An Adjoint Consistency Analysis for a Class of Hybrid Mixed Methods

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AN ADJOINT CONSISTENCY ANALYSIS FOR A CLASS OF HYBRID MIXED METHODS

JOCHEN SCHÜTZ* AND GEORG MAY†

Abstract. Hybrid methods represent a classic discretization paradigm for elliptic equations. More recently, hybrid methods have been formulated for convection-diffusion problems, in particular compressible fluid flow. In [25], we have introduced a hybrid mixed method for the compressible Navier-Stokes equations as a combination of a hybridized DG scheme for the convective terms, and an $H(div, \Omega)-$method for the diffusive part. Since hybrid methods are based on Galerkin's principle, the adjoint of a given hybrid discretization may be used for PDE-constraint optimal control problems, or error estimation, provided that the discretization is adjoint consistent. In the present paper, we extend the adjoint consistency analysis, previously reported for many DG schemes to the more complex hybrid methods. We show that a class of Hybrid Mixed schemes, including the scheme by Nguyen et al. [19] and our recently proposed method [25], is adjoint consistent.

Key words. Hybrid Mixed discretizations, Hybridized Discontinuous Galerkin discretizations, Adjoint Consistency, compressible Navier-Stokes equations

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1. Introduction.

Recent years have seen tremendous development of solution strategies for high-order consistent discretization of the compressible Navier-Stokes equations [2, 13, 21]. Many well-known discretization methods are based on the Discontinuous Galerkin (DG) paradigm [23, 10, 9, 8, 7, 11, 2, 3, 1]. Such schemes achieve high order of consistency very easily by locally adding degrees of freedom. One drawback of these methods is the high amount of storage that is often needed when implicit methods are used for the computation of an approximate solution. One approach to reduce the amount of unknowns is to not express the solution in a cell-based fashion, but rather on the edges of the elements. This leads to *hybrid* methods [5, 25, 6, 19].

In general terms, the Navier-Stokes equations are (systems of) convection-diffusion equations. These are written in mixed form as

$$\begin{aligned} \sigma - f_v(w, \nabla w) &= 0 \quad \forall x \in \Omega \\ \nabla \cdot (f(w) - \sigma) &= 0 \quad \forall x \in \Omega. \end{aligned}$$

Based on the work by Egger and Schöberl [12], we have recently proposed a hybrid-mixed scheme for equations of this type [25]. The scheme was designed to have the following properties:

- In the viscous limit, i.e. $f(w) \to 0$, it should reduce to a standard dual mixed scheme, meaning that $\sigma_h \in H(div, \Omega)$.
- In the convective limit, i.e. $f_v(w, \nabla w) \to 0$, it should reduce to a standard DG scheme, meaning the approximate solution w_h coincides with that obtained by a DG discretization.

As such, the scheme lies somewhere between a hybridized DG and a mixed method. However, in [26], we have shown similarities between our scheme and a previously developed hybridized DG [19] scheme. This is interesting from the point of view that

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we can identify a class of Hybrid Mixed methods, containing a subset of the hybrid methods defined in [6], for which the analysis to be presented applies.

From an engineering point of view, it is often not an approximation w_h to the unknown solution w that is of interest per se, but one is rather interested in an approximation $J(w_h)$ to $J(w)$, where $J : X \to \mathbb{R}$ is a functional that acts on the solution to produce only one real number. The solution space X is usually a suitable Hilbert space. Typical examples of J in aerodynamic studies include lift and drag. In the context of adjoint error-control [4], one important task is to estimate the difference $J(w) - J(w_h)$ in order to obtain an accurate approximation to $J(w)$. A first-order Taylor's expansion yields

$$e_h := J(w) - J(w_h) = J'[w_h](w - w_h) + O(\|w - w_h\|^2). \tag{1.1}$$

A reasonable approximation for e_h is thus given by the quantity $J'(w_h)(w-w_h)$ (which is, however, due to the unavailability of w not directly computable). The action of the derivative on the quantity $(w-w_h)$ can be approximated by an adjoint procedure. Similar procedures are also used in shape optimization to estimate variation of target functionals with respect to shape variation [16]. Roughly speaking, the principle is as follows: Assume that the original (primal) partial differential equation is given in the weak formulation as

$$N(w, v) = 0 \quad \forall v \in X. \tag{1.2}$$

We furthermore assume that there exists an adjoint solution z such that

$$N'[w](\mathrm{dw}, z) = J'[w](\mathrm{dw}) \quad \forall \mathrm{dw} \in X, \tag{1.3}$$

where $N'[w](\mathrm{dw}, z)$ denotes the derivative of N with respect to the first argument in direction dw. Small solution disturbances dw fulfill

$$J(w + \mathrm{dw}) - J(w) + O(\|\mathrm{dw}\|^2) = J'[w](\mathrm{dw}) = N'[w](\mathrm{dw}, z).$$

The essence of this is thus that the adjoint solution relates linearized changes in the functional to linearized changes in the residual. This is at the root of the analytical process we pursue in the following.

There are two obvious ways of solving the adjoint problem:

- Discretizing the adjoint equations independently with a method of choice. This is called *continuous adjoint procedure.*
- Building the adjoint of the discretization that was used to obtain the approximate solution w_h. This is called *discrete adjoint procedure.*

The continuous adjoint approach can be, and in fact is, used in many applications [17]. It might even have advantages in some cases where either the adjoint equation has a certain structure that can be exploited [27] or the method is not based on Galerkin's principle [17]. However, given a Galerkin method, the discrete adjoint procedure offers the advantage that a significant amount of the data structure, which may already be available in a numerical code to solve the primal problem, can often be re-used in the computation of the adjoint solution z_h. However, it is not trivial that such a discrete adjoint approach is viable. Depending on the discretization of the primal problem, the adjoint equations produced in this manner may not be a consistent approximation of the correctly posed adjoint differential equations. One quality measure of the discretization of the primal problem is thus *adjoint consistency*, which means precisely the property that the discrete adjoint is automatically consistent with the adjoint PDE. In the context of DG schemes approximating compressible flow problems, adjoint consistency has been discussed for example by Lu [18], Hartmann [14], and Oliver and Darmofal [20]. Another very important aspect of adjoint consistency is that it allows, under certain conditions, superconvergence of target functionals, as well as optimal L^2-norm convergence. Hence this property is also useful if one is not interested in the adjoint solution. In this paper, we show adjoint consistency of a class of hybrid mixed methods, including our newly-developed hybrid mixed scheme [25], and the hybridized Discontinuous Galerkin scheme introduced by Nguyen et al. [19].

The paper is outlined as follows. In section 2, we introduce the governing equations, while section 3 treats the associated adjoint equations. In section 4, we briefly introduce a class of hybrid mixed methods and give the underlying Ansatz spaces. Section 5, which is the main part of this work, shows that the given class of hybrid mixed method is adjoint consistent. To make the ideas more transparent, these sections are each subdivided into two parts, treating first the simple, scalar convection-diffusion equation, and subsequently the more complex Navier-Stokes equations. Section 6 offers conclusions.

