Reduced Basis a Posteriori Error Bounds for Parametrized Linear–Quadratic Elliptic Optimal Control Problems

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Reduced basis a posteriori error bounds for parametrized linear-quadratic elliptic optimal control problems

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Abstract

We employ the reduced basis method as a surrogate model for the solution of optimal control problems governed by parametrized partial differential equations (PDEs) and develop rigorous *a posteriori* error bounds for the error in the optimal control and the associated error in the cost functional. The proposed bounds can be efficiently evaluated in an offline-online computational procedure. We present numerical results that confirm the validity of our approach.

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Résumé

Bornes d'erreur a posterori par bases réduites pour des problèmes linéaires-quadratiques elliptiques de contrôle optimal. Nous employons la méthode des bases réduites comme modèle de substitution pour la solution de problèmes de contrôle optimal, qui sont régis par des équations aux dérivées partielles paramétrisée, et nous développons ainsi des bornes d'erreur rigoureuses *a posteriori* pour l'erreur dans le contrôle optimal et l'erreur associée dans la fonctionnelle du coût. Les bornes proposées peuvent être efficacement évaluées avec une procédure de calcul en-ligne/hors-ligne. Nous présentons des résultats numériques qui confirment le bien-fondé de notre méthode.

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Version française abrégée

Nous considérons le problème de contrôle optimal défini en (1). Dans ce cas, la fonctionnelle du coût est exprimée ainsi $J(y,u;\mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}|u - u_d|^2$; $y_d \in Y$ et $u_d \in \mathcal{U} \equiv \mathbb{R}$ sont respectivement l'état et le contrôle souhaités ; $\lambda > 0$ est le paramètre de régularisation donné ; $\mu \in \mathcal{D}$ est le paramètre ; Y est un espace de Hilbert approprié ; et $u \in \mathcal{U}$ est le contrôle scalaire. Supposons que l'équation aux dérivées partielles paramétrisée du deuxième ordre coercive et elliptique – pour un $\mu \in \mathcal{D}$ donné, $y \in Y$ satisfait $a(y,v;\mu) = b(v;\mu)\,u,\ \forall v \in Y$ – soit bien posée. Dans le problème de contrôle optimal approché par bases réduites (BR), nous remplaçons simplement la contrainte donnée par l'équation aux dérivées partielles par son approximation BR : pour un $\mu \in \mathcal{D}$ donné, l'approximation BR $y_N \in Y_N$ est la solution de $a(y_N,v;\mu) = b(v;\mu)\,u,\ \forall v \in Y_N$, où Y_N est l'espace BR et la fonctionnelle du coût BR est donnée par $J_N(y_N,u_N;\mu) = \frac{1}{2}\|y_N - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}|u_N - u_d|^2$.

Considérons d'abord l'erreur dans le contrôle optimal. Nous utilisons deux résultats intermédiaires, donnés dans les Lemmes 3.2 et 3.3. En nous basant sur le résultat obtenu en [6], nous pouvons alors prouver que notre nouvelle borne d'erreur a posteriori, $\Delta_N^{u,*}(\mu)$, définie en (12), est une borne d'erreur supérieure rigoureuse (et efficacement évaluable) pour l'erreur véritable dans le contrôle optimal. Nous déduisons ensuite notre nouvelle borne d'erreur a posteriori, $\Delta_N^{J,*}(\mu)$, définie en (15) pour la fonctionnelle du coût associée. Notre démonstration est basée sur le résultat standard en [1], qui nous permet d'estimer l'erreur dans la fonctionnelle du coût par $|J^* - J_N^*| \le \frac{1}{2}\left(\|r^{\mathrm{du}}(\cdot;\mu)\|_{Y'}\,\|e^{\mathrm{pr},*}\|_Y + \|r^{\mathrm{pr}}(\cdot;\mu)\|_{Y'}\,\|e^{\mathrm{du},*}\|_Y\right)$, $\forall \mu \in \mathcal{D}$. En utilisant les bornes d'erreur pour les erreurs d'optimisation primales et duales définies en (13) et (14), nous en déduisons immédiatement la borne que nous proposons en (15).

