C^1[s_1,s_2]$, $[s_1,s_2]\subset[0,1]$, such that

$$\begin{aligned} T^0 &= \{\psi(s,R(s)) : s\in[s_1,s_2]\}\,, \\ S &:= \{\psi(s,r) : s\in[s_1,s_2],\, r\in[0,R(s)]\} \subset T^\square, \end{aligned}$$

compare Fig. 8(b). Introducing $I_s := (0,R(s))$, similar to our treatment for polar coordinates, we can estimate

$$\begin{aligned} R(s)^2 v(\psi(s,R(s)))^2 &= \int_{I_s}\frac{\partial}{\partial r}\left(r^2 v(\psi(s,r))^2\right) dr \\ &\leq 2\,C_{(64)}\,(p+1)^2\int_{I_s} r\,v(\psi(s,r))^2\,dr \\ &\leq 8\,C_{(64)}\,p^2 R(s)\int_{I_s} v(\psi(s,r))^2\,dr. \end{aligned}$$

Similar to (31), we define

$$R_{min} \quad = \quad \min_{s \in [s_1, s_2]} R(s)$$

and assuming again (32), (33), we obtain

$$v(\psi(s, R(s)))^2 \quad \leq \quad \frac{8\, C_{(64)} M_3\, p^2}{h_T} \int_{I_s} v(\psi(s,r))^2\, dr \qquad \forall\; s \in [0,1].$$

We have

$$\nabla\psi \;=\; (N_2 - N_1, n\,)\,, \qquad |det\,\nabla\psi| \;=\; ||N_2 - N_1||_2$$

and assuming further (24) to hold, we estimate

$$\begin{aligned} \|v\|^2_{0,T^0} &= \int_{s_1}^{s_2} v(\psi(s,R(s)))^2\, ||\psi_s + \psi_r\, R'||_2\, ds \\ &\leq \int_{s_1}^{s_2} v(\psi(s,R(s)))^2\, (1 + M_2) h_T\, ds \\ &\leq \frac{8\, C_{(64)} M_3\, (1+M_2)\, p^2}{||N_2 - N_1||_2} \int_{s_1}^{s_2} \int_{I_s} v(\psi(s,r))^2\, |det\,\nabla\psi|\, dr\, ds, \\ &\leq \frac{8\, C_{(64)} M_3\, (1+M_2)\, p^2}{2\, r_T}\, \|v\|^2_{0,S}\,, \end{aligned} \tag{36}$$

which yields

$$\|v\|^2_{0,T^0} \quad \leq \quad C_{(37)}\, \frac{p^2}{h_T}\, \|v\|^2_{0,T^\square} \qquad \forall\; v \in P^p(T)\,, \tag{37}$$

where $C_{(37)} = 4\,\varrho\, C_{(64)} M_3\, (1 + M_2)$.

The reason for introducing the two cases (B)-(C) and (B)-(D) for $T^\square$, and not just to stick to only one of the two cases, is the following: In Fig. 9 we have two harmless looking interface sections, with moderate curvatures. If we used case (B)-(D) for $T^\square$ given in Fig. 9(a), then (33) would be violated, since T^0 gets very close to N_2. If we used case (B)-(C) for $T^\square$ given in Fig. 9(b), then (32) would be violated, since T^0 is tangential to a line of constant polar angle.

The above analysis reveals the assumptions under which we can obtain the inverse estimate (35), respectively (37). We will see that it suffices that the inverse estimate holds either for $T^\triangle$ or for $T^\square$. Thus we demand:

Assumption 3.3 *For the geometrical features of the interface to be well resolved by the grid, we require that positive constants M_2, M_3 are given, such that for each $T \in \mathcal{T}_h^0$ according to the choice of local coordinates, i.e. case (A), (B)-(C) or (B)-(D), we have that for $T^\triangle$ or for $T^\square$ the local coordinates are well-defined and both* (32) *and* (33) *hold.*

As a consistency check for Assumption 3.3 we determine in Lemma 3.6 the constants M_2, M_3 for the case that T^0 is a straight line that cuts arbitrarily through T. Note that the better we resolve Γ, the less T^0 deviates from T^0_1, which is a straight line approximation for T^0. The constants M_2, M_3 given in Lemma 3.6 may also be valuable for an implementation of the method, namely we may choose M_2, M_3 as given in Lemma 3.6, or larger, then check approximately if the conditions (32) and (33) are satisfied for $T^\triangle$ or for $T^\square$, and if the conditions are not satisfied for both subsets of T, then we refine T.

But first, we formulate what results from (35) and (37) in Lemma 3.4, and deduce a local trace inequality in Lemma 3.5, which is proved with techniques similar to the ones used for Lemma 3.4. Note that in Lemma 3.4, the choice $\lambda_T^\pm = |T^\pm|/|T|$ is of no significance. It merely constitutes a

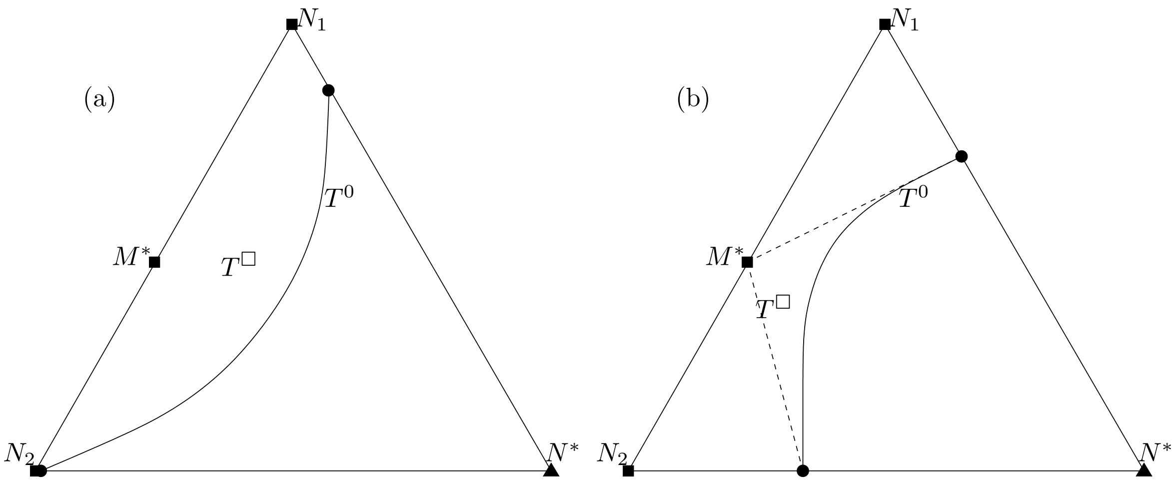


Figure 9: Neccessity for introducing the two cases (B)-(C) and (B)-(D).

smooth transition between the extreme cases $\lambda_T^\pm = 0$ and $\lambda_T^\pm = 1$. In [11] only $\lambda_T^\pm = |T^\pm|/|T|$ was used, since there it was sufficient to prove the inverse estimate $\lambda_T^\pm \, ||v||^2_{0,T^0} \leq \dfrac{C}{h_T} ||v||^2_{0,T^\pm}$ for $v \in P^0(T)$.

Lemma 3.4 (Inverse estimates at the interface) *Let Assumption 3.3 hold and let $T \in \mathcal{T}_h^0$ satisfy* (24)*. Then setting*

$$\lambda_T^\pm \;=\; \begin{cases} |T^\pm|/|T| & \textit{if } (32) \textit{ and } (33) \textit{ hold for both } T^\pm \textit{ and } T^\mp \\ 1 & \textit{if } (32) \textit{ and } (33) \textit{ hold only for } T^\pm \\ 0 & \textit{if } (32) \textit{ and } (33) \textit{ hold only for } T^\mp, \end{cases}$$

we have

$$\lambda_T^\pm \, ||v||^2_{0,T^0} \;\leq\; C_{(3.4)} \, \frac{p^2}{h_T} ||v||^2_{0,T^\pm} \qquad \forall \, v \in P^p(T),$$

where $C_{(3.4)} = \max\{8\, C_{(64)} M_3^2 \sqrt{M_2^2+1},\, 4\, \varrho\, C_{(64)} M_3\, (1+M_2)\}$.

Proof: The lemma is an immediate consequence of the estimates (35) and (37). □

Lemma 3.5 (Local trace inequality at the interface) *Let Assumption 3.3 hold and let $T \in \mathcal{T}_h^0$ satisfy* (24)*. Then, using $\lambda_T^\pm$ as in Lemma 3.4, we have*

$$\lambda_T^\pm \, ||v||^2_{0,T^0} \;\leq\; C_{(3.5)} \left(\frac{p}{h_T} ||v||^2_{0,T^\pm} + \frac{h_T}{p} |v|^2_{1,T^\pm} \right) \qquad \forall \; p \geq 1, \; v \in H^1(T^\pm),$$

where $C_{(3.5)} = \max\{4\, M_3^2 \sqrt{M_2^2+1},\, 2\, \varrho\, M_3 (1+M_2)\}$.

Proof: We consider again (A), (B), (C) together, using the polar coordinates (30), but now we estimate

$$\begin{aligned}
R(\theta)^2 v(\psi(\theta, R(\theta)))^2 &= \int_{I_\theta} \frac{\partial}{\partial r}\left(r^2 v(\psi(\theta, r))^2\right) dr \\
&= 2\int_{I_\theta} r^2\, v\, \frac{\partial v}{\partial r} + r\, v^2\, dr \\
&= 2\int_{I_\theta} r^2\, v\, \nabla v \cdot (\cos(\theta), \sin(\theta))^T + r\, v^2\, dr \\
&\leq 2\int_{I_\theta} r^2\, |v|\, ||\nabla v||_2 + r\, v^2\, dr \\
&\leq 2\int_{I_\theta} r\,(h_T\, |v|\, ||\nabla v||_2 + v^2)\, dr\,.
\end{aligned}$$

Assuming (32), (33) to hold, from (34) we continue estimating

$$\begin{aligned}
\|v\|_{0,T^0}^2 &\leq 2\sqrt{M_2^2+1}\,\frac{h_T}{R_{min}^2} \int_{\theta_1}^{\theta_2}\int_{I_\theta} r\,(h_T\, |v|\, ||\nabla v||_2 + v^2)\, dr\, d\theta \\
&\leq 2\sqrt{M_2^2+1}\,\frac{h_T}{R_{min}^2}\left(h_T\, \|v\|_{0,S} |v|_{1,S} + \|v\|_{0,S}^2\right) \\
&\leq 2\sqrt{M_2^2+1}\,\frac{h_T^2}{R_{min}^2}\left(\frac{p}{h_T}\|v\|_{0,S}^2 + \frac{h_T}{p}|v|_{1,S}^2 + \frac{1}{h_T}\|v\|_{0,S}^2\right) \\
&\leq 4\, M_3^2\sqrt{M_2^2+1}\left(\frac{p}{h_T}\|v\|_{0,S}^2 + \frac{h_T}{p}|v|_{1,S}^2\right).
\end{aligned}$$

For (D) we have

$$\begin{aligned}
R(s)^2 v(\psi(s, R(s)))^2 &= 2\int_{I_s} r^2\, v\, \nabla v \cdot n + r\, v^2\, dr \\
&\leq 2\, R(s) \int_{I_s} h_T\, |v|\, ||\nabla v||_2 + v^2\, dr\,,
\end{aligned}$$

and assuming (32), (33) to hold, from (36) we continue estimating

$$\begin{aligned}
\|v\|_{0,T^0}^2 &\leq \frac{(1+M_2)\, h_T}{||N_2 - N_1||_2} \int_{s_1}^{s_2}\int_{I_s} v(\psi(s,r))^2\, |det\, \nabla\psi|\, dr\, ds \\
&\leq \frac{2\,(1+M_2)\, h_T}{||N_2 - N_1||_2\, R_{min}} \int_{s_1}^{s_2}\int_{I_s} (h_T\, |v|\, ||\nabla v||_2 + v^2)\, |det\, \nabla\psi|\, dr\, ds \\
&\leq \frac{4\,(1+M_2)\, h_T^2}{||N_2 - N_1||_2\, R_{min}}\left(\frac{p}{h_T}\|v\|_{0,S}^2 + \frac{h_T}{p}|v|_{1,S}^2\right) \\
&\leq 2\,\varrho\, M_3 (1+M_2)\left(\frac{p}{h_T}\|v\|_{0,S}^2 + \frac{h_T}{p}|v|_{1,S}^2\right). \quad \square
\end{aligned}$$

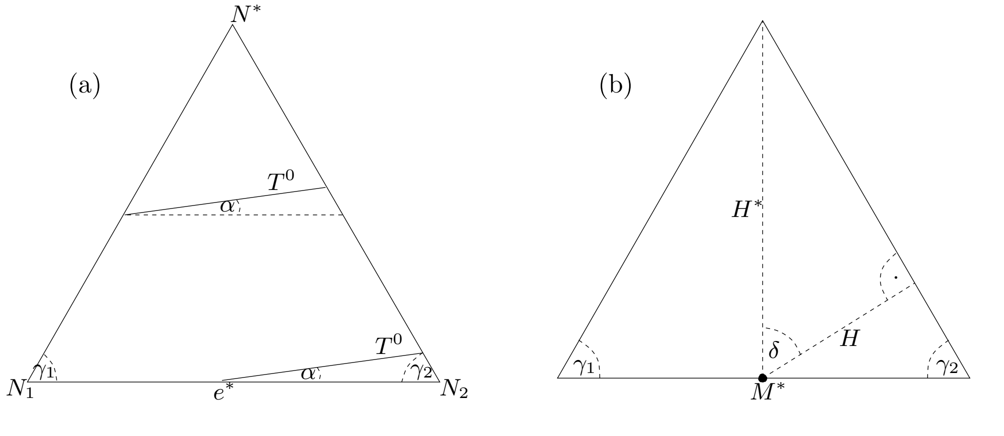


Figure 10: Illustrations for proof of Lemma 3.6. (a): Angles $\alpha, \gamma_1, \gamma_2$. (b): Case (B)-(C).

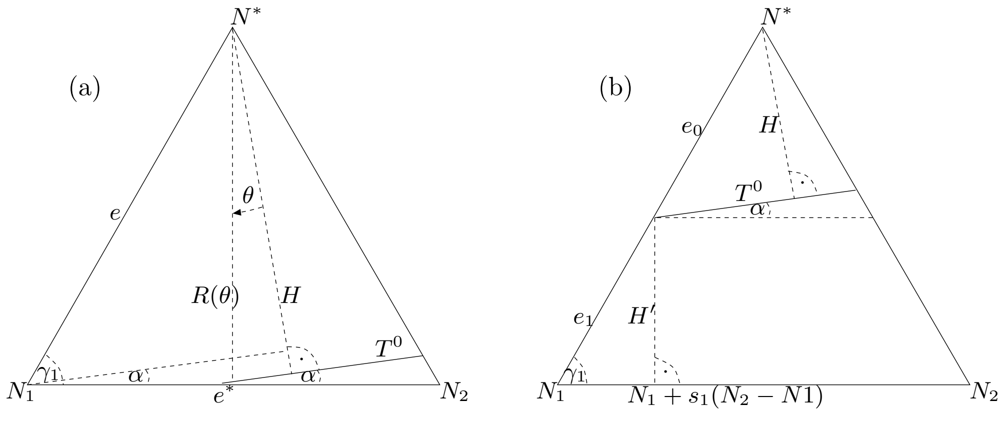


Figure 11: Illustrations for proof of Lemma 3.6. (a): Cases (A). (b): Case (B)-(D).

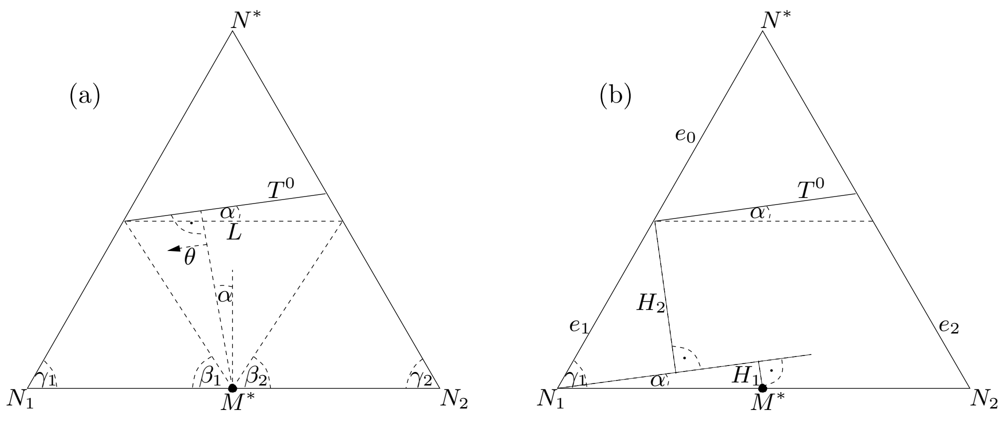


Figure 12: Illustrations for case (B)-(C) in proof of Lemma 3.6.

