Certified Reduced Basis Methods for Nonaffine Linear Time–Varying Partial Differential Equations

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Abstract

We present reduced basis approximations and associated *a posteriori* error bounds for nonaffine linear time-varying parabolic partial differential equations. We employ the Empirical Interpolation Method in order to construct "affine" coefficient-function approximations of the "nonaffine" parametrized functions. To this end, we extend previous work on time-invariant functions to time-varying functions and introduce a new sampling approach to generate the function approximation space for the latter case. Our *a posteriori* error bounds take both error contributions explicitly into account — the error introduced by the reduced basis approximation *and* the error induced by the coefficient function interpolation. We present an efficient offline-online computational procedure for the calculation of the reduced basis approximation and associated error bound. Numerical results are presented to confirm and test our approach.

1 Introduction

Our focus here is on parabolic PDEs. For simplicity, we will directly consider a time-discrete framework associated to the time interval $I \equiv]0, t_f]$ ($\bar{I} \equiv [0, t_f]$). We divide $\bar{I}$ into K subintervals of equal length $\Delta t = \frac{t_f}{K}$ and define $t^k \equiv k\Delta t$, $0 \le k \le K \equiv \frac{t_f}{\Delta t}$, and $\mathbb{I} \equiv \{t^0, \ldots, t^K\}$; for notational convenience, we also introduce $\mathbb{K} \equiv \{1, \ldots, K\}$. We shall consider Euler-Backward for the time integration although higher-order schemes such as Crank-Nicolson can also be readily treated [14]. We refer to [33] for a reduced basis approach for

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parabolic problems using arbitrary-order Discontinuous-Galerkin temporal schemes. The abstract formulation can be stated as follows: given any $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we evaluate the output $s^{\mathrm{e}\,k}(\mu) \equiv s^{\mathrm{e}}(t^k;\mu) = \ell(y^{\mathrm{e}\,k}(\mu))$, $\forall k \in \mathbb{K}$, where $y^{\mathrm{e}\,k}(\mu) \equiv y^{\mathrm{e}}(t^k;\mu) \in X^{\mathrm{e}}$ satisfies

$$m(y^{\mathrm{e}\,k}(\mu), v) + \Delta t\, a(y^{\mathrm{e}\,k}(\mu), v; \mu) = m(y^{\mathrm{e}\,k-1}(\mu), v) + \Delta t\, f(v;\mu)\, u(t^k),$$
$$\forall v \in X^{\mathrm{e}},\ \forall k \in \mathbb{K}, \quad (1)$$

with initial condition (say) $y^{\mathrm{e}}(t^0;\mu) = y^{\mathrm{e}}_0(\mu) = 0$. Here, $\mathcal{D}$ is the parameter domain in which our P-tuple (input) parameter μ resides, X^{e} is an appropriate Hilbert space, and $\Omega \subset \mathbb{R}^d$ is our spatial domain, a point in which shall be denoted x. Furthermore, $a(\cdot,\cdot;\mu)$ and $m(\cdot,\cdot)$ are X^{e}-continuous and Y^{e}-continuous ($X^{\mathrm{e}} \subset Y^{\mathrm{e}}$) bounded bilinear forms, respectively; $f(\cdot;\mu)$, $\ell(\cdot)$ are Y^{e}-continuous bounded linear functionals; and $u(t^k)$ is the "control input" at time $t = t^k$. We assume here that $\ell(\cdot)$, and $m(\cdot,\cdot)$ do not depend on the parameter; parameter dependence, however, is readily admitted [17].

Since the exact solution is usually unavailable, numerical solution techniques must be employed to solve (1). Classical approaches such as the finite element method can not typically satisfy the requirements of *real-time certified* prediction of the outputs of interest. In the finite element method, the infinite dimensional solution space is replaced by a finite dimensional "truth" approximation space $X \subset X^{\mathrm{e}}$ of size $\mathcal{N}$: for any $\mu \in \mathcal{D}$, we evaluate the output

$$s^k(\mu) = \ell(y^k(\mu)), \quad \forall k \in \mathbb{K}, \quad (2)$$

where $y^k(\mu) \in X$ satisfies

$$m(y^k(\mu), v) + \Delta t\, a(y^k(\mu), v; \mu) = m(y^{k-1}(\mu), v) + \Delta t\, f(v;\mu)\, u(t^k),$$
$$\forall v \in X,\ \forall k \in \mathbb{K}, \quad (3)$$

with initial condition $y(\mu, t^0) = y_0(\mu) = 0$. We shall assume — hence the appellation "truth" — that the approximation space is sufficiently rich such that the FEM approximation $y^k(\mu)$ (respectively, $s^k(\mu)$) is indistinguishable from the analytic, or exact, solution $y^{\mathrm{e}\,k}(\mu)$ (respectively, $s^{\mathrm{e}\,k}(\mu)$).

Unfortunately, for any reasonable error tolerance, the dimension $\mathcal{N}$ needed to satisfy this condition is typically extremely large, and in particular much too large to satisfy the condition of real-time response or the need for numerous solutions. Our goal is the development of numerical methods that permit the *efficient* and *reliable* evaluation of this PDE-induced input-output relationship *in real-time* or *in the limit of many queries* — that is, in the design, optimization, control, and characterization contexts. To achieve this goal we pursue the reduced basis method; see [34] for a recent review of contributions to the methodology.

In this paper we focus on time-varying problems, i.e, we assume that the bilinear form a is given by

$$a_t^k(w, v; \mu) = a_0(w, v) + a_1(w, v, g_t^k(x; \mu)), \quad \forall k \in \mathbb{K} \tag{4}$$

and

$$f(v; g_t^k(x; \mu)) = \int_\Omega v\, g_t^k(x; \mu), \quad \forall k \in \mathbb{K}. \tag{5}$$

where $g_t^k(x; \mu) = g_t(x, t^k; \mu)$ is a parametrized nonaffine time-varying function. In [16] we developed efficient offline-online strategies for *reduced basis approximations* of time-invariant nonaffine (and certain classes of nonlinear) elliptic and parabolic PDEs. Our approach is based on the Empirical Interpolation Method (EIM) [3] — a technique that recovers the efficient offline-online decomposition even in the presence of nonaffine parameter dependence. *A posteriori* error bounds for nonaffine linear and certain classes of nonaffine nonlinear elliptic problems have been proposed in Ref. [26] and Ref. [5], respectively. In this paper, we shall consider the extension of these techniques and develop *a posteriori* error bounds for nonaffine linear time-varying parabolic problems. We recall that the computational cost to generate the collateral reduced basis space for the function approximation is very high in the parabolic case if the function g is time-varying either through an explicit dependence on time or an implicit dependence *via* the field variable $y(t^k; \mu)$ [16]. We therefore propose a novel more efficient approach to generate the collateral reduced basis space which is based on a POD(in time)/Greedy(in parameter space) search [18].

A large number of model order reduction (MOR) techniques [1, 6, 7, 25, 27, 32, 36, 40] have been developed to treat (nonlinear) time-dependent problems. One approach is linearization [40] and polynomial approximation [7, 27]: however, due to a lack of efficient representations of nonlinear terms and fast exponential growth (with the degree of the nonlinear approximation order) of computational complexity, these methods are quite expensive and do not address strong nonlinearities efficiently. Other approaches for highly nonlinear systems (such as piecewise-linearization) have also been proposed [32, 35] but at the expense of high computational cost and little control over model accuracy. Furthermore, although *a priori* error bounds to quantify the error due to model reduction have been derived in the linear case, *a posteriori* error bounds have not yet been adequately considered even for the linear case, let alone the nonlinear case, for most MOR approaches. Finally, it is important to note that most MOR techniques focus mainly on reduced order modeling of dynamical systems in which time is considered the *only* "parameter;" the development of reduced order models for problems with a simultaneous dependence of the field variable on parameter and time — our focus here — is much less common [4, 8].

This paper is organized as follows: In Section 2 we first present a short review of the empirical interpolation method and then extend these ideas to treat nonaffine time-varying functions. The abstract problem formulation, reduced basis approximation, associated *a posteriori* error estimation, and computational considerations for linear time-varying parabolic problems with nonaffine parameter dependence are discussed in Section 3. Numerical results are used throughout to test and confirm our theoretical results. We offer concluding remarks in Section 4.

2 Empirical Interpolation Method

The Empirical Interpolation Method, introduced in [3], serves to construct "affine" coefficient-function approximations of "non-affine" parametrized functions. The method is frequently applied in reduced basis approximations of parametrized partial differential equations with nonaffine parameter dependence [3, 16, 16]; the affine approximation of the equations is crucial for computational efficiency. Here, we briefly summarize the results for the interpolation procedure and the estimator for the interpolation error and subsequently extend these ideas to treat nonaffine time-varying functions.

2.1 Time-invariant parametrized functions

2.1.1 Coefficient-function approximation

We are given a function $g : \Omega \times \mathcal{D} \to \mathbb{R}$ such that, for all $\mu \in \mathcal{D}$, $g(\cdot;\mu) \in L^\infty(\Omega)$. Here, $\mathcal{D} \subset \mathbb{R}^P$ is the parameter domain, $\Omega \subset \mathbb{R}^2$ is the spatial domain – a point in which shall be denoted by $x = (x_{(1)}, x_{(2)})$ – and $L^\infty(\Omega) \equiv \{v \,|\, \operatorname{ess\,sup}_{v\in\Omega} |v(x)| < \infty\}$.

We first define the nested sample sets $S^g_M \equiv \{\mu^g_1 \in \mathcal{D}, \ldots, \mu^g_M \in \mathcal{D}\}$, associated reduced basis spaces $W^g_M = \text{span}\,\{\xi_m \equiv g(x;\mu^g_m), 1 \le m \le M\}$, and nested sets of interpolation points $T^g_M = \{x_1, \ldots, x_M\}$, $1 \le M \le M_{\max}$. We present here a generalization for the construction of the EIM which allows a simultaneous definition of the generating functions W^g_M and associated interpolation points T^g_M [24]. The construction is based on a greedy algorithm [38] and is required for our POD/Greedy-EIM algorithm which we will introduce in Section 2.2.1.