Nous présentons des résultats numériques pour la conduction stationnaire de la chaleur dans un bloc thermique, composé de trois sous-domaines. Les paramètres sont les conductivités de deux de ces sous-domaines, qui sont normalisées par rapport à la conductivité dans le troisième sous-domaine qui est unitaire. Les côtés du bloc thermique sont isolés et le sommet est maintenu à température constante. Le contrôle optimal est le flux de chaleur entrant à la base du bloc. Nous considérons la fonctionnelle du coût définie ci-dessus avec $y_d = 0$, $u_d = 1$ et $\lambda = 1$. Les résultats pour l'erreur maximale relative et pour la borne d'erreur, ainsi que l'efficience moyenne pour le contrôle optimal et la fonctionnelle du coût associée sont montrés dans les Tableaux 1 et 2. Nous observons que les efficiences pour la borne de l'erreur de contrôle se dégradent lorsque N augmente, en raison de la convergence quadratique de l'erreur véritable. Notre borne d'erreur pour la fonctionnelle du coût est cependant précise pour toutes les valeurs de N. Dans la pratique, nous pouvons donc légitimement et avec assurance remplacer le contrôle optimal véritable par le contrôle optimal BR au prix d'un tout petit effet négatif sur le coût optimal.

1. Introduction

The solution of optimal control problems governed by PDEs using classical discretization techniques such as finite elements is computationally expensive and time-consuming since the PDE must be solved many times. One way to decrease the computational burden is the surrogate model approach, where the original high-dimensional model is replaced by its reduced order approximation. However, the solution of the reduced order optimal control problem is suboptimal and reliable error estimation is therefore crucial.

A posteriori estimates for the error in the optimal control and the associated cost functional have been proposed in [6] and [2], respectively. However, the bounds in [6], although rigorous, require solution of the high-dimensional problem and are thus online-inefficient; whereas the estimates in [2], although efficient, are not rigorous upper bounds for the error. In this paper we develop new rigorous *and* efficiently evaluable *a posteriori* error bounds for the optimal control *and* the associated cost functional. Our approach thus allows not only the efficient real-time solution of the reduced optimal control problem, but also the efficient real-time evaluation of the quality of the suboptimal solution.

We first recall the reduced basis (RB) recipe for second-order coercive elliptic PDEs (see [4] for a recent review and further references): Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we determine $y^{\rm e} \equiv y^{\rm e}(\mu) \in Y^{\rm e}$ from $a(y^{\rm e}, v; \mu) = b(v;\mu)\, u,\ \forall v \in Y^{\rm e}$. Here, μ and $\mathcal{D}$ are the parameter and parameter domain, respectively; $Y^{\rm e}$ is a Hilbert space with associated inner product $(w,v)_{Y^{\rm e}}$ and norm $\|\cdot\|_{Y^{\rm e}}$; $\Omega \subset \mathbb{R}^d,\ d = 1,2,3$, is our spatial domain, a point in which shall be denoted $(x_1,\ldots,x_d)$; $u \in \mathcal{U} \equiv \mathbb{R}$ is the control; b is a linear bounded functional on $Y^{\rm e}$; and, for any $\mu \in \mathcal{D}$, $a(\cdot,\cdot;\mu) : Y^{\rm e} \times Y^{\rm e} \to \mathbb{R}$ is a coercive continuous bilinear form.