Lemma 3.6 (Well-resolvedness in case of straight interface sections) *Let* $T \in \mathcal{T}_h^0$ *satisfy* (24) *and let the interface section* T^0 *be straight, i.e.* $T^0 = T_1^0$. *Then Assumption 3.3 is satisfied with*

$$\begin{aligned} M_2 &= \max\left\{\frac{\sin(\theta^{**})}{\cos^2(\theta^{**})}, 1\right\}, \\ M_3 &= \frac{\varrho}{\sin(\vartheta(\varrho)/2)}, \end{aligned}$$

where

$$\begin{aligned} \theta^* &= \frac{\pi - \vartheta(\varrho)}{2}, \\ \delta^* &= \arccos\left(\frac{\sin(\vartheta(\varrho))}{\varrho}\right), \\ \theta^{**} &= \max\{\delta^*, \theta^*\} \ < \ \pi/2. \end{aligned}$$

and $\vartheta(\varrho)$ *is the smallest angle that can occur in a* ϱ*-regular triangle, see* (58).

Proof: Let us first have a look at Fig. 10(a) and introduce the angles α, γ_1, γ_2, which will be made use of throughout the proof. The face e^*, with end points N_1 and N_2, is the face of T selected by the angle condition. Fig. 10(a) displays both possibilities that may occur, namely that $T^0 \cap e^* = \emptyset$ or $T^0 \cap e^* \neq \emptyset$. In both cases we define α as the angle under which the line that contains T^0 and the line which contains e^* intersect. If they do not intersect, then $\alpha := 0$. The angle γ_1, respectively γ_2, is the interior angle of T at N_1, respectively N_2. Here N_1, N_2 are such that $N_2 \in T^\triangle$ if $T^0 \cap e^* \neq \emptyset$, and such that the angle α is centered around a point on $\overline{N_1 N^*}$ if $T^0 \cap e^* = \emptyset$. Then we have

$$\begin{aligned} &\vartheta(\varrho) \ \le \ \gamma_1, \gamma_2 \ \le \ \pi/2 \\ &0 \ \le \ \alpha \ \le \ \gamma_1/2 \ \le \ \pi/4 \\ &\pi/2 \ \ge \ \gamma_1 - \alpha \ \ge \ \gamma_1/2 \ \ge \ \vartheta(\varrho)/2, \end{aligned} \tag{38}$$

where we have used (24).

<u>Case (A):</u> Apart from the angles defined as above, in this case we use the notations according to Fig. 11(a): For $T^\square$ we may assume, that $\theta = 0$ corresponds to the line through N^*, which is perpendicular to T^0, and that the distance from N^* to the line that contains T^0 is denoted by H. Note that the line which corresponds to $\theta = 0$ may ly outside of T. We find

$$\begin{aligned} |\theta| &\le \max\{\pi/2 - (\gamma - \alpha), \pi - (\pi/2 + \alpha) - \gamma'\} \\ &\le \max\{\pi/2 - \vartheta(\varrho)/2, \pi/2 - \alpha - \vartheta(\varrho)\} \ \le \ \theta^* \\ R(\theta) &= \frac{H}{\cos(\theta)}, \qquad \theta \in [\theta_1, \theta_2] \subset [-\theta^*, \theta^*] \qquad (39) \\ h_T \ \ge \ R(\theta) &\ge H \\ R'(\theta) &= H\,\frac{\sin(\theta)}{\cos^2(\theta)} \qquad (40) \\ |R'(\theta)| &\le h_T\,\frac{\sin(\theta^*)}{\cos^2(\theta^*)} \ \le \ M_2 h_T \qquad (41) \end{aligned}$$

Note that (41) holds along T^0 as well as along $\overline{N_1 N_2} \cap \partial T^\square$, since the case $\alpha = 0$ has been included in all steps. But we have to take care of the fact, that along T^0 and $\overline{N_1 N_2} \cap \partial T^\square$ we have different values for H. Thus we have

$$\begin{aligned} R_{min} &\geq \min\{|e|\sin(\gamma-\alpha),\, |e|\sin(\gamma)\} \;\geq\; |e|\,\sin(\gamma/2) \\ &\geq 2\,r_T\,\sin(\vartheta(\varrho)/2) \;\geq\; \frac{2\,h_T}{\varrho}\sin(\vartheta(\varrho)/2) \;\geq\; h_T/M_3\,. \end{aligned}$$

We see that (32), (33) are always satisfied for $T^{\square}$, so that we do not have to examine $T^{\triangle}$ in case (A).

Case (B)-(D): In this case we use the notations according to Fig. 11(b): Similar to case (A), H is the distance from N^* to the line that contains T^0, and for $T^{\triangle}$ we obtain $R(\theta)$ as in (39) and we find that (40), (41) hold again and

$$R_{min} \;\geq\; H \;=\; |e_0|\sin(\gamma-\alpha) \;\geq\; |e_0|\sin(\vartheta(\varrho)/2)\,.$$

Furthermore, we have

$$\begin{aligned} |e_0|+|e_1| &\geq 2\,r_T \geq 2\,h_T/\varrho \\ \max\{|e_0|,|e_1|\} &\geq h_T/\varrho\,. \end{aligned} \tag{42}$$

Now if $|e_0| \geq |e_1|$, then $R_{min} \geq h_T/M_3$. If $|e_1| \geq |e_0|$, we examine $T^{\square}$: We have

$$\begin{aligned} R(s) &= H' + (s-s_1)\,||N_2-N_1||_2\tan(\alpha) \\ R'(s) &= ||N_2-N_1||_2\tan(\alpha) \\ |R'(s)| &\leq h_T\,\tan(\alpha) \;\leq\; h_T\,, \end{aligned}$$

since we have again (38). Furthermore, using $|e_1| \geq h_T/\varrho$, we find

$$R_{min} \;=\; H' \;=\; |e_1|\sin(\gamma) \;\geq\; h_T\,\sin(\vartheta(\varrho))/\varrho \;\geq\; h_T/M_3\,.$$

Case (B)-(C): In this case we use the notations according to Fig. 12(a),(b): As in case (B)-(D), we can distinguish between the cases $|e_0| \geq |e_1|$ and $|e_1| \geq |e_0|$ and have (42). If $|e_0| \geq |e_1|$, we can proceed for $T^{\triangle}$ as in case (B)-(D). If $|e_1| \geq |e_0|$, we examine $T^{\square}$, see Fig. 12(a): Let $\theta = 0$ correspond to the line through M^*, which is perpendicular to T^0, then we find

$$\begin{aligned} L &\leq |e^*|/2 \\ \beta_2 &\geq \gamma_1 \;\geq\; \vartheta(\varrho) \\ \beta_1 &\geq \gamma_2 \;\geq\; \vartheta(\varrho) \\ |\theta| &\leq \max\{\pi/2-\alpha-\beta_1,\, \pi/2+\alpha-\beta_2\} \;\leq\; \max\{\pi/2-\vartheta(\rho),\, \pi/2+\gamma_1/2-\gamma_1\} \\ &\leq \pi/2-\vartheta(\varrho)/2 \;=\; \theta^*. \end{aligned}$$

In Fig. 12(b) we find along T^0, that

$$\begin{aligned} H &= H_1+H_2 \\ R_{min} &\geq H \;\geq\; H_2 \;=\; |e_0|\sin(\gamma_1-\alpha) \;\geq\; \frac{h_T}{\varrho}\sin(\vartheta(\varrho)/2) \\ R(\theta) &= \frac{H}{\cos(\theta)}\,, \qquad \theta\in[\theta_1,\theta_2]\subset[-\theta^*,\theta^*] \\ h_T \;\geq\; R(\theta) &\geq H. \end{aligned}$$

We finally have to examine $R(\theta)$ on $\partial T^{\Box} \setminus (\overline{N_1 N_2} \cup T^0)$, i.e. on e_0 and e_2: We consider e_2, for e_0 the arguments are the same. According to Fig. 10(b) we have

$$\begin{aligned} R_{min} &= H = \frac{|e^*|}{2} \sin(\gamma_2) \geq \frac{h_T}{\varrho} \sin(\vartheta(\varrho)) \\ H^* &\leq h_T \\ \cos(\delta) &= \frac{H}{H^*} \geq \frac{\sin(\vartheta(\varrho))}{\varrho} = \cos(\delta^*) \\ |\theta| &\leq \max\{\delta^*, \pi/2 - \gamma_2\} \leq \theta^{**} \\ R(\theta) &= \frac{H}{\cos(\theta)}, \qquad \theta \in [\theta_1, \theta_2] \subset [-\theta^{**}, \theta^{**}] \\ h_T \geq R(\theta) &\geq H. \end{aligned}$$

Thus we obtain (40) again, and alltogether for $T^{\Box}$ the estimates (32), (33) hold with M_2 and M_3 as given in the Lemma. □

From Lemma 3.5 and 3.6, we can deduce for the interpolation operator π_p of Theorem C.4 the following approximation property:

Lemma 3.7 *Let $T \in \mathcal{T}_h^0$ satisfy (24), $u \in H^s(T)$ and let j be a multi-index with $0 \leq |j| \leq s-1$. Then we have*

$$||\partial^j (u - \pi_p(u))||_{0,e} \leq C_{(3.7)} \frac{h_T^{\mu-|j|-1/2}}{p^{s-|j|-1/2}} ||u||_{s,T}$$

for any face $e \subset \partial T$. Additionally let Assumption 3.3 hold. Then

$$||\partial^j (u - \pi_p(u))||_{0,T^0} \leq C_{(3.7)} \frac{h_T^{\mu-|j|-1/2}}{p^{s-|j|-1/2}} ||u||_{s,T}$$

where $\mu = \min(p+1, s)$ and $C_{(3.7)} = \sqrt{2\, C_{(3.5)}}\, C_{(\mathrm{C.4})}$.

Proof: Using $v \equiv \partial^j (u - \pi_p(u)) \in H^1(T)$, from Lemma 3.5 and Theorem C.4 we get

$$\begin{aligned} ||v||_{0,T^0}^2 &= \lambda_T^+ ||v||_{0,T^0}^2 + \lambda_T^- ||v||_{0,T^0}^2 \\ &\leq C_{(3.5)} \cdot \left(\frac{p}{h_T} ||v||_{0,T}^2 + \frac{h_T}{p} |v|_{1,T}^2 \right) \\ &\leq C_{(3.5)}\, C_{(\mathrm{C.4})}^2 \cdot \left(\frac{h_T^{2(\mu-|j|)-1}}{p^{2(s-|j|)-1}} ||u||_{s,T}^2 + \frac{h_T^{2(\mu-|j|-1)+1}}{p^{2(s-|j|-1)+1}} ||u||_{s,T}^2 \right). \end{aligned}$$

This proves the second estimate of the lemma. Since $T^0 = e \subset \partial T$ is contained in Lemma 3.6 as a limiting case, we can use the constants M_2, M_3 from Lemma 3.6 in the definition of $C_{(3.5)}$, and the first estimate of the lemma is obtained in the same way as the second estimate of the lemma. □

3.3 Lifting operators

This section will provide us with bounds for the terms in (22) that contain fluxes across faces of the triangulation and across the interface. Furthermore, these bounds will enable us to introduce lifting operators, which will then be used to extend (22) to a variational formulation on $H^1(\Omega^+) \times H^1(\Omega^-)$.

As auxiliary Hilbert spaces we introduce

$$M_h^\pm = (V_h^\pm)^2, \qquad (\boldsymbol{q},\boldsymbol{p})_{M_h^\pm} = \int_{\Omega^\pm} \kappa^\pm \cdot (q_1\,p_1 + q_2\,p_2)\;dx\,,$$

$$M_h = M_h^+ \times M_h^-, \qquad (\boldsymbol{r},\boldsymbol{s})_{M_h} = (\boldsymbol{r}^+,\boldsymbol{s}^+)_{M_h^+} + (\boldsymbol{r}^-,\boldsymbol{s}^-)_{M_h^-}\,,$$

and denote the corresponding norms by $||\cdot||_{M_h^\pm}$ and $||\cdot||_{M_h}$. For $v^\pm \in V_h^\pm$, respectively $v \in V_h$, the linear functionals

$$l^\pm(\boldsymbol{q}) \equiv \sum_{e\in\mathcal{E}_h^\pm} (\kappa^\pm\,\overline{\boldsymbol{n}\cdot\boldsymbol{q}}, [v^\pm])_{e^\pm}\,, \qquad \text{respectively} \qquad l^0(\boldsymbol{r}) \equiv \sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}}, [v])_{T^0}\,,$$

appear in (19) & (20) or (22), with $\boldsymbol{q} \equiv \nabla u_h^\pm$, respectively $\boldsymbol{r} \equiv (\nabla u_h^+, \nabla u_h^-)$. We will next determine the bounds of these functionals when

$$v^\pm \equiv v_h^\pm + v_0^\pm \in W_h^\pm \equiv V_h^\pm + V^\pm, \qquad v \equiv (v^+,v^-) \in W_h \equiv V_h + V.$$

Lemma 3.8 (Boundedness and definition of the Lifting Operators) *Let Assumption 2.3, (24), (27), (28), (29), and Assumption 3.3 be satisfied and define $\lambda_e^\pm$, respectively $\lambda_T^\pm$, according to Lemma 3.2, repectively Lemma 3.4. Then for all $v \equiv v_h + v_0 \in W_h$ the estimates*

$$\sum_{e\in\mathcal{E}_h^\pm} (\kappa^\pm\,\overline{\boldsymbol{n}\cdot\boldsymbol{q}}, [v^\pm])_{e^\pm} \le \left(C_{(3.8)} \cdot \sum_{e\in\mathcal{E}_h^\pm} \kappa^\pm \frac{(p_e^\pm)^2}{|e^\pm|} ||[v_h^\pm]||^2_{0,e^\pm} \right)^{1/2} \cdot ||\boldsymbol{q}||_{M_h^\pm} \qquad \forall\, \boldsymbol{q} \in M_h^\pm\,,$$

$$\sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}}, [v])_{T^0} \le \left(C_{(3.8)} \cdot \sum_{T\in\mathcal{T}_h^0} \kappa^0 \frac{(p_T^0)^2}{h_T} ||[v]||^2_{0,T^0} \right)^{1/2} \cdot ||\boldsymbol{r}||_{M_h} \qquad \forall\, \boldsymbol{r} \in M_h\,.$$

hold, i.e. there exist bounded linear operators $\boldsymbol{L}_h^\pm : W_h^\pm \to M_h^\pm$, $\boldsymbol{L}_h^0 : W_h \to M_h$ satisfying

$$(\boldsymbol{L}_h^\pm(v),\boldsymbol{q})_{M_h^\pm} = \sum_{e\in\mathcal{E}_h^\pm} (\kappa^\pm\,\overline{\boldsymbol{n}\cdot\boldsymbol{q}}, [v^\pm])_{e^\pm} \qquad \forall\, \boldsymbol{q} \in M_h^\pm, \tag{43}$$

$$(\boldsymbol{L}_h^0(v),\boldsymbol{r})_{M_h} = \sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}}, [v])_{T^0} \qquad \forall\, \boldsymbol{r} \in M_h\,. \tag{44}$$

Using the seminorms introduced in (49), we have

$$||\boldsymbol{L}_h^+(v^+)||^2_{M_h^+} + ||\boldsymbol{L}_h^-(v^-)||^2_{M_h^-} \le \frac{C_{(3.8)}}{\gamma}\,|||[v]|||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \qquad \forall\, v \in W_h\,,$$

$$||\boldsymbol{L}_h^0(v)||^2_{M_h} \le \frac{C_{(3.8)}}{\gamma}\,|||[v]|||^2_{0,h,\Gamma} \qquad \forall\, v \in W_h\,.$$

The constant is $C_{(3.8)} = \max\left\{G_2 \max\left\{C_{(67)}, 2\,C_{(3.2)}\right\}, C_{(3.4)}\right\}$.

