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# A NOTE ON ADJOINT ERROR ESTIMATION FOR ONE-DIMENSIONAL STATIONARY CONSERVATION LAWS WITH SHOCKS.

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**Abstract.** We consider one-dimensional steady-state conservation laws with discontinuous solutions. Giles and Pierce [7] realized that a shock leads to a new term in the adjoint error representation for target functionals. This term disappears if and only if the adjoint solution satisfies an *internal boundary condition*. Curiously, most computer codes implementing adjoint error estimation ignore the new term in the functional, as well as the internal adjoint boundary condition. The purpose of this note is to justify this omission as follows: if one represents the exact forward and adjoint solutions as vanishing viscosity limits of the corresponding viscous problems, then the internal boundary condition is naturally satisfied in the limit.

**Key words.** Adjoint Error Control, Conservation Laws, Discontinuities, Interior Boundary Condition

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**1. Introduction.** We consider stationary one-dimensional conservation laws with source term, also called balance laws [10]

$$f(w)_x + S(w) = 0 \quad \forall x \in \Omega, \quad (1.1)$$

equipped with in- and outflow boundary conditions. A prototype is one-dimensional nozzle flow [1]. In many applications, the user is interested in the value of so-called *target functionals*, such as lift and drag coefficients in aerodynamics. In this context, for a given smooth function  $p$  of the solution  $w$ , we consider the functional  $J(w) := \int_{\Omega} p(w) dx$ . For smooth exact, respectively approximate, solutions  $w$  and  $v$ , the error

$$\mathcal{E}(v, w) := J(v) - J(w) \quad (1.2)$$

in the target functional  $J$  is given by

$$\mathcal{E}(v, w) = \mathcal{R}(z, v) + \mathcal{H}(z, v, w), \quad (1.3)$$

where  $\mathcal{H}(z, v, w)$  is a higher order term to be discussed in section 2 below, and

$$\mathcal{R}(z, v) := \int_{\Omega} z^T (f(v)_x + S(v)) dx \quad (1.4)$$

is the inner product of the residual of the approximate solution  $v$  and an adjoint solution  $z \equiv z(w)$ , which is implicitly defined by the equation

$$-f'(w)^T z_x + S'(w)^T z = p'(w) \quad \forall x \in \Omega, \quad (1.5)$$

subject to suitable boundary conditions. The clue of this error representation is that  $\mathcal{R}(z, v)$  does not (directly) depend on the unknown solution  $w$  and can thus be evaluated numerically, given one approximates  $z$  suitably. Therefore, neglecting  $\mathcal{H}$ , and localizing the terms in the inner product  $\mathcal{R}$ , one obtains an a-posteriori error estimate, which can be used to refine the grid in such a way that the target functional is computed accurately at low cost. This strategy has been used by many authors,

in particular for steady state computations (see, e.g., [2, 8, 13] and the references therein).

Giles and Pierce [6, 7] generalized this framework to solutions  $w$  and  $v$  with shocks, and found the error representation

$$\mathcal{E}(v, w) = \mathcal{R}(z, v) + \bar{\alpha} \mathcal{I}(z, w) + \mathcal{H}(z, v, w) \quad (1.6)$$

where the new term consists of the product of the error in the shock location  $\bar{\alpha} \equiv \bar{\alpha}(w, v)$  and the jump term

$$\mathcal{I}(z, w) := -z(\alpha)^T [f(w)_x] - [p(w)]. \quad (1.7)$$

Here  $\alpha \equiv \alpha(w)$  is the shock location, and  $[\cdot]$  denotes the jump of a quantity across the shock.

It is not obvious whether (1.7) is zero. In the context of time-dependent conservation laws, Giles and Ulbrich [4, 5] proved convergence of a numerically computed  $z_h$  towards  $z$  in the framework of a Finite Volume scheme, while in the stationary case, Schütz et al. [11] studied convergence in the context of a Discontinuous Galerkin scheme. The last result indicates that at least  $\mathcal{I}(z_h, w_h)$  converges to zero. In this paper, we will show that under certain conditions,  $\mathcal{I}(z, w)$  is indeed zero.

The internal error term presents a serious obstacle to a-posteriori adjoint error control, since the exact solution, the shock position and the error in the shock location are not known from the data of the computation, and hence the internal error cannot be evaluated. Therefore, in practical computations,  $\bar{\alpha} \mathcal{I}$  is usually neglected. Perhaps surprisingly, this leads to successful adaptive schemes. The aim of this paper is to show analytically that this omission is justified.