We next introduce a truth finite element approximation space $Y \subset Y^{\rm e}$ of very large dimension $\mathcal{N}$; note that Y shall inherit the inner product and norm from $Y^{\rm e}$. Our truth approximation is then: Given $\mu \in \mathcal{D}$, $y \equiv y(\mu) \in Y$ satisfies $a(y,v;\mu) = b(v;\mu)\, u,\ \forall v \in Y$. We next define the nested parameter sample $S_N \equiv \{\mu_1,\ldots,\mu_N\}$ and associated RB space, $Y_N = \mathrm{span}\{y(\mu_n),\ 1 \le n \le N\}$. Given $\mu \in \mathcal{D}$, the RB approximation $y_N \equiv y_N(\mu) \in Y_N$ satisfies $a(y_N,v;\mu) = b(v;\mu)\, u,\ \forall v \in Y_N$. We may readily derive *a posteriori* error bounds for the error between the RB and truth approximation: the error satisfies $\|y - y_N\|_Y \le \Delta_N(\mu) \equiv \|r(\cdot;\mu)\|_{Y'}/\alpha_{\rm LB}(\mu),\ \forall \mu \in \mathcal{D}$. Here, the dual norm of the residual is defined as $\|r(\cdot;\mu)\|_{Y'} \equiv \sup_{v \in Y} r(v;\mu)/\|v\|_Y$; for given $u \in \mathcal{U}$ the residual is given by $r(v;\mu) = b(v;\mu)u - a(y_N,v;\mu),\ \forall v \in Y$; and $\alpha_{\rm LB}(\mu) : \mathcal{D} \to \mathbb{R}_+$ is a lower bound for the coercivity constant $\alpha(\mu) \equiv \inf_{v\in Y} a(v,v;\mu)/\|v\|_Y^2$. If a and b are affine parameter dependent, e.g., $a(w,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(w,v)$, an efficient offline-online computational procedure can be developed to evaluate y_N and $\Delta_N(\mu)$.

2. Optimal control problem

We first consider the truth formulation. The quadratic cost functional $J(\cdot,\cdot;\mu) : Y \times \mathcal{U} \to \mathbb{R}$ is given by $J(y,u;\mu) = \frac{1}{2}\,\|y - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\,|u - u_d|^2$, where $y_d \in Y$ and $u_d \in \mathcal{U}$ are the desired state and control, respectively; and $\lambda > 0$ is the given regularization parameter. The truth optimal control problem is then

$$\min J(y,u;\mu) \quad \text{s.t.} \quad (y,u) \in Y \times \mathcal{U} \ \text{ solves } \ a(y,v;\mu) = b(v;\mu)\, u,\ \forall v \in Y. \tag{1}$$

It follows from our assumptions that there exists a unique optimal solution to (1) [5]. Employing a Lagrangian approach we obtain the first-order optimality system consisting of the primal and dual equation, and the optimality condition: Given $\mu \in \mathcal{D}$, the optimal solution $(y^*,p^*,u^*) \in Y \times Y \times \mathcal{U}$ satisfies

$$a(y^*,\phi;\mu) = b(\phi;\mu)\, u^*, \quad \forall \phi \in Y, \tag{2}$$
$$a(\varphi,p^*;\mu) = (y^* - y_d, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in Y, \tag{3}$$
$$b(p^*;\mu) + \lambda\,(u^* - u_d) = 0. \tag{4}$$

Here, p is the dual variable and the superscript * denotes optimality.

We next define the RB cost functional as $J_N(\cdot,\cdot;\mu) : Y_N \times \mathcal{U} \to \mathbb{R}$ given by $J_N(y_N,u_N;\mu) = \frac{1}{2}\,\|y_N - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\,|u_N - u_d|^2$. The RB optimal control problem is then

$$\min J_N(y_N,u_N;\mu) \quad \text{s.t.} \quad (y_N,u_N) \in Y_N \times \mathcal{U} \ \text{ solves } \ a(y_N,v;\mu) = b(v;\mu)\, u_N,\ \forall v \in Y_N, \tag{5}$$

and the first order optimality condition is: Given $\mu \in \mathcal{D}$, find $(y_N^*,p_N^*,u_N^*) \in Y_N \times Y_N \times \mathcal{U}$ such that

$$a(y_N^*,\phi;\mu) = b(\phi;\mu)\, u_N^*, \quad \forall \phi \in Y_N, \tag{6}$$
$$a(\varphi,p_N^*;\mu) = (y_N^* - y_d, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in Y_N, \tag{7}$$
$$b(p_N^*;\mu) + \lambda\,(u_N^* - u_d) = 0. \tag{8}$$

We note that the usual offline-online computational decomposition applies to the RB optimality system (6)-(8). If $N \ll \mathcal{N}$, we anticipate that the solution of (5) is computationally much more effecient than the solution of (1). However, we need to rigorously and efficiently assess the error introduced.