Proof:

$$\sum_{e\in\mathcal{E}_h^\pm} (\kappa^\pm\,\overline{\boldsymbol{n}\cdot\boldsymbol{q}}, [v])_{e^\pm} \le \left(\sum_{e\in\mathcal{E}_h^\pm} \frac{\kappa^\pm|e^\pm|}{(p_e^\pm)^2} ||\overline{\boldsymbol{n}\cdot\boldsymbol{q}}||^2_{0,e^\pm}\right)^{1/2} \left(\sum_{e\in\mathcal{E}_h^\pm} \frac{\kappa^\pm(p_e^\pm)^2}{|e^\pm|} ||[v]||^2_{0,e^\pm}\right)^{1/2}$$

Depending on whether (13), (14), (15) applies we have with $T, \tilde{T} \in \omega(e)$ that

$$\frac{|e^\pm|}{(p_e^\pm)^2}\, ||\overline{\boldsymbol{n}\cdot\boldsymbol{q}}||^2_{0,e^\pm} \;\leq\; \begin{cases} \frac{1}{2}\,\frac{h_T}{(p_T^\pm)^2}||\boldsymbol{n}\cdot\boldsymbol{q}|_T||^2_{0,e} \;+\; \frac{1}{2}\,\frac{h_{\tilde{T}}}{(p_{\tilde{T}}^\pm)^2}||\boldsymbol{n}\cdot\boldsymbol{q}|_{\tilde{T}}||^2_{0,e} \\ 2\,\frac{(\lambda_e^\pm)^2|e^\pm|}{(p_T^\pm)^2}||\boldsymbol{n}\cdot\boldsymbol{q}|_T||^2_{0,e^\pm} + 2\,\frac{(\tilde{\lambda}_e^\pm)^2|e^\pm|}{(p_{\tilde{T}}^\pm)^2}||\boldsymbol{n}\cdot\boldsymbol{q}|_{\tilde{T}}||^2_{0,e^\pm} \\ \frac{h_T}{(p_T^\pm)^2}||\boldsymbol{n}\cdot\boldsymbol{q}|_T||^2_{0,e} \end{cases}$$

and thus according to (67) and Lemma 3.2

$$\frac{|e^\pm|}{(p_e^\pm)^2}\; ||\overline{\boldsymbol{n}\cdot\boldsymbol{q}}||^2_{0,e^\pm} \;\leq\; \begin{cases} \frac{C_{(67)}}{2}\left(||\boldsymbol{n}\cdot\boldsymbol{q}||^2_{0,T^\pm} + ||\boldsymbol{n}\cdot\boldsymbol{q}||^2_{0,\tilde{T}^\pm}\right) \\ 2\,C_{(3.2)}\,\left(||\boldsymbol{n}\cdot\boldsymbol{q}||^2_{0,T^\pm} + ||\boldsymbol{n}\cdot\boldsymbol{q}|||^2_{0,\tilde{T}^\pm}\right) \\ C_{(67)}\,||\boldsymbol{n}\cdot\boldsymbol{q}||^2_{0,T^\pm}\,, \end{cases}$$

which can be combined to

$$\kappa^\pm\,\frac{|e^\pm|}{(p_e^\pm)^2}\,||\overline{\boldsymbol{n}\cdot\boldsymbol{q}}||^2_{0,e^\pm} \;\leq\; C_{(45)}\,\kappa^\pm \sum_{T\in\omega(e)} ||\boldsymbol{q}||^2_{L^2(T^\pm)^2}\,, \tag{45}$$

where $C_{(45)} = \max\{C_{(67)}, 2\,C_{(3.2)}\}$. Thus we have

$$\begin{aligned} \sum_{e\in\mathcal{E}_h^\pm}(\kappa^\pm\,\overline{\boldsymbol{n}\cdot\boldsymbol{q}},[v])_{e^\pm} \;&\leq\; \left(\sum_{T\in\mathcal{T}_h^\pm} G_2 C_{(45)}\kappa^\pm||\boldsymbol{q}||^2_{L^2(T^\pm)^2}\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h^\pm}\frac{\kappa^\pm(p_e^\pm)^2}{|e^\pm|}\,||[v_h^\pm]||^2_{0,e^\pm}\right)^{1/2} \\ &=\; \sqrt{G_2 C_{(45)}}\;||\boldsymbol{q}||_{M_h^\pm}\left(\sum_{e\in\mathcal{E}_h^\pm}\frac{\kappa^\pm(p_e^\pm)^2}{|e^\pm|}\,||[v_h^\pm]||^2_{0,e^\pm}\right)^{1/2}. \end{aligned}$$

This proves the first estimate of the Lemma. The proof of the second estimate is similar:

$$\begin{aligned} \sum_{T\in\mathcal{T}_h^0}(\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}},[v])_{T^0} \;&=\; \sum_{T\in\mathcal{T}_h^0}\left(\kappa^+\lambda_T^+\boldsymbol{n}^+\cdot\boldsymbol{r}^+ + \kappa^-\lambda_T^-\,\boldsymbol{n}^+\cdot\boldsymbol{r}^-,\,[v]\right)_{T^0} \\ &\leq\; \left(\sum_{T\in\mathcal{T}_h^0} I_T^+ + I_T^-\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h^0}\frac{\kappa^0\,(p_T^0)^2}{h_T}||[v]||^2_{0,T^0}\right)^{1/2} \end{aligned}$$

where

$$I_T^\pm \;=\; \frac{(\kappa^\pm)^2}{\kappa^0}\,(\lambda_T^\pm)^2\,\frac{h_T}{(p_T^0)^2}\,||\boldsymbol{n}^+\cdot\boldsymbol{r}^\pm||^2_{0,T^0} \;\leq\; C_{(3.4)}\,\kappa^\pm\,||\boldsymbol{n}^+\cdot\boldsymbol{r}^\pm||^2_{0,T^\pm} \;\leq\; C_{(3.4)}\,\kappa^\pm\,||\boldsymbol{r}^\pm||^2_{L^2(T^\pm)^2}\,.$$

Thus

$$\begin{aligned} \sum_{T\in\mathcal{T}_h^0} I_T^+ + I_T^- \;&\leq\; C_{(3.4)}\left(\kappa^+||\boldsymbol{r}^+||^2_{L^2(\Omega^+)^2} + \kappa^-||\boldsymbol{r}^-||^2_{L^2(\Omega^-)^2}\right) \\ &=\; C_{(3.4)}\,||\boldsymbol{r}||^2_{M_h}\,, \end{aligned}$$

so that the second estimate of the lemma is also proven. The remaining assertions made in the lemma are then a consequence of the Riesz representation theorem. □

3.4 Primal Formulation

From now on, we use λ_e as given in Lemma 3.2 to evaluate $\overline{v^\pm}$ in (14) and λ_T as given in Lemma 3.4 to evaluate $\overline{v}$ in (21). For $j_D \in H^{1/2}(\Gamma)$, let $\hat{j}_D^+ \in H^1(\Omega^+)$ be an extension of j_D to Ω^+ and define $\hat{j}_D^- = 0 \in H^1(\Omega^-)$ and $\hat{j}_D = \hat{j}_D^\pm$ on $\Omega^\pm$. Then

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}}, j_D)_{T^0} &= \sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\,\boldsymbol{n}^+\cdot\boldsymbol{r}}, [\hat{j}_D])_{T^0} \\ &= \int_{\Omega^+} \kappa^+ \boldsymbol{L}_h^0(\hat{j}_D)^+\cdot\boldsymbol{r}^+\,dx \;+\; \int_{\Omega^-} \kappa^- \boldsymbol{L}_h^0(\hat{j}_D)^-\cdot\boldsymbol{r}^-\,dx\,.\end{aligned}$$

We extend $\mathcal{B}_h : V_h\times V_h \to \mathbb{R}$ to a bilinear form $\mathcal{B}_h : W_h\times W_h\to\mathbb{R}$ by defining

$$\begin{aligned}\mathcal{B}_h(v,\phi) \;\equiv\; & \sum_{T\in\mathcal{T}_h}\langle\kappa\,\nabla v,\nabla\phi\rangle_T + \sum_{e\in\mathcal{E}(\mathcal{T}_h)}\langle\sigma_e\,\kappa\,[v],[\phi]\rangle_e + \sum_{T\in\mathcal{T}_h^0}\sigma_T^0\cdot(\kappa^0[v],[\phi])_{T^0} \\ & - \big(\boldsymbol{L}_h^+(\phi^+) + \boldsymbol{L}_h^0(\phi)^+, \kappa^+\nabla v^+\big)_{\Omega^+} - \big(\boldsymbol{L}_h^+(v^+) + \boldsymbol{L}_h^0(v)^+, \kappa^+\nabla\phi^+\big)_{\Omega^+} \\ & - \big(\boldsymbol{L}_h^-(\phi^-) + \boldsymbol{L}_h^0(\phi)^-, \kappa^-\nabla v^-\big)_{\Omega^-} - \big(\boldsymbol{L}_h^-(v^-) + \boldsymbol{L}_h^0(v)^-, \kappa^-\nabla\phi^-\big)_{\Omega^-}\end{aligned} \tag{46}$$

and similarly we extend $\mathcal{F}_h : V_h\to\mathbb{R}$ to a linear functional $\mathcal{F}_h : W_h\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{F}_h(\phi) \;\equiv\; & \sum_{T\in\mathcal{T}_h}\langle f,\phi\rangle_T + \sum_{T\in\mathcal{T}_h^0}\lambda_T^-(j_N,\phi^+)_{T^0} + \lambda_T^+(j_N,\phi^-)_{T^0} + \sigma_T^0\cdot(\kappa^0 j_D,[\phi])_{T^0} \\ & -\big(\boldsymbol{L}_h^0(\hat{j}_D)^+,\kappa^+\nabla\phi^+\big)_{\Omega^+} - \big(\boldsymbol{L}_h^0(\hat{j}_D)^-,\kappa^-\nabla\phi^-\big)_{\Omega^-}\,.\end{aligned} \tag{47}$$

Our DG Method (22) still reads as follows: Find $u_h\in V_h$ such that

$$\mathcal{B}_h(u_h,\phi) \;=\; \mathcal{F}_h(\phi) \qquad \forall\,\phi\in V_h\,. \tag{48}$$

Let us introduce the following seminorms and the grid-dependent energy norm $\|\cdot\|_h$ on W_h:

$$\begin{aligned}|v|^2_{1,h,\Omega} &= \sum_{T\in\mathcal{T}_h}\langle\kappa\,\nabla v,\nabla v\rangle_T\,, && v\in W_h \\ ||v||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} &= \sum_{e\in\mathcal{E}(\mathcal{T}_h)}\langle\sigma_e\,\kappa\,v,v\rangle_e\,, && v\equiv(v^+,v^-), v^\pm\in L^2(\Omega^\pm_{\mathcal{E}}) \\ ||v||^2_{0,h,\Gamma} &= \sum_{T\in\mathcal{T}_h^0}\sigma_T^0\kappa^0(v,v)_{T^0}\,, && v\in L^2(\Gamma) \\ ||v||^2_h &= |v|^2_{1,h,\Omega} + ||[v]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} + ||[v]||^2_{0,h,\Gamma}\,, && v\in W_h\end{aligned} \tag{49}$$

Here we have employed $\Omega^\pm_{\mathcal{E}} \equiv \{x\in\Omega^\pm : x\in e \text{ for a face } e\in\mathcal{E}(\mathcal{T}_h)\}$.

Lemma 3.9 (Boundedness of $\mathcal{B}_h$) *Let Assumption 2.3,* (24)-(29)*, and Assumption 3.3 be satisfied. Then we have*

$$|\mathcal{B}_h(v,\phi)| \;\le\; C_{(3.9)}\cdot||v||_h\cdot||\phi||_h \qquad \forall\,v,\phi\in W_h\,,$$

where $C_{(3.9)} = 1 + 2\,C_{(3.8)}/\gamma$.

Proof:

$$\begin{aligned}
|\mathcal{B}_h(v,\phi)| \;\; &\leq \;\; |v|_{1,h,\Omega}\,|\phi|_{1,h,\Omega} \;+\; ||[v_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)}\,||[\phi_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)} \;+\; ||[v]||_{0,h,\Gamma}\,||[\phi]||_{0,h,\Gamma} \\
&\quad + |(\boldsymbol{L}_h^+(\phi^+), \nabla v^+)_{M_h^+}| \;+\; |(\boldsymbol{L}_h^-(\phi^-), \nabla v^-)_{M_h^-}| \;+\; |(\boldsymbol{L}_h^0(\phi), \nabla v)_{M_h}| \\
&\quad + |(\boldsymbol{L}_h^+(v^+), \nabla \phi^+)_{M_h^+}| \;+\; |(\boldsymbol{L}_h^-(v^-), \nabla \phi^-)_{M_h^-}| \;+\; |(\boldsymbol{L}_h^0(v), \nabla \phi)_{M_h}| \\
&\leq \;\; |v|_{1,h,\Omega}\,|\phi|_{1,h,\Omega} \;+\; ||[v_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)}\,||[\phi_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)} \;+\; ||[v]||_{0,h,\Gamma}\,||[\phi]||_{0,h,\Gamma} \\
&\quad + \frac{C_{(3.8)}}{\gamma}\,\big(||[\phi_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)}\,|v|_{1,h,\Omega} \;+\; ||[\phi]||_{0,h,\Gamma}\,|v|_{1,h,\Omega} \\
&\qquad\qquad + ||[v_h]||_{0,h,\mathcal{E}(\mathcal{T}_h)}\,|\phi|_{1,h,\Omega} \;+\; ||[v]||_{0,h,\Gamma}\,|\phi|_{1,h,\Omega}\big) \\
&\leq \;\; \left(1 \,+\, 2\,\frac{C_{(3.8)}}{\gamma}\right)\cdot ||v||_h \cdot ||\phi||_h\,. \qquad \square
\end{aligned}$$

Lemma 3.10 (Coercivity of $\mathcal{B}_h$) *Let Assumption 2.3,* (24)-(29), *and Assumption 3.3 be satisfied. Then for* $\gamma \geq 4\,C_{(3.8)}$ *we have*

$$\mathcal{B}_h(v,v) \;\; \geq \;\; \frac{1}{2}\,||v||_h^2 \qquad \forall\, v \in W_h\,.$$

Proof:

$$\begin{aligned}
\mathcal{B}_h(v,v) \;\; &= \;\; |v|^2_{1,h,\Omega} \,+\, ||[v_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \,+\, ||[v]||^2_{0,h,\Gamma} \\
&\quad - 2\,\big(\boldsymbol{L}_h^+(v^+) \,+\, \boldsymbol{L}_h^0(v)^+, \kappa^+\,\nabla v^+\big)_{\Omega^+} \,-\, 2\,\big(\boldsymbol{L}_h^-(v^-) \,+\, \boldsymbol{L}_h^0(v)^-, \kappa^-\,\nabla v^-\big)_{\Omega^-} \\
&\geq \;\; |v|^2_{1,h,\Omega} \,+\, ||[v_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \,+\, ||[v]||^2_{0,h,\Gamma} \\
&\quad - \varepsilon^{-1}\cdot\Big(||\boldsymbol{L}_h^+(v^+)||^2_{M_h^+} \,+\, ||\boldsymbol{L}_h^-(v^-)||^2_{M_h^-} \,+\, ||\boldsymbol{L}_h^0(v)||^2_{M_h}\Big) \,-\, \varepsilon\cdot|v|^2_{1,h,\Omega} \\
&\geq \;\; |v|^2_{1,h,\Omega} \,+\, ||[v_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \,+\, ||[v]||^2_{0,h,\Gamma} \\
&\quad - \frac{C_{(3.8)}}{\gamma\,\varepsilon}\cdot\Big(||[v_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \,+\, ||[v]||^2_{0,h,\Gamma}\Big) \,-\, \varepsilon\cdot|v|^2_{1,h,\Omega} \\
&= \;\; (1-\varepsilon)\cdot|v|^2_{1,h,\Omega} \,+\, \left(1 \,-\, \frac{C_{(3.8)}}{\gamma\,\varepsilon}\right)\cdot\Big(||[v_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \,+\, ||[v]||^2_{0,h,\Gamma}\Big)\,,
\end{aligned}$$

for any $\varepsilon > 0$. We choose $\varepsilon = 1/2$ and $\gamma \geq 4\,C_{(3.8)}$. $\square$

Furthermore, we easily prove that $\mathcal{F}_h$ is bounded on V_h by estimating

$$\begin{aligned}|\mathcal{F}_h(\phi)| \;\le\; & ||f||_{0,\Omega^+}\cdot||\phi^+||_{0,\Omega^+} + ||f||_{0,\Omega^-}\cdot||\phi^-||_{0,\Omega^-} + \sum_{T\in\mathcal{T}_h^0}||j_N||_{0,T^0}\cdot\left(||\phi^+||_{0,T^0}+||\phi^-||_{0,T^0}\right)\\ & + ||j_D||_{0,h,\Gamma}\cdot||[\phi]||_{0,h,\Gamma} + \frac{C_{(3.8)}}{\gamma}||j_D||_{0,h,\Gamma}\cdot|\phi|_{1,h,\Omega}\\ \le\; & \left(||f||^2_{0,\Omega^+} + ||f||^2_{0,\Omega^-} + \left(1+\frac{C^2_{(3.8)}}{\gamma^2}\right)||j_D||^2_{0,h,\Gamma} + \sum_{T\in\mathcal{T}_h^0}||j_N||^2_{0,T^0}\right)^{1/2}\times\\ & \left(||\phi^+||^2_{0,\Omega^+} + ||\phi^-||^2_{0,\Omega^-} + ||\phi^+||^2_{0,\Gamma} + ||\phi^-||^2_{0,\Gamma} + ||[\phi]||^2_{0,h,\Gamma} + |\phi|^2_{1,h,\Omega}\right)^{1/2}\end{aligned}$$

and noting that the last line constitutes a norm on the finite dimensional space V_h. Employing techniques as in Lemma 2.1 of [2], we can also prove that $\mathcal{F}_h$ is bounded on V with respect to $\|\cdot\|_h$. Then the variational problem (48) can also be considered on V and has a unique solution in V. Although a nice property, this is of no significance to our convergence analysis.