The paper proceeds as follows: In section 2 we present an alternative, and more detailed, derivation of the rather subtle error representation (1.6) for piecewise smooth solutions. In section 3, we prove that vanishing viscosity solutions  $w$  and  $z$  satisfy the so-called *internal boundary condition*

$$z(\alpha)^T [f(w)_x] = -[p(w)]. \quad (1.8)$$

Therefore, the internal error term  $\bar{\alpha} \mathcal{I}$  vanishes identically, and the a-posteriori error representation is justified for stationary conservation laws with shocks.

**2. Adjoint Error Control in the Discontinuous Case.** In this section, we give an alternative derivation of Giles' and Pierce's [6, 7] adjoint error representation (1.6) for a non-smooth solution  $w$  and a non-smooth function  $v$  approximating  $w$  in a certain sense we make more precise below. Giles and Pierce use a very short, formal calculus. But the differentiation of nonsmooth solutions, whose discontinuities are in different locations, is rather subtle. Here we confirm their calculation by a more detailed argument: we introduce a one-parameter family of coordinates, which links the smooth regions of both solutions. This helps us to formulate the distance of two such solutions, and to differentiate with respect to the new grid parameter.

We consider the domain  $\Omega = [0, 1]$ . For suitable boundary conditions, it is well-known that solutions to nonlinear conservation laws exhibit jump discontinuities. The location of such a discontinuity (the so-called *shock location*) is denoted by  $x = \alpha$ . We assume that  $w$  is discontinuous in  $x = \alpha$ , while it is sufficiently smooth away from  $\alpha$ , which means that especially  $\lim_{\varepsilon \rightarrow 0^+} w(\alpha \pm \varepsilon) =: w^\pm$  exists. This is a standard setting and in no way a restriction. Now assume  $w$  is perturbed in such a way that the resulting function  $v := w + \bar{w}$  has one (and only one) discontinuity at  $x = \beta =:$

$\alpha + \bar{\alpha}$ , and is also smooth away from  $\beta$ . In the case of a non-smooth function  $w$ , being *approximated* by some other non-smooth function  $v$ , we cannot simply say that  $\|v - w\|_\infty$  is small, say  $O(\nu)$  for some small parameter  $\nu$ , because if  $\alpha$  and  $\beta$  do not coincide, we always have an  $O(1)$  approximation error in the  $\infty$ -norm in the region between  $\alpha$  and  $\beta$ .

As a consequence, in the following definition, we state what we mean by a *sufficiently small* perturbation  $\bar{w}$ :

DEFINITION 2.1 (*Sufficiently close approximation of a discontinuous function*).

We say that  $w$  is approximated by  $v$  to order  $\nu$  if

- There exist smooth, invertible functions

$$\xi_1 : [0, \alpha] \rightarrow [0, \beta], \quad (2.1)$$

$$\xi_2 : [\alpha, 1] \rightarrow [\beta, 1] \quad (2.2)$$

such that we have

$$w(x) = v(\xi_1(x))^- + O(\nu), \quad x < \alpha \quad (2.3)$$

$$w(x) = v(\xi_2(x))^+ + O(\nu), \quad x > \alpha, \quad (2.4)$$

with the  $O(\nu)$ -bound assumed to be uniform.

- The  $\xi_i$  have to fulfill the properties

$$\frac{d}{dx} \xi_1 = 1 + O(\nu), \quad (2.5)$$

$$\frac{d}{dx} \xi_2 = 1 + O(\nu), \quad (2.6)$$

$$\xi_1(\alpha) = \xi_2(\alpha) = \beta \quad (2.7)$$

and the second derivatives of  $\xi_i$  are bounded.

- The residual  $r(v)$  is sufficiently small, meaning that we have the property

$$r(v) := f(v)_x + S(v) = O(\mu) \quad (2.8)$$

pointwise except at the discontinuity of  $v$ , where  $\mu$  is another parameter going to zero. Usually,  $\mu$  tends much slower to zero than  $\nu$  does. (We actually only need the somewhat weaker formulation  $\int_\alpha^\beta f(v)_x + S(v) dx = o(\nu)$ , but keep the preceding for the ease of presentation.)

ASSUMPTION 2.2. The results we obtain in this section are independent of the relative position of  $\alpha$  and  $\beta$ . However, for the sake of simplicity, we assume without loss of generality that  $\alpha < \beta$ .