We first choose $\mu_1^g \in \mathcal{D}$, compute $\xi_1 \equiv g(x; \mu_1^g)$, define $W_1^g \equiv \operatorname{span}\{\xi_1\}$, and set $x_1 = \arg \operatorname{ess\,sup}_{x\in\Omega} |\xi_1(x)|$, $q_1 = \xi_1(x)/\xi_1(x_1)$, and $B_{11}^1 = 1$. We then proceed by induction to generate S_M^g, W_M^g, and T_M^g: for $1 \leq M \leq M_{\max} - 1$, we determine $\mu_{M+1}^g \equiv \arg\max_{\mu\in\Xi_{\text{train}}^g} \|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)}$, compute $\xi_{M+1} \equiv g(x; \mu_{M+1}^g)$, and define $W_{M+1}^g \equiv \operatorname{span}\{\xi_m\}_{m=1}^{M+1}$. To generate the interpolation points we solve the linear system $\sum_{j=1}^M \sigma_j^M \, q_j(x_i) = \xi_{M+1}(x_i)$, $1 \leq i \leq M$ and we set $r_{M+1}(x) = \xi_{M+1}(x) - \sum_{j=1}^M \sigma_j^M \, q_j(x)$, $x_{M+1} = \arg \text{ ess } \sup_{x\in\Omega} |r_{M+1}(x)|$, and $q_{M+1}(x) = r_{M+1}(x)/r_{M+1}(x_{M+1})$. Here, $\Xi_{\text{train}}^g \subset \mathcal{D}$ is a finite but suitably large train sample which shall serve as our $\mathcal{D}$ surrogate, and $g_M(\cdot;\mu) \in W_M^g$ is the EIM interpolant of $g(\cdot;\mu)$ over the set T_M^g for any $\mu \in \mathcal{D}$. Specifically

$$g_M(x;\mu) \equiv \sum_{m=1}^{M} \varphi_{M\,m}(\mu) q_m, \tag{6}$$

where

$$\sum_{j=1}^{M} B_{ij}^M \varphi_{M\,j}(\mu) = g(x_i;\mu), \quad 1 \leq i \leq M, \tag{7}$$

and the matrix $B^M \in \mathbb{R}^{M\times M}$ is defined such that $B_{ij}^M = q_j(x_i)$, $1 \leq i, j \leq M$. We note that the determination of the coefficients $\varphi_{M\,m}(\mu)$ requires only $\mathcal{O}(M^2)$ computational cost since B^M is lower triangular with unity diagonal and that $\{q_m\}_{m=1}^M$ is a basis for W_M^g [3, 16].

Finally, we define a "Lebesgue constant" [30] $\Lambda_M \equiv \sup_{x\in\Omega} \sum_{m=1}^M |V_m^M(x)|$, where $V_m^M(x) \in W_M^g$ are the characteristic functions of W_M^g satisfying $V_m^M(x_n) \equiv \delta_{mn}$, $1 \leq m, n \leq M$; here, δ_{mn} is the Kronecker delta symbol. We recall that *(i)* the set of all characteristic functions $\{V_m^M\}_{m=1}^M$ is a basis for W_M^g, and *(ii)* the Lebesgue constant Λ_M satisfies $\Lambda_M \leq 2^M - 1$ [3, 16]. In applications, the actual asymptotic behavior of Λ_M is much lower, as we shall observe subsequently.

2.1.2 *A posteriori* error estimation

Given an approximation $g_M(x;\mu)$ for $M \leq M_{\max} - 1$, we define $\mathcal{E}_M(x;\mu) \equiv \hat{\varepsilon}_M(\mu) \, q_{M+1}(x)$, where $\hat{\varepsilon}_M(\mu) \equiv |g(x_{M+1};\mu) - g_M(x_{M+1};\mu)|$. We also define the interpolation error as

$$\varepsilon_M(\mu) \equiv \|g(\,\cdot\,;\mu) - g_M(\,\cdot\,;\mu)\|_{L^\infty(\Omega)}. \tag{8}$$

In general, $\varepsilon_M(\mu) \geq \hat{\varepsilon}_M(\mu)$, since $\varepsilon_M(\mu) \geq |g(x;\mu) - g_M(x;\mu)|$ for all $x \in \Omega$, and thus also for $x = x_{M+1}$. However, we can prove (see [3, 16, 24]).

Proposition 1. *If $g(\cdot;\mu) \in W^g_{M+1}$, then (i) $g(x;\mu) - g_M(x;\mu) = \pm\mathcal{E}_M(x;\mu)$ (either $\mathcal{E}_M(x;\mu)$ or $-\mathcal{E}_M(x;\mu)$), and (ii) $\|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)} = \hat{\varepsilon}_M(\mu)$.*

Of course, in general $g(\,\cdot\,;\mu) \notin W^g_{M+1}$, and hence our estimator $\hat{\varepsilon}_M(\mu)$ is indeed a lower bound. However, if $\varepsilon_M(\mu) \to 0$ very fast, we expect that the effectivity,

$$\eta_M(\mu) \equiv \frac{\hat{\varepsilon}_M(\mu)}{\varepsilon_M(\mu)}\,, \tag{9}$$

shall be close to unity. Furthermore, the estimator is very inexpensive – *one additional evaluation* of $g(\,\cdot\,;\mu)$ at a single point in Ω.

Finally, we note that we can readily improve the rigor of our bound by relaxing the condition $g(\,\cdot\,;\mu) \in W^g_{M+1}$. In fact, since the space W^g_M is hierarchical, i.e., $W^g_1 \subset W^g_2 \subset \ldots \subset W^g_{M_{\max}}$, the assumption $g(\,\cdot\,;\mu) \in W^g_M$ is more likely to hold as we increase the dimension M of the approximation space. Thus, given an approximation $g_M(x;\mu)$ for $M \leq M_{\max} - k$, we can show that if $g(\cdot;\mu) \in W^g_{M+k}$, then $\tilde{\varepsilon}_M(\mu) = 2^{k-1}\max_{i\in\{1,\ldots,k\}}|g(x_{M+i};\mu) - g_M(x_{M+i};\mu)|$ is an upper bound for interpolation error $\varepsilon_M(\mu)$ [16]. This relaxation of the assumption on $g(x;\mu)$ only comes at a modest additional cost – we need to evaluate $g(\,\cdot\,;\mu)$ at k additional points in Ω.

2.1.3 Numerical results

We consider the function $g(\cdot;\mu) = G(\cdot;\mu)$, where

$$G(x;\mu) \equiv \frac{1}{\sqrt{(x_{(1)} - \mu_1)^2 + (x_{(2)} - \mu_2)^2}}, \tag{10}$$

for $x \in \Omega =]0,1[^2 \in \mathbb{R}^2$ and $\mu \in \mathcal{D} \equiv [-1,-0.01]^2$. From a physical point of view, $G(x;\mu)$ describes the gravity potential of a unit mass located at the position (μ_1,μ_2) in the spatial domain.

We introduce a triangulation of Ω with $\mathcal{N} = 2601$ vertices over which we realize $G(\cdot;\mu)$ as a piecewise linear function. We choose for $\Xi_{\text{train}} \subset \mathcal{D}$ a deterministic grid of 40×40 parameter points over $\mathcal{D}$ and we take $\mu^g_1 = (-0.01,-0.01)$. Next, we pursue the empirical interpolation procedure described in Section 2.1.1 to construct S^g_M, W^g_M, T^g_M, and B^M, $1 \leq M \leq M_{\max}$, for $M_{\max} = 57$.

We now introduce a parameter test sample Ξ_{Test} of size $Q_{\text{Test}} = 225$, and define the maximum error $\varepsilon_{M,\max} = \max_{\mu\in\Xi_{\text{Test}}}\varepsilon_M(\mu)$, the maximum error estimator $\hat{\varepsilon}_{M,\max} = \max_{\mu \in \Xi_{\text{Test}}} \hat{\varepsilon}_M(\mu)$, the average effectivity $\bar{\eta}_M = Q_{\text{Test}}^{-1} \sum_{\mu \in \Xi_{\text{Test}}} \eta_M(\mu)$, where $\eta_M(\mu)$ is the effectivity defined in (9), and $\varkappa_M$ is the condition number of B^M. We present in Table 2 $\varepsilon_{M,\max}$, $\hat{\varepsilon}_{M,\max}$, $\bar{\eta}_M$, Λ_M, and $\varkappa_M$ as a function of M. We observe that $\varepsilon_{M,\max}$ and the bound $\hat{\varepsilon}_{M,\max}$ converge rapidly with M and that the error estimator effectivity is less than but reasonably close to unity. We also note that the Lebesgue constant grows very slowly and that B^M is quite well-conditioned for our choice of basis.

Table 1: Numerical results for empirical interpolation of $G(x;\mu)$: $\varepsilon_{M,\max}$, $\hat{\varepsilon}_{M,\max}$, $\bar{\eta}_M$, Λ_M, and $\varkappa_M$ as a function of M.

M	$\varepsilon_{M,\max}$	$\hat{\varepsilon}_{M,\max}$	$\bar{\eta}_M$	Λ_M	$\varkappa_M$
8	2.05 E−01	1.62 E−01	0.17	1.98	3.73
16	8.54 E−03	8.54 E−03	0.85	2.26	6.01
24	6.53 E−04	6.49 E−04	0.50	3.95	8.66
32	1.29 E−04	1.28 E−05	0.73	5.21	12.6
40	1.37 E−05	1.35 E−06	0.43	5.18	16.6
48	4.76 E−06	1.78 E−07	0.19	10.2	20.0

2.2 Time-varying parametrized functions

2.2.1 Coefficient-function approximation

We extend the previous results and consider parametrized nonaffine time-varying functions $g_t : \Omega \times I \times \mathcal{D} \to \mathbb{R}$. We assume that g_t is smooth in time; for simplicity here, we suppose that for all $\mu \in \mathcal{D}$, $g_t(\cdot,\cdot;\mu) \in C^\infty(I, L^\infty(\Omega))$. Note that we use the subscript t to signify the dependence on time. We consider the time-discretization introduced in Section 1 and — analogous to the notation used for the field variable $y^k(\mu)$ – write $g_t^k(x;\mu) = g_t(x,t^k;\mu)$.