3.5 Approximation error

Let us define the following domain and spaces,

$$\begin{aligned}&\Omega_h^\pm \equiv \{x\in\Omega : x\in T \text{ for a triangle } T\in\mathcal{T}_h^\pm\},\\ &\overline{V}_h^\pm \equiv \mathcal{S}^{p^\pm}(\mathcal{T}_h^\pm), && \overline{V}_h \equiv \overline{V}_h^+\times\overline{V}_h^-,\\ &\overline{V}^\pm \equiv H^1(\Omega_h^\pm), && \overline{V}\equiv\overline{V}^+\times\overline{V}^-.\end{aligned} \tag{50}$$

Note that since $\Omega^\pm\subset\Omega_h^\pm$ we can restrict functions from $\overline{V}_h^\pm$, respectively $\overline{V}^\pm$, to $\Omega^\pm$ and obtain functions belonging to $V_h^\pm$, respectively $V^\pm$. Thus it is well-defined to plug functions from $\overline{V}_h$ or $\overline{V}$ into $\mathcal{B}(\cdot,\cdot)$ or $\|\cdot\|_h$. Note that V_h and $\overline{V}_h$ can be identified and while $\|\cdot\|_h$ is still a norm on $\overline{V}_h$ it is not a norm on $\overline{V}$.

Lemma 3.11 *Let Assumptions 2.3, (24), 3.3 be satisfied and let π_p be the operator given in Theorem C.4. Then we define the interpolation operator $\Pi:\overline{V}\to\overline{V}_h$ by*

$$\Pi(v)^\pm|_T \;=\; \pi_{p_T^\pm}(v^\pm|_T) \qquad \forall\, T\in\mathcal{T}_h^\pm$$

and we have for all $v=(v^+,v^-)\in\overline{V}=\overline{V}^+\times\overline{V}^-$ with $v^\pm|_T\in H^{s_T^\pm}(T)$, $s_T^\pm\ge 2$, the estimate

$$||v-\Pi(v)||_h^2 \;\le\; C_{(3.11)}\cdot\left(\sum_{T\in\mathcal{T}_h^+}\frac{h_T^{2\mu_T^+-2}}{(p_T^+)^{2s_T^+-3-\chi(|T^-|)}}||v^+||^2_{s_T^+,T} + \sum_{T\in\mathcal{T}_h^-}\frac{h_T^{2\mu_T^--2}}{(p_T^-)^{2s_T^--3-\chi(|T^+|)}}||v^-||^2_{s_T^-,T}\right),$$

where $C_{(3.11)}=C^2_{(C.4)}\,\kappa^0\,(1+2\gamma G_1^2G_2\max\{2G_3C_{(3.5)},1\})$ and $\mu_T^\pm=\min\{p_T^\pm+1,s_T^\pm\}$. The characteristic function χ is given by $\chi(t)=1$ for $t>0$ and $\chi(t)=0$ for $t\le 0$.

Proof: Using Theorem C.4 we find

$$|v^\pm-\Pi(v)^\pm|^2_{1,T^\pm} \;\le\; |v^\pm-\Pi(v)^\pm|^2_{1,T} \;\le\; C^2_{(C.4)}\,\frac{h_T^{2\mu_T^\pm-2}}{(p_T^\pm)^{2s_T^\pm-2}}||v^\pm||^2_{s_T^\pm,T} \qquad \forall\, T\in\mathcal{T}_h^\pm\,.$$

Now let $e \in \mathcal{E}_h^{\pm}$ and $T \in \omega(e)$. If $e = e^{\pm}$, then

$$\begin{aligned}\sigma_e^{\pm}\,||(v^{\pm} - \Pi(v)^{\pm})|_T||_{0,e^{\pm}}^2 &= \gamma\,\frac{(p_e^{\pm})^2}{|e|}\,||(v^{\pm} - \Pi(v)^{\pm})|_T||_{0,e}^2 \\ &\leq \gamma\,G_3 G_1^2\,\frac{(p_T^{\pm})^2}{h_T}\,||(v^{\pm} - \Pi(v)^{\pm})|_T||_{0,e}^2 \\ &\leq \gamma\,G_3 G_1^2 C_{(3.7)}^2\,\frac{h_T^{2\mu_T^{\pm}-2}}{(p_T^{\pm})^{2s_T^{\pm}-3}}\,||v^{\pm}||_{s_T^{\pm},T}^2\,.\end{aligned}$$

If $e \neq e^{\pm}$, then

$$\begin{aligned}\sigma_e^{\pm}\,||(v^{\pm} - \Pi(v)^{\pm})|_T||_{0,e^{\pm}}^2 &\leq \gamma\,(p_e^{\pm})^2\,||(v^{\pm} - \Pi(v)^{\pm})|_T||_{L^{\infty}(e^{\pm})}^2 \\ &\leq \gamma\,C_{(\mathrm{C.4})}^2 G_1^2\,\frac{h_T^{2\mu_T^{\pm}-2}}{(p_T^{\pm})^{2s_T^{\pm}-4}}\,||v^{\pm}||_{s_T^{\pm},T}^2\,.\end{aligned}$$

Combining both cases, for any $e \in \mathcal{E}_h^{\pm}$ with $\{T, \tilde{T}\} = \omega(e)$ we can write

$$\sigma_e^{\pm}\,||[v^{\pm} - \Pi(v)^{\pm}]||_{0,e^{\pm}}^2 \leq C\left(\frac{h_T^{2\mu_T^{\pm}-2}}{(p_T^{\pm})^{2s_T^{\pm}-3-\chi(|e^{\mp}|)}}\,||v^{\pm}||_{s_T^{\pm},T}^2 + \frac{h_{\tilde{T}}^{2\mu_{\tilde{T}}^{\pm}-2}}{(p_{\tilde{T}}^{\pm})^{2s_{\tilde{T}}^{\pm}-3-\chi(|e^{\mp}|)}}\,||v^{\pm}||_{s_{\tilde{T}}^{\pm},\tilde{T}}^2\right),$$

where $C = \gamma G_1^2 G_2 \max\{G_3 C_{(3.7)}^2, C_{(\mathrm{C.4})}^2\} \leq \gamma G_1^2 G_2 C_{(\mathrm{C.4})}^2 \max\{2G_3 C_{(3.5)}, 1\}$. Similarly, due to Lemma 3.7, for all $T \in \mathcal{T}_h^0$ we find

$$\sigma_T^0\,||v^{\pm} - \Pi(v)^{\pm}||_{0,T^0}^2 \leq C_{(3.7)}^2\,\gamma\,\frac{(G_1\,p_T^{\pm})^2}{h_T}\,\frac{h_T^{2\mu_T^{\pm}-1}}{(p_T^{\pm})^{2s_T^{\pm}-1}}\,||v^{\pm}||_{s_T^{\pm},T}^2\,.$$

Thus the lemma is proven. □

3.6 Consistency error

Lemma 3.12 *Let $u^{\pm} \in H^2(\Omega^{\pm})$, such that $u = (u^+, u^-)$ is the solution of Problem 1.1. Then for all $\phi \in W_h$ we have*

$$\begin{aligned}\mathcal{B}_h(u,\phi) - \mathcal{F}_h(\phi) &= \sum_{e\in\mathcal{E}(\mathcal{T}_h)} \langle \kappa\overline{\partial_n u}, [\phi]\rangle_e + \sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\partial_{n^+} u}, [\phi])_{T^0} \\ &\quad - \big(\boldsymbol{L}_h^+(\phi^+) + \boldsymbol{L}_h^0(\phi)^+, \kappa^+\,\nabla u^+\big)_{\Omega^+} - \big(\boldsymbol{L}_h^-(\phi^-) + \boldsymbol{L}_h^0(\phi)^-, \kappa^-\,\nabla u^-\big)_{\Omega^-}\,.\end{aligned}$$

In particular for all $\phi \in V$ with $[\phi] = 0$ along Γ, we have

$$\mathcal{B}_h(u,\phi) = \mathcal{F}_h(\phi)\,. \tag{51}$$

Proof: Partial integration yields

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \langle \kappa\,\nabla u, \nabla\phi\rangle_T &= \sum_{T\in\mathcal{T}_h} \langle f, \phi\rangle_T + \sum_{e\in\mathcal{E}(\mathcal{T}_h)} \langle \kappa\overline{\partial_n u}, [\phi]\rangle_e \\ &\quad + \sum_{T\in\mathcal{T}_h^0} (\kappa^+\partial_{n^+} u^+, \phi^+)_{T^0} - (\kappa^-\partial_{n^+} u^-, \phi^-)_{T^0}.\end{aligned}$$

Along each interface section T^0 we obtain from (6) the identity

$$\overline{\kappa\partial_{n^+}u} \;=\; \kappa^+\partial_{n^+}u^+ \,-\, \lambda_T^- j_N \;=\; \kappa^-\partial_{n^+}u^- \,+\, \lambda_T^+ j_N\,,$$

so that

$$\sum_{T\in\mathcal{T}_h}\langle\kappa\,\nabla u,\nabla\phi\rangle_T \;=\; \sum_{T\in\mathcal{T}_h}\langle f,\phi\rangle_T \,+\, \sum_{e\in\mathcal{E}(\mathcal{T}_h)}\langle\overline{\kappa\partial_n u},[\phi]\rangle_e$$
$$+\sum_{T\in\mathcal{T}_h^0}(\overline{\kappa\partial_{n^+}u},[\phi])_{T^0} \,+\, \lambda_T^-(j_N,\phi^+)_{T^0} \,+\, \lambda_T^+(j_N,\phi^-)_{T^0}\,,$$

and thus the first identity of the lemma follows from (46) and (47). The second identity of the lemma is an immediate consequence of the definition of the lifting operators. □

Lemma 3.13 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, where m is the number characterizing the smoothness of Γ in (23) and let $E^\pm$ be the strong m-extension operator of Theorem C.5 for $\Omega^\pm$. Let $u^\pm\in H^s(\Omega^\pm)$, with $2\le s\le m$, such that $u=(u^+,u^-)$ is the solution of Problem 1.1 and let $(E^\pm u^\pm)|_T\in H^{s_T^\pm}(T)$ for $T\in\mathcal{T}_h{}^\pm$, $2\le s_T^\pm\le m$. Then we have*

$$|\mathcal{B}_h(u,\phi)-\mathcal{F}_h(\phi)| \;\le$$
$$C_{(3.13)}\cdot\left(\sum_{T\in\mathcal{T}_h^+}\frac{\kappa^+ h_T^{2\mu_T^+-2}}{(p_T^+)^{2s_T^+-2}}\,||E^+u^+||^2_{s_T^+,T} \;+\; \frac{\kappa^- h_T^{2\mu_T^--2}}{(p_T^-)^{2s_T^--2}}\,||E^-u^-||^2_{s_T^-,T}\right)^{1/2}\cdot\;||\phi||_h \qquad \forall\,\phi\in W_h,$$

where $C_{(3.13)}=\sqrt{\dfrac{2}{\gamma}\left((G_2+1)\,C^2_{(3.7)}\,+\,2\,C_{(3.8)}\,C^2_{(\mathrm{C}.4)}\right)}$ *and* $\mu_T^\pm=\min\{p_T^\pm+1,s_T^\pm\}$.

Proof: From Lemma 3.12 we have

$$\mathcal{B}_h(u,\phi)-\mathcal{F}_h(\phi) \;=\; \sum_{e\in\mathcal{E}(\mathcal{T}_h)}\langle\overline{\kappa\partial_n u},[\phi]\rangle_e \,+\, \sum_{T\in\mathcal{T}_h^0}(\overline{\kappa\partial_{n^+}u},[\phi])_{T^0}$$
$$-\,\big(\boldsymbol{L}_h^+(\phi^+)\,+\,\boldsymbol{L}_h^0(\phi)^+,\kappa^+\,\nabla\,u^+\big)_{\Omega^+} \,-\, \big(\boldsymbol{L}_h^-(\phi^-)\,+\,\boldsymbol{L}_h^0(\phi)^-,\kappa^-\,\nabla\,u^-\big)_{\Omega^-}\,.$$

Defining $\tilde u \;=\; \Big((E^+u^+)|_{\Omega_h^+},\,(E^-u^-)|_{\Omega_h^-}\Big)\;\in\;\overline{V}$, using the operator Π from Lemma 3.11 and the definition of the lifting operators in Lemma 3.8, we have

$$\big(\boldsymbol{L}_h^\pm(\phi^\pm),\kappa^\pm\nabla(\Pi(\tilde u)^\pm)\big)_{\Omega^\pm} \;=\; \sum_{e\in\mathcal{E}_h^\pm}(\kappa^\pm\,\overline{\boldsymbol{n}\cdot\nabla(\Pi(\tilde u)^\pm)},[\phi^\pm])_{e^\pm}$$
$$\big(\boldsymbol{L}_h^0(\phi)^+,\kappa^+\nabla(\Pi(\tilde u)^+)\big)_{\Omega^+} \,+\, \big(\boldsymbol{L}_h^0(\phi)^-,\kappa^-\nabla(\Pi(\tilde u)^-)\big)_{\Omega^-} \;=\; \sum_{T\in\mathcal{T}_h^0}(\overline{\kappa\,\boldsymbol{n}^+\cdot\nabla(\Pi(\tilde u))},[\phi])_{T^0}\,,$$

which yields

$$\mathcal{B}_h(u,\phi)-\mathcal{F}_h(\phi) \;=\; \sum_{e\in\mathcal{E}(\mathcal{T}_h)}\langle\overline{\kappa\partial_n(u-\Pi(\tilde u))},[\phi]\rangle_e \,+\, \sum_{T\in\mathcal{T}_h^0}(\overline{\kappa\partial_{n^+}(u-\Pi(\tilde u))},[\phi])_{T^0}$$
$$-\,\big(\boldsymbol{L}_h^+(\phi^+),\kappa^+\,\nabla\,(u^+-\Pi(\tilde u)^+)\big)_{\Omega^+} \,-\, \big(\boldsymbol{L}_h^-(\phi^-),\kappa^-\,\nabla\,(u^--\Pi(\tilde u)^-)\big)_{\Omega^-}$$
$$-\,\big(\boldsymbol{L}_h^0(\phi)^+,\kappa^+\,\nabla\,(u^+-\Pi(\tilde u)^+)\big)_{\Omega^+} \,-\, \big(\boldsymbol{L}_h^0(\phi)^-,\kappa^-\,\nabla\,(u^--\Pi(\tilde u)^-)\big)_{\Omega^-}\,.$$