A visualization of the relevant quantities can be seen in figure 2.1. Let us make the remark that functions  $w$  and  $v$  according to definition 2.1 fulfill  $\|w(\cdot) - v(\xi(\cdot))\|_\infty = O(\nu)$ .

LEMMA 2.3. We have

$$\xi_i(x) - x = O(\nu) \quad \forall i = 1, 2, \quad (2.9)$$

which, as a special case, implies  $\beta - \alpha = O(\nu)$ .

*Proof.* (We consider only the case  $i = 1$ . The case  $i = 2$  is completely analogous with the obvious interchange of 0 and 1.) We have that

$$\xi_1(0) - 0 = 0 \quad \text{due to the invertibility of } \xi_1 \text{ and}$$

$$\frac{d}{dx}(\xi_1(x) - x) = O(\nu) \quad \text{due to (2.5).}$$

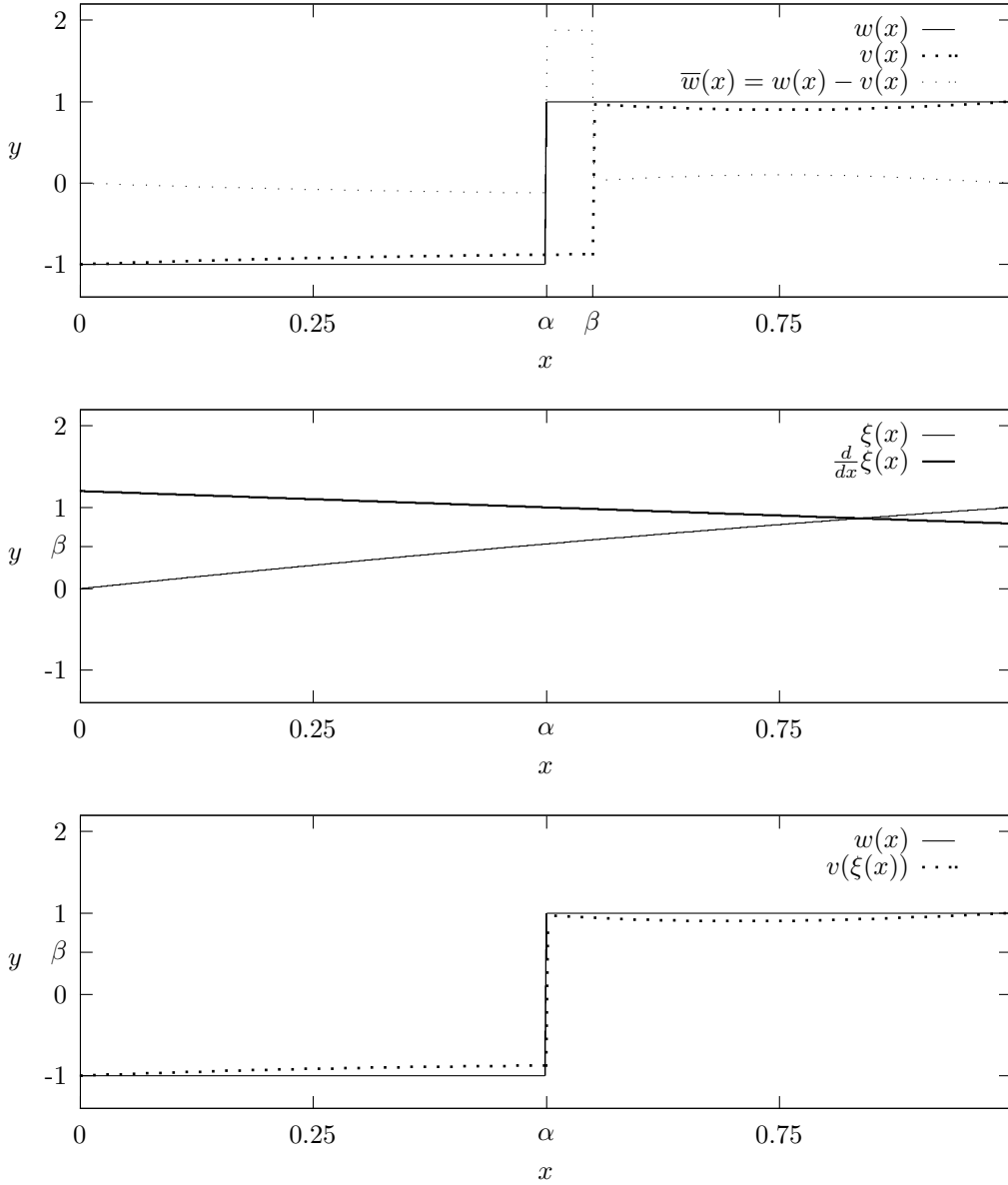


FIG. 2.1.  $v$  approximates  $w$  sufficiently close in the sense of definition 2.1. First plot: The given functions  $w$  and  $v$ . Second plot: The coordinate transformation  $\xi$  and its derivative. Third plot:  $v(\xi(x))$  approximates  $w(x)$  pointwise.