We first consider the construction of the nested sample sets $S_M^{g_t}$, associated reduced basis spaces $W_M^{g_t}$, and nested sets of interpolation points $T_M^{g_t}$. To this end, we propose a new POD/Greedy-EIM procedure which combines the greedy selection procedure in parameter space, described in Section 2.1.1 for nonaffine time-invariant functions, with the Proper Orthogonal Decomposition (POD) in time.

Let $\text{POD}_Y(\{g_t^k(\cdot;\mu), 1 \le k \le K\}, R)$ return the R largest POD modes, $\{\chi_i, 1 \le i \le R\}$, with respect to the $(\cdot,\cdot)_Y$ inner product. We recall that the POD modes, χ_i, are mutually Y-orthogonal such that $\mathcal{P}_R = \text{span}\{\chi_i, 1 \le$

$i \le R\}$ satisfies the optimality property

$$\mathcal{P}_R = \arg \inf_{Y_R \subset \text{span}\{g_t^k(\cdot;\mu), 1 \le k \le K\}} \left(\frac{1}{K} \sum_{k=1}^{K} \inf_{w \in Y_R} \| g_t^k(\cdot;\mu) - w \|_Y^2 \right), \qquad (11)$$

where Y_R denotes a linear space of dimension R. Here, we are only interested in the largest POD mode which we obtain using the method of snapshots [36]. To this end, we solve the eigenvalue problem $C\psi^i = \lambda^i \psi^i$ for $(\psi^1 \in \mathbb{R}^K, \lambda^1 \in \mathbb{R})$ associated with the largest eigenvalue of C, where $C_{ij} = (g_t^i(\cdot;\mu), g_t^j(\cdot;\mu))_Y,\ 1 \le i,j \le K$. We then obtain the first POD mode from $\chi_1 = \sum_{k=1}^{K} \psi_k^1 \, g_t^k(\cdot;\mu)$.

Before summarizing the POD/Greedy-EIM procedure, we define the EIM interpolant in the time-varying case as

$$g_{t,M}^k(x;\mu) \equiv \sum_{m=1}^{M} \varphi_{M\,m}^k(\mu)\, q_m, \quad \forall k \in \mathbb{K}, \qquad (12)$$

where

$$\sum_{j=1}^{M} B_{ij}^M \varphi_{M\,j}^k(\mu) = g_t^k(x_i;\mu), \quad 1 \le i \le M,\ \forall k \in \mathbb{K}. \qquad (13)$$

We note that the computational cost to determine the time-varying coefficients $\varphi_{M\,m}^k(\mu),\ 1 \le m \le M$, for all timesteps is $\mathcal{O}(KM^2)$. The POD/Greedy-EIM procedure is summarized in Algorithm 1.

2.2.2 *A posteriori* error estimation

The *a posteriori* error estimation procedure for the time-varying case directly follows from the time-invariant case of Section 2.1.2. We first define the time-varying interpolation error as

$$\varepsilon_{t,M}^k(\mu) \equiv \| g_t^k(x;\mu) - g_{t,M}^k(x;\mu) \|_{L^\infty(\Omega)}, \quad \forall k \in \mathbb{K}, \qquad (14)$$

and the estimator $\hat{\varepsilon}_{t,M}^k(\mu) \equiv |g_t^k(x_{M+1};\mu) - g_{t,M}^k(x_{M+1};\mu)|$, $\forall k \in \mathbb{K}$. The estimator for the interpolation error at each timestep follows from Proposition 1 and is stated in

Corollary 2.1. *If $g_t^k(\,\cdot\,;\mu) \in W_{M+1}^{g_t}$ for all $k \in \mathbb{K}$, then (i) $g_t^k(x;\mu) - g_{t,M}^k(x;\mu) = \pm \hat{\varepsilon}_{t,M}^k(\mu) q_{M+1}(x)$, $\forall k \in \mathbb{K}$, and (ii) $\| g_t^k(\cdot;\mu) - g_{t,M}^k(\cdot;\mu) \|_{L^\infty(\Omega)} = \hat{\varepsilon}_{t,M}^k(\mu)$, $\forall k \in \mathbb{K}$.*

Algorithm 1: POD/Greedy-EIM Algorithm

specify $\Xi^{g}_{\text{train}} \subset \mathcal{D}$, $M_{\max}$, $\mu_1^{g_t} \in \mathcal{D}$ (arbitrary).

$\xi_1 \equiv \text{POD}_Y(\{g_t^k(\cdot;\mu_1^{g_t}), 1 \le k \le K\}, 1)$.

set $M = 1$, $S_1^{g_1} = \{\mu_1^{g_t}\}$, $W_1^{g_t} \equiv \text{span}\{\xi_1\}$.

set $x_1 = \arg \text{ ess } \sup_{x\in\Omega} |\xi_1(x)|$, $q_1 = \xi_1(x)/\xi_1(x_1)$, and $B_{11}^1 = 1$.

while $M \le M_{\max} - 1$ **do**

$\mu_{M+1}^{g_t} = \arg\max_{\mu\in\Xi_{\text{train}}} \Delta t \sum_{k=1}^{K} \|g_t^k(\cdot;\mu) - g_{t,M}^k(\cdot;\mu)\|_{L^\infty(\Omega)}$,
where $g_{t,M}^k$ is calculated from (12) and (13);

$e^k_{M,\text{EIM}}(\mu) = g_t^k(x;\mu_{M+1}^{g_t}) - g_{t,M}^k(x;\mu_{M+1}^{g_t})$, $1 \le k \le K$;

$\xi_{M+1} = \text{POD}_Y(\{e^k_{M,\text{EIM}}(\mu_{M+1}^{g_t}), 1 \le k \le K\}, 1)$;

$W_{M+1}^{g_t} \leftarrow W_M^{g_t} \oplus \text{span}\{\xi_{M+1}\}$;

$S_{M+1}^{g_t} \leftarrow S_M^{g_t} \cup \mu_{M+1}^{g_t}$;

solve for σ_j^M from $\sum_{j=1}^{M} \sigma_j^M\, q_j(x_i) = \xi_{M+1}(x_i)$, $1 \le i \le M$;

set $r_{M+1}(x) = \xi_{M+1}(x) - \sum_{j=1}^{M} \sigma_j^M\, q_j(x)$;

set $x_{M+1} = \arg \text{ ess } \sup_{x\in\Omega} |r_{M+1}(x)|$;

set $q_{M+1}(x) = r_{M+1}(x)/r_{M+1}(x_{M+1})$;

update $B_{i\,j}^{M+1} = q_j(x_i)$, $1 \le i, j \le M+1$;

$M \leftarrow M + 1$;

end

We note that the condition $g_t^k(\cdot\,;\mu) \in W_{M+1}^{g_t}$ has to hold for all timesteps and is thus more restrictive than in the time-invariant case. In general, $\hat{\varepsilon}_{t,M}^k(\mu)$ is a lower bound for the interpolation error at each timestep. Finally, we may also define the effectivity

$$\eta_{t,M}^k(\mu) \equiv \frac{\hat{\varepsilon}_{t,M}^k(\mu)}{\varepsilon_{t,M}^k(\mu)}\,, \quad \forall k \in \mathbb{K}. \tag{15}$$

Again, our estimator is very inexpensive – at each individual timestep we have to perform only *one additional evaluation* of $g_t^k(\,\cdot\,;\mu)$ at a single point in Ω.

2.2.3 Numerical results

We consider the nonaffine time-varying function $g_t^k(\cdot;\mu) = G_t^k(\cdot;\mu)$, where

$$G_t^k(x;\mu) \equiv \frac{1}{\sqrt{\left(x_{(1)} - (\mu_1 - t^k/2)\right)^2 + \left(x_{(2)} - (\mu_2 - t^k/2)\right)^2}}, \tag{16}$$

for $x \in \Omega =]0,1[^2 \in \mathbb{R}^2$, $t^k \in \mathbb{I}$, and $\mu \in \mathcal{D} \equiv [-1,-0.01]^2$. Compared to the stationary problem (10), $G_t(x;\mu)$ describes the gravity potential of a unit mass which is now moving in the spatial domain, i.e., the mass is initially located at the position (μ_1, μ_2) and then moving with velocity $(-1/2, -1/2)$ as time proceeds.

We use the triangulation of Ω and Ξ_{train} from Section 2.1.3 and take $\mu_1^{g_t} = (-0.01, -0.01)$. Next, we employ Algorithm 1 to construct $S_M^{g_t}$, $W_M^{g_t}$, $T_M^{g_t}$, and B^M, $1 \le M \le M_{\max}$, for $M_{\max} = 49$. We define the maximum error $\varepsilon_{t,M,\max} = \max_{\mu \in \Xi_{\text{Test}}} \max_{k \in \mathbb{K}} \varepsilon_{t,M}^k(\mu)$, maximum error bound $\hat{\varepsilon}_{t,M,\max} = \max_{\mu \in \Xi_{\text{Test}}} \max_{k \in \mathbb{K}} \hat{\varepsilon}_{t,M}^k(\mu)$, and the average effectivity $\bar{\eta}_{t,M} = (KQ_{\text{Test}})^{-1} \sum_{\mu \in \Xi_{\text{Test}}} \sum_{k \in \mathbb{K}} \eta_{t,M}^k(\mu)$. Here, we use the parameter test sample Ξ_{Test} of size $Q_{\text{Test}} = 225$ from Section 2.1.3.

We present in Table 2 $\varepsilon_{t,M,\max}$, $\hat{\varepsilon}_{t,M,\max}$, $\bar{\eta}_{t,M}$, the Lebesgue constant Λ_M, and the condition number of B^M, $\varkappa_M$, as a function of M. Similar to the time-invariant case, the maximum error and bound converge rapidly with M and the error estimator effectivity is less than but still reasonably close to unity. However, $\varepsilon_{t,M,\max}$ and $\hat{\varepsilon}_{t,M,\max}$ are always larger than the corresponding time-invariant quantities in Table 1 for the same value of M. This is to be expected since time acts like an additional, albeit special, parameter. In fact, for the numerical example considered here the time-dependence effectively increases the admissible parameter domain $\mathcal{D}$ of the time-invariant problem. Finally, we note that the Lebesgue constant grows very slowly and that B^M remains well-conditioned also for the time-varying case.