3. *A posteriori* error estimation

We now develop the new rigorous *a posteriori* error bounds for the error in the optimal control, $|u^* - u_N^*|$, and the error on the optimal cost functional, $|J^*(y^*, u^*; \mu) - J_N^*(y_N^*, u_N^*; \mu)|$. We only consider linear coercive elliptic PDEs with a scalar unconstrained control here; however, the subsequent analysis can be extended to optimal control problems with distributed controls, control constraints, and also time-dependent problems [3].

3.1. *Error bounds for the optimal control*

We first consider the error in the optimal control. Our derivation is based on the following result from [6].

Theorem 3.1 *Let u^* and u_N^* be the optimal solutions to the truth and RB optimal control problems, respectively. The error in the optimal control then satisfies*

$$|u^* - u_N^*| \le \frac{1}{\lambda} \left| \lambda(u_N^* - u_d) + b\left(p\left(y(u_N^*)\right); \mu\right) \right|, \quad \forall \mu \in \mathcal{D}. \tag{9}$$

We note that $y(u_N^*)$ is the solution to the primal problem (2) with control u_N^* instead of u^* and $p\left(y(u_N^*)\right)$ is the solution to the dual problem (3) with $y(u_N^*)$ instead of $y^*(u^*)$ on the right hand side. Evaluation of the bound (9) thus requires a solution of the truth approximation and is computationally expensive. In contrast, our new bound is online-efficient, i.e, its evaluation is independent of $\mathcal{N}$. We first require the following two intermediate results and then state the main result.

Lemma 3.2 *The primal predictability error, $\tilde{e}^{\mathrm{pr}} = y_N^*(u_N^*) - y(u_N^*)$, is bounded by*

$$\|\tilde{e}^{\mathrm{pr}}\|_Y \le \tilde{\Delta}_N^{\mathrm{pr}}(\mu) \equiv \frac{\|r^{\mathrm{pr}}(\cdot; \mu)\|_{Y'}}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}, \tag{10}$$

where $r^{\mathrm{pr}}(\phi; \mu) = b(\phi; \mu) u_N^* - a(y_N^*, \phi; \mu)$, $\forall \phi \in Y$, $y_N^*(u_N^*)$ is the solution of (6), and $y(u_N^*)$ is the solution of the truth primal equation (2) with control u_N^*.

This is the standard *a posteriori* error bound for coercive elliptic PDEs, for a proof see e.g. [4].

Lemma 3.3 *The dual predictability error, $\tilde{e}^{\mathrm{du}} = p\left(y(u_N^*)\right) - p_N^*\left(y_N^*(u_N^*)\right)$, is bounded by*

$$\|\tilde{e}^{\mathrm{du}}\|_Y \le \tilde{\Delta}_N^{\mathrm{du}}(\mu) \equiv \frac{1}{\alpha_{\mathrm{LB}}(\mu)} \left(\|r^{\mathrm{du}}(\cdot; \mu)\|_{Y'} + C_{\mathrm{eq}}^2 \, \tilde{\Delta}_N^{\mathrm{pr}}(\mu) \right), \ \forall \mu \in \mathcal{D}, \tag{11}$$

where $r^{\mathrm{du}}(\varphi; \mu) = (y_N^* - y_d, \varphi)_{L^2(\Omega)} - a(\varphi, p_N^*; \mu)$, $\forall \varphi \in Y$, $C_{\mathrm{eq}} \equiv \sup_{v \in Y} \frac{\|v\|_{L^2(\Omega)}}{\|v\|_Y}$, $p_N^*\left(y_N^*(u_N^*)\right)$ is the solution of (7), and $p\left(y(u_N^*)\right)$ is the solution of the truth dual equation (3) with $y(u_N^*)$ on the right-hand side.