2. Underlying Equations.

The analysis presented in this paper is done first on the conceptually and technically simple case of the scalar convection-diffusion equation, and is then extended to the compressible Navier-Stokes equations.

2.1. Convection-Diffusion Equation.

As a simple, scalar example, we consider the (nonlinear) convection-diffusion equation given on a domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. In *primal* form, this equation can be written as

$$\begin{aligned} \nabla \cdot f(w) - \varepsilon \Delta w &= h \quad \forall x \in \Omega, \\ w &= g \quad \forall x \in \partial\Omega. \end{aligned} \tag{2.1}$$

Many hybridized methods, including the recently proposed HDG methods [19], and the DG/mixed method presented in [25], start from the *mixed* form

$$\begin{aligned} \sigma - \varepsilon \nabla w &= 0 \quad \forall x \in \Omega, \\ \nabla \cdot (f(w) - \sigma) &= h \quad \forall x \in \Omega, \\ w &= g \quad \forall x \in \partial\Omega. \end{aligned} \tag{2.2}$$

We assume that the quantities f, g and h are smooth. The diffusion coefficient $0 < \varepsilon \in \mathbb{R}$ is assumed to be constant. Otherwise, all quantities are, of course, functions of $x \in \Omega$. This simple equation will be used as a prototype for the more complicated compressible Navier-Stokes equations. To formulate approximation spaces for the method, we have to define the dimension d of the system under consideration. As the advection-diffusion equation is a scalar example, we have $d = 1$.

2.2. Navier-Stokes Equations.

The compressible Navier-Stokes equations describe viscous, compressible flow in a domain $\Omega \subset \mathbb{R}^2$. We write these equations as

$$\begin{aligned} \sigma - f_v(w, \nabla w) &= 0 \quad \forall x \in \Omega, \\ \nabla \cdot (f(w) - \sigma) &= 0 \quad \forall x \in \Omega, \end{aligned} \tag{2.3}$$

subject to suitable boundary conditions to be explained later. The state variable w is given by the vector of conserved variables $w = (\rho, \rho u, \rho v, E)$. Here ρ is the density, (u,v) is the velocity vector, and E is the total specific energy. The functions $f = (f_1, f_2)$ and $f_v(w, \nabla w) = (f_{v,1}, f_{v,2})$ are the convective and diffusive fluxes, respectively, given as

$$f_1 = (\rho u, p + \rho u^2, \rho uv, u(E+p))^T, \qquad f_2 = (\rho v, \rho uv, p + \rho v^2, v(E+p))^T,$$
$$f_{v,1} = (0, \tau_{11}, \tau_{21}, \tau_{11}u + \tau_{12}v + kT_{x_1})^T, \quad f_{v,2} = (0, \tau_{12}, \tau_{22}, \tau_{21}u + \tau_{22}v + kT_{x_2})^T.$$

The dimension of the system is thus $d = 4$. Boundary conditions imposed on the velocity vector (u,v) are the *no-slip* boundary conditions

$$u = v = 0 \quad \forall x \in \partial\Omega,$$

while conditions on the temperature T are either *adiabatic* boundary conditions,

$$n \cdot \nabla T = 0 \quad \forall x \in \partial\Omega,$$

or *isothermal* boundary conditions

$$T - T_{wall} = 0 \quad \forall x \in \partial\Omega$$

for a given value $T_{wall} > 0$. Using the ideal gas law, temperature T and pressure p can be related to the conserved variables as

$$T = \frac{\mu\gamma}{k \cdot Pr}\left(\frac{E}{\rho} - \frac{1}{2}(u^2+v^2)\right) = \frac{1}{(\gamma-1)c_v}\frac{p}{\rho},$$

where $Pr = \frac{\mu c_p}{k}$ is the Prandtl number, which for air at moderate conditions is constant, with a value of $Pr = 0.72$. The thermal conductivity coefficient is denoted by k, while c_p and c_v are specific heats at constant pressure and constant volume, respectively. These are related via $\gamma = \frac{c_p}{c_v}$, where $\gamma = 1.4$ is again a constant for air at moderate conditions. Given a Newtonian fluid and assuming that the Stokes hypothesis holds, the viscous stress tensor τ can be written as

$$\tau = \mu\left(\nabla\widehat{w} + (\nabla\widehat{w})^T - \frac{2}{3}(\nabla\cdot\widehat{w})Id\right),$$

where we have set $\widehat{w} := (u,v)^T$. The dynamic viscosity μ is taken, using Sutherland's law [28], as

$$\mu = \frac{C_1 T^{3/2}}{T + C_2}$$

with C_1 and C_2 that can, for air at moderate temperatures, assumed to be constant. Let us note that the adiabatic boundary condition $n \cdot \nabla T = 0$ can, in combination with the no-slip condition, be equivalently written as

$$(\sigma \cdot n)_4 = (f_v(w, \nabla w)\cdot n)_4 = 0 \quad \forall x \in \partial\Omega. \tag{2.4}$$

The viscous fluxes f_v are linear functions of ∇w, and hence allow a decoupling as

$$f_{v,i}(w) = \sum_{j=1}^{2} B_{ij}(w) w_{x_j} =: B(w)\nabla w \tag{2.5}$$

with (nonlinear) matrices $B_{ij}(w)$.

The non-dimensionalized equations depend on the flow conditions only through the Mach number M, the (constant) Prandtl number Pr, the constant γ, and the Reynolds number Re. The latter is defined as

$$Re := \frac{UL\rho_0}{\mu_0},$$

where U is a reference speed, L a reference length (for example the chord length of an airfoil), ρ_0 is a reference density, and μ_0 a reference viscosity.

3. The Adjoint Equations. In this section, we derive the adjoint equations for both the convection-diffusion and the Navier-Stokes equations. This means essentially deriving a concrete expression for (1.3) for the weak formulation of both these equations. This will then be used as the correctly posed adjoint problem with which we require consistency when analyzing the discrete adjoint approach.

3.1. Adjoint Convection-Diffusion Equation. The adjoint equation is determined by testing the linearized convection-diffusion equation (2.1) with a smooth function z, which yields

$$\int_\Omega (\nabla \cdot (f'(w)\mathrm{dw}) - \varepsilon\Delta\mathrm{dw}) \cdot z \, dx = \int_\Omega \mathrm{dw}(-f'(w)^T \nabla z - \varepsilon\Delta z) \, dx - \int_{\partial\Omega} z(\varepsilon\nabla\mathrm{dw} \cdot n) \, d\sigma, \tag{3.1}$$

given that the directions of linearization, dw, are such that $w+\mathrm{dw}$ fulfills the boundary conditions, which in this simple case is equivalent to stating $\mathrm{dw} = 0$ on $\partial\Omega$. This respects the boundary conditions, as

$$w + \mathrm{dw} = g \quad \forall x \in \partial\Omega. \tag{3.2}$$

In the nonlinear case, this is weakened to respecting the linearized boundary conditions.