This proves the claim because we can write

$$\xi_1(x) - x = \xi_1(0) - 0 + \int_0^x \frac{d}{d\tau} (\xi_1(\tau) - \tau) d\tau = O(\nu).$$

□

LEMMA 2.4. *Due to the definition of sufficiently close, we have that*

$$v(x) - w(x) = O(\nu) \quad \forall x \in \Omega \setminus [\alpha, \beta], \quad (2.10)$$

given that  $v$  is a sufficiently close approximation to  $w$  in the sense of definition 2.1.

*Proof.* (Without loss of generality,  $x < \alpha$ )

$$\begin{aligned} v(x) - w(x) &= v(\xi_1(x)) - w(x) + v(x) - v(\xi_1(x)) \\ &\stackrel{(2.3)}{=} O(\nu) + v(x) - v(\xi_1(x)) \\ &= O(\nu) + v'(\xi_1(x)) \cdot (x - \xi_1(x)) + O(\|x - \xi_1(x)\|^2) \stackrel{(2.9)}{=} O(\nu). \end{aligned}$$

□

**2.1. Linearization of the Rankine-Hugoniot Condition.** Not every discontinuity of  $w$  is permissible. A very basic restriction following directly from the weak formulation of a hyperbolic conservation law is the Rankine-Hugoniot condition, which states that the flux has a (weak) divergence, even in the vicinity of a shock, more precisely,

$$[f(w)] := f(w(\alpha))^+ - f(w(\alpha))^- = 0. \quad (2.11)$$

Let us again assume that we are interested in a perturbed solution  $v = w + \bar{w}$  (in the sense of definition 2.1) which has its only shock at  $x = \beta = \alpha + \bar{\alpha}$ . In this section, we investigate how the Rankine-Hugoniot condition changes for such a  $v$ .

Let us first state the following lemma:

LEMMA 2.5. *Given  $v$  approximates  $w$  sufficiently close in the sense of definition 2.1, and  $f \equiv f(w)$  is a smooth quantity, it holds that*

$$\begin{aligned} [f(v)] &= [f(w)] + f'(w(\beta))(v(\beta) - w(\beta))^+ - f'(w(\alpha))(v(\alpha) - w(\alpha))^- \\ &\quad + \bar{\alpha} \left[ \frac{d}{dx} f(w(x)) \right] + o(\nu). \end{aligned} \quad (2.12)$$

Note that  $[f(v)]$  denotes a jump at  $x = \beta$ , while  $[f(w)]$  and  $[\frac{d}{dx} f(w(x))]$  denote jumps at  $x = \alpha$ .

*Proof.* We have that

$$\begin{aligned} f(v(\beta))^- &= f(w(\alpha))^- + f(v(\beta))^- - f(w(\alpha))^- \\ &= f(w(\alpha))^- + f(v(\alpha))^- - f(w(\alpha))^- + f(v(\beta))^- - f(v(\alpha))^- \\ &= f(w(\alpha))^- + f'(w(\alpha))^- (v(\alpha) - w(\alpha))^- + \frac{d(f \circ v)}{dx}(\alpha)^- \cdot \bar{\alpha} + O(\nu^2) \\ &= f(w(\alpha))^- + f'(w(\alpha))^- (v(\alpha) - w(\alpha))^- + \frac{d(f \circ w)}{dx}(\alpha)^- \cdot \bar{\alpha} + o(\nu). \end{aligned}$$

The last step is true because by replacing  $\frac{d(f \circ v)}{dx}$  by  $\frac{d(f \circ w)}{dx}$  we make an  $O(\mu)$  error which is augmented to  $O(\mu\nu) = o(\nu)$  by multiplying it with  $\bar{\alpha}$ . By treating  $f(v(\beta))^+$  in an analog manner, and then subtracting  $f(v(\beta))^-$  from  $f(v(\beta))^+$ , we get the claimed identity (2.12). □