Table 2: Numerical results for empirical interpolation of $G_t^k(x;\mu)$, $1 \le k \le K$: $\varepsilon_{t,M,\max}$, $\hat{\varepsilon}_{t,M,\max}$, $\bar{\eta}_{t,M}$, Λ_M, and $\varkappa_M$ as a function of M.

M	$\varepsilon_{t,M,\max}$	$\hat{\varepsilon}_{t,M,\max}$	$\bar{\eta}_{t,M}$	Λ_M	$\varkappa_M$
8	4.79 E−01	4.79 E−01	0.58	3.67	6.26
16	2.76 E−02	2.39 E−02	0.56	5.11	13.0
24	3.09 E−03	3.09 E−03	0.77	6.47	18.8
32	1.99 E−04	1.15 E−04	0.60	11.4	27.7
40	9.13 E−05	9.13 E−05	0.34	8.84	45.8
48	1.11 E−05	5.16 E−06	0.11	9.05	54.4

3 Nonaffine Linear Time-varying Parabolic Equations

In this section we consider reduced basis approximations and associated *a posteriori* error estimation procedures for linear parabolic PDEs with nonaffine parameter dependence. We derive the theoretical results for linear time-varying (LTV) problems and occasionally comment on the simplifications that arise for linear time-invariant (LTI) problems. Numerical results are presented for both the LTI and LTV problem.

3.1 Problem statement

3.1.1 Abstract formulation

We first recall the Hilbert spaces $X^{\mathrm{e}} \equiv H_0^1(\Omega)$ — or, more generally, $H_0^1(\Omega) \subset X^{\mathrm{e}} \subset H^1(\Omega)$ — and $Y^{\mathrm{e}} \equiv L^2(\Omega)$, where $H^1(\Omega) \equiv \{v \mid v \in L^2(\Omega), \nabla v \in (L^2(\Omega))^d\}$, $H_0^1(\Omega) \equiv \{v \mid v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$, and $L^2(\Omega)$ is the space of square integrable functions over Ω [31]. Here Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. The inner product and norm associated with X^{e} (Y^{e}) are given by $(\cdot,\cdot)_{X^{\mathrm{e}}}$ ($(\cdot,\cdot)_{Y^{\mathrm{e}}}$) and $\|\cdot\|_{X^{\mathrm{e}}} = (\cdot,\cdot)_{X^{\mathrm{e}}}^{1/2}$ ($\|\cdot\|_{Y^{\mathrm{e}}} = (\cdot,\cdot)_{Y^{\mathrm{e}}}^{1/2}$) , respectively; for example, $(w,v)_{X^{\mathrm{e}}} \equiv \int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\, v,\ \forall\, w,v \in X^{\mathrm{e}}$, and $(w,v)_{Y^{\mathrm{e}}} \equiv \int_\Omega w\, v,\ \forall\, w,v \in Y^{\mathrm{e}}$. The truth approximation subspace $X \subset X^{\mathrm{e}} (\subset Y^{\mathrm{e}})$ shall inherit this inner product and norm:

$(\cdot;\cdot)_X \equiv (\cdot;\cdot)_X^{\mathrm{e}}$ and $\|\cdot\|_X \equiv \|\cdot\|_X^{\mathrm{e}}$; we further define $Y \equiv Y^{\mathrm{e}}$.

We directly consider the truth approximation statement defined in (3) with the output given by (2), where the bilinear form a is given by

$$a_t^k(w,v;\mu) = a_0(w,v) + a_1(w,v,g_t^k(x;\mu)), \quad \forall k \in \mathbb{K} \tag{17}$$

and

$$f(v;g_t^k(x;\mu)) = \int_\Omega v\, g_t^k(x;\mu), \quad \forall k \in \mathbb{K}. \tag{18}$$

Here, $a_0(\cdot,\cdot)$ is a continuous (and, for simplicity, parameter-independent) bilinear form and $a_1 : X \times X \times L^\infty(\Omega)$ is a trilinear form. We shall use the subscript "t" notation to signify the dependence of the bilinear form a_t^k on time. We obtain the LTI problem simply by replacing $g_t^k(x;\mu)$ with the nonaffine time-invariant function $g(x;\mu)$ in (17) and (18).

We shall further assume that $a_t^k(\cdot,\cdot;\mu)$, $\forall k \in \mathbb{K}$, and $m(\cdot,\cdot)$ are continuous

$$\begin{aligned} a_t^k(w,v;\mu) &\leq \gamma_a(\mu)\|w\|_X\|v\|_X \leq \gamma_a^0\|w\|_X\|v\|_X, \quad \forall\, w,v \in X,\ \forall\, \mu \in \mathcal{D}, & (19)\\ m(w,v) &\leq \gamma_m^0\|w\|_Y\|v\|_Y, \quad \forall\, w,v \in X; & (20) \end{aligned}$$

coercive,

$$\begin{aligned} 0 &< \alpha_a^0 \leq \alpha_a(\mu) \equiv \inf_{w\in X} \frac{a_t^k(w,w;\mu)}{\|w\|_X^2}, \quad \forall\, \mu \in \mathcal{D}, & (21)\\ 0 &< \alpha_m^0 \equiv \inf_{v\in X} \frac{m(v,v)}{\|v\|_Y^2}; & (22) \end{aligned}$$

and symmetric, $a_t^k(v,w;\mu) = a_t^k(w,v;\mu)$, $\forall w,v \in X$, $\forall \mu \in \mathcal{D}$, and $m(v,w) = m(w,v)$, $\forall\, w,v \in X$, $\forall\, \mu \in \mathcal{D}$. (We (plausibly) suppose that γ_a^0, γ_m^0, α_a^0, α_m^0 may be chosen independent of $\mathcal{N}$.) We also assume that the trilinear form a_1 satisfies

$$a_1(w,v,z) \leq \gamma_{a_1}^0 \|w\|_X\, \|v\|_X\, \|z\|_{L^\infty(\Omega)}, \quad \forall\, w,v \in X,\ \forall z \in L^\infty(\Omega). \tag{23}$$

Next, we require that the linear forms $f(\cdot;g_t^k(x;\mu)) :\ X \to \mathbb{R}$, $\forall k \in \mathbb{K}$, and $\ell(\cdot) :\ X \to \mathbb{R}$ be bounded with respect to $\|\cdot\|_Y$. It follows that a solution to (3) exists and is unique [13, 37]; also see [31] for the LTI case.

3.1.2 Model problem

As a numerical test case for the LTI and LTV problem we consider the following nonaffine diffusion problem defined on the unit square, $\Omega =]0,1[^2 \in \mathbb{R}^2$:

Given $\mu \equiv (\mu_1, \mu_2) \in \mathcal{D} \equiv [-1, -0.01]^2 \subset \mathbb{R}^{P=2}$, we evaluate $y^k(\mu) \in X$ from (3), where $X \subset X^{\mathrm{e}} \equiv H_0^1(\Omega)$ is a linear finite element truth approximation subspace of dimension $\mathcal{N} = 2601$,

$$m(w,v) \equiv \int_\Omega w\, v,\ a_0(w,v) \equiv \int_\Omega \nabla w \cdot \nabla v,\ a_1(w,v,z) \equiv \int_\Omega z\, w\, v,\ f(v;z) \equiv \int_\Omega z\, v, \tag{24}$$

and z is given by $G(x;\mu)$ defined in (10) for the LTI problem and by $G_t^k(x;\mu)$ defined in (16) for the LTV problem. The output can be written in the form (2), $s^k(\mu) = \ell(y^k(\mu))$, $\forall\, k \in \mathbb{K}$, where $\ell(v) \equiv |\Omega|^{-1} \int_\Omega v$ — clearly a very smooth functional. We shall consider the time interval $\bar{I} = [0, 2]$ and a timestep $\Delta t = 0.01$; we thus have $K = 200$. We also presume the periodic control input $u(t^k) = \sin(2\pi t^k)$, $t^k \in \mathbb{I}$.

We first present results for the LTI problem (note that this problem is similar to the one used in [16]). Two snapshots of the solution $y^k(\mu)$ at time $t^k = 25\Delta t$ are shown in Figures 1(a) and (b) for $\mu = (-1, -1)$ and $\mu = (-0.01, -0.01)$, respectively. The solution oscillates in time and the peak is offset towards $x = (0, 0)$ for μ near the "corner" $(-0.01, -0.01)$. In Figure 2 we plot the output $s^k(\mu)$ as a function of time for these two parameter values.

We next turn to the LTV problem and present the output $s^k(\mu)$ for $\mu = (-0.01, -0.01)$ also in Figure 2 (dashed line). The LTV output shows a transition in time from the LTI output for $\mu = (-0.01, -0.01)$ to the LTI output for $\mu = (-1, -1)$. This behavior is plausible by comparing the definitions of the nonaffine functions $G(x;\mu)$ and $G_t^k(x;\mu)$.

3.2 Reduced basis approximation

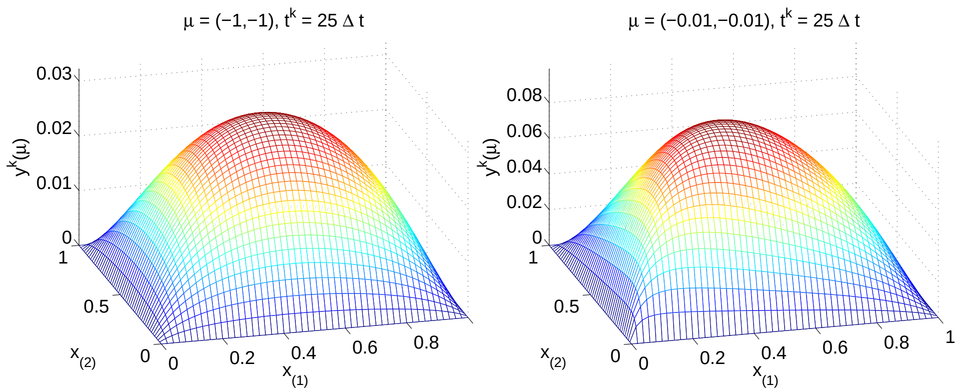
3.2.1 Formulation

We suppose that we are given the nested Lagrangian [28] reduced basis spaces

$$W_N^y = \mathrm{span}\{\zeta_n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max}, \tag{25}$$

where the $\zeta_n,\ 1 \le n \le N$, are mutually $(\cdot,\cdot)_X$-orthogonal basis functions. We comment on the POD/Greedy algorithm for constructing the basis functions in Section 3.4.