PROOF. We note from the definition of the dual residual, $r^{\mathrm{du}}(\varphi; \mu)$, and (3) that the error, $\tilde{e}^{\mathrm{du}}$, satisfies $a(\varphi, \tilde{e}^{\mathrm{du}}; \mu) = r^{\mathrm{du}}(\varphi; \mu) + (y(u_N^*) - y_N^*(u_N^*), \varphi)_{L^2(\Omega)}$, $\forall \varphi \in Y$. We now choose $\varphi = \tilde{e}^{\mathrm{du}}$, invoke the coercivity of a, the coercivity lower bound, the definition of the dual norm of the residual, and the Cauchy-Schwarz inequality to obtain $\alpha_{\mathrm{LB}}(\mu) \|\tilde{e}^{\mathrm{du}}\|_Y^2 \le \|r^{\mathrm{du}}(\cdot; \mu)\|_{Y'} \|\tilde{e}^{\mathrm{du}}\|_Y + \|y(u_N^*) - y_N^*(u_N^*)\|_{L^2(\Omega)} \|\tilde{e}^{\mathrm{du}}\|_{L^2(\Omega)}$. The desired result directly follows from the definition of C_{eq} and Lemma 3.2.

Proposition 3.4 *Let u^* and u_N^* be the optimal solutions to the truth and RB optimal control problems, respectively. Given $\tilde{\Delta}_N^{\mathrm{du}}(\mu)$ defined in (11), the error in the optimal control satisfies*

$$|u^* - u_N^*| \le \Delta_N^{u,*}(\mu) \equiv \frac{1}{\lambda} \|b(\cdot; \mu)\|_{Y'} \, \tilde{\Delta}_N^{\mathrm{du}}(\mu), \quad \forall \mu \in \mathcal{D}. \tag{12}$$

PROOF. We append $\pm b\left(p_N^*\left(y_N^*(u_N^*)\right);\mu\right)$ to the bound in (9) and invoke (8) to obtain $|u^* - u_N^*| \le \frac{1}{\lambda}|b\left(p\left(y(u_N^*)\right) - p_N^*\left(y_N^*(u_N^*)\right);\mu\right)|$, $\forall\mu \in \mathcal{D}$. The desired result direclty follows from Lemma 3.3.

3.2. *Error bounds for the cost functional*

Given the error bound $\Delta_N^{u,*}(\mu)$ for the optimal control we may readily derive a bound for the error in the cost functional. We require the following two preparatory results.

Lemma 3.5 *The primal optimality error, $e^{\mathrm{pr},*} = y^*(u^*) - y_N^*(u_N^*)$, is bounded by*

$$\|e^{\mathrm{pr},*}\|_Y \le \Delta_N^{\mathrm{pr},*}(\mu) \equiv \frac{1}{\alpha_{\mathrm{LB}}(\mu)}\left(\|r^{\mathrm{pr}}(\cdot;\mu)\|_{Y'} + \|b(\cdot;\mu)\|_{Y'}\,\Delta_N^{u,*}(\mu)\right), \quad \forall\mu\in\mathcal{D}. \tag{13}$$

PROOF. We note from the definition of the primal residual, $r^{\mathrm{pr}}(\phi;\mu)$, and (2) that the error, $e^{\mathrm{pr},*}$, satisfies $a(e^{\mathrm{pr},*},\phi;\mu) = r^{\mathrm{pr}}(\phi;\mu) + b(\phi;\mu)(u^* - u_N^*)$, $\forall\phi \in Y$. We now choose $\phi = e^{\mathrm{pr},*}$, invoke the coercivity of a, the coercivity lower bound, the definition of the dual norm of the residual and the linear functional $b(\phi;\mu)$ to obtain $\alpha_{\mathrm{LB}}(\mu)\|e^{\mathrm{pr},*}\|_Y^2 \le \|r^{\mathrm{pr}}(\cdot;\mu)\|_{Y'}\|e^{\mathrm{pr},*}\|_Y + \|b(\cdot;\mu)\|_{Y'}\|e^{\mathrm{pr},*}\|_Y|u^* - u_N^*|$. We invoke Proposition 3.4 to obtain the desired result.