Let us, for the ease of notation, define

$$[f'(w)\bar{w}] := f'(w(\beta))(v(\beta) - w(\beta))^+ - f'(w(\alpha))(v(\alpha) - w(\alpha))^- . \quad (2.13)$$

Inserting (2.13) into (2.12), exploiting the Rankine-Hugoniot condition as given in (2.11), and assuming that  $w$  solves (1.1), the jump in  $f(v)$  can be linearized as

$$[f(v)] = [f'(w)\bar{w}] + \bar{\alpha}\left[\frac{d}{dx}f(w(x))\right] + o(\nu). \quad (2.14)$$

**2.2. Linearization of the Functional.** We are interested in computing the changes in the functional

$$J(w) := \int_{\Omega} p(w) dx, \quad (2.15)$$

with  $p$  being sufficiently regular. Again, we assume that  $v$  is sufficiently close to  $w$ . We can then compute

$$\begin{aligned} J(v) - J(w) &= \int_{\Omega} p(v) - p(w) dx \\ &= \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) dx + O(\nu^2) + \int_{\alpha}^{\beta} p(v) - p(w) dx \\ &= \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) dx + O(\nu^2) + (\beta - \alpha)(p(v(\alpha)) - p(w(\alpha))^+) \\ &\quad + O(\nu^2) \\ &\stackrel{(2.10)}{=} \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) dx + \bar{\alpha}(p(w(\alpha))^- - p(w(\alpha))^+) + O(\nu^2) \\ &= \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) dx - \bar{\alpha}[p(w)] + O(\nu^2), \end{aligned}$$

and in summary, we have the following lemma:

LEMMA 2.6. *Given that  $J$  is as in (2.15), we can write*

$$J(v) - J(w) = \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) dx - \bar{\alpha}[p(w)] + O(\nu^2). \quad (2.16)$$

**2.3. Adjoint Approach.** In this section, we put together the information from the previous subsections, and show that the adjoint error control works under suitable assumptions as usual. We make the following *consistent* modification to the functional and consider

$$J(w) = \int_{\Omega} p(w) dx - z_{\alpha}^T [f(w)] \quad (2.17)$$

instead of  $J$  as in (2.15).  $z_{\alpha} \in \mathbb{R}^3$  is a parameter that will be determined later. For the ease of presentation, we also denote this functional by  $J$ , which is reasonable as  $[f(w)]$  vanishes for a solution  $w$  to (1.1). The same modification has already been done in [7].

We assume that the dual solution  $z$  is given as in (1.5), and we additionally assume that it is at least Lipschitz-continuous. This is in good agreement with both our numerical experiences and Tadmor's theory for scalar conservation laws proposed in [12]. In this section, we do not care about boundary conditions at all, as the focus is just on the behavior of the adjoint in the shock. We thus assume that all terms occurring at the (physical) boundary vanish, more precisely,

$$v(0) - w(0) = v(1) - w(1) = 0. \quad (2.18)$$

Putting all our information together, we can state the following theorem:

**THEOREM 1.** *Let  $w$  be an exact solution to (1.1), and  $v$  be an approximation to  $w$  in the sense of definition 2.1, for which additionally holds  $v = w$  at the boundary, i.e.,  $\bar{w} = v - w$  vanishes at  $x = 0$  and  $x = 1$ . Let furthermore be  $z$  a smooth (at least Lipschitz-continuous) solution to (1.5). The functional  $J$  is defined as in (2.17) for a sufficiently smooth function  $p \equiv p(w)$ . Upon choosing  $z_\alpha := z(\alpha)$ , we can write*

$$J(v) - J(w) = \int_{\Omega} z^T (f(v)_x + S(v)) \, dx + \bar{\alpha} \left( -z(\alpha)^T \left[ \frac{d}{dx} f(w) \right] - [p(w)] \right) + o(\nu). \quad (2.19)$$

*Proof.* The proof is a direct computation, it exploits the already known linearizations of both  $[f(w)]$  and  $J(w)$ :