Equating (3.1) with $J'[w](\mathrm{dw})$, as stated in (1.3), yields the adjoint convection-diffusion equation. However, a target functional has to be compatible (see e.g. [14]) to fit into the adjoint approach. This is stated in the following remark:

REMARK 3.1. *Eq. (3.1) reveals that, for this type of equation, the only suitable functionals that allow for a well-posed adjoint equation, are given as*

$$J(w) = \int_\Omega \zeta w \, dx + \int_{\partial\Omega} \xi(\varepsilon\nabla w \cdot n) \, d\sigma,$$

with $\zeta \in L^2(\Omega)$*, and* $\xi \in L^2(\partial\Omega)$.

We consider for simplicity such a compatible functional defined only on the boundary of Ω:

$$J(w) = \int_{\partial\Omega} \xi(\varepsilon\nabla w \cdot n) \, d\sigma. \tag{3.3}$$

Note that, in the mixed formulation (2.2), the target functional can equally well be written as

$$J(\sigma) = \int_{\partial\Omega} \xi(\sigma \cdot n) \, d\sigma.$$

The derivative of (3.3) can be written as

$$J'[w](\mathrm{dw}) = \int_{\partial\Omega} \xi(\varepsilon \nabla \mathrm{dw} \cdot n)\, d\sigma. \tag{3.4}$$

As a result of the above analysis, we obtain the adjoint equations for z as

$$\begin{aligned} -f'(w)^T \nabla z - \varepsilon \Delta z = 0, &\qquad \forall x \in \Omega \\ z = -\xi, &\quad \forall x \in \partial\Omega. \end{aligned} \tag{3.5}$$

3.2. Adjoint Navier-Stokes Equations. In this section, we give a short overview on the adjoint Navier-Stokes equations, only repeating those details that are necessary for the adjoint consistency analysis. For a more thorough investigation, we refer to the work of Hartmann [14], Jameson [15] and Schütz [24].

The functional of interest is defined on the wall-boundary $\partial\Omega$ (which in our two-dimensional examples is the boundary of an airfoil), and has the form

$$J(w) = \int_{\partial\Omega} p(w)\beta \cdot n - (\tau\beta) \cdot n\ d\sigma. \tag{3.6}$$

where $\beta \in \mathbb{R}^2$ is a constant vector, while n denotes the normal into the surface of the airfoil. The remaining quantities are defined as in section 2.2. Note that upon suitably choosing β, (viscous) lift and drag are of this form. More precisely, upon choosing β as β_d for drag or β_l for lift, given by

$$\begin{aligned} \beta_d &= \frac{1}{C_\infty}(\cos(\alpha), \sin(\alpha))^T, \\ \beta_l &= \frac{1}{C_\infty}(-\sin(\alpha), \cos(\alpha))^T, \end{aligned}$$

respectively, $J(w)$ is exactly the functional for computing drag and lift coefficient. As usual, α denotes the angle of attack while C_∞ is a normalized reference value defined as

$$C_\infty = \frac{1}{2}\left(\gamma p_\infty M_\infty^2 l\right).$$

Here l is the chord length of the airfoil, while p_∞ and M_∞ are the values of pressure and Mach number at free-stream conditions, i.e., values in the far field.

The linearization of (3.6), i.e., $\frac{d}{dw}J(w)$ dw can be written as (recall that $\tau \equiv \tau(w, \nabla w)$!)

$$J'[w](\mathrm{dw}) = \int_{\partial\Omega} \left(\frac{d}{dw}p(w)\mathrm{dw}\right)\beta \cdot n\, d\sigma - \int_{\partial\Omega} \frac{d}{dw}(\tau\beta \cdot n)\mathrm{dw} - \frac{d}{d\nabla w}(\tau\beta \cdot n)\nabla\mathrm{dw}\, d\sigma. \tag{3.7}$$

Note that this only involves the boundary, so the contribution to the dual equation in the interior of Ω is zero.

We proceed with the linearized version of (the primal form of) (2.3), meaning that we consider the directional derivative of equations (2.3) at point w in direction dw, which results in the term

$$\nabla \cdot \left(\frac{d}{dw}f(w)\mathrm{dw}\right) - \nabla \cdot \left(\frac{d}{dw}f_v(w, \nabla w)\mathrm{dw}\right) - \nabla \cdot \left(\frac{d}{d\nabla w}f_v(w, \nabla w)\nabla\mathrm{dw}\right). \tag{3.8}$$

Integrating (3.8) versus a smooth test-function z, and equating the result to (3.7) leads to

$$J'[w](\mathrm{dw}) = \int_\Omega z^T \left(\nabla \cdot \left(\frac{d}{dw} f(w)\mathrm{dw} - \frac{d}{dw} f_v(w, \nabla w)\mathrm{dw} - \frac{d}{d\nabla w} f_v(w, \nabla w)\nabla \mathrm{dw}\right)\right) dx \tag{3.9}$$

for all test functions dw not disturbing the boundary conditions. After integration by parts and careful treatment of the boundary terms one may show that (3.9) is in fact fulfilled, provided that z fulfills the adjoint Navier-Stokes equations,

$$-\frac{d}{dw} f(w)^T \nabla z + \left(\frac{d}{dw} B(w)\nabla w\right)^T \nabla z - \nabla \cdot \left(B(w)^T \nabla z\right) = 0, \quad \forall x \in \Omega \tag{3.10}$$

$$U_\Gamma^*(z, \nabla z) = 0, \quad \forall x \in \partial\Omega, \tag{3.11}$$

where U_Γ^* is defined for isothermal boundary conditions as

$$U_\Gamma^*(z, \nabla z) = (z_2 - \beta_1, z_3 - \beta_2, z_4)$$

and for adiabatic boundary conditions in the original problem as

$$U_\Gamma^*(z, \nabla z) = (z_2 - \beta_1, z_3 - \beta_2, \nabla z_4 \cdot n).$$

While we have omitted the details in the derivation of the boundary conditions, we note the following remark, which leads to the boundary condition for the adjoint energy variable z_4, and is also important for the adjoint consistency analysis to be presented below.

REMARK 3.2. *Consider the term*

$$\varphi \cdot (B(w)^T \nabla z)n = \varphi \cdot \left(\left(B_{11}^T z_{x_1} + B_{21}^T z_{x_2}\right) n_1 + \left(B_{12}^T z_{x_1} + B_{22}^T z_{x_2}\right) n_2\right),$$

where $\varphi \in \mathbb{R}^4$ is such that $\varphi_2 = \varphi_3 = 0$. A standard computation yields

$$\begin{aligned} \varphi \cdot (B(w)^T \nabla z)n &= \frac{\mu}{\rho} \begin{pmatrix} -\frac{\gamma}{Pr}\frac{E}{\rho}\nabla z_4 \cdot n \\ \frac{4}{3}n_1(z_2)_{x_1} - \frac{2n_1}{3}(z_3)_{x_2} + n_2(z_2)_{x_2} + n_2(z_3)_{x_1} \\ n_1(z_2)_{x_2} + n_1(z_3)_{x_1} - \frac{2n_2}{3}(z_2)_{x_1} + \frac{4n_2}{3}(z_3)_{x_2} \\ \frac{\gamma}{Pr}\nabla z_4 \cdot n \end{pmatrix} \cdot \varphi \\ &= \frac{\mu}{\rho}\frac{\gamma}{Pr}\nabla z_4 \cdot n \left(-\frac{E}{\rho}\varphi_1 + \varphi_4\right) = \frac{\mu}{\rho^2}\frac{\gamma}{Pr}\nabla z_4 \cdot n \left(-E\varphi_1 + \rho\varphi_4\right). \end{aligned}$$

For an adiabatic boundary, with $\nabla z_4 \cdot n = 0$, $\varphi \cdot (B(w)^T \nabla z)n = 0$. For an isothermal boundary, provided that $-E\varphi_1 + \rho\varphi_4 = 0$, one also obtains $\varphi \cdot (B(w)^T \nabla z)n = 0$.