Our reduced basis approximation $y_{N,M}^k(\mu)$ to $y^k(\mu)$ is then: given $\mu \in \mathcal{D}$,



(a) (b)

Figure 1: Solution $y^k(\mu)$ of LTI problem at $t^k = 25\Delta t$ for (a) $\mu = (-1,-1)$ and (b) $\mu = (-0.01,-0.01)$.

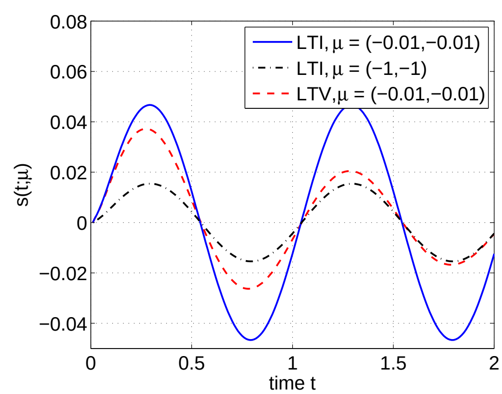


Figure 2: Output $s^k(\mu)$ of the LTI problem for $\mu = (-0.01,-0.01)$ and $\mu = (-1,-1)$, and output of the LTV problem for $\mu = (-0.01,-0.01)$.

$y^k_{N,M}(\mu) \in W^y_N,\ \forall k \in \mathbb{K}$, satisfies

$$\begin{aligned} m(y^k_{N,M}(\mu), v) + \Delta t\ (a_0(y^k_{N,M}(\mu), v) + a_1(y^k_{N,M}(\mu), v; g^k_{t,M}(x;\mu))) \\ = m(y^{k-1}_{N,M}(\mu), v) + \Delta t\ f(v; g^k_{t,M}(x;\mu))\ u(t^k), \quad \forall\, v \in W^y_N, \end{aligned} \tag{26}$$

with initial condition $y^0_{N,M}(\mu) = 0$. We then evaluate the output estimate, $s^k_{N,M}(\mu),\ \forall k \in \mathbb{K}$, from

$$s^k_{N,M}(\mu) \equiv \ell(y^k_{N,M}(\mu)). \tag{27}$$

Note that we directly replaced $g^k_t(x;\mu)$ in (17) by its affine approximation $g^k_{t,M}(x;\mu)$ defined in (12).

We now express $y^k_{N,M}(\mu) = \sum_{n=1}^N\ y^k_{N,Mn}(\mu)\ \zeta_n$, choose as test functions $v = \zeta_j$, $1 \le j \le N$, and invoke (6) to obtain

$$\begin{aligned} &\sum_{i=1}^N \left\{ m(\zeta_i, \zeta_j) + \Delta t \left(a_0(\zeta_i, \zeta_j) + \sum_{m=1}^M \varphi^k_{M\,m}(\mu)\, a_1(\zeta_i, \zeta_j, q_m) \right) \right\}\ y^k_{N,M\,i}(\mu) \\ &= \sum_{i=1}^N m(\zeta_i, \zeta_j)\ y^{k-1}_{N,M\,i}(\mu) + \Delta t \sum_{m=1}^M \varphi^k_{M\,m}(\mu)\, f(\zeta_j; q_m)\ u(t^k), \quad 1 \le j \le N. \end{aligned} \tag{28}$$

where $\varphi^k_{M\,m}(\mu)$, $1 \le m \le M$, $1 \le k \le K$, is determined from (13). We note that (28) is well-posed if M is large enough such that the interpolation error satisfies $\varepsilon^k_{t,M}(\mu) \le \alpha_a(\mu)/\gamma^0_{a_1},\ \forall k \in \mathbb{K}$. This condition on the interpolation error directly follows from (21) and (23). We can thus recover online $\mathcal{N}$-independence even for nonaffine problems: the quantities $m(\zeta_i, \zeta_j)$, $a_0(\zeta_i, \zeta_j)$, $a_1(\zeta_i, \zeta_j, q_m)$, and $f(\zeta_i; q_m)$ are all *parameter independent* and can thus be pre-computed offline, as discussed in the next section.

3.2.2 Computational procedure

We summarize here the offline-online procedure [2, 20, 23, 29]. We first express $y^k_{N,M}(\mu)$ as

$$y^k_{N,M}(\mu) = \sum_{n=1}^N\ y^k_{N,Mn}(\mu)\ \zeta_n, \tag{29}$$

and choose as test functions $v = \zeta_j$, $1 \leq j \leq N$ in (26). It then follows from (28) that $\underline{y}^k_{N,M}(\mu) = [y^k_{N,M\,1}(\mu) \;\; y^k_{N,M\,2}(\mu) \ldots y^k_{N,M\,N}(\mu)]^T \in \mathbb{R}^N$ satisfies

$$\Big(M_N + \Delta t\; A^k_N(\mu)\Big)\; \underline{y}^k_{N,M}(\mu) = M_N\; \underline{y}^{k-1}_{N,M}(\mu) + \Delta t\; F^k_N(\mu)\; u(t^k), \quad \forall\, k \in \mathbb{K}, \tag{30}$$

with initial condition $y_{N,M\,n}(\mu, t^0) = 0$, $1 \leq n \leq N$. Given $\underline{y}^k_{N,M}(\mu)$, $\forall k \in \mathbb{K}$, we finally evaluate the output estimate from

$$s^k_{N,M}(\mu) = L^T_N\; \underline{y}^k_{N,M}(\mu), \quad \forall\, k \in \mathbb{K}. \tag{31}$$

Here, $M_N \in \mathbb{R}^{N\times N}$ is a *parameter-independent* SPD matrix with entries

$$M_{N\,i,j} = m(\zeta_i, \zeta_j), \quad 1 \leq i, j \leq N. \tag{32}$$

Furthermore, we obtain from (6) and (17) that $A^k_N(\mu) \in \mathbb{R}^{N\times N}$ and $F^k_N(\mu) \in \mathbb{R}^N$ can be expressed as

$$A^k_N(\mu) \;=\; A_{0,N} + \sum_{m=1}^{M} \varphi^k_{M\,m}(\mu)\; A^m_{1,N}, \tag{33}$$

$$F^k_N(\mu) \;=\; \sum_{m=1}^{M} \varphi^k_{M\,m}(\mu)\; F^m_N, \tag{34}$$

where $\varphi^k_{M\,m}(\mu)$, $1 \leq m \leq M$, is calculated from (13) at each timestep, and the *parameter-independent* quantities $A_{0,N} \in \mathbb{R}^{N\times N}$, $A^m_{1,N} \in \mathbb{R}^{N\times N}$, and $F^m_N \in \mathbb{R}^N$ are given by

$$\begin{aligned} A_{0,N\,i,j} &= a_0(\zeta_i, \zeta_j), && 1 \leq i, j \leq N, \\ A^m_{1,N\,i,j} &= a_1(\zeta_i, \zeta_j, q_m), && 1 \leq i, j \leq N,\; 1 \leq m \leq M, \\ F^m_{N\,j} &= f(\zeta_j; q_m), && 1 \leq j \leq N,\; 1 \leq m \leq M, \end{aligned} \tag{35}$$

respectively. Finally, $L_N \in \mathbb{R}^N$ is the output vector with entries $L_{N\,i} = \ell(\zeta_i)$, $1 \leq i \leq N$. We note that these quantities must be computed in a stable fashion which is consistent with the finite element quadrature points (see [14], p. 132).

The offline-online decomposition is now clear. In the offline stage — performed only *once* — we first construct the nested approximation spaces $W^{g_t}_M$ and sets of interpolation points $T^{g_t}_M$, $1 \leq M \leq M_{\max}$. We then solve for the ζ_n, $1 \leq n \leq N_{\max}$ and compute and store the μ-independent quantities in (32), (35), and L_N. The computational cost — without taking into account the construction of $W_M^{g_t}$ and $T_M^{g_t}$ — is therefore $O(KN_{\max})$ solutions of the underlying $\mathcal{N}$-dimensional "truth" finite element approximation and $O(M_{\max}N_{\max}^2)$ $\mathcal{N}$-inner products; the storage requirements are also $O(M_{\max}N_{\max}^2)$. In the online stage — performed many times, for each new parameter value μ — we compute $\varphi_{M\,m}^k(\mu)$, $1 \leq m \leq M$, from (13) at cost $O(M^2)$ per timestep by multiplying the pre-computed inverse of B^M with the vector $g_t^k(x_m;\mu)$, $1 \leq m \leq M$. We then assemble the reduced basis matrix (33) and vector (34); this requires $O(MN^2)$ operations per timestep. We then solve (30) for $\underline{y}_{N,M}^k(\mu)$; since the reduced basis matrices are in general full, the operation count is $O(N^3)$ per timestep. The total cost to evaluate $\underline{y}_{N,M}^k(\mu)$, $\forall k \in \mathbb{K}$, is thus $O(K(M^2 + MN^2 + N^3))$. Finally, given $\underline{y}_{N,M}^k(\mu)$ we evaluate the output estimate $s_{N,M}^k(\mu)$, $\forall k \in \mathbb{K}$, from (31) at a cost of $O(KN)$.

Concerning the LTI problem, we note that the overall cost to evaluate $\underline{y}_{N,M}^k(\mu)$, $\forall k \in \mathbb{K}$, in the online stage reduces to $O(M^2+MN^2+N^3+KN^2)$. We need to evaluate the time-independent coefficients $\varphi_{M\,m}(\mu)$, $1 \leq m \leq M$, from (7) and subsequently assemble the reduced basis matrices *only once*, we may then use LU decomposition for the time stepping.

Hence, as required in the many-query or real-time contexts, the online complexity is *independent* of $\mathcal{N}$, the dimension of the underlying "truth" finite element approximation space. Since N, $M \ll \mathcal{N}$ we expect significant computational savings in the online stage relative to classical discretization and solution approaches.