$$\begin{aligned} J(v) - J(w) &\stackrel{(2.16),(2.14)}{=} \int_{\Omega \setminus [\alpha, \beta]} p'(w)(v - w) \, dx - \bar{\alpha}[p(w)] \\ &\quad - z_\alpha^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) + o(\nu) \\ &\stackrel{(1.5)}{=} \int_{\Omega \setminus [\alpha, \beta]} (-f'(w)^T z_x + S'(w)^T z)(v - w) \, dx - \bar{\alpha}[p(w)] \\ &\quad - z_\alpha^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) + o(\nu) \\ &= \int_{\Omega \setminus [\alpha, \beta]} (z_x^T (f(w) - f(v)) + z^T (S(v) - S(w))) \, dx - \bar{\alpha}[p(w)] \\ &\quad - z_\alpha^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) + o(\nu) \\ &\stackrel{(1.1)}{=} \int_{\Omega \setminus [\alpha, \beta]} z^T (f(v)_x + S(v)) \, dx \\ &\quad - z(\alpha)^T f'(w(\alpha))(v(\alpha) - w(\alpha))^- + z^T(\beta) f'(w(\beta))(v(\beta) - w(\beta))^+ \\ &\quad - \bar{\alpha}[p(w)] - z_\alpha^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) + o(\nu) \\ &\stackrel{(2.13)}{=} \int_{\Omega \setminus [\alpha, \beta]} z^T (f(v)_x + S(v)) \, dx + z(\alpha)^T [f'(w)\bar{w}] \\ &\quad - z_\alpha^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) - \bar{\alpha}[p(w)] + o(\nu), \end{aligned}$$

where the last step is allowed due to the assumed Lipschitz-continuity of  $z$  and the fact that  $[f'(w)\bar{w}]$  is of order  $\nu$ . Furthermore, as  $f(v)_x + S(v)$  is of order  $\mu$ , we can



augment the first integral on the whole domain to conclude that we have

$$J(v) - J(w) = \int_{\Omega} z^T (f(v)_x + S(v)) dx - \int_{\alpha}^{\beta} z^T (f(v)_x + S(v)) dx + z(\alpha)^T [f'(w)\bar{w}] \quad (2.20)$$

$$\begin{aligned} & - z_{\alpha}^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) - \bar{\alpha} [p(w)] + o(\nu) \\ & = \int_{\Omega} z^T (f(v)_x + S(v)) dx + z(\alpha)^T [f'(w)\bar{w}] \quad (2.21) \\ & - z_{\alpha}^T \left( [f'(w)\bar{w}] + \bar{\alpha} \left[ \frac{d}{dx} f(w) \right] \right) - \bar{\alpha} [p(w)] + o(\nu) + O(\nu\mu). \end{aligned}$$

Now upon choosing  $z_{\alpha} := z(\alpha)$ , we proved our claim (2.19) as the terms involving  $[f'(w)\bar{w}]$  cancel each other.  $\square$

Let us now make the following definition of what we mean by *interior boundary condition*:

**DEFINITION 2.7.** *A function  $z$  fulfills the interior boundary condition with respect to  $p$  and the shock position  $\alpha$ , given that*

$$z(\alpha)^T \left[ \frac{d}{dx} f(w) \right] = -[p(w)]. \quad (2.22)$$

In the next section (sec. 3), we prove that the dual solution to the quasi one-dimensional Euler equations, under standard assumptions, fulfills (2.22).

**COROLLAR 2.8.** *Given that the adjoint solution  $z$  as given in (1.5) fulfills the interior boundary condition (2.22), we have under the assumptions of Theorem 1 the usual adjoint error representation*

$$J(v) - J(w) = \int_{\Omega} z^T (f(v)_x + S(v)) dx + o(\nu).$$

**2.4. Interior Boundary Condition for the Euler Equations.** A prototype of (1.1) with  $d = 3$  are the steady-state quasi one-dimensional Euler equations, which are a model for compressible nozzle flow.  $w$ ,  $f$  and  $S$  are defined as

$$\begin{aligned} w &= (\rho, \rho u, E)^T, \\ f(w) &= (\rho u, \rho u^2 + p, u(E + p))^T, \quad (2.23) \\ S(w) &= \frac{A'}{A} (\rho u, \rho u^2, u(E + p))^T, \end{aligned}$$

respectively. The so-called conservative variables  $\rho, \rho u, E$  are density, momentum (which equals density times velocity) and total energy. Furthermore,  $A \equiv A(x)$  describes the nozzle geometry (assumed to be rotational-symmetric, so  $A(x)$  does in fact describe the diameter) and

$$p := (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right) \quad (2.24)$$

is the pressure, where we have used a specific equation of state for  $p$  that holds for a polytropic ideal gas, and  $\gamma$  is the ratio of specific heats, a gas-specific constant, which takes  $\gamma = 1.4$  for an ideal di-atomic gas, of which air is a specific example.