Finally we can estimate

$$
\begin{aligned}
|\mathcal{B}_h(u,\phi) - \mathcal{F}_h(\phi)| \;\leq\; & \sum_{e \in \mathcal{E}_h^+} \kappa^+ \, ||\overline{\partial_n (u^+ - \Pi(\tilde{u})^+)}||_{0,e^+} ||[\phi^+]||_{0,e^+} \\
& + \sum_{e \in \mathcal{E}_h^-} \kappa^- \, ||\overline{\partial_n (u^- - \Pi(\tilde{u})^-)}||_{0,e^-} ||[\phi^-]||_{0,e^-} \\
& + \sum_{T \in \mathcal{T}_h^0} \left(\frac{\kappa^+ \lambda_T^+}{\sqrt{\kappa^0}} ||\partial_{n^+} (u^+ - \Pi(\tilde{u})^+)||_{0,T^0} + \right. \\
& \qquad \left. \frac{\kappa^- \lambda_T^-}{\sqrt{\kappa^0}} ||\partial_{n^+} (u^- - \Pi(\tilde{u})^-)||_{0,T^0} \right) \cdot \sqrt{\kappa^0} ||[\phi]||_{0,T^0} \\
& + \sqrt{2} \left(||\boldsymbol{L}_h^+(\phi^+)||^2_{M_h^+} + ||\boldsymbol{L}_h^-(\phi^-)||^2_{M_h^-} + ||\boldsymbol{L}_h^0(\phi)||^2_{M_h} \right)^{1/2} \times \\
& \quad \left(\sum_{T \in \mathcal{T}_h^+} \kappa^+ |u^+ - \Pi(\tilde{u})^+|^2_{1,T^+} + \sum_{T \in \mathcal{T}_h^-} \kappa^- |u^- - \Pi(\tilde{u})^-|^2_{1,T^-} \right)^{1/2} \\
\leq \; & C_{(3.7)} \left(\frac{G_2}{\gamma} \sum_{T \in \mathcal{T}_h^+} \frac{\kappa^+ h_T^{2\mu_T^+ - 2}}{(p_T^+)^{2 s_T^+ - 1}} ||E^+ u^+||^2_{s_T^+, T} \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h^+} \frac{\gamma \, \kappa^+ (p_e^+)^2}{|e^+|} ||[\phi^+]||^2_{0,e^+} \right)^{1/2} \\
& + C_{(3.7)} \left(\frac{G_2}{\gamma} \sum_{T \in \mathcal{T}_h^-} \frac{\kappa^- h_T^{2\mu_T^- - 2}}{(p_T^-)^{2 s_T^- - 1}} ||E^- u^-||^2_{s_T^-, T} \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h^-} \frac{\gamma \, \kappa^- (p_e^-)^2}{|e^-|} ||[\phi^-]||^2_{0,e^-} \right)^{1/2} \\
& + \frac{C_{(3.7)}}{\sqrt{\gamma}} \left(\left(\sum_{T \in \mathcal{T}_h^0} \frac{\kappa^+ h_T^{2\mu_T^+ - 2}}{(p_T^+)^{2 s_T^+ - 1}} ||E^+ u^+||^2_{s_T^+, T} \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_h^0} \frac{\kappa^- h_T^{2\mu_T^- - 2}}{(p_T^-)^{2 s_T^- - 1}} ||E^- u^-||^2_{s_T^-, T} \right)^{1/2} \right) \\
& \quad \times \left(\sum_{T \in \mathcal{T}_h^0} \gamma \, \frac{\kappa^0 (p_T^0)^2}{h_T} ||[\phi]||^2_{0,T^0} \right)^{1/2} \\
& + \sqrt{2} \, C_{(\mathrm{C}.4)} \left(||\boldsymbol{L}_h^+(\phi^+)||^2_{M_h^+} + ||\boldsymbol{L}_h^-(\phi^-)||^2_{M_h^-} + ||\boldsymbol{L}_h^0(\phi)||^2_{M_h} \right)^{1/2} \\
& \quad \times \left(\sum_{T \in \mathcal{T}_h^+} \frac{\kappa^+ h_T^{2\mu_T^+ - 2}}{(p_T^+)^{2 s_T^+ - 2}} ||E^+ u^+||^2_{s_T^+, T} + \sum_{T \in \mathcal{T}_h^-} \frac{\kappa^- h_T^{2\mu_T^- - 2}}{(p_T^-)^{2 s_T^- - 2}} ||E^- u^-||^2_{s_T^-, T} \right)^{1/2} .
\end{aligned}
$$

Using Lemma 3.8 we are done. □

3.7 Error estimate of the DG method

The error estimate in the energy norm $||\cdot||_h$ can now be deduced from Strang's second lemma, and using the typical duality argument then leads to an error estimate in L_2-norm. These are given in the following theorems.

Theorem 3.14 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, and let $\gamma \geq 4\,C_{(3.8)}$. Let u_h be the solution of* (48) *and let m, $E^{\pm}$, u, $s_T^{\pm}$, $\mu_T^{\pm}$ be as in Lemma 3.13. Then the following hp-error estimate holds:*

$$
\begin{aligned}
||u-u_h||_h \;&=\; \Bigg(\sum_{T\in\mathcal{T}_h^+} \kappa^+ |u^+ - u_h^+|^2_{1,T^+} \;+\; \sum_{T\in\mathcal{T}_h^-} \kappa^- |u^- - u_h^-|^2_{1,T^-} \\
&\qquad + \; ||[u_h]||^2_{0,h,\mathcal{E}(\mathcal{T}_h)} \;+\; \sum_{T\in\mathcal{T}_h^0} \sigma_T^0\,\kappa^0\,||[u-u_h]||^2_{T^0} \Bigg)^{1/2} \\
&\leq\; C_{(3.14)} \cdot \Bigg(\sum_{T\in\mathcal{T}_h^+} \frac{h_T^{2\mu_T^+-2}}{(p_T^+)^{2s_T^+-3-\chi(|T^-|)}}\,||E^+u^+||^2_{s_T^+,T} \\
&\qquad + \; \sum_{T\in\mathcal{T}_h^-} \frac{h_T^{2\mu_T^--2}}{(p_T^-)^{2s_T^--3-\chi(|T^+|)}}\,||E^-u^-||^2_{s_T^-,T} \Bigg)^{1/2},
\end{aligned}
$$

where $C_{(3.14)} = \max\left\{ C_{(3.11)} \cdot \left(3 + \frac{4\,C_{(3.8)}}{\gamma}\right),\; 2\,C_{(3.13)} \right\} \;\leq\; \max\left\{4\,C_{(3.11)},\; 2\,C_{(3.13)}\right\}$.

Proof: Applying Theorem C.1, we obtain

$$
||u-u_h||_h \;\leq\; \left(1 + 2\,C_{(3.9)}\right) \inf_{\phi\in V_h} ||u-\phi||_h \;+\; 2\;\sup_{\phi\in V_h} \frac{|\mathcal{B}_h(u,\phi) - \mathcal{F}_h(\phi)|}{||\phi||_h}
$$

and making use of Lemma 3.11, Lemma 3.13, and the definition of $C_{(3.9)}$, the Theorem is proven. □

Theorem 3.15 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, where m is the number characterizing the smoothness of Γ in* (23) *and let $\gamma \geq 4\,C_{(3.8)}$. Let u_h be the solution of* (48) *and let $u^{\pm} \in H^s(\Omega^{\pm})$, with $2 \leq s \leq m$, such that $u = (u^+, u^-)$ is the solution of Problem 1.1. Then the following error estimate holds:*

$$
\left(\|u^+ - u_h^+\|^2_{0,\Omega^+} + \|u^- - u_h^-\|^2_{0,\Omega^-}\right)^{1/2} \;\leq\; C_{(3.15)}\,\frac{h^{\min\{p+1,s\}}}{p^{s-2}} \left(\|u^+\|^2_{s,\Omega^+} + \|u^-\|^2_{s,\Omega^-}\right)^{1/2},
$$

where $C_{(3.15)} > 0$ depends on $\kappa^0, C_{(3.9)}, C_{(3.11)}, C_{(3.13)}, C_{(3.14)}, C^2_{(C.5)}, C_{(C.6)}$.

Proof: Consider Problem 1.1 with $f^{\pm} \equiv u^{\pm} - u_h^{\pm}|_{\Omega^{\pm}}$ and $g_D = 0$, $j_D = j_N = 0$. According to [8] this problem has a unique solution $v = (v^+, v^-)$ with $v^{\pm} \in H^2(\Omega^{\pm})$. We denote the right hand side functional of the corresponding variational formulation by

$$
\mathcal{G}_h : W_h \to \mathbb{R}, \qquad \mathcal{G}_h(\phi) \;=\; \sum_{T\in\mathcal{T}_h} \langle u-u_h, \phi\rangle_T\,.
$$

Then, using the symmetry of $\mathcal{B}_h$, (48) and (51) we find for any $v_h \in V_h$

$$
\begin{aligned}
\|u^+ - u_h^+\|^2_{0,\Omega^+} + \|u^- - u_h^-\|^2_{0,\Omega^-} \;&=\; \mathcal{G}_h(u-u_h) \\
&=\; \mathcal{B}(v-v_h, u-u_h) \\
&\quad - \left(\mathcal{B}_h(u, v-v_h) - \mathcal{F}_h(v-v_h)\right) \\
&\quad - \left(\mathcal{B}_h(v, u-u_h) - \mathcal{G}_h(u-u_h)\right)
\end{aligned}
$$

Introducing $h = \max_{T\in\mathcal{T}_h} h_T$, $p = \min\left\{\min_{T\in\mathcal{T}_h^+} p_T^+, \min_{T\in\mathcal{T}_h^-} p_T^-\right\}$, yields

$$
\begin{aligned}
&\|u^+ - u_h^+\|_{0,\Omega^+}^2 + \|u^- - u_h^-\|_{0,\Omega^-}^2 \\
&\qquad \le C_{(3.9)}\|v - v_h\|_h \|u - u_h\|_h \\
&\qquad\quad + C_{(3.13)} \frac{h^{\min\{p+1,s\}-1}}{p^{s-1}} \left(\kappa^+\|E^+u^+\|_{s,\Omega_h^+}^2 + \kappa^-\|E^-u^-\|_{s,\Omega_h^-}^2\right)^{1/2} \|v - v_h\|_h \\
&\qquad\quad + C_{(3.13)} \frac{h}{p} \left(\kappa^+\|E^+v^+\|_{2,\Omega_h^+}^2 + \kappa^-\|E^-v^-\|_{2,\Omega_h^-}^2\right)^{1/2} \|u - u_h\|_h
\end{aligned}
$$

Due to Theorem C.5 and the regularity result of Theorem C.6, we can estimate

$$
\begin{aligned}
\kappa^+\|E^+v^+\|_{2,\Omega_h^+}^2 + \kappa^-\|E^-v^-\|_{2,\Omega_h^-}^2 &\le C_{(C.5)}^2\, \kappa^0 \left(\|v^+\|_{2,\Omega^+}^2 + \|v^-\|_{2,\Omega^-}^2\right) \\
&\le C_{(C.6)}^2\, C_{(C.5)}^2\, \kappa^0 \left(\|u^+ - u_h^+\|_{\Omega^+}^2 + \|u^- - u_h^-\|_{\Omega^-}^2\right)
\end{aligned}
$$

and taking $v_h = \Pi(\tilde{v})$ according to Lemma 3.11, we similarly obtain

$$
\begin{aligned}
\|v - v_h\|_h^2 &\le C_{(3.11)}\, h^2 \left(\|E^+v^+\|_{2,\Omega_h^+}^2 + \|E^-v^-\|_{2,\Omega_h^-}^2\right) \\
&\le C_{(C.6)}^2\, C_{(C.5)}^2\, C_{(3.11)}\, h^2 \left(\|u^+ - u_h^+\|_{\Omega^+}^2 + \|u^- - u_h^-\|_{\Omega^-}^2\right)
\end{aligned}
$$

and

$$
\|u - u_h\|_h^2 \le C_{(C.5)}^2\, C_{(3.14)}^2\, \frac{h^{2\min\{p+1,s\}-2}}{p^{2s-4}} \left(\|u^+\|_{s,\Omega^+}^2 + \|u^-\|_{s,\Omega^-}^2\right).
$$

This yields

$$
\begin{aligned}
&\left(\|u^+ - u_h^+\|_{0,\Omega^+}^2 + \|u^- - u_h^-\|_{0,\Omega^-}^2\right)^{1/2} \\
&\qquad \le C_{(3.9)}\, C_{(C.6)}\, C_{(3.11)}^{1/2}\, C_{(C.5)}^2\, C_{(3.14)}\, \frac{h^{\min\{p+1,s\}}}{p^{s-2}} \left(\|u^+\|_{s,\Omega^+}^2 + \|u^-\|_{s,\Omega^-}^2\right)^{1/2} \\
&\qquad\quad + C_{(3.13)}\, C_{(C.5)} \sqrt{\kappa^0 C_{(C.6)} C_{(3.11)}}\, \frac{h^{\min\{p+1,s\}}}{p^{s-1}} \left(\|u^+\|_{s,\Omega^+}^2 + \|u^-\|_{s,\Omega^-}^2\right)^{1/2} \\
&\qquad\quad + C_{(3.13)}\, C_{(C.5)}^2\, C_{(C.6)}\, C_{(3.14)} \sqrt{\kappa^0}\, \frac{h^{\min\{p+1,s\}}}{p^{s-1}} \left(\|u^+\|_{s,\Omega^+}^2 + \|u^-\|_{s,\Omega^-}^2\right)^{1/2}
\end{aligned}
$$

and the Theorem is proven. □

3.8 An additional penalization

The error estimates found so far do not control pointwise errors. This should not concern us too much for the numerical solution in triangles away from the interface. But near the interface $T^\pm$ may be very small and although not making an essential contribution to L_2-errors, within these triangles large pointwise errors may occur. In particular we observe large pointwise errors for the derivatives in such small $T^\pm$, see Section 4. This may be very annoying, if the elliptic interface problem is coupled to a transport equation, as for example in Hele-Shaw flow, where $\partial_n u$ on the interface has to be evaluated in order to determine the velocity field, see [12].

With the purpose of controlling L_∞-errors near the interface, we add another penalty to the DG method (48). This additional penalty also alters the energy norm, but we will show that with the new energy norm we can prove the same error estimate as in Theorem 3.14. In Section 4 we will clearly see that due to this additional penalty L_∞-errors of the derivatives behave very well. In the new bilinear form $\overline{\mathcal{B}}_h$ we additionally penalize jumps of derivatives $\partial^j u_h^\pm$ along $e^\mp$ for faces $e \in \mathcal{E}_h^0$. The idea is to enforce a smooth extension of $u_h^\pm$ along face sections sticking out of $\Omega^\pm$. The new bilinear form is defined as follows. The role played before by the spaces denoted with a V will now be played by the spaces denoted with a $\mathcal{V}$, which are provided with more smoothness:

$$
\begin{aligned}
p^\pm &\equiv \max_{T\in\mathcal{T}_h^\pm} p_T^\pm \\
\overline{\mathcal{V}}^\pm &\equiv H^{p^\pm+1}(\Omega_h^\pm) \;\subset\; \overline{V}^\pm \\
\overline{\mathcal{V}} &\equiv \overline{\mathcal{V}}^+ \times \overline{\mathcal{V}}^- \;\subset\; \overline{V} \\
\overline{W}_h &\equiv \overline{V}_h + \overline{\mathcal{V}}, \text{ where } \overline{V}_h \text{ has been defined in (50)} \\
\partial^j v &\equiv \frac{\partial^{j_1}}{\partial x_1}\frac{\partial^{j_2}}{\partial x_2}, \text{ with multi-index } j=(j_1,j_2) \\
D^l v &\equiv (\partial^j v)_{|j|=l}, \text{ where } |j| = j_1 + j_2 \\
\langle v, w\rangle_{0,p,\mathcal{E}_h^0} &\equiv \sum_{e\in\mathcal{E}_h^0} \Bigg\{ \sum_{l=0}^{p_e^+} \frac{\kappa^+}{p_e^+} \cdot \left(\frac{(p_e^+)^2}{|e|}\right)^{1-2l} \cdot \Big([D^l v^+], [D^l w^+] \Big)_{e^-} \\
&\qquad + \sum_{l=0}^{p_e^-} \frac{\kappa^-}{p_e^-} \cdot \left(\frac{(p_e^-)^2}{|e|}\right)^{1-2l} \cdot \Big([D^l v^-], [D^l w^-] \Big)_{e^+} \Bigg\} \\
\|v\|_{0,p,\mathcal{E}_h^0}^2 &\equiv \langle v, v\rangle_{0,p,\mathcal{E}_h^0} \\
\overline{\mathcal{B}}_h(v,\phi) &\equiv \mathcal{B}_h(v,\phi) \;+\; \langle v,\phi\rangle_{0,p,\mathcal{E}_h^0} \;:\; \overline{W}_h \times \overline{W}_h \to \mathbb{R} \\
\|v\|_{h,p}^2 &\equiv \|v\|_h^2 \;+\; \|v\|_{0,p,\mathcal{E}_h^0}^2
\end{aligned}
$$

Our DG method is now to find $u_h \in V_h$ such that

$$\overline{\mathcal{B}}_h(u_h,\phi) \;=\; \mathcal{F}(\phi) \qquad \forall\, \phi \in V_h\,. \tag{52}$$

We will see in the analysis of the approximation error below, that the jumps appearing in $\langle v, w\rangle_{0,p,\mathcal{E}_h^0}$ are weighted in terms of $p_e^\pm$ and $|e|$ in such a way that $\|v - \Pi(v)\|_{0,p,\mathcal{E}_h^0}$ falls with the optimal rate in both h and p.