3.3 *A posteriori* error estimation

We will now develop *a posteriori* error estimators which will help us to (i) assess the error introduced by our reduced basis approximation (relative to the "truth" finite element approximation); and (ii) devise an efficient procedure for generating the reduced basis space W_N^y. We recall that *a posteriori* error estimates have been developed for reduced basis approximations of linear *affine* parabolic problems using a finite element truth discretization in [17]. Subsequently, extensions to finite volume disretizations including bounds for the error in the $L^2(\Omega)$-norm have also been considered [18].

3.3.1 Preliminaries

To begin, we specify the inner products $(v,w)_X \equiv a_0(v,w)$, $\forall\, v,w \in X$ and $(v,w)_Y \equiv m(v,w)$, $\forall\, v,w \in X$. We next assume that we are given a positive lower bound for the coercivity constant $\alpha_a(\mu)$: $\hat{\alpha}_a(\mu) : \mathcal{D} \to \mathbb{R}_+$ satisfies

$$\alpha_a(\mu) \geq \hat{\alpha}_a(\mu) \geq \hat{\alpha}_a^0 > 0, \quad \forall\, \mu \in \mathcal{D}. \tag{36}$$

We note that if $g_t^k(x;\mu) > 0$, $\forall k \in \mathbb{K}$, we may readily use $\hat{\alpha}_a(\mu) = 1$ as a lower bound. In general, however, we may need to develop a lower bound, $\hat{\alpha}_{a_1,M}(\mu)$ for the coercivity constant of the perturbed weak form $a_1(\cdot,\cdot; g_{t,M}^k(x;\mu))$ using the Successive Constraint Method (SCM) [19]. In this case we directly obtain from (17), (21) and (23) the additional requirement that the interpolation error has to satisfy $\varepsilon_{t,M}^k(\mu) < (1+\hat{\alpha}_{a_1,M}(\mu))/\gamma_{a_1}^0$. In some instances, simpler recipes may also suffice [29, 39].

We next introduce the dual norm of the residual

$$\varepsilon_{N,M}^k(\mu) \equiv \sup_{v\in X} \frac{R^k(v;\mu)}{\|v\|_X}, \quad \forall\, k \in \mathbb{K}, \tag{37}$$

where

$$\begin{aligned} R^k(v;\mu) \equiv f(v; g_{t,M}^k(x;\mu))\, u(t^k) - a_0(y_{N,M}^k(\mu), v) - a_1(y_{N,M}^k(\mu), v, g_{t,M}^k(x;\mu)) \\ - \frac{1}{\Delta t} m(y_{N,M}^k(\mu) - y_{N,M}^{k-1}(\mu), v), \quad \forall\, v \in X,\ \forall\, k \in \mathbb{K}. \end{aligned} \tag{38}$$

We also introduce the dual norm

$$\Phi_M^{\mathrm{na}\,k}(\mu) \equiv \sup_{v\in X} \frac{f(v; q_{M+1})\ u(t^k) - a_1(y_{N,M}^k(\mu), v, q_{M+1})}{\|v\|_X}, \quad \forall\, k \in \mathbb{K}, \tag{39}$$

which reflects the contribution of the nonaffine terms. Finally, we define the "spatio-temporal" energy norm, $|||v^k(\mu)|||^2 \equiv m(v^k(\mu), v^k(\mu)) + \sum_{k'=1}^{k} a_t^{k'}(v^{k'}(\mu), v^{k'}(\mu);\mu)\,\Delta t$, $\forall k \in \mathbb{K}$.

3.3.2 Error bound formulation

We obtain the following result for the error bound.

Proposition 2. *Suppose that $g_t^k(x;\mu) \in W_{M+1}^{g_t}$ for $1 \leq k \leq K$. The error, $e^k(\mu) \equiv y^k(\mu) - y_{N,M}^k(\mu)$, is then bounded by*

$$|||e^k(\mu)||| \leq \Delta_{N,M}^{y\,k}(\mu), \quad \forall\, \mu \in \mathcal{D},\ \forall\, k \in \mathbb{K}, \tag{40}$$

where the error bound $\Delta_{N,M}^{y\,k}(\mu) \equiv \Delta_{N,M}^{y}(t^k;\mu)$ is defined as

$$\Delta_{N,M}^{y\,k}(\mu) \equiv \left(\frac{2\Delta t}{\hat{\alpha}_a(\mu)} \sum_{k'=1}^{k} \varepsilon_{N,M}^{k'}(\mu)^2 + \frac{2\Delta t}{\hat{\alpha}_a(\mu)} \sum_{k'=1}^{k} \left((\hat{\varepsilon}_{t,M}^{k'}(\mu)\ \Phi_M^{\mathrm{na}\,k'}(\mu) \right)^2 \right)^{\frac{1}{2}}. \tag{41}$$

Proof. The proof is an extension of the result in [17]. We thus focus on the new bits – the nonaffine parameter dependence. Following the steps in [17], we obtain

$$\begin{aligned} &m(e^k(\mu), e^k(\mu)) - m(e^{k-1}(\mu), e^{k-1}(\mu)) + \Delta t \ a_t^k(e^k(\mu), e^k(\mu); \mu) \\ &\quad \le \Delta t \ \varepsilon_{N,M}^k(\mu) \ \|e^k(\mu)\|_X + \Delta t \ \big(f(e^k(\mu); g_t^k(x;\mu) - g_{t,M}^k(x;\mu)) \ u(t^k) \\ &\qquad\qquad - a_1(y_{N,M}^k(\mu), e^k(\mu), g_t^k(x;\mu) - g_{t,M}^k(x;\mu)) \big). \quad (42) \end{aligned}$$

Using Young's inequality, the first term on the right hand side can be bound by

$$2 \ \varepsilon_N^k(\mu) \ \|e^k(\mu)\|_X \le \frac{2}{\hat{\alpha}_a(\mu)} \ \varepsilon_{N,M}^k(\mu)^2 + \frac{\hat{\alpha}_a(\mu)}{2} \ \|e^k(\mu)\|_X^2. \quad (43)$$

From our assumption, $g_t^k(x;\mu) \in W_{M+1}^{g_t}$ for $1 \le k \le K$, Corollary 2.1, and (39) it directly follows that

$$\begin{aligned} &f(e^k(\mu); g_t^k(x;\mu) - g_{t,M}^k(x;\mu)) \ u(t^k) - a_1(y_{N,M}^k(\mu), e^k(\mu), g_t^k(x;\mu) - g_{t,M}^k(x;\mu)) \\ &\qquad \le \ \hat{\varepsilon}_{t,M}^k(\mu) \ \sup_{v \in X} \frac{f(v; q_{M+1}) \ u(t^k) - a_1(y_{N,M}^k(\mu), v, q_{M+1})}{\|v\|_X} \ \|e^k(\mu)\|_X \\ &\qquad \le \ \hat{\varepsilon}_{t,M}^k(\mu) \ \Phi_M^{\mathrm{na}\,k}(\mu) \ \|e^k(\mu)\|_X, \quad (44) \end{aligned}$$

and again from Young's inequality that

$$2 \, \hat{\varepsilon}_{t,M}^k(\mu) \, \Phi_M^{\mathrm{na}\,k}(\mu) \, \|e^k(\mu)\|_X \le \frac{2}{\hat{\alpha}_a(\mu)} \ \Big(\hat{\varepsilon}_{t,M}^k(\mu) \ \Phi_M^{\mathrm{na}\,k}(\mu) \Big)^2 + \frac{\hat{\alpha}_a(\mu)}{2} \ \|e^k(\mu)\|_X^2. \quad (45)$$

The desired results then directly follows from (42)-(45), invoking (21) and (36), and finally summing from $k' = 1$ to k with $e(\mu, t^0) = 0$. □

We note from (41) that our error bound comprises the affine as well as the nonaffine error contributions. We may thus choose N and M such that both contributions balance, i.e., neither N nor M should be chosen unnecessarily high. We also recall that our (crucial) assumption $g_t^k(x;\mu) \in W_{M+1}^{g_t}$, $1 \le k \le K$, cannot be confirmed in actual practice — in fact, we generally have $g_t^k(x;\mu) \notin W_{M+1}^{g_t}$ and hence our error bound (41) is *not* completely rigorous, since $\hat{\varepsilon}_{t,M}^k(\mu) \le \epsilon_{t,M}^k(\mu)$. We comment on both of these issues again in detail in Section 3.5 when discussing numerical results.

Finally, we note that the bound for the LTI case slightly simplifies due to the fact that the error estimator $\hat{\varepsilon}_M(\mu)$ is independent of time and can thus be pulled out of the summation.

We can now define the (simple) output bound in

Proposition 3. *Suppose that $g_t^k(x;\mu) \in W_{M+1}^{g_t}$ for $1 \le k \le K$. The error in the output of interest is then bounded by*

$$|s^k(\mu) - s_{N,M}^k(\mu)| \le \Delta_{N,M}^{s\,k}(\mu), \quad \forall\, k \in \mathbb{K},\ \forall\, \mu \in \mathcal{D}, \tag{46}$$

where the output bound $\Delta_{N,M}^{s\,k}(\mu)$ is defined as

$$\Delta_{N,M}^{s\,k}(\mu) \equiv \sup_{v \in X} \frac{\ell(v)}{\|v\|_Y} \Delta_{N,M}^{y\,k}(\mu)\,. \tag{47}$$

Proof. From (2) and (27) we obtain

$$\begin{aligned} |s^k(\mu) - s_{N,M}^k(\mu)| &= |\ell(y^k(\mu)) - \ell(y_{N,M}^k(\mu))| \\ &= |\ell(e^k(\mu))| \le \sup_{v \in X} \frac{\ell(v)}{\|v\|_Y}\, \|e^k(\mu)\|_Y. \end{aligned}$$

The result immediately follows since $\|e^k(\mu)\|_Y \le \Delta_{N,M}^{y\,k}(\mu)$, $1 \le k \le K$. $\square$

3.3.3 Computational procedure

We now turn to the development of offline-online computational procedures for the calculation of $\Delta_{N,M}^{y\,k}(\mu)$ and $\Delta_{N,M}^{s\,k}(\mu)$. The necessary computations for the offline and online stage are detailed in [15]. Here, we only summarize the computational costs involved.