Boundary conditions  $U_{\partial\Omega}$  can, for example, be set as

$$0 = U_{\partial\Omega}(w) := \begin{cases} p - p_0 & \text{on the outflow boundary} \\ (s, h) - (s_0, h_0) & \text{on the inflow boundary} \end{cases} \quad (2.25)$$

where  $s = \alpha_0 \log(\frac{p}{\rho^\gamma}) + \alpha_1$  denotes entropy and  $h = \frac{c^2}{\gamma-1} + \frac{u^2}{2}$  total enthalpy, i.e., one prescribes the pressure  $p_0$  at the outflow, enthalpy  $h_0$  and entropy  $s_0$  at the inflow. Here  $\alpha_i$  are constants, and  $c$  denotes the speed of sound.

For the Euler equations, i.e., eq. (1.1) with  $f$  and  $S$  defined as in (2.23), one can make (2.22) more explicit as follows (of course, this has already been done, for example in [6]): Due to the underlying equation (1.1), we have for  $w = (w_1, w_2, w_3) = (\rho, \rho u, E)$

$$[\frac{d}{dx} f(w)] = -[S(w)], \quad (2.26)$$

$$[S(w)] = \frac{A'(\alpha)}{A(\alpha)}([\rho u], [\rho u^2], [u(E + p(w))]) \quad (2.27)$$

and due to Rankine-Hugoniot, we have that

$$[f(w)] = ([\rho u], [\rho u^2 + p(w)], [u(E + p(w))]) = 0 \quad (2.28)$$

which yields

$$[S(w)] = \frac{A'(\alpha)}{A(\alpha)}(0, [\rho u^2], 0). \quad (2.29)$$

Substituting all this information into the interior boundary condition (2.22), we get for  $z = (z_1, z_2, z_3)^T$

$$z_2(\alpha) \frac{-A'(\alpha)}{A(\alpha)}[\rho u^2] = -[p(w)], \quad (2.30)$$

which yields

$$z_2(\alpha) = \frac{A(\alpha)}{A'(\alpha)} \frac{[p(w)]}{[\rho u^2]}. \quad (2.31)$$

Again, thanks to Rankine-Hugoniot, we have

$$[\rho u^2] = -[p(w)], \quad (2.32)$$

which in all yields the internal adjoint boundary condition for the Euler equations,

$$z_2(\alpha) = -\frac{A(\alpha)}{A'(\alpha)}. \quad (2.33)$$

Usually, (2.33) is of course not enforced in a numerical procedure, as for example  $\alpha$  is in general not known. Due to the fact that numerical schemes in general approximate the solution  $w$  by a viscous regularization, it has been argued that neglecting (2.33) is reasonable, because in the vanishing viscosity limit,  $z$  is supposed to fulfill (2.33). This, however, has to our knowledge not been proven. In the following chapter, we therefore show that, given the adjoint solution can be seen as a limit of a viscous adjoint (to be defined below), the exact adjoint fulfills the interior boundary condition.

**3. Convergence of the Interior Boundary Condition.** In this section, we assume that the exact adjoint solution can be given as the small-viscosity limit of a viscous adjoint solution. This is a reasonable assumption as already indicated in [3]. Using a viscosity parameter  $\varepsilon > 0$ , the viscous primal equation can be written as

$$f(w^\varepsilon)_x + S(w^\varepsilon) = \varepsilon w^\varepsilon_{xx} \quad (3.1)$$

including again boundary conditions which are not relevant to this investigation here. Standard theory [9] shows that in the scalar case, given that  $\varepsilon \rightarrow 0$ , one has  $w^\varepsilon \rightarrow w$  in  $L^1$ . The corresponding dual equation is then

$$-f'(w^\varepsilon)^T z^\varepsilon_x + S'(w^\varepsilon)^T z^\varepsilon = \varepsilon z^\varepsilon_{xx} + p'(w^\varepsilon). \quad (3.2)$$