If $|e^+| \ll |e|$, then along e^+ the term $\sigma_e^+ \cdot (\kappa^+[u_h],[\phi])_{e^+}$ dominates in (19) due to our choice $\sigma_e^+ = \gamma\cdot(p_e^+)^2\cdot|e^+|^{-1}$. Thus, (19) will only enforce continuity of u_h^+ across e, and no further smoothness is guaranteed. In this situation $|e^-| \approx |e|$, so that the additional penalty is of the same magnitude as $\sigma_e^+ \cdot (\kappa^+[u_h],[\phi])_{e^+}$ and thus more smoothness is enforced.

We easily conclude, that replacing $\mathcal{B}_h$, W_h, $\|\cdot\|_h$ by $\overline{\mathcal{B}}_h$, $\overline{W}_h$, $\|\cdot\|_{h,p}$ in Lemma 3.9 and Lemma 3.10, the lemmas remain valid without changing the constants. But note that while $\|\cdot\|_{h,p}$ is a norm on $\overline{V}_h$ it is not a norm on $\overline{\mathcal{V}}$. Thus we can only speak of coercivity of $\overline{\mathcal{B}}_h$ on $\overline{V}_h$. But this suffices to apply Strang's second lemma. Boundedness of $\mathcal{F}_h$ on $\overline{V}_h$ with respect to $\|\cdot\|_{h,p}$ follows as-well. Again we have to find estimates for the approximation error and the consistency error in order to apply Strang's second lemma.

Approximation error: Let $v \in \overline{\mathcal{V}}$ and consider a multi-index j with $0 \le |j| = l \le p_e^\pm$, then

$$
\begin{aligned}
\left(\frac{(p_e^\pm)^2}{|e|}\right)^{-2l} \|\partial^j(v^\pm - \Pi(v)^\pm)|_T\|_{0,e^\mp}^2 \;&\le\; C_{(3.7)}^2 \left(\frac{(p_T^\pm)^2}{h_T}\right)^{-2l} \frac{h_T^{2(p_T^\pm+1)-2l-1}}{(p_T^\pm)^{2(p^\pm+1)-2l-1}} \,||v^\pm||_{p^\pm+1,T}^2 \\
&\le\; C_{(3.7)}^2\,(p_T^\pm)^{-2l}\,\frac{h_T^{2p_T^\pm+1}}{(p_T^\pm)^{2p^\pm+1}}\,||v^\pm||_{p^\pm+1,T}^2
\end{aligned}
$$

and thus, since $l+1$ different partial derivatives of order l exist,

$$\sum_{l=0}^{p_e^\pm} \left(\frac{(p_e^\pm)^2}{|e|}\right)^{1-2l} \|D^l(v^\pm - \Pi(v)^\pm)|_T\|_{0,e^\mp}^2 \quad\leq\quad \frac{(p_e^\pm)^2}{|e|} C_{(3.7)}^2 \frac{h_T^{2p_T^\pm+1}}{(p_T^\pm)^{2p^\pm+1}} ||v^\pm||_{p^\pm+1,T}^2 \sum_{l=0}^{p_e^\pm} \frac{l+1}{(p_T^\pm)^{2l}} .$$

We have

$$\sum_{l=0}^{p_e^\pm} \frac{l+1}{(p_T^\pm)^{2l}} \quad\leq\quad \begin{cases} \displaystyle\sum_{l=0}^{p_e^\pm} l+1 \;=\; \frac{(p_e^\pm+2)(p_e^\pm+1)}{2} \;\leq\; \frac{(G_1+2)(G_1+1)}{2} & \text{if } p_T^\pm = 1 \\ \displaystyle\sum_{l=0}^{\infty} \frac{l+1}{4^l} \;=\; \frac{16}{9} & \text{if } p_T^\pm \geq 2 \end{cases}$$

and thus,

$$\sum_{l=0}^{p_e^\pm} \frac{1}{p_e^\pm} \left(\frac{(p_T^\pm)^2}{|e|}\right)^{1-2l} \|D^l(v^\pm - \Pi(v)^\pm)|_T\|_{0,e^\mp}^2 \quad\leq\quad C_{(53)} \frac{h_T^{2p_T^\pm}}{(p_T^\pm)^{2p^\pm}} ||v^\pm||_{p^\pm+1,T}^2 , \tag{53}$$

where $C_{(53)} = G_1 G_3 C_{(3.7)}^2 \max\{(G_1+2)(G_1+1)/2, 16/9\}$. Finally, we obtain

$$||v^\pm - \Pi(v)\|_{0,p,\mathcal{E}_h^0}^2 \quad\leq\quad C \sum_{T\in\mathcal{T}_h^0} \frac{h_T^{2p_T^+}}{(p_T^+)^{2p^+}} ||v^+||_{p^++1,T}^2 + \frac{h_T^{2p_T^-}}{(p_T^-)^{2p^-}} ||v^-||_{p^-+1,T}^2 ,$$

where $C = 2\,\kappa^0 C_{(53)}$. Combining with Lemma 3.11, we deduce the following result.

Lemma 3.16 *Let the assumptions of Lemma 3.11 hold. Then for all* $v = (v^+, v^-) \in \overline{\mathcal{V}}$ *we have*

$$||v - \Pi(v)||_{h,p}^2 \quad\leq$$

$$C_{(3.16)} \cdot \left(\sum_{T\in\mathcal{T}_h^+} \frac{h_T^{2p_T^+}}{(p_T^+)^{2p^+-1-\chi(|T^-|)}} ||v^+||_{p^++1,T}^2 + \sum_{T\in\mathcal{T}_h^-} \frac{h_T^{2p_T^-}}{(p_T^-)^{2p^--1-\chi(|T^+|)}} ||v^-||_{p^-+1,T}^2 \right),$$

where $C_{(3.16)} = C_{(3.11)} + 2\,\kappa^0 C_{(53)}$.

Consistency error: From Lemmas 3.12, 3.18 we immediately deduce the following analogue results.

Lemma 3.17 *Let* (23) *hold with* $m \geq 1 + \max\{p^+, p^-\}$ *and let* $E^\pm$ *be the strong* m*-extension operator of Theorem C.5 for* $\Omega^\pm$*. Let* $u^\pm \in H^{p^\pm+1}(\Omega^\pm)$*, such that* $u = (u^+, u^-)$ *is the solution of Problem 1.1 and set* $\tilde{u} \equiv \Big((E^+u^+)|_{\Omega_h^+}, (E^-u^-)|_{\Omega_h^-}\Big) \in \overline{\mathcal{V}}$*. Then for all* $\phi \in \overline{W}_h$ *we have*

$$\overline{\mathcal{B}}_h(\tilde{u},\phi) - \mathcal{F}_h(\phi) \quad=\quad \sum_{e\in\mathcal{E}(\mathcal{T}_h)} \langle \kappa\overline{\partial_n u}, [\phi]\rangle_e + \sum_{T\in\mathcal{T}_h^0} (\overline{\kappa\partial_{n^+}u}, [\phi])_{T^0}$$
$$- \big(\boldsymbol{L}_h^+(\phi^+) + \boldsymbol{L}_h^0(\phi)^+, \kappa^+\nabla u^+\big)_{\Omega^+} - \big(\boldsymbol{L}_h^-(\phi^-) + \boldsymbol{L}_h^0(\phi)^-, \kappa^-\nabla u^-\big)_{\Omega^-} .$$

In particular for all $\phi \in \overline{\mathcal{V}}$ *with* $[\phi] = 0$ *along* Γ*, we have*

$$\overline{\mathcal{B}}_h(\tilde{u},\phi) \quad=\quad \mathcal{F}_h(\phi) . \tag{54}$$

Lemma 3.18 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, and let* m, $E^{\pm}$, u, $\tilde{u}$ *be as in Lemma 3.17. Then we have*

$$|\overline{\mathcal{B}}_h(\tilde{u},\phi) - \mathcal{F}_h(\phi)| \;\leq\; C_{(3.13)} \left(\sum_{T\in\mathcal{T}_h^+} \frac{\kappa^+ h_T^{2p_T^+}}{(p_T^+)^{2p^+}} \,||E^+u^+||^2_{p^++1,T} \;+\; \frac{\kappa^- h_T^{2p_T^-}}{(p_T^-)^{2p^-}} \,||E^-u^-||^2_{p^-+1,T} \right)^{1/2} \cdot\; ||\phi||_{h,p} \qquad \forall\, \phi \in \overline{W}_h\,.$$

<u>Error estimate:</u> Also, from Theorem 3.14 we immediately deduce the following error estimate.

Theorem 3.19 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, and let* $\gamma \geq 4\,C_{(3.8)}$. *Let* u_h *be the solution of* (52) *and let* m, $E^{\pm}$, u, $\tilde{u}$ *be as in Lemma 3.17. Then the following error estimate holds:*

$$||\tilde{u} - u_h||_{h,p} \;\leq\; C_{(3.19)} \cdot \left(\sum_{T\in\mathcal{T}_h^+} \frac{h_T^{2p_T^+}}{(p_T^+)^{2p^+-1-\chi(|T^-|)}} \,||E^+u^+||^2_{p^++1,T} \right.$$
$$\left. +\; \sum_{T\in\mathcal{T}_h^-} \frac{h_T^{2p_T^-}}{(p_T^-)^{2p^--1-\chi(|T^+|)}} \,||E^-u^-||^2_{p^-+1,T} \right)^{1/2},$$

where $C_{(3.19)} = \max\left\{ C_{(3.16)} \cdot \left(3 + \frac{4\,C_{(3.8)}}{\gamma}\right),\; 2\,C_{(3.13)} \right\} \;\leq\; \max\left\{ 4\,C_{(3.16)},\; 2\,C_{(3.13)} \right\}$.

Proof: Applying Theorem C.1, we obtain

$$||\tilde{u} - u_h||_{h,p} \;\leq\; \left(1 + 2\,C_{(3.9)}\right) \inf_{\phi\in\overline{V}_h} ||\tilde{u} - \phi||_{h,p} \;+\; 2 \sup_{\phi\in\overline{V}_h} \frac{|\overline{\mathcal{B}}_h(\tilde{u},\phi) - \mathcal{F}_h(\phi)|}{||\phi||_{h,p}}$$

and making use of Lemma 3.16, Lemma 3.18, and the definition of $C_{(3.9)}$, the Theorem is proven. □

Trying to carry over the proof of Theorem 3.14 to obtain an L_2-error estimate for the DG Method (52), we note that since $v^{\pm} \in H^2(\Omega^{\pm})$ in the proof of Theorem 3.14, plugging $\tilde{v} \equiv \left((E^+v^+)|_{\Omega_h^+},\, (E^-v^-)|_{\Omega_h^-}\right)$ into the bilinear form $\overline{\mathcal{B}}_h$ works only if $p^+ = p^- = 1$. Thus we only obtain an L_2-error estimate for the case $p^+ = p^- = 1$.

Theorem 3.20 *Let Assumptions 2.3, 3.1, 3.3 be satisfied, where* $m = 2$ *is the number characterizing the smoothness of* Γ *in* (23) *and let* $\gamma \geq 4\,C_{(??)}$. *Let* u_h *be the solution of* (52), *using* $p_T^{\pm} = 1$ *for all* $T \in \mathcal{T}_h^{\pm}$ *and let* $u^{\pm} \in H^2(\Omega^{\pm})$, $u = (u^+, u^-)$ *is the solution of Problem 1.1. Then the following error estimate holds:*

$$\left(\|u^+ - u_h^+\|^2_{0,\Omega^+} + \|u^- - u_h^-\|^2_{0,\Omega^-}\right)^{1/2} \;\leq\; C_{(3.20)}\, h^2 \left(\|u^+\|^2_{2,\Omega^+} + \|u^-\|^2_{2,\Omega^-}\right)^{1/2},$$

where $C_{(3.20)} > 0$ *depends on* $\kappa^0, C_{(3.9)}, C_{(3.16)}, C_{(3.13)}, C_{(3.19)}, C^2_{(C.5)}, C_{(C.6)}$.

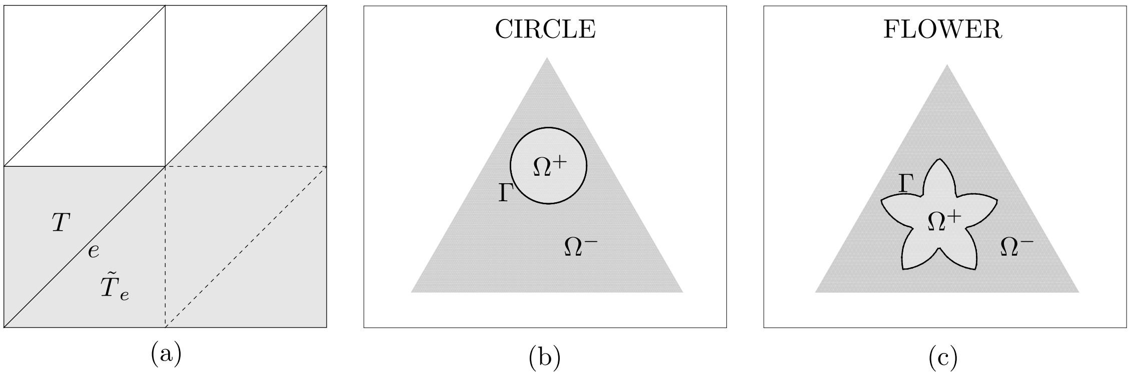


Figure 13: (a): Definition of $\tilde{T}_e$. (b), (c): Domains for examples CIRCLE and FLOWER.

4 Numerical experiments

In this section we display the behaviour of (48) and (52) when applied to Problem 1.1. For this purpose we utilize further approximation steps which are shortly described now. The details will be published elsewhere.

First, let us assume that Γ is defined as the zero-level set of a function $\varphi : \Omega \to \mathbb{R}$, which does not have to be a signed distance function, and let $\Omega^\pm = \{x \in \Omega : \pm\varphi(x) > 0\}$. We approximate T^0 by a polynomial of degree p_T^0, where efficient iterations are employed to approximately determine $p_T^0 + 1$ points which lie in T and satisfy $\varphi = 0$. This also results in corresponding approximations of $T^\pm$. Furthermore efficient quadrature formulas are used to integrate on $T^\pm$. By $T_1^\pm$ we denote the approximations of $T^\pm$, which are due to the straight interface approximation T_1^0, compare Section 3.2. Finally, let us assume that on each face of $T \in \mathcal{T}_h$ there is at most one hanging node. Let face $e \in \mathcal{E}(\mathcal{T}_h)$ possess an end point which is a hanging node for $\tilde{T} \in \omega(e)$, as in Fig. 1(a). Then considering the children of $\tilde{T}$ obtained by refinement, we denote by $\tilde{T}_e$ the child for which e is a face, compare Fig. 13(a). With these definitions we approximate λ_e and λ_T in our computations as follows:

$$\lambda_e^\pm = \frac{|T_1^\pm|}{|T_1^\pm| + |\tilde{T}_1^\pm|}, \qquad \tilde{\lambda}_e^\pm = \frac{|\tilde{T}_1^\pm|}{|T_1^\pm| + |\tilde{T}_1^\pm|}, \qquad \text{if } \{T, \tilde{T}\} = \omega(e) \text{ and } e \text{ is a face of both } T \text{ and } \tilde{T},$$

$$\lambda_e^\pm = \frac{|T_1^\pm|}{|T_1^\pm| + |(\tilde{T}_e)_1^\pm|}, \quad \tilde{\lambda}_e^\pm = \frac{|(\tilde{T}_e)_1^\pm|}{|(T_e)_1^\pm| + |(\tilde{T}_e)_1^\pm|}, \qquad \text{if } \{T, \tilde{T}\} = \omega(e) \text{ and } e \text{ is not a face of } \tilde{T},$$

$$\lambda_T^\pm = \begin{cases} 1 & \text{if } \frac{|T_1^\pm|}{|T_1^+|+|T_1^-|} > 0.8 \\ 0 & \text{if } \frac{|T_1^\pm|}{|T_1^+|+|T_1^-|} < 0.2 \\ \frac{|T_1^\pm|}{|T_1^+|+|T_1^-|} & \text{otherwise}. \end{cases}$$

The stabilization parameter is set to $\gamma = 10$, which seems to be a suitable choice in general. We use constant $p = p_T^\pm$ for all $T \in \mathcal{T}_h^\pm$ in our calculations and display the following norms and seminorms of the error:

$$
\begin{aligned}
e_2^0 &= \log\left(||u^+ - u_h^+||_{0,\Omega^+}^2 + ||u^- - u_h^-||_{0,\Omega^+}^2\right)^{1/2} \\
e_\infty^0 &= \log\left(\max\{||u^+ - u_h^+||_{L_\infty(\Omega^+)},\, ||u^- - u_h^-||_{L_\infty(\Omega^-)}\}\right) \\
e_2^1 &= \log\left(\sum_{T\in\mathcal{T}_h^+} ||\nabla(u^+ - u_h^+)||_{0,T^+}^2 + \sum_{T\in\mathcal{T}_h^-} ||\nabla(u^- - u_h^-)||_{0,T^-}^2\right)^{1/2} \\
e_\infty^1 &= \log\left(\max\left\{\max_{T\in\mathcal{T}_h^+} ||\nabla(u^+ - u_h^+)||_{L_\infty(T^+)},\, \max_{T\in\mathcal{T}_h^-} ||\nabla(u^- - u_h^-)||_{L_\infty(T^-)}\right\}\right)
\end{aligned}
$$

CIRCLE: In this example Γ is a circle and is contained in a triangle Ω, see Fig. 13(b). Let parameters $a, b, c, r_0 \in \mathbb{R}$ be given, where $b, r_0 > 0$. Setting $r(x,y) = \sqrt{x^2+y^2}$, we define the level set function by $\varphi(x,y) = r_0 - r(x,y)$. Furthermore, we specify the diffusion coefficients $\kappa^+ = 1$, $\kappa^- = b$, the interface data $j_D = a$, $j_N = -b\gamma/r_0$ and $f = 0$, $g_D(x,y) = 1 + c\ln(r(x,y)/r_0)$. The exact solution of Problem 1.1 is then given by $u^+(x,y) = 1 + a$, $u^-(x,y) = 1 + c\ln(r(x,y)/r_0)$. We set the parameters to $a = -1$, $b = 10$, $c = 10$, $r_0 = 0.28$.