In the offline stage we require solution of several Poisson problems and inner products. These operations requires (to leading order) $O(M_{\max} N_{\max})$ expensive "truth" finite element solutions, and $O(M_{\max}^2 N_{\max}^2)$ $\mathcal{N}$-inner products. In the online stage — given a new parameter value μ and associated reduced basis solution $\underline{y}_{N,M}^k(\mu)$, $1 \le k \le K$ — the computational cost to evaluate $\Delta_{N,M}^{y\,k}(\mu)$ and $\Delta_{N,M}^{s\,k}(\mu)$, $1 \le k \le K$, is $O(KM^2N^2)$. Thus, all online calculations needed are *independent* of $\mathcal{N}$.

Concerning the LTI problem, we note that we can slightly lessen the computational cost by performing the M-dependent sums once before evaluating the dual norm at each timestep; the computational cost is thus $O(M^2N^2 + KN^2)$.

3.4 Sampling procedure

The sampling procedure is a two stage process. We first construct the sample set $S_M^{g_t}$, associated space $W_M^{g_t}$, and set of interpolation points $T_M^{g_t}$ for the nonaffine function as described in Section 2. We then invoke a

POD/Greedy sampling procedure — a combination of the Proper Orthogonal Decomposition (POD) in time with a Greedy selection procedure in parameter space [18, 21] — to generate W_N^y. We first recall the function $\mathrm{POD}_X(\{y^k(\mu), 1 \le k \le K\}, R)$ which returns the R largest POD modes, $\{\chi_i, 1 \le i \le R\}$, now with respect to the X inner product.

The POD/Greedy procedure proceeds as follows: we first choose a $\mu^* \in \mathcal{D}$ and set $S_0^y = \{0\}$, $W_0^y = \{0\}$, $N = 0$. Then, for $1 \le N \le N_{\max}$, we first compute the projection error $e_{N,\mathrm{proj}}^k(\mu) = y^k(\mu^*) - \mathrm{proj}_{X,W_{N-1}^y}(y^k(\mu^*))$, $1 \le k \le K$, where $\mathrm{proj}_{X,W_N}(w)$ denotes the X-orthogonal projection of $w \in X$ onto W_N, and we expand the parameter sample $S_N^y \leftarrow S_{N-1}^y \cup \{\mu^*\}$ and the reduced basis space $W_N^y \leftarrow W_{N-1}^y \cup \mathrm{POD}_X(\{e_{N,\mathrm{proj}}^k(\mu^*), 1 \le k \le K\}, 1)$, and set $N \leftarrow N + 1$. Finally, we choose the next parameter value from $\mu^* \leftarrow \arg\max_{\mu \in \Xi_{\mathrm{train}}} \Delta_{N,M_{\max}}^{y\,K}(\mu)/|||y_N^K(\mu)|||$, i.e., we perform a greedy search over Ξ_{train} for the largest relative *a posteriori* error bound at the final time. Here, $\Xi_{\mathrm{train}} \subset \mathcal{D}$ is a finite but suitably large train sample. In general, we may also specify a desired error tolerance, $\epsilon_{\mathrm{tol,min}}$, and stop the procedure as soon as $\max_{\mu \in \Xi_{\mathrm{train}}} \Delta_{N,M_{\max}}^{y\,K}(\mu)/|||y_N^K(\mu)||| \le \epsilon_{\mathrm{tol,min}}$ is satisfied; $N_{\max}$ is then indirectly determined through the stopping criterion.

During the POD/Greedy sampling procedure we shall use the "best" possible approximation $g_{t,M}^k(x;\mu)$ of $g_t^k(x;\mu)$ so as to minimize the error induced by the empirical interpolation procedure, i.e., we set $M = M_{\max}$.

For the model problem introduced in Section 3.1.2 the control input $u(t^k)$ was assumed to be known. In many instances, however, the control input may not be known a priori — a typical example is the application of reduced order models in a control setting. If the problem is linear time-invariant, we can appeal to the LTI property and generate the reduced basis space based on an impulse input in such cases [17]. Unfortunately, this approach will not work for LTV problems and nonlinear problems, i.e., a reduced basis space trained on an impulse response will, in general, not yield good approximation properties for arbitrary control inputs $u(t^k)$. One possible approach proposed in the literature is to train the reduced order model on a "generalized" impulse input, see [22]. The idea here is to use a collection of "representative" control inputs, e.g., impulse and step functions of different magnitude shifted in time, in order to capture a richer dynamic behavior of the system. These ideas are, of course, heuristic and the treatment of unknown control inputs in the model reduction context is an open problem. However, our *a posteriori* error bound serves as a measure of fidelity especially in the online stage and we can thus detect an unacceptable deviation from the truth approximation in real-time.

3.5 Numerical results

We next present results for the model problem introduced in Section 3.1.2. We first consider the LTI problem and subsequently the LTV problem.

3.5.1 The time-invariant case

We construct the reduced basis space W_N^y according to the POD/Greedy sampling procedure in Section 3.4. To this end, we sample on Ξ_{train} with $M = M_{\max}$ and obtain $N_{\max} = 45$ for $\epsilon_{\text{tol,min}} = 1\text{E}{-6}$. We note from the definition of the X-inner product and the fact that $G(x;\mu) > 0, \forall \mu \in \mathcal{D}$, that we can simply use $\hat{\alpha}_a(\mu) = 1$ as a lower bound for the coercivity constant.

In Figure 3 we plot, as a function of N and M, the maximum relative error in the energy norm $\epsilon^y_{N,M,\max,\text{rel}} = \max_{\mu \in \Xi_{\text{Test}}} |||e^K(\mu)|||/|||y^K(\mu_y)|||$, where $\mu_y \equiv \arg\max_{\mu \in \Xi_{\text{Test}}} |||y^K(\mu)|||$. We observe that the reduced basis approximation converges very rapidly. We also note the "plateau" in the curves for M fixed and the "drops" in the $N \to \infty$ asymptotes as M increases: for fixed M the error due to the coefficient function approximation, $g_M(x;\mu) - g(x;\mu)$, will ultimately dominate for large N; increasing M renders the coefficient function approximation more accurate, which in turn leads to a drop in the error. We further note that the separation points, or "knees," of the N-M-convergence curves reflect a balanced contribution of both error terms. At these points neither N nor M limit the convergence of the reduced basis approximation.

In Table 3 we present, as a function of N and M, $\epsilon^y_{N,M,\max,\text{rel}}$, the maximum relative error bound $\Delta^y_{N,M,\max,\text{rel}}$, and the average effecitivity $\bar{\eta}^y_{N,M}$; here, $\Delta^y_{N,M,\max,\text{rel}}$ is the maximum over Ξ_{Test} of $\Delta^{y\,k}_{N,M}(\mu)/|||y^K(\mu_y)|||$ and $\bar{\eta}^y_{N,M}$ is the average over $\Xi_{\text{Test}} \times \mathbb{I}$ of $\Delta^{y\,k}_{N,M}(\mu)/|||y^k(\mu) - y^k_{N,M}(\mu)|||$. Note that the tabulated (N, M) values correspond roughly to the "knees" of the N-M-convergence curves. We observe very rapid convergence of the reduced basis approximation and error bound.

The effectivity serves as a measure of rigour and sharpness of the error bound: we would like $\bar{\eta}^y_{N,M} \geq 1$, i.e., $\Delta^{y\,k}_{N,M}(\mu)$ be a true upper bound for the error in the energy-norm, and ideally we have $\bar{\eta}^y_{N,M} \approx 1$ so as to obtain a sharp bound for the error. We recall, however, that in actual practice we cannot confirm the assumption $g(x;\mu) \in W^g_{M+1}$ from Proposition 2 and thus $\bar{\eta}^y_{N,M} \geq 1$ may not hold. Specifically, if we choose (N, M) such that the function interpolation limits the convergence we do obtain effectivities less than 1, e.g., for $(N, M) = (25, 24)$ (instead of $(25, 32)$ in Table 2) we obtain

$\bar{\eta}^y_{N,M} = 0.83$. A judicious choice for N and M is thus important for rigour and safety.

We next turn to the output estimate and present, in Table 3, the maximum relative output error $\epsilon^s_{N,M,\max,\mathrm{rel}}$, the maximum relative output bound $\Delta^s_{N,M,\max,\mathrm{rel}}$, and the average output effectivity $\bar{\eta}^s$. Here, $\epsilon^s_{N,M,\max,\mathrm{rel}}$ is the maximum over Ξ_{Test} of $|s(\mu, t^k_s(\mu)) - s_{N,M}(\mu, t^k_s(\mu))|/|s(\mu, t^k_s(\mu))|$, $\Delta^s_{N,M,\max,\mathrm{rel}}$ is the maximum over Ξ_{Test} of $\Delta^s_{N,M}(\mu, t^k_s(\mu))/|s(\mu, t^k_s(\mu))|$, and $\bar{\eta}^s$ is the average over Ξ_{Test} of $\Delta^s_{N,M}(\mu, t_\eta(\mu))/|s(\mu, t_\eta(\mu)) - s_{N,M}(\mu, t_\eta(\mu))|$, where $t^k_s(\mu) \equiv \arg\max_{t^k \in \mathbb{I}} |s(\mu, t^k)|$ and $t_\eta(\mu) \equiv \arg\max_{t^k \in \mathbb{I}} |s(\mu, t^k) - s_{N,M}(\mu, t^k)|$. Again, we observe very rapid convergence of the reduced basis output approximation and output bound — for only $N = 15$ and $M = 24$ the output error bound is already less than 0.3%. The output effectivities are still acceptable for smaller values of (N, M), but deteriorate for larger values.