Standard assumptions, which can be proven in the scalar one-dimensional case, on the behavior of  $w^\varepsilon$  are that it is smooth all over the domain, albeit having in a transition region  $[\alpha^-, \alpha^+] := [\alpha - \bar{\alpha}, \alpha + \bar{\alpha}]$  a gradient that scales as  $\frac{1}{\varepsilon}$ . Outside this region, we state that the gradient is of order unity, i.e., its order of magnitude is independent of  $\varepsilon$ . Here  $\bar{\alpha}$  is a parameter that goes, in dependency of  $\varepsilon$ , to zero. We furthermore assume, in the spirit of Tadmor [12], that the adjoint solution is Lipschitz-continuous at  $x = \alpha$ . This is in good agreement with the results found by Giles and Pierce [6] for the quasi one-dimensional Euler equations. With respect to the interior boundary conditions, it is thus interesting what happens with the expression

$$\lim_{\varepsilon \rightarrow 0^+} ([p(w^\varepsilon)] - [z^\varepsilon S(w^\varepsilon)]), \quad (3.3)$$

which, in the limit, should be equivalent to (2.22) and thus yield zero. Of course, only involving smooth functions, (3.3) does not make sense unless we define what we mean by a jump. A reasonable definition is

$$[w^\varepsilon] := \int_{\alpha^-}^{\alpha^+} \left( \frac{d}{dx} w^\varepsilon \right) dx, \quad (3.4)$$

which, if  $w^\varepsilon$  converges towards a function  $w$  that is discontinuous at  $x = \alpha$ , it also converges towards the jump of  $w$ .

Let us state the following theorem:

**THEOREM 2.** *Given that both  $w^\varepsilon$  and  $z^\varepsilon$ , solutions to (3.1) and (3.2), respectively, are smooth, and that outside the transition region  $[\alpha^-, \alpha^+]$ , both  $z^\varepsilon_x$  and  $w^\varepsilon_x$  have orders of magnitude independent of  $\varepsilon$ , it holds that*

$$\lim_{\varepsilon \rightarrow 0^+} ([p(w^\varepsilon)] - [z^\varepsilon S(w^\varepsilon)]) = 0. \quad (3.5)$$

*Proof.* The proof exploits both the equations defining  $w^\varepsilon$  and  $z^\varepsilon$ , and can in

principle in a straightforward manner be written as

$$\begin{aligned}
[p(w^\varepsilon)] - [z^\varepsilon S(w^\varepsilon)] &\stackrel{(3.4)}{=} \int_{\alpha^-}^{\alpha^+} \frac{d}{dx} (p(w^\varepsilon) - z^\varepsilon S(w^\varepsilon)) dx \\
&= \int_{\alpha^-}^{\alpha^+} p'(w^\varepsilon) w_x^\varepsilon - z_x^\varepsilon S(w^\varepsilon) - z^\varepsilon S'(w^\varepsilon) w_x^\varepsilon dx \\
&= \int_{\alpha^-}^{\alpha^+} (p'(w^\varepsilon) - S'(w^\varepsilon)^T z^\varepsilon) w_x^\varepsilon - z_x^\varepsilon S(w^\varepsilon) dx \\
&\stackrel{(3.2)}{=} \int_{\alpha^-}^{\alpha^+} (-f'(w^\varepsilon)^T z_x^\varepsilon - \varepsilon z_{xx}^\varepsilon) w_x^\varepsilon - z_x^\varepsilon S(w^\varepsilon) dx \\
&= \int_{\alpha^-}^{\alpha^+} (-z_x^\varepsilon (f'(w^\varepsilon) w_x^\varepsilon) - \varepsilon z_{xx}^\varepsilon w_x^\varepsilon) - z_x^\varepsilon S(w^\varepsilon) dx \\
&= \int_{\alpha^-}^{\alpha^+} z_x^\varepsilon (-f(w^\varepsilon)_x + \varepsilon w_{xx}^\varepsilon - S(w^\varepsilon)) dx - [\varepsilon z_x^\varepsilon w_x^\varepsilon]_{\alpha^-}^{\alpha^+} \\
&\stackrel{(3.1)}{=} -\varepsilon [z_x^\varepsilon w_x^\varepsilon]_{\alpha^-}^{\alpha^+} = O(\varepsilon).
\end{aligned}$$

Because we are outside the transition region, the term  $[z_x^\varepsilon w_x^\varepsilon]$  scales independently of  $\varepsilon$ . This proves our claim.  $\square$

**COROLLAR 3.1.** *Under the assumptions that both  $w^\varepsilon$  and  $z^\varepsilon$  converge towards  $w$  and  $z$  pointwise,  $z$  fulfills the interior boundary condition in the sense of Definition 2.7, given that the conditions of Theorem 2 are met.*

**4. Conclusions and Outlook.** We have given a general framework for the derivation of both the adjoint equation and the interior boundary condition in the presence of shocks. We have furthermore proven that under certain reasonable assumptions, the adjoint equation fulfills this interior boundary condition. In particular, this could explain why low-order, i.e., very diffusive schemes, have no problem in converging towards the correct adjoint solution.

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