4. Formulation of the Hybrid Mixed Method. In this section, we introduce both the discretization spaces and a class of Hybrid Mixed methods. The analysis has been motivated by the method as defined in [25], however, it extends to a broader class of methods which will be introduced in this section. We formulate the methods for both the convection-diffusion equation and the Navier-Stokes equations.

4.1. Preliminaries. Let us assume that our domain Ω is triangulated as $\{\Omega_k\}_{k=1}^N$, where

$$\bigcup_{k=1}^N \overline{\Omega}_k = \Omega, \qquad \Omega_k \cap \Omega_{k'} = \emptyset \quad \forall k \neq k'.$$

Based on these quantities, we define Γ as the set of both interior and boundary edges. Following standard nomenclature, we define an interior edge e as an intersection of two neighboring element boundaries $\partial\Omega_k \cap \partial\Omega_{k'}$ having a positive 1-dimensional measure. A boundary edge e is defined as the intersection of an element boundary $\partial\Omega_k$ with the physical boundary $\partial\Omega$. Let us furthermore define $\Gamma_0 \subset \Gamma$ to be the set of all internal edges. Note that due to the definition of Ω_k, we implicitly assume that the physical domain Ω is such that the boundary edges align with the physical boundary $\partial\Omega$, more precisely:

$$\Gamma \backslash \Gamma_0 = \partial\Omega.$$

We need this (rather standard) assumption in our analysis. However, as we allow arbitrary Ω_k, this is no restriction. Assuming that $\Gamma = \{\Gamma_k\}_{k=1}^{\widehat{N}}$, and the Γ_k are equipped with an orientation given by the direction of the corresponding normal vectors n, we define for a function v, and $x \in \Gamma_k$,

$$v(x)^{\pm} := \lim_{\tau \to 0^+} v(x \pm \tau n).$$

Average and jump operators are defined in a standard way as

$$\{v\} = \frac{v^+ + v^-}{2}, \quad [\![v]\!] = v^- n - v^+ n.$$

Similar definitions hold when considering a function on an element boundary $\partial\Omega_k$.

Let us define the Ansatz spaces

$$\begin{aligned}
V_h &:= \{\varphi \in L^2(\Omega) \ |\varphi_{|\Omega_k} \in V_{loc}(\Omega_k) \quad \forall k = 1, \dots N\}^d \\
H_h &:= \{\tau \in L^2(\Omega)^2 |\tau_{|\Omega_k} \in H_{loc}(\Omega_k) \quad \forall k = 1, \dots N\}^d \\
M_h &:= \{\mu \in L^2(\Gamma) \ |\mu_{|e} \ \in M_{loc}(e) \quad \forall e \in \Gamma\}^d
\end{aligned}$$

where the local spaces differ for different methods. The spaces are needed as follows:

- The approximate solution w_h is a function in V_h.
- The approximate viscous flux σ_h is a function in H_h.
- The hybrid variable, which is an approximation of w on Γ, is a function in M_h.

Note that the definition of the spaces depends on the dimension of the system. We recall that $d = 1$ for the scalar convection-diffusion equation and $d = 4$ for the two-dimensional compressible Navier-Stokes equations.

Let us comment on the choices of the local spaces. Hybridized DG methods as presented by Nguyen et al. [19] use the choice

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := \Pi^p(\Omega_k)^2, \qquad M_{loc}(e) := \Pi^p(e)$$

while the method in [25] relies on the choice

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := \Pi^{p+1}(\Omega_k)^2, \quad M_{loc}(e) := \Pi^{p+1}(e).$$

Π^p denotes the space of polynomials up to degree p. Yet another method is obtained if one chooses, as in [12],

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := RT_p(\Omega_k), \quad M_{loc}(e) := \Pi^p(e).$$

where RT_p is the Raviart-Thomas space of order p [22].

4.2. Convection-Diffusion Equation. In [19, 25], Hybrid Mixed methods were proposed for the convection-diffusion equation. Both methods can be formulated as the task of finding $(\sigma_h, w_h, \lambda_h) \in H_h \times V_h \times M_h$, such that

$$\begin{aligned}
&\sum_{k=1}^{N} \left(\int_{\Omega_k} \sigma_h \cdot \tau_h + \varepsilon w_h \nabla \cdot \tau_h \, dx - \varepsilon \int_{\partial\Omega_k \setminus \partial\Omega} \lambda_h \tau_h^- \cdot n \, d\sigma - \varepsilon \int_{\partial\Omega_k \cap \partial\Omega} g \tau_h^- \cdot n \, d\sigma \right) \qquad (4.1)\\
&\quad + \sum_{k=1}^{N} \left(- \int_{\Omega_k} f(w_h) \cdot \nabla \varphi_h \, dx + \int_{\partial\Omega_k \setminus \partial\Omega} \varphi_h^- \left(f(\lambda_h) \cdot n - \alpha(\lambda_h - w_h^-) \right) \, d\sigma \right.\\
&\qquad\qquad \left. + \int_{\partial\Omega_k \cap \partial\Omega} \varphi_h^- \left(f(g) \cdot n - (\lambda_h - g) \right) \, d\sigma - \int_{\Omega_k} \nabla \cdot \sigma_h \varphi_h \, dx \right)\\
&\quad + \int_{\Gamma_0} \mu_h \left(\sigma_h^- \cdot n - \sigma_h^+ \cdot n + \alpha(2\lambda_h - w_h^- - w_h^+) \right) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\lambda_h - g) \, d\sigma = \int_{\Omega} h \varphi_h \, dx.
\end{aligned}$$

holds for all $(\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h$. Note that in the third row, the term $-(\lambda_h - g)$ was not present in the original formulation in [25]. However, due to the fact that the last equation enforces $\lambda_h = g$ weakly on the boundary, this term does not change the original method. It is, however, needed for the adjoint consistency analysis. The choice of the parameter α depends on the method at hand. For the method as defined in [25], it denotes a Lax-Friedrichs-type coefficient, while for the method proposed in [19], it denotes the sum of a Lax-Friedrichs-type coefficient and an LDG-stabilization parameter stemming from the viscous discretization [1]. While we treat it as constant, we note in remark 5.5 that the analysis is not substantially changed if α depends on, for example, λ_h. As a definition, note that one can write (4.1) as

$$N(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h) = \int_{\Omega} h \varphi_h \, dx \quad \forall (\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h. \qquad (4.2)$$