We apply both (48) and (52) and find in Fig. 14-16 that both methods behave as expected with respect to e_2^1. The convergence rates indicated are determined from the errors measured on the coarsest and the finest grid. Both methods also behave almost identically with respect to e_2^0 and e_∞^0. Major differences occur for e_∞^1. Here (52) significantly improves the results compared to (48). One clearly observes how, due to the additional penalty, pointwise errors in the gradient are controlled very well by (52). In Fig. 17(a) we display how e_2^0 behaves for (52) on a hierarchy of grids when increasing p and infer from Fig. 17(b) that increasing p also pays off in terms of degrees of freedom (DOF). As is well-know, $u_h^\pm|_T \in \mathcal{P}^p(T)$ possesses $1 + 2 + ... + (p+1) = \frac{(p+1)\cdot(p+2)}{2}$ DOF.

In order to envision the differences of the numerical solutions obtained with (52) for various p, we consider the same problem again, changing only the parameter $a = 0$, so that the exact solution is continuous at the interface. In Fig. 18 the numerical solutions are displayed for $p = 1$ and $p = 3$ on a coarse grid. Note that the visualization tool can only represent the numerical solution piecewise linearly, using the solution values in the triangle corners, even if the underlying numerical solution is a piecewise polynomial of higher degree. Nethertheless the improvements of the solution with $p = 3$ compared to the one with $p = 1$ is clearly visible. Note that the redundant part of the numerical solution, i.e. $u_h^\pm$ for $x \in \Omega_h^\pm \setminus \Omega^\pm$, is also shown, which makes u_h^+ and u_h^- interpenetrate each other at the interface. The interface is projected onto the solution graphs and displayed as a white curve. Here we have $e_\infty^0 = -1.33$ for $p = 1$, employing 857 DOF, and $e_\infty^0 = -3.37$ for $p = 3$, employing 2860 DOF. In order to obtain $e_\infty^0 = -3.37$ with $p = 1$ requires around 100000 DOF.

FLOWER: In this example Γ looks like a flower and is contained in a triangle Ω, see Fig. 13(c). The polar-coordinates are given by $r(x,y) = \sqrt{x^2+y^2}$,

$$
\theta(x,y) = \begin{cases} \arccos(x/r(x,y)) & \text{for } y \geq 0 \\ -\arccos(x/r(x,y)) & \text{for } y < 0 \end{cases}
$$

and the continuous 1-periodic function $z : \mathbb{R} \to \mathbb{R}$ is defined by the following conditions: $z(0) = z(0.5) = z(1) = 0$, $z(0.25) = -z(0.75) = 1$ and z is a polynomial of degree 1 on each of the intervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$. The interface Γ is given as a parametrized curve

$$
\left.\begin{aligned}
x_0(\theta) &= (0.5 + 0.2\, z(2.5\,\theta/\pi)) \cdot \cos(\theta) \\
y_0(\theta) &= (0.5 + 0.2\, z(2.5\,\theta/\pi)) \cdot \sin(\theta)
\end{aligned}\right\} \quad \theta \in [0, 2\pi], \tag{55}
$$

with distance $r_0(\theta) = \sqrt{x_0(\theta)^2 + y_0(\theta)^2}$ from the origin. A corresponding level set function is

$$
\varphi(x,y) = 1 - \frac{r(x,y)}{r_0(\theta(x,y))}.
$$

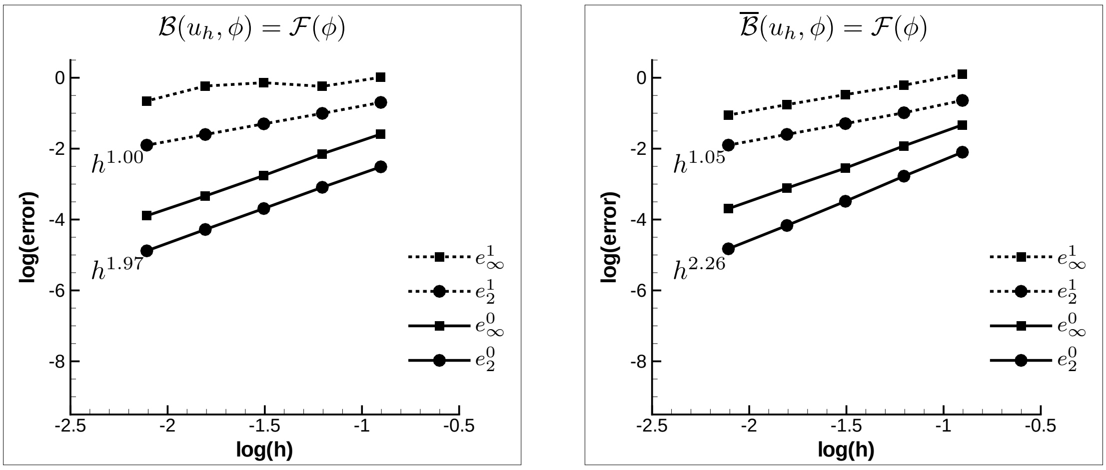


Figure 14: CIRCLE: h-convergence of (48) and (52) for $p = 1$.

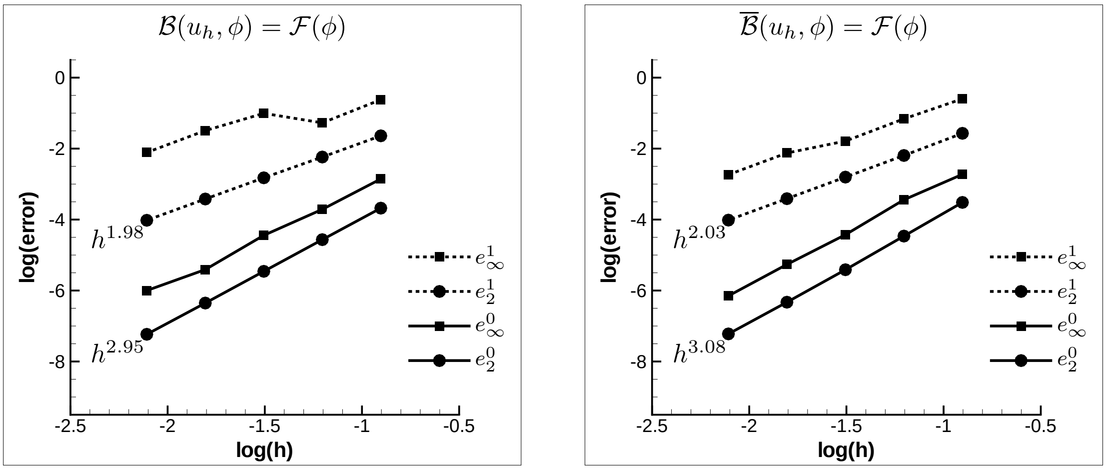


Figure 15: CIRCLE: h-convergence of (48) and (52) for $p = 2$.

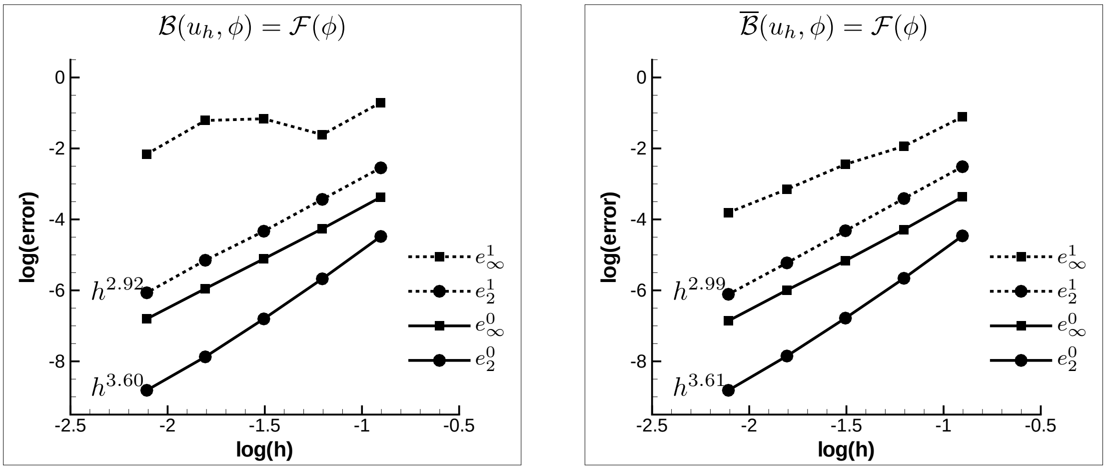


Figure 16: CIRCLE: h-convergence of (48) and (52) for $p = 3$.

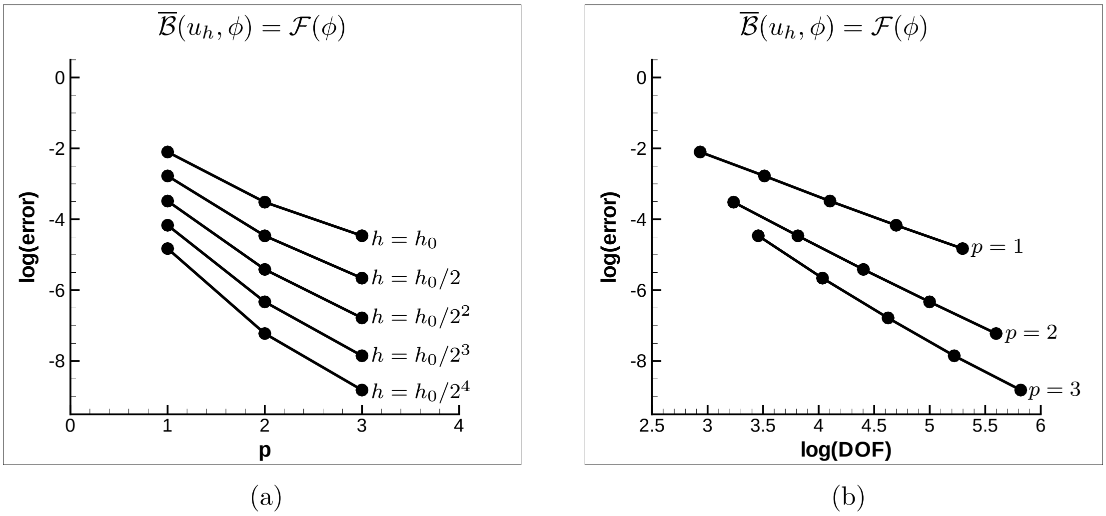
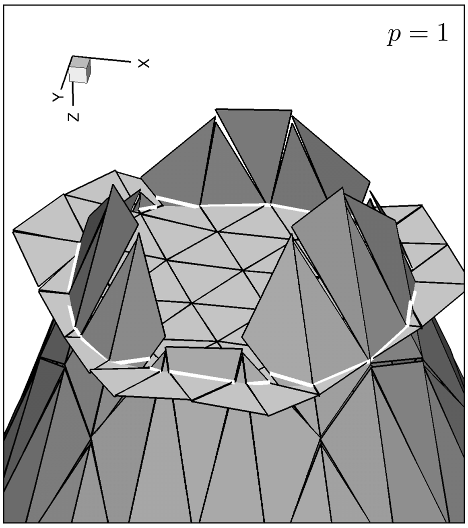


Figure 17: CIRCLE: p-convergence and DOF-convergence in L_2-norm.



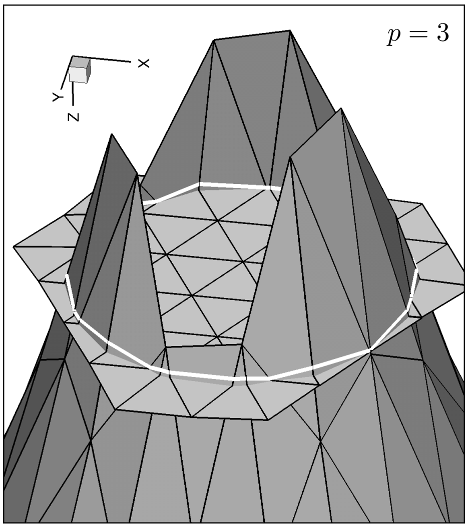


Figure 18: CIRCLE: Numerical solutions for $p = 1$ and $p = 3$ on same grid.

Now we consider Problem 1.1 with the following data and exact solution: $\kappa^+ = \kappa^- = 1$, $u^+(x, y) = 1 + \sin(x)\, e^{-y}$, $u^-(x, y) = \sin(x)\, e^{-y}$. Here $j_D = 1$, $j_N = 0$, $f = 0$.

Applying (52) we find in Fig. 19 that the L_2-error behaves as expected, and again we see how increasing the degree p pays off in terms of DOF. In Fig. 20 numerical solutions are shown on a coarse grid for $p = 1$ and $p = 3$. The linear approximation of the zero level set of φ on a triangle can cut the triangle boundary only twice. Thus Assumption 2.3 is always satisfied for $p = 1$. If we have $p > 1$ and our algorithm detects that Assumption 2.3 is not satisfied for a triangle, i.e. the interface cuts through a face twice, then the triangles adjacent to the face are refined. This detection and refinement strategy is repeated until Assumption 2.3 is satisfied throughout the grid. Of course this can only guarantee Assumption 2.3 to hold up to a finite precision. The details of this will be provided elsewhere. For problem FLOWER on coarse grids only a few triangles need to be refined in order to satisfy Assumption 2.3, as can be seen for $p = 3$ in Fig. 20. Note that this makes diamond-shaped holes appear in the grids $\mathcal{T}_h^{\pm}$. Again Γ is the white curve, projected onto both the graphs of u_h^+ and u_h^-.

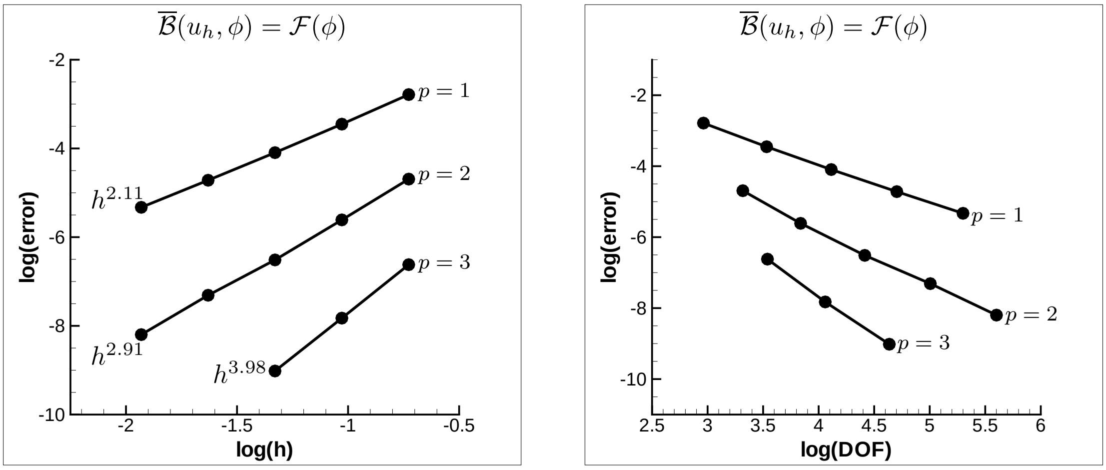


Figure 19: FLOWER: e_2^0 versus h and versus DOF.