Lemma 3.6 *The dual optimality error, $e^{\mathrm{du},*} = p^*\left(y^*(u^*)\right) - p_N^*\left(y_N^*(u_N^*)\right)$, is bounded by*

$$\|e^{\mathrm{du},*}\|_Y \le \Delta_N^{\mathrm{du},*}(\mu) \equiv \frac{1}{\alpha_{\mathrm{LB}}(\mu)}\left(\|r^{\mathrm{du}}(\cdot;\mu)\|_{Y'} + C_{\mathrm{eq}}^2\,\Delta_N^{\mathrm{pr},*}(\mu)\right),\ \forall\mu\in\mathcal{D}. \tag{14}$$

The proof is analogous to the proof of Lemma 3.3 and therefore omitted. We can now state

Proposition 3.7 *Let $J^* = J^*(y^*,u^*;\mu)$ and $J_N^* = J_N^*(y_N^*,u_N^*;\mu)$ be the cost functionals of the truth and RB optimal control problems, respectively. The error then satisfies*

$$|J^* - J_N^*| \le \Delta_N^{J,*}(\mu) \equiv \frac{1}{2}\left(\|r^{\mathrm{du}}(\cdot;\mu)\|_{Y'}\,\Delta_N^{\mathrm{pr},*}(\mu) + \|r^{\mathrm{pr}}(\cdot;\mu)\|_{Y'}\,\Delta_N^{\mathrm{du},*}(\mu)\right),\ \forall\mu\in\mathcal{D}. \tag{15}$$

PROOF. We use the standard result from [1] to estimate the error in the cost functional by $|J^* - J_N^*| \le \frac{1}{2}\left(\|r^{\mathrm{du}}(\cdot;\mu)\|_{Y'}\,\|e^{\mathrm{pr},*}\|_Y + \|r^{\mathrm{pr}}(\cdot;\mu)\|_{Y'}\,\|e^{\mathrm{du},*}\|_Y\right)$, $\forall\mu\in\mathcal{D}$. The result follows from Lemma 3.5 and 3.6.

We first note that our bounds defined in (12) and (15) are rigorous upper bounds for the true errors and that the standard RB offline-online computational procedure directly applies — evaluation of $\Delta_N^{u,*}(\mu)$ and $\Delta_N^{J,*}(\mu)$ is *independent* of $\mathcal{N}$. Furthermore, our *a posteriori* error bounds are not only crucial to confirm the fidelity of the RB optimal control solution, but are also an essential ingredient in the Greedy procedure [4] to generate the RB space Y_N in the first place.

4. Numerical results

We consider steady heat conduction in a two-dimensional thermal block, $\Omega =]0,1[\times]0,1[$, consisting of three subdomains, $\Omega_1 =]0,\frac{1}{3}[\times]0,1[$, $\Omega_2 =]\frac{1}{3},\frac{2}{3}[\times]0,1[$ $\Omega_3 =]\frac{2}{3},1[\times]0,1[$. We consider $\mu = (\mu_1,\mu_2) \in \mathcal{D} \equiv [0.5,5]^2$, where μ_1 and μ_2 are the thermal conductivities in the subdomains Ω_1 and Ω_2, respectively; the reference conductivity in Ω_3 is unity. The temperature satisfies Laplace's equation in Ω with continuity of temperature and heat flux across subdomain interfaces, zero Neumann conditions on the left and right boundaries, zero Dirichlet conditions on the top boundary, Γ_{top}, and imposed heat flux of strength u into the domain on the bottom boundary, Γ_{base}. We use a linear finite element approximation space $Y \subset$

Table 1
$\epsilon^u_{\text{max,rel}}$, $\Delta^u_{\text{max,rel}}$, and $\bar{\eta}^u$ as a function of N.