4.3. Navier-Stokes Equations. Similar to the previous subsection, we only state the definition of the method for the compressible Navier-Stokes equations (2.3). In [21], the authors have extended the scheme from [19] to the compressible Navier-Stokes equations. Contrary to the method by the present authors, the additional variable σ is defined as $\sigma = \nabla w$, instead of $\sigma = f_v(w, \nabla w)$, resulting thus in a different scheme. However, defining $\sigma = f_v(w, \nabla w)$, and using the hybridized DG spaces, results in a hybridized DG scheme for the Navier-Stokes equations, slightly different from [21] . Both methods can thus be written in the unifying expression, given as the task of finding $(\sigma_h, w_h, \lambda_h) \in H_h \times V_h \times M_h$, such that

$$
\begin{aligned}
N(\sigma_h, w_h, \lambda_h) :=& \int_\Omega \sigma_h \cdot \tau_h \; dx + \int_\Omega w_h \nabla \cdot (B(w_h)^T \tau_h) \, dx - \int_{\Gamma_0} \lambda_h \cdot [\![B(w_h)^T \tau_h]\!] \, d\sigma \\
& - \int_{\Gamma \setminus \Gamma_0} w_{\partial\Omega}(\lambda_h) \cdot (B(w)^T \tau_h) n \, d\sigma - \int_\Omega f(w_h) \nabla \varphi_h \; dx - \int_\Omega \nabla \cdot \sigma_h \varphi_h \, dx \\
& + \int_{\Gamma_0} \left(\varphi_h^- - \varphi_h^+ \right) f(\lambda_h) \cdot n \; d\sigma - \int_{\Gamma_0} \varphi_h^- \left(\alpha(\lambda_h - w_h^-) \right) + \varphi_h^+ \left(\alpha(\lambda_h - w_h^+) \right) \, d\sigma \\
& + \int_{\Gamma \setminus \Gamma_0} \varphi_h^- \left(f(w_{\partial\Omega}(\lambda_h)) \cdot n - \alpha(\lambda_h - w_{\partial\Omega}(w_h^-)) \right) \; d\sigma \\
& + \int_{\Gamma_0} \mu_h \alpha(2\lambda_h - w_h^- - w_h^+) + \int_{\Gamma_0} \mu_h (\sigma_h^- \cdot n - \sigma_h^+ \cdot n) \, d\sigma \\
& + \int_{\Gamma \setminus \Gamma_0} \mu_h \alpha(\lambda_h - w_{\partial\Omega}(w_h^-)) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\sigma_h^- \cdot n - \sigma_{\partial\Omega}(\sigma_h^-) \cdot n) \, d\sigma = 0 \quad (4.3)
\end{aligned}
$$

holds for all $(\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h$. The boundary operators $w_{\partial\Omega} \equiv w_{\partial\Omega}(w_h^-)$ and $\sigma_{\partial\Omega} \equiv \sigma_{\partial\Omega}(\sigma_h^-)$ are defined differently for adiabatic and isothermal boundary conditions. For *adiabatic* boundary conditions, one defines

$$
\begin{aligned}
w_{\partial\Omega}(w_h) \equiv w_{\partial\Omega}((\rho_h, \rho_h u_h, \rho_h v_h, E_h)) &= (\rho_h, 0, 0, E_h), \\
\sigma_{\partial\Omega}(\sigma_h) \cdot n &= (0, \sigma_{h,2} \cdot n, \sigma_{h,3} \cdot n, 0),
\end{aligned}
$$

while for *isothermal* boundary conditions, one defines

$$
\begin{aligned}
w_{\partial\Omega}(w_h) \equiv w_{\partial\Omega}((\rho_h, \rho_h u_h, \rho_h v_h, E_h)) &= (\widehat{\rho}, 0, 0, \widehat{E}), \\
\sigma_{\partial\Omega}(\sigma_h) \cdot n &= (0, \sigma_{h,2} \cdot n, \sigma_{h,3} \cdot n, \sigma_{h,4} \cdot n),
\end{aligned}
$$

where $\widehat{\rho}$ and $\widehat{E}$ are such that $T \equiv T(\widehat{\rho}, \widehat{E}) = T_{wall}$, thereby incorporating the isothermal boundary conditions.

5. Adjoint Consistency Analysis. In this section, we show that the class of Hybrid Mixed methods as defined in section 4 is in fact adjoint consistent. Let us briefly introduce the concept of adjoint consistency. In a straightforward manner, as we have seen in the introduction, we formulate the discrete adjoint, as in the continuous case, just replacing the weak formulation by the discrete Galerkin formulation. This procedure yields the *discrete adjoint equations* as the task of finding $(\tau_h^z, \varphi_h^z, \mu_h^z) \in H_h \times V_h \times M_h$ such that

$$
N'[\sigma_h, w_h, \lambda_h](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda; \tau_h^z, \varphi_h^z, \mu_h^z) = J'[\sigma_h, w_h, \lambda_h](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \qquad (5.1)
$$

holds for all $(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h$. The tuple $(\sigma_h, w_h, \lambda_h)$ represents the approximate solution computed in the primal problem, given by the approximation to the viscous flux, the unknown solution and the unknown solution on the edges of the triangulation. The tuple $(\tau_h^z, \varphi_h^z, \mu_h^z)$ denotes approximations to quantities related to the adjoint solution z. Our analysis will show that they are, given (5.1) is well-posed, related to $(-\nabla z, z, z_{|\Gamma})$.

Adjoint Consistency is defined in a way similar to 'normal' consistency: Substituting the exact solutions into (5.1), meaning to substitute $(\sigma_h, w_h, \lambda_h)$ by the corresponding exact quantities $(\sigma, w, w_{|\Gamma})$ and substituting $(\tau_h^z, \varphi_h^z, \mu_h^z)$ by $(-\nabla z, z, z_{|\Gamma})$, we must have

$$
N'[\sigma, w, w_{|\Gamma}](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda; -\nabla z, z, z_{|\Gamma}) = J'[\sigma, w, w_{|\Gamma}](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda), \qquad (5.2)
$$

again for all $(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h$.

REMARK 5.1. *Computing not only w as in a standard DG scheme for example, but also derived quantities, makes it not so obvious which quantities are approximated by $(\tau_h^z, \varphi_h^z, \mu_h^z)$. This is why for the convection-diffusion case, we start with the expression*

$$N'[\sigma, w, w_{|\Gamma}](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda; \tau, \varphi, \mu) = J'[\sigma, w, w_{|\Gamma}](\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \tag{5.3}$$

and derive conditions on the quantities (τ, φ, μ) such that (5.3) holds for all quantities $(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda)$ in the space $H_h \times V_h \times M_h$. These of course turn out to be those as defined in (5.2).

The section is subdivided in the usual way, treating the conceptually simple convection-diffusion equation first, and then extending the analysis to the compressible Navier-Stokes equations.