In Table 4 we present, as a function of N and M, the online computational times to calculate $s^k_{N,M}(\mu)$ and $\Delta^{s\,k}_{N,M}(\mu)$ for $1 \le k \le K$. The values are normalized with respect to the computational time for the direct calculation of the truth approximation output $s^k(\mu) = \ell(y^k(\mu))$, $1 \le k \le K$. The computational savings for an accuracy of less than 0.3 percent ($N = 15$, $M = 24$) in the output bound is approximately a factor of 30. We note that the time to calculate $\Delta^{s\,k}_{N,M}(\mu)$ exceeds that of calculating $s^k_N(\mu)$ — this is due to the higher computational cost, $O(M^2N^2 + KN^2)$, to evaluate $\Delta^{y\,k}_{N,M}(\mu)$. Thus, although our previous observation suggests to choose M large so that the error contribution due to the nonaffine function approximation is small, we should choose M as small as possible to retain the computational efficiency of our method. We emphasize that the reduced basis entry does *not* include the extensive offline computations — and is thus only meaningful in the real-time or many-query contexts.

Table 3: LTI problem: convergence rates and effectivities as a function of N and M.

N	M	$\epsilon^y_{N,M,\max,\mathrm{rel}}$	$\Delta^y_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^y_{N,M}$	$\epsilon^s_{N,M,\max,\mathrm{rel}}$	$\Delta^s_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^s_{N,M}$
5	16	1.22 E−02	1.74 E−02	1.42	3.30 E−03	1.01 E−01	29.1
15	24	3.32 E−04	4.75 E−04	1.09	1.57 E−04	2.77 E−03	27.5
25	32	2.91 E−05	4.30 E−05	1.44	1.88 E−05	2.50 E−04	85.4
35	40	3.78 E−06	3.50 E−06	1.11	3.22 E−06	2.04 E−05	137
45	48	5.66 E−07	8.17 E−07	1.39	8.14 E−08	4.76 E−06	553

Table 4: LTI problem: online computational times (normalized with respect to the time to solve for $s^k(\mu),\ 1 \le k \le K$).

N	M	$s^k_{N,M}(\mu),\ \forall k \in \mathbb{K}$	$\Delta^{s\,k}_{N,M}(\mu),\ \forall k \in \mathbb{K}$	$s^k(\mu),\ \forall k \in \mathbb{K}$
5	16	2.70 E−03	1.84 E−02	1
15	24	3.18 E−03	3.01 E−02	1
25	32	3.96 E−03	4.57 E−02	1
35	40	4.71 E−03	7.16 E−02	1
45	48	5.52 E−03	1.02 E−01	1

3.5.2 The time-varying case

We next consider the LTV problem and first recall the results for the non-affine time-varying function approximation in Section 2.2.3. We perform the POD/Greedy sampling procedure from Section 3.4 to generate the reduced basis space. To this end, we sample on Ξ_{train} with $M = M_{\max}$ and obtain $N_{\max} = 39$ for $\epsilon_{\text{tol,min}} = 1\,\text{E}{-}6$. We may again use $\hat{\alpha}_a(\mu) = 1$ as a lower bound for the coercivity constant. Note that the quantities presented here are defined analogous to the quantities presented for the LTI problem.

In Figure 4 we plot, as a function of N and M, the maximum relative error in the energy norm $\epsilon^y_{N,M,\max,\text{rel}}$. We observe that the reduced basis approximation converges very rapidly and the curves show the same behavior as in the LTI case. A balanced contribution of both error terms is important to not limit the convergence of the approximation and thus to guarantee computational efficiency.

In Table 5 we present, as a function of N and M, $\epsilon^y_{N,M,\max,\text{rel}}$, the maximum relative error bound $\Delta^y_{N,M,\max,\text{rel}}$, the average effecitivity $\bar{\eta}^y_{N,M}$, the maximum relative output error $\epsilon^s_{N,M,\max,\text{rel}}$, the maximum relative output bound $\Delta^s_{N,M,\max,\text{rel}}$, and the average output effectivity $\bar{\eta}^s$. Again, the tabulated (N, M) values correspond roughly to the "knees" of the N-M-convergence curves. We observe very rapid convergence of the reduced basis (output) approximation and (output) error bound. We obtain an average effectivity of less than one for $(N, M) = (5, 8)$, showing that our assumption in Corollary 2.1 is not satisfied in general. However, for all other values of N and M tabulated our a posteriori error bounds do provide an upper bound for the true error. Furthermore, our bounds for the error in the energy norm and the output are very sharp for all values of (N, M).

Finally, we present the online computational times to calculate $s^k_{N,M}(\mu)$ and $\Delta^{s\,k}_{N,M}(\mu)$ for $1 \le k \le K$ in Table 6. The values are normalized with

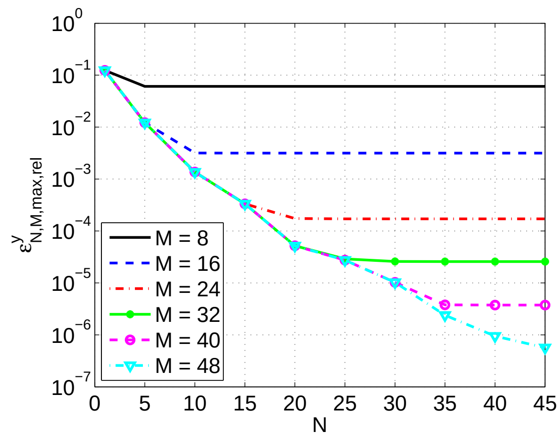


Figure 3: LTI problem: convergence of the maximum relative error, $\epsilon^y_{N,M,\max,\mathrm{rel}}$.

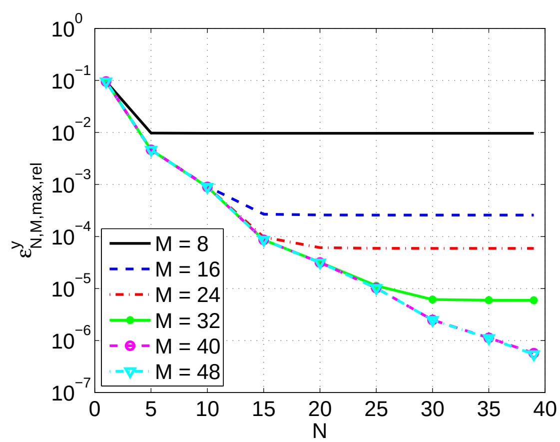


Figure 4: LTV problem: convergence of the maximum relative error $\epsilon^y_{N,M,\max,\mathrm{rel}}$.

respect to the computational time for the direct calculation of the truth approximation output $s^k(\mu) = \ell(y^k(\mu)),\ 1 \le k \le K$. The computational savings for an accuracy of approximately 0.5 percent ($N = 10,\ M = 16$) in the output bound is approximately a factor of 120. As we already observed in the LTI case, the time to calculate $\Delta^{s\,k}_{N,M}(\mu)$ exceeds that of calculating $s^k_N(\mu)$ due to the higher computational cost for the bound calculation. Even though the computational cost to evaluate the error bound in the LTV case is higher than in the LTI case (see Section 3.3.3), the savings with respect to the truth approximation are still larger here. The reason is that solving the truth approximation requires a matrix assembly of the nonaffine terms at every timestep.

Table 5: LTV problem: convergence rates and effectivities as a function of N and M.

N	M	$\epsilon^y_{N,M,\max,\mathrm{rel}}$	$\Delta^y_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^y_{N,M}$	$\epsilon^s_{N,M,\max,\mathrm{rel}}$	$\Delta^s_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^s_{N,M}$
5	8	9.72 E−03	6.27 E−02	0.64	1.17 E−02	2.46 E−02	1.64
10	16	9.13 E−04	1.29 E−03	1.44	2.68 E−04	5.05 E−03	34.8
15	24	9.75 E−05	1.28 E−04	1.32	6.48 E−05	5.03 E−04	16.7
25	32	1.13 E−05	1.54 E−05	1.19	6.99 E−06	6.03 E−05	7.28
35	40	1.13 E−06	1.64 E−06	1.44	1.38 E−07	6.44 E−06	64.0

Table 6: LTV problem: online computational times (normalized with respect to the time to solve for $s^k(\mu),\ 1 \le k \le K$).

N	M	$s^k_{N,M}(\mu),\ \forall k \in \mathbb{K}$	$\Delta^{s\,k}_{N,M}(\mu),\ \forall k \in \mathbb{K}$	$s^k(\mu),\ \forall k \in \mathbb{K}$
5	8	1.35 E−04	2.18 E−03	1
10	16	1.92 E−04	8.18 E−03	1
15	24	2.86 E−04	1.99 E−02	1
25	32	4.48 E−04	3.85 E−02	1
35	40	7.07 E−04	6.52 E−02	1

4 Conclusions

We have presented *a posteriori* error bounds for reduced basis approximations of nonaffine linear time-varying parabolic partial differential equations. We employed the Empirical Interpolation Method to construct affine coefficient-function approximations of the nonaffine parametrized functions, thus permitting an efficient offline-online computational procedure for the calculation of the reduced basis approximation and the associated error bounds. The error bounds take both error contributions — the error introduced by the reduced basis approximation *and* the error induced by the coefficient function interpolation — explicitly into account and are rigorous upper bounds under certain conditions on the function approximation. The POD/Greedy sampling procedure is commonly used to generate the reduced basis space for time-dependent problems. Here, we extended these ideas to the Empirical Interpolation Method and introduced a new sampling approach to construct the collateral reduced basis space for time-varying functions. The new sampling approach is more efficient than our previous approach and thus also allows to consider higher parameter dimensions.

We presented numerical results that showed the very fast convergence of the reduced basis approximations and associated error bounds. We note that there exists an optimal, i.e., most online-efficient, choice of N vs. M where neither error contribution limits the convergence of the reduced basis approximation. Although our results showed that we can obtain upper bounds for the error with a judicious choice of N and M, our error bounds are, unfortunately, provably rigorous only under a very restrictive condition on the function interpolation. However, we can easily lift this restriction by replacing our current bound for the interpolation error with a new rigorous bound proposed in a recent note [9].

Our results also showed that the computational savings to calculate the output estimate and bound in the online stage compared to direct calculation of the truth output are considerable. Recently, *hp* techniques have been proposed for reduced basis approximations [10, 11] and also for the EIM [12]. These ideas help limit the online cost by reducing the size of N and M required to achieve a desired accuracy at the expense of a higher offline cost. Combining the *hp* reduced basis and EIM ideas into a unified approach would result in a further significant speed-up and is currently under investigation.

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