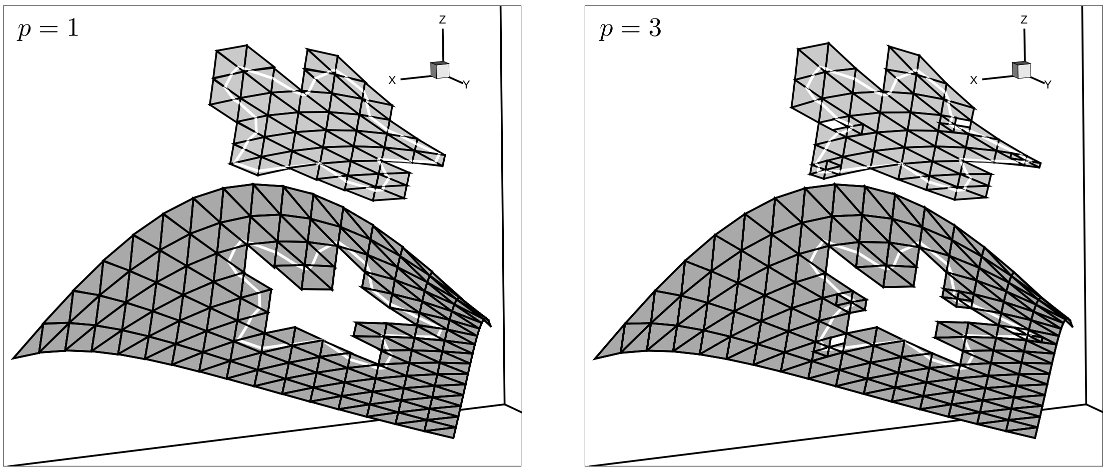


Figure 20: FLOWER: Numerical solutions, $p = 1$ and $p = 3$.

A Norms

We use the following notations:

$$
\begin{aligned}
||x||_2 &\equiv \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} && \text{for } x \in \mathbb{R}^n \\
(v,w)_\Omega &\equiv \int_\Omega v \cdot w \, dx && \text{for } v, w \in L_2(\Omega)^n \\
H^l(\Omega) &\equiv W^{l,2}(\Omega) && \text{for } l \in \mathbb{N} \\
\partial^j v &\equiv \frac{\partial^{j_1}}{\partial x_1}\frac{\partial^{j_2}}{\partial x_2}\cdots\frac{\partial^{j_n}}{\partial x_n} && \text{with multi-index } j = (j_1, j_2, ..., j_n) \\
|v|_{l,\Omega} &\equiv \left(\sum_{|j|=l} \int_\Omega |\partial^j v|^2 \, dx\right)^{1/2} && \text{for } v \in H^l(\Omega) \\
\|v\|_{l,\Omega} &\equiv \left(\sum_{k=0} |v|^2_{k,\Omega}\right)^{1/2} && \text{for } v \in H^l(\Omega)
\end{aligned}
$$

B Triangles

For a triangle T we define

$$
\begin{aligned}
r_T &\equiv \text{radius of largest circle contained in } T, \\
R_T &\equiv \text{radius of smallest circle containing } T, \\
h_T &\equiv \text{diameter of } T \; = \; \text{length of largest edge of } \; T.
\end{aligned}
$$

We always have

$$
r_T \;\leq\; \frac{h_T}{2} \;\leq\; R_T
$$

and, particularly for the reference triangle $\hat{T} \equiv \{(x,y) \in \mathbb{R}^2 \,:\, 0 \leq x, y \leq 1,\, x+y \leq 1\}$, we have $r_{\hat{T}} = (2+\sqrt{2})^{-1} \geq 2/7,\ R_{\hat{T}} = 2^{-1/2},\ h_{\hat{T}} = \sqrt{2}$. Assuming that a bijective affine mapping $F_T : \hat{T} \to T$ maps the reference triangle $\hat{T}$ onto T, i.e. $F_T(\hat{x}) \equiv x_0 + B_T\,\hat{x}$ with $x_0 \in \mathbb{R}^2$ and an invertible matrix B_T,

$$
\begin{aligned}
||B_T||_2 &\leq R_T / r_{\hat{T}} && (56) \\
||B_T^{-1}||_2 &\leq R_{\hat{T}} / r_T && (57) \\
2\,\pi\, r_T^2 &\leq |\det B_T| \;=\; 2\,|T| \;\leq\; h_T^2
\end{aligned}
$$

follow. For a proof of (56) and (57) we refer to [6], pg. 76. If $\varrho \cdot r_T \geq h_T$ holds, we call T a *ϱ-regular* triangle. In this case, the smallest interior angle of T can be bounded from below by

$$
\vartheta(\varrho) \;=\; \max\{\arctan(2/\varrho),\, 2\,\arctan(1/\varrho),\, 2\,\arcsin(1/\varrho)\} \;=\; 2\,\arcsin(1/\varrho) \qquad (58)
$$

and for ϱ-regular triangles we obtain

$$\begin{aligned} R_T &\leq \frac{h_T}{2\sin(\vartheta(\varrho))}, \\ ||B_T||_2 \cdot ||B_T^{-1}||_2 &\leq \frac{\varrho}{2\sin(\vartheta(\varrho))} \cdot \frac{R_{\hat{T}}}{r_{\hat{T}}} = \frac{(\sqrt{2}+1)\,\varrho}{2\sin(\vartheta(\varrho))}, \\ 2\pi\, h_T^2/\varrho^2 &\leq |\det B_T| \leq h_T^2, \\ \pi\, h_T^2/\varrho^2 &\leq |T|. \end{aligned}$$

C Theorems from the literature

C.1 Nonconforming Galerkin–Approximations

Theorem C.1 (Strang's Second Lemma) *Let V and V_h be linear spaces. Set $W_h \equiv V_h + V$ and let $||\cdot||_h$ be a norm on V_h and a seminorm on W_h. Let $a_h(\cdot,\cdot) : W_h \times V_h \to \mathbb{R}$ be a continuous bilinear form, which is coercive on V_h and let $f : V_h \to \mathbb{R}$ be a continuous linear functional, i.e. there exist h–dependent positive constants c_h, C_h, K_h, such that*

$$|a_h(w_h, v_h)| \leq C_h \cdot ||w_h||_h \cdot ||v_h||_h \qquad \forall\, w_h \in W_h,\, v_h \in V_h, \tag{59}$$

$$a_h(v_h, v_h) \geq c_h \cdot ||v_h||_h^2 \qquad \forall\, v_h \in V_h, \tag{60}$$

$$|f_h(v_h)| \leq K_h \cdot ||v_h||_h \qquad \forall\, v_h \in V_h. \tag{61}$$

Then the solution u_h of the variational equation

$$a_h(u_h, v_h) = f_h(v_h) \qquad \forall\, v_h \in V_h \tag{62}$$

is unique and satisfies

$$||v - u_h||_h \leq \left(1 + \frac{C_h}{c_h}\right) \inf_{v_h \in V_h} ||v - v_h||_h + \frac{1}{c_h} \sup_{v_h \in V_h} \frac{|a_h(v, v_h) - f_h(v_h)|}{||v_h||_h} \qquad \forall\, v \in V. \tag{63}$$

Proof: For all $v_h \in V_h, v \in V$ we have

$$\begin{aligned} c_h \cdot ||u_h - v_h||_h^2 &\leq a_h(u_h - v_h, u_h - v_h) \\ &= f_h(u_h - v_h) + a_h(v - v_h, u_h - v_h) - a_h(v, u_h - v_h) \\ &\leq f_h(u_h - v_h) + C_h\, ||v - v_h||_h \cdot ||u_h - v_h||_h - a_h(v, u_h - v_h), \end{aligned}$$

where we have used (60), (62), (59). Thus we have

$$\begin{aligned} ||u_h - v_h||_h &\leq \frac{C_h}{c_h} \cdot ||v - v_h||_h + \frac{1}{c_h} \cdot \frac{|a_h(v, u_h - v_h) - f_h(u_h - v_h)|}{||u_h - v_h||_h} \\ &\leq \frac{C_h}{c_h} \cdot ||v - v_h||_h + \frac{1}{c_h} \sup_{z_h \in V_h} \frac{|a_h(v, z_h) - f_h(z_h)|}{||z_h||_h}. \end{aligned}$$

Employing the triangle inequality, we obtain

$$||v - u_h||_h \leq \left(1 + \frac{C_h}{c_h}\right) \cdot ||v - v_h||_h + \frac{1}{c_h} \sup_{z_h \in V_h} \frac{|a_h(v, z_h) - f_h(z_h)|}{||z_h||_h},$$

which holds for all $v_h \in V_h, v \in V$. Thus the Lemma is proven. □

C.2 Inverse estimates

Theorem C.2 ([20], Theorem 3.91, 3.92) *Let* $I \equiv (a,b) \subset \mathbb{R}$, $h \equiv b-a$, *then*

$$||v'||_{0,I} \;\leq\; C_{(64)}\,\frac{p^2}{h}\,||v||_{0,I} \qquad \forall\; v \in P^p(I),\; p \geq 0, \tag{64}$$

where $C_{(64)} = 2\sqrt{3}$ *and*

$$||v||_{L_\infty(I)} \;\leq\; C_{(65)}\,\frac{p}{\sqrt{h}}\,||v||_{0,I} \qquad \forall\; v \in P^p(I),\; p \geq 1, \tag{65}$$

where $C_{(65)} = 4\sqrt{2}$.

Theorem C.3 ([20], Theorem 4.76) *Let* $\hat{T} \subset \mathbb{R}^2$ *denote the reference triangle and* $\hat{e}$ *any face of* $\hat{T}$, *then there exists a constant* $C_{(C.3)} > 0$, *such that*

$$||\hat{v}||_{0,\hat{e}} \;\leq\; C_{(C.3)}\,p\,||\hat{v}||_{0,\hat{T}} \qquad \textit{for all } \hat{v} \in P^p(\hat{T}),\; p \geq 1.$$

By a scaling argument we immediately deduce for any triangle T and any of its faces e, that

$$||v||^2_{0,e} \;\leq\; C_{(66)}\,\frac{p^2|e|}{|T|}\,||v||^2_{0,T} \qquad \text{for all } v \in P^p(T), \tag{66}$$

where $C_{(66)} = C^2_{(C.3)}/2$ and in the particular case that T is ϱ-regular,

$$||v||^2_{0,e} \;\leq\; C_{(67)}\,\frac{p^2}{h_T}\,||v||^2_{0,T} \qquad \text{for all } v \in P^p(T), \tag{67}$$

where $C_{(67)} = C^2_{(C.3)}\,\varrho^2/(2\,\pi)$.

C.3 hp–Interpolation

Theorem C.4 ([4], Lemma 4.5) *Let* T *be a* ϱ*-regular triangle and* $u \in H^s(T)$. *Then there exists a positive constant* $C_{(C.4)} > 0$ *depending on* s *and* ϱ *but independent of* u, p *and* h_T, *and a sequence* $\pi_p(u) \in P^p(K)$, $p = 1, 2, \ldots$, *such that for an arbitrary* q *with* $0 \leq q \leq s$

$$\|u - \pi_p(u)\|_{q,T} \;\leq\; C_{(C.4)}\,\frac{h_T^{\mu-q}}{p^{s-q}}\,||u||_{s,T} \qquad \textit{for } s \geq 0,$$

$$\|u - \pi_p(u)\|_{L_\infty(T)} \;\leq\; C_{(C.4)}\,\frac{h_T^{\mu-1}}{p^{s-1}}\,||u||_{s,T} \qquad \textit{for } s > 1,$$

where $\mu := \min(p+1, s)$.

C.4 Extension result for Sobolev spaces

In the following result the C^m*–regularity property* of a domain $\Omega \subset \mathbb{R}^n$ is due to [1], page 84.

Theorem C.5 ([1], Theorem 5.22) *Let $\Omega \subset \mathbb{R}^n$ have the uniform C^m–regularity property with $m \in \mathbb{N}$ and a bounded boundary. Then there exists a strong m-extension operator $E : H^0(\Omega) \to H^0(\mathbb{R}^n)$, i.e. an operator satisfying $(E\,v)\big|_\Omega = v$ for all $v \in H^0(\Omega)$, and a constant $C_{(C.5)} > 0$ such that*

$$\|Ev\|_{s,\mathbb{R}^n} \;\leq\; C_{(C.5)}\,\|v\|_{s,\Omega} \quad \forall\, v \in H^s(\Omega) \quad \textit{and} \quad s \in \{0,1,...,m\}.$$

C.5 Regularity of an elliptic interface problem

Theorem C.6 ([8], Theorem 2.1) *Let $g_D = 0$ and $j_D = j_N = 0$ in Problem 1.1. Then Problem 1.1 has a unique solution $u \equiv (u^+, u^-)$. The solution satisfies $u^\pm \in H^2(\Omega^\pm)$ and there exists a constant $C_{(C.6)} > 0$, such that*

$$\Big(\|u^+\|^2_{2,\Omega^+} + \|u^-\|^2_{2,\Omega^-}\Big)^{1/2} \;\leq\; C_{(C.6)}\,\|f\|_{0,\Omega}\,.$$

References

[1] R. Adams, J. Fournier, *Sobolev Spaces.* Second Edition, Academic Press (2003)

[2] D.N. Arnold, *An interior penalty finite element method with discontinuous elements.* SIAM J. Numer. Anal., Vol. 19 (1982), 742–760

[3] D. Babuška, *The finite element method for elliptic equations with discontinuous coefficients.* Computing, Vol. 5 (1970), 207–213

[4] D. Babuška, M. Suri, *The h-p Version of the Finite Element Method with Quasiuniform Meshes.* Math. Model. Num. Ana., Vol. 21 (1987), 199–238

[5] J. Barrett, C. Elliott, *Fitted and Unfitted Finite-Element Methods for Elliptic Equations with smooth Interfaces.* IMA Journal of Numerical Analysis, Vol. 7 (1987), 283–300

[6] D. Braess, *Finite Elemente.* Springer (1997)

[7] T. Chen, J. Strain, *Finite Element methods and their convergence for elliptic and parabolic interface problems.* J. Comput. Phys., Vol. 227 (2008), 7503–7542

[8] Z. Chen, J. Zou, *Piecewise–polynomial discretization and Krylov–accelerated multigrid for elliptic interface problems.* Numer. Math., Vol. 79 (1998), 175–202

[9] R. Fedkiw, T. Aslam, B. Merriman, S. Osher, *A non-oscillatory Eulerian approach to interfaces in multimaterial flows.* J. Comput. Phys., Vol. 152 (1999), 457

[10] Y. Gang, B. Li, Z. Li, *Immersed-interface Finite-element methods for elliptic interface problems with nonhomogeneous jump conditions.* SIAM J. Numer. Anal., Vol. 46(1) (1998), 472–495

[11] A. Hansbo, P. Hansbo, *An unfitted finite element method, based on Nitsche's method, for elliptic interface problems.* Comp. Meth. Appl. Mech. Eng., Vol. 191 (2002), 5537–5552

[12] T. Hou, Z. Li, S. Osher, H. Zhao, *A Hybrid Method for Moving Interface Problems with Application to the Hele-Shaw Flow.* J. Comput. Phys., Vol. 134 (1997), 236–252

[13] M. Kang, R. Fedkiw, X. Liu, *A Boundary Condition Capturing Method for Multiphase Incompressible Flow.* J. Sci. Comput., Vol. 15, No.3 (2000), 323–360

[14] R. LeVeque, Z. Li, *The immersed interface method for elliptic equations with discontinuous coefficients and singular sources*. SIAM J. Num. Anal., Vol. 31, No. 4 (1994), 1019–1044

[15] Z. Li, M. Lai, *The Immersed Interface Method for the Navier-Stokes Equations with Singular Forces*. J. Comput. Phys., Vol. 171 (2001), 822–842

[16] X. Liu, R. Fedkiw, M. Kang, *A Boundary Condition Capturing Method for Poisson's Equation on Irregular Domains*. J. Comput. Phys., Vol. 160 (2000), 151–178

[17] S. Osher, R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*. Springer (2003)

[18] I. Perugia, D. Schötzau, *An hp-Analysis of the Local Discontinuous Galerkin Method for Diffusion Problems*. J. Sci. Comput., Vol. 17, No. 1–4 (2002), 561–671

[19] S. Prudhomme, F. Pascal, J. Oden, A. Romkes, *Review of Apriori Error Estimation for Discontinuous Galerkin Methods*. TICAM Report 00–27 (2000)

[20] Ch. Schwab, *p– and hp–Finite Element Methods*. Oxford University Press (1998)