5.1. Convection-Diffusion Equation. Given that $\sigma := \varepsilon \nabla w$, a consistent discretization of (3.3) is

$$J_h(\sigma_h) := \int_{\partial\Omega} \xi(\sigma_h \cdot n)\, d\sigma, \tag{5.4}$$

with the derivative

$$J_h'(\mathrm{d}\sigma) = \int_{\partial\Omega} \xi(\mathrm{d}\sigma \cdot n)\, d\sigma. \tag{5.5}$$

From now on, we refrain from writing out the arguments that are not associated to the direction of differentiation. Adjoint consistency means that we have to show that (5.3) holds for all $(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h$ for a suitable choice of τ, φ and μ, meaning that

$$J_h'(\mathrm{d}\sigma) = N'(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \quad \forall (\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h, \tag{5.6}$$

where τ, φ and μ should in some way relate to the exact adjoint z as given implicitly by (3.5). We will derive conditions on these quantities in the course of the analysis. Let us begin with a very first assumption, which is present in all adjoint consistency analysis.

ASSUMPTION 5.2. *We assume that w, τ, φ and μ are smooth.*

To show (5.6), we note that it is proved as soon as

$$N'(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) = N'(\mathrm{d}\sigma, 0, 0) + N'(0, \mathrm{dw}, 0) + N'(0, 0, \mathrm{d}\lambda) = J_h'(\mathrm{d}\sigma) + 0 + 0, \tag{5.7}$$

holds, which clearly follows from the linearity of the derivative.

We start with the terms in (5.7) that belong to $\mathrm{d}\sigma$, (which also includes the target functional J_h, which only depends on $\mathrm{d}\sigma$), yielding

$$\begin{aligned} N'(\mathrm{d}\sigma, 0, 0) &= \int_\Omega \mathrm{d}\sigma \cdot \tau\, dx - \int_\Omega \nabla \cdot \mathrm{d}\sigma \varphi\, dx + \int_{\Gamma_0} \mu [\![\mathrm{d}\sigma]\!]\, d\sigma \\ &= \int_\Omega \mathrm{d}\sigma \cdot (\tau + \nabla\varphi)\, dx - \int_{\Gamma_0} [\![\mathrm{d}\sigma]\!] \varphi\, d\sigma - \int_{\Gamma \setminus \Gamma_0} (\mathrm{d}\sigma \cdot n)\varphi\, d\sigma + \int_{\Gamma_0} \mu [\![\mathrm{d}\sigma]\!]\, d\sigma \\ &= \int_\Omega \mathrm{d}\sigma \cdot (\tau + \nabla\varphi)\, dx - \int_{\Gamma \setminus \Gamma_0} (\mathrm{d}\sigma \cdot n)\varphi\, d\sigma + \int_{\Gamma_0} (\mu - \varphi) [\![\mathrm{d}\sigma]\!]\, d\sigma \end{aligned}$$

ASSUMPTION 5.3. *We assume that* $\tau = -\nabla\varphi$, $\varphi_{\Gamma_0} = \mu$ *and* $\varphi_{|\partial\Omega} = z_{\partial\Omega}$ $(= -\xi)$.

Given that Assumption 5.3 holds, we have

$$N'(\mathrm{d}\sigma, 0, 0) = J_h'(\mathrm{d}\sigma).$$

Now it remains to show that all the other terms reduce to zero (again, under suitable conditions on τ, φ and μ). We continue with the terms depending on dw, exploiting the assumptions we have already made:

$$\begin{aligned} N'(0, \mathrm{dw}, 0) &= \int_\Omega -\varepsilon \mathrm{dw} \nabla \cdot \nabla\varphi - f'(w)\mathrm{dw} \cdot \nabla\varphi \, dx + \sum_{k=1}^N \int_{\partial\Omega_k \setminus \partial\Omega} \alpha\varphi \mathrm{dw}^- \, d\sigma - \int_{\Gamma_0} \alpha\mu(\mathrm{dw}^- + \mathrm{dw}^+) \, d\sigma \\ &= \int_\Omega \mathrm{dw} \left(-\varepsilon\Delta\varphi - f'(w)^T \nabla\varphi\right) \, dx + \int_{\Gamma_0} \alpha(\varphi - \mu)(\mathrm{dw}^- + \mathrm{dw}^+) \, dx \\ &= \int_\Omega \mathrm{dw} \left(-\varepsilon\Delta\varphi - f'(w)^T \nabla\varphi\right) \, dx. \end{aligned}$$

ASSUMPTION 5.4. *We assume that* $\varphi = z$. *(Due to Assumption 5.3, we have the identities* $\tau = -\nabla z$ *and* $\mu = z_{|\Gamma}$.*)*

Under Assumption 5.4, there holds

$$N'(0, \mathrm{dw}, 0) = 0,$$

which is due to the definition of the adjoint equation (3.5). The last part involves those terms containing $\mathrm{d}\lambda$:

$$\begin{aligned} N'(0, 0, \mathrm{d}\lambda) = & \sum_{k=1}^N \left(\varepsilon \int_{\partial\Omega_k \setminus \partial\Omega} \mathrm{d}\lambda \nabla z \cdot n + z\left(f'(w) \cdot n \mathrm{d}\lambda - \alpha \mathrm{d}\lambda\right) \, d\sigma - \int_{\partial\Omega_k \cap \partial\Omega} z \mathrm{d}\lambda \, d\sigma \right) \\ & + \int_{\Gamma_0} 2z\alpha \mathrm{d}\lambda \, d\sigma + \int_{\Gamma \setminus \Gamma_0} z \mathrm{d}\lambda \, d\sigma \\ = & \int_{\Gamma_0} \varepsilon \mathrm{d}\lambda [\![\nabla z \cdot n]\!] + \mathrm{d}\lambda [\![f'(w)^T z]\!] - 2\alpha z \mathrm{d}\lambda \, d\sigma - \int_{\Gamma \setminus \Gamma_0} z \mathrm{d}\lambda \, d\sigma \\ & + \int_{\Gamma_0} 2z\alpha \mathrm{d}\lambda \, d\sigma + \int_{\Gamma \setminus \Gamma_0} z \mathrm{d}\lambda \, d\sigma \\ = & \; 0 \end{aligned}$$

This, together with (5.7) proves that the method is adjoint consistent. We have thus proven that (5.6) holds with τ, φ, μ as defined by the assumptions. To summarize, we have proved that

$$J_h'(\mathrm{d}\sigma) = N'(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \quad \forall (\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h. \tag{5.8}$$

REMARK 5.5. *Usually,* α *is not a constant, but a nonlinear function of* λ_h. *(For the method defined in [25], it is a Lax-Friedrichs-type constant, and could therefore be defined as* $\alpha = \max\{|c|\}$, *where c is an eigenvalue of* $f'(\lambda_h) \cdot n$.*) Such a choice does not destroy the adjoint consistency property. This can be easily seen when considering those terms in (4.1) where* α *appears, i.e. terms of the form* $\alpha(\lambda_h - w_h^-)$. *Upon differentiation, and using the product rule, one obtains* $\alpha'(\lambda_h - w_h^-) + \alpha(\lambda_h - w_h^-)'$. *The second term has been treated in our analysis, and the first term is zero, given one substitutes the exact solution for* λ_h *and* w_h. *The same observation holds true for the discretization of the Navier-Stokes equations.*