N	$\epsilon^u_{\text{max,rel}}$	$\Delta^u_{\text{max,rel}}$	$\bar{\eta}^u$
2	5.69 E − 2	1.19 E + 0	20.6
4	1.65 E − 2	8.63 E − 1	50.5
8	4.19 E − 3	8.45 E − 2	961
12	3.19 E − 6	5.09 E − 3	3.99 E + 04
16	9.73 E − 9	3.19 E − 4	1.24 E + 06

Table 2
$\epsilon^J_{\text{max,rel}}$, $\Delta^J_{\text{max,rel}}$, and $\bar{\eta}^J$ as a function of N.

N	$\epsilon^J_{\text{max,rel}}$	$\Delta^J_{\text{max,rel}}$	$\bar{\eta}^J$
2	1.26 E − 1	3.07 E + 0	8.99
4	3.65 E − 2	1.60 E + 0	10.7
8	9.26 E − 3	4.60 E − 2	8.91
12	7.06 E − 6	4.89 E − 5	25.6
16	2.15 E − 8	2.83 E − 7	55.6

$Y^{\rm e} \equiv \{v \in H^1(\Omega) \,|\, v|_{\Gamma_{\rm top}} = 0\}$ with $\mathcal{N} = 3660$; the linear and bilinear forms are given by $b(v) = \int_{\Gamma_{\rm base}} v$ and $a(w, v; \mu) = \mu_1 \int_{\Omega_1} \nabla w \nabla v + \mu_2 \int_{\Omega_2} \nabla w \nabla v + \int_{\Omega_3} \nabla w \nabla v$; the inner product is $(w, v)_Y = a(w, v; \mu_{\rm ref})$ for $\mu_{\rm ref} = (\sqrt{2.5}, \sqrt{2.5})$; we hence choose $\alpha_{\rm LB}(\mu) = \min(\mu_1/\mu_{{\rm ref},1}, \mu_2/\mu_{{\rm ref},2}, 1)$ [4]. We consider the cost functional defined in Section 2 with $y_d = 0$, $u_d = 1$, and $\lambda = 1$. To generate Y_N, we introduce a parameter train sample $\Xi_{\rm train}$ of size $n_{\rm train} = 1600$ and perform a Greedy sampling procedure on $\Delta_N^{J,*}(\mu)$ to obtain $S_N = \{\mu_1, \ldots, \mu_N\}$ and our "integrated" RB space $Y_N = {\rm span}\{(y(\mu_n), p(\mu_n)), 1 \le n \le N\}$.

We now introduce a test sample $\Xi_{\rm test}$ of size $n_{\rm test} = 225$, and define $\epsilon^u_{\rm max,rel} = \max_{\mu \in \Xi_{\rm test}} |u^* - u^*_N|/u^*_{\max}$, $\Delta^u_{\rm max,rel} = \max_{\mu \in \Xi_{\rm test}} \Delta_N^{u,*}(\mu)/u^*_{\max}$, and $\bar{\eta}^u = (n_{\rm test})^{-1} \sum_{\mu \in \Xi_{\rm test}} \Delta_N^{u,*}(\mu)/|u^* - u^*_N|$, where $u^*_{\max} = \max_{\mu \in \Xi_{\rm test}} |u^*|$; the quantities for the cost are defined analogously. In Table 1 and 2 we present maximum relative errors, error bounds, and the associated average effectivities as a function of N. We observe that both the error and error bound converge very fast for the control as well as the cost functional. However, the effectivity for the control bound deteriorates considerably as N increases. This is related to the (problem specific) quadratic convergence of the control error which our bound cannot capture. For more complicated problems, e.g., distributed controls, the error usually does not converge quadratically and our bounds remain sharp for all values of N [3]. Furthermore, our cost functional error bound is sharp for all values of N. In practice we may thus legitimately and reliably replace the truth optimal control by the RB optimal control with very little detriment to the optimal cost.

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