5.2. Navier-Stokes Equations. A consistent modification of the target functional given in (3.6) is achieved by considering

$$J(\lambda,\sigma) := \int_{\partial\Omega} p(w_{\partial\Omega}(\lambda))\beta\cdot n - (\sigma_2\cdot n, \sigma_3\cdot n)\beta \, d\sigma. \tag{5.9}$$

In order to show adjoint consistency, we have to show that the following expression holds:

$$N'(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) = J'(\mathrm{d}\lambda, \mathrm{d}\sigma) \tag{5.10}$$

for all $(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h$. We make the same choice as before and substitute $\sigma_h = f_v(w, \nabla w)$, $w_h = w$, $\lambda_h = w$, $\tau = -\nabla z$, $\varphi_h = z$ and $\mu_h = z$. The procedure is similar to that used in the previous section, so let us begin with the expression

$$\begin{aligned}
N'(\mathrm{d}\sigma, 0, 0) =& -\int_\Omega \mathrm{d}\sigma \nabla z \, dx - \int_\Omega \nabla\cdot \mathrm{d}\sigma \; z \, dx + \int_{\Gamma_0} z[\![\mathrm{d}\sigma]\!] \, d\sigma \\
&+ \int_{\Gamma\setminus\Gamma_0} z\mathrm{d}\sigma\cdot n - z\cdot \frac{d}{d\sigma}(0, \sigma_2\cdot n, \sigma_3\cdot n, (\sigma_{\partial\Omega})_4\cdot n)\mathrm{d}\sigma \, d\sigma \\
=& -\int_{\Gamma\setminus\Gamma_0} z\cdot \frac{d}{d\sigma}(0, \sigma_2\cdot n, \sigma_3\cdot n, (\sigma_{\partial\Omega})_4\cdot n)\mathrm{d}\sigma \, d\sigma \\
=& J'(0, \mathrm{d}\sigma)
\end{aligned}$$

Note that the term $z_4(\sigma_{\partial\Omega})_4 \cdot n$ is always zero, as either $(\sigma_{\partial\Omega})_4 \cdot n = 0$ (adiabatic boundary) or $z_4 = 0$ (isothermal boundary). The remaining steps follow from the boundary conditions imposed on z.

To simplify matters, we split $N'(0, \mathrm{dw}, 0) = N'_\Omega(0, \mathrm{dw}, 0) + N'_\Gamma(0, \mathrm{dw}, 0)$ into two parts, one consisting of all volume integrals, and one consisting of all face integrals. For the volume part, there holds

$$\begin{aligned}
N'_\Omega(0, \mathrm{dw}, 0) =& -\int_\Omega \mathrm{dw}\nabla\cdot(B(w)^T\nabla z)dx - \int_\Omega w\nabla\cdot(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla z)dx - \int_\Omega f'(w)\mathrm{dw}\nabla z \, dx \\
=& \int_\Omega \mathrm{dw}\left((\frac{d}{dw}B(w)\nabla w)^T\nabla z - \nabla\cdot(B(w)^T\nabla z) - f'(w)^T\nabla z \right) dx \\
&- \int_{\Gamma_0} w\left(\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+ \right)\nabla zn \right) d\sigma - \int_{\Gamma\setminus\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla zn \right) d\sigma \\
=& -\int_{\Gamma_0} w\left(\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+ \right)\nabla zn \right) d\sigma - \int_{\Gamma\setminus\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla zn \right) d\sigma,
\end{aligned}$$

while the face part can be written as

$$\begin{aligned}
N'_\Gamma(0, \mathrm{dw}, 0) =& \int_{\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+ \right)\nabla zn \, d\sigma + \int_{\Gamma\setminus\Gamma_0} w\cdot(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla z)n \, d\sigma \\
&+ \int_{\Gamma_0} z\alpha(\mathrm{dw}^- + \mathrm{dw}^+) \, d\sigma + \int_{\Gamma\setminus\Gamma_0} z\alpha w'_{\partial\Omega}(w)\mathrm{dw} \, d\sigma \\
&- \int_{\Gamma_0} z\alpha(\mathrm{dw}^- + \mathrm{dw}^+) \, d\sigma - \int_{\Gamma\setminus\Gamma_0} z\alpha w'_{\partial\Omega}(w)\mathrm{dw} \, d\sigma \\
=& \int_{\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+ \right)\nabla zn \, d\sigma + \int_{\Gamma\setminus\Gamma_0} w\cdot(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla z)n \, d\sigma.
\end{aligned}$$

Summarizing, one obtains

$$N'(0, \mathrm{dw}, 0) = N'_\Omega(0, \mathrm{dw}, 0) + N'_\Gamma(0, \mathrm{dw}, 0) = 0.$$

The remaining term is $N'(0,0,\mathrm{d}\lambda)$. To simplify the notational workload, let us note that due to the (assumed) smoothness of the quantities z and w, both $B(w^-)\nabla z^- - B(w^+)\nabla z^+)$ and $z^- - z^+$ vanish. This leaves us with

$$\begin{aligned}
N'(0,0,\mathrm{d}\lambda) =& \int_{\Gamma_0} -2\alpha \mathrm{d}\lambda z \, d\sigma + \int_{\Gamma_0} z\alpha 2\mathrm{d}\lambda \, dx + \int_{\Gamma\setminus\Gamma_0} -\alpha \mathrm{d}\lambda z \, d\sigma + \int_{\Gamma\setminus\Gamma_0} z\alpha \mathrm{d}\lambda \, dx \\
&+ \int_{\Gamma\setminus\Gamma_0} w'_{\partial\Omega}(w)\mathrm{d}\lambda \cdot (B(w)^T \nabla z) n \, d\sigma + \int_{\Gamma\setminus\Gamma_0} z \frac{d}{dw} f(w_{\partial\Omega}(w)) n \mathrm{d}\lambda \, d\sigma \\
=& \int_{\Gamma\setminus\Gamma_0} z \frac{d}{dw}(0, p(w_{\partial\Omega}(w))n_1, p(w_{\partial\Omega}(w))n_2, 0)\mathrm{d}\lambda \, d\sigma \\
=& J'_h(\mathrm{d}\lambda, 0)
\end{aligned}$$

The last step requires some explanation:

- The quantity $f(w_{\partial\Omega}(w))$ is, for all w, equal to the expression $(0, pn_1, pn_2, 0)$, which is due to the fact that $w_{\partial\Omega}(w)_2 = w_{\partial\Omega}(w)_3 = 0$. Evaluating boundary fluxes in this manner is a modification that has already been done by Lu [18] and Hartmann [14]. (The quantity p is of course evaluated with the discrete quantities $w_{\partial\Omega}(w)$.)
- Adiabatic boundary: Due to remark 3.2, the quantity $\varphi \cdot (B(w)^T \nabla z)n$ is zero given that $\varphi_2 = \varphi_3 = 0$. It is easily seen that setting $\varphi = w'_{\partial\Omega}(w)\mathrm{d}\lambda$ fulfills this claim.
- Isothermal boundary: The quantity $\varphi = w'_{\partial\Omega}(w)\mathrm{d}\lambda$ still fulfills $\varphi_2 = \varphi_3 = 0$. Furthermore, due to the fact that $T(w_{\partial\Omega}(w)) = T_{wall}$ as claimed in sec. 4.3, one easily computes

$$\begin{aligned}
0 &= \frac{d}{dw} T(w_{\partial\Omega}(w))\mathrm{d}\lambda = T'(w_{\partial\Omega}(w)) w'_{\partial\Omega}(w)\mathrm{d}\lambda = T'(w_{\partial\Omega}(w))\varphi \\
&= T'(w)\varphi = \frac{1}{c_v}(-\frac{E}{\rho^2}, 0, 0, \frac{1}{\rho})\varphi \\
\Leftrightarrow 0 &= -E\varphi_1 + \varphi_4.
\end{aligned}$$

  As stated in remark 3.2, for isothermal walls this is precisely the condition for $\varphi \cdot (B(w)^T \nabla z)n = 0$.

In summary, we have thus shown that

$$N'(\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) = J'_h(\mathrm{d}\lambda, \mathrm{d}\sigma) \quad \forall (\mathrm{d}\sigma, \mathrm{dw}, \mathrm{d}\lambda) \in H_h \times V_h \times M_h,$$

which is clearly the adjoint consistency property.

6. Conclusions. We have presented an adjoint consistency analysis for a class of Hybrid Mixed methods, including the Hybridized Discontinuous Galerkin method [19] and the method developed in [25]. In contrast to [14], we do not need to include additional terms in the functional to make the method adjoint consistent.

Adjoint methods are standard tools in the context of Discontinuous Galerkin methods. Future work includes showing that adjoint methods in the context of Hybrid Mixed methods work equally well in practice.

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REFERENCES

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of Discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2002.

[2] F. Bassi and S. Rebay. A high-order accurate discontinuous Finite-Element method for the numerical solution of the compressible Navier-Stokes equations. *Journal of Computational Physics*, 131:267–279, 1997.

[3] C. Baumann and J. Oden. An adaptive-order Discontinuous Galerkin method for the solution of the Euler equations of gas dynamics. *International Journal for Numerical Methods in Engineering*, 47:61–73, 2000.

[4] R. Becker and R. Rannacher. An optimal control approach to a posteriori error estimation in finite element methods. *Acta Numerica*, 10:1–102, 2001.

[5] F. Brezzi, J. Douglas, and L. D. Marini. Two families of mixed Finite Elements for second order elliptic problems. *Numerische Mathematik*, 47:217–235, 1985.

[6] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of Discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47:1319–1365, 2009.

[7] B. Cockburn, S. Hou, and C.-W. Shu. The Runge–Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws IV: The multidimensional case. *Mathematics of Computation*, 54:545–581, 1990.

[8] B. Cockburn and S. Y. Lin. TVB Runge-Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws III: One dimensional systems. *Journal of Computational Physics*, 84:90–113, 1989.

[9] B. Cockburn and C.-W. Shu. TVB Runge-Kutta local projection Discontinuous Galerkin finite element method for Conservation Laws II: General framework. *Mathematics of Computation*, 52:411–435, 1988.

[10] B. Cockburn and C.-W. Shu. The Runge-Kutta local projection p^1-Discontinuous Galerkin Finite Element Method for Scalar Conservation Laws. *RAIRO - Modélisation mathématique et analyse numérique*, 25:337–361, 1991.

[11] B. Cockburn and C.-W. Shu. The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V: Multidimensional Systems. *Mathematics of Computation*, 141:199–224, 1998.

[12] H. Egger and J. Schöberl. A hybrid mixed Discontinuous Galerkin Finite Element method for convection-diffusion problems. *IMA Journal of Numerical Analysis*, 30:1206–1234, 2010.

[13] R. Hartmann. Adaptive Discontinuous Galerkin methods with shock-capturing for the compressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 51:1131–1156, 2006.

[14] R. Hartmann. Adjoint consistency analysis of Discontinuous Galerkin discretizations. *SIAM J. Numer. Anal.*, 45:2671–2696, 2007.

[15] A. Jameson. Aerodynamic design via control theory. *Journal of Scientific Computing*, 3:233–260, 1988.

[16] A. Jameson. Optimum aerodynamic design using control theory. In M. Hafez and K. Oshima, editors, *Computational Fluid Dynamics Review*, pages 495–528. Wiley, 1995.

[17] L. Y. Li, Y. Allaneau, and A. Jameson. Continuous adjoint approach for adaptive mesh refinement. *AIAA Paper 11-3982*, 2011.

[18] J. Lu. *An a posteriori error control framework for adaptive precision optimization using Discontinuous Galerkin Finite Element method.* PhD thesis, Massachusetts Institute of Technology, 2005.

[19] N. C. Nguyen, J. Peraire, and B. Cockburn. An implicit high-order hybridizable Discontinuous Galerkin method for nonlinear convection-diffusion equations. *Journal of Computational Physics*, 228:8841–8855, 2009.

[20] T. A. Oliver and D. L. Darmofal. Analysis of dual consistency for Discontinuous Galerkin discretizations of source terms. *SIAM J. Numer. Anal.*, 47:3507–3525, 2008.

[21] J. Peraire, N.C.Nguyen, and B. Cockburn. A hybridizable Discontinuous Galerkin method for the compressible Euler and Navier-Stokes equations. *AIAA Paper 2010-363*, 2010.

[22] P. A. Raviart and J. M. Thomas. A mixed Finite Element method for 2nd order elliptic problems. In A. Dold and B. Eckmann, editors, *Mathematical Aspects of Finite Element Methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer-Verlag, 1977.

[23] W. Reed and T. Hill. Triangular mesh methods for the neutron transport equation. *Los Alamos Report LA-UR-73-479*, 1973.

[24] J. Schütz. *A Hybrid Mixed Finite Element Scheme for the Compressible Navier-Stokes Equa-*

tions and Adjoint-Based Error Control for Target Functionals. PhD thesis, RWTH Aachen University, 2011.

[25] J. Schütz and G. May. A Hybrid Mixed Method for the Compressible Navier-Stokes Equations. AICES Technical Report 2011/12-01, Aachen Institute for Advanced Study in Computational Engineering Science. Submitted to the Journal of Computational Physics, 2011.

[26] J. Schütz, M. Woopen, and G. May. A Hybridized DG/Mixed Scheme for Nonlinear Advection-Diffusion Systems, Including the Compressible Navier-Stokes Equations. *AIAA Paper 2012-0729*, 2012.

[27] C. Steiner and S. Noelle. On adaptive timestepping for weakly instationary solutions of hyperbolic Conservation Laws via adjoint error control. *International Journal for Numerical Methods in Biomedical Engineering*, 26:790–806, 2010.

[28] W. Sutherland. The viscosity of gases and molecular force. *Philosophical Magazine*, 5:507–531, 1893.