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Bericht Nr. 351 **Dezember 2012**

Key words: heterogeneous media, acoustic waves,
finite volume methods, evolution Galerkin scheme,
hyperbolic conservation laws, bicharacteristics.

Math. Subject Classifications : Primary: 35L05, 35L40, 35L65, 35L67
Secondary: 65M08, 65M25, 76M12

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The first author is supported by the Alexander–von–Humboldt Foundation through a postdoctoral fellowship. The second and third authors are partially supported by the Deutsche Forschungsgemeinschaft (DFG) under grant number NO 361/3-2. The second author is also partially supported by the National Natural Science Foundation of China (No. 11001211, 51178359).

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Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

A FINITE VOLUME EVOLUTION GALERKIN SCHEME FOR ACOUSTIC WAVES IN HETEROGENEOUS MEDIA

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Abstract. In this paper, we present a numerical scheme for the propagation of acoustic waves in a heterogeneous medium in the context of the finite volume evolution Galerkin (FVEG) method (M. Lukáčová-Medviďová et al. *J. Comput. Phys.*, 183:533–562, 2002). As a mathematical model we consider a wave equation system with space dependent wave-speed and impedance, which is used to study the wave propagation in a complex media. A main building block of our scheme is a genuinely multidimensional evolution operator based on the bicharacteristic theory of hyperbolic systems under the assumption of space dependent Jacobian matrices. We employ a novel approximation of the evolution operator, resulting from quadratures, in the flux evaluation stage of a finite volume scheme. The results of several numerical case studies clearly demonstrate the efficiency and robustness of the new FVEG scheme.

1. **Introduction.** Hyperbolic conservation laws with spatially varying flux functions model acoustic or elastic waves in a heterogeneous medium [2]. In exploration seismology, e.g., one studies the propagation of small amplitude man-made waves in earth and their reflection off geological structures. The hope is to determine the geological structure (for example oil reservoirs) from measurements at the surface.

2000 *Mathematics Subject Classification.* Primary: 35L05, 35L40, 35L65, 35L67; Secondary: 65M08, 65M25, 76M12.

Key words and phrases. Heterogeneous Media, Acoustic Waves, Finite volume methods, Evolution Galerkin scheme, Hyperbolic conservation laws, bicharacteristics.

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A similar principle as in seismological exploration of the earth is used also in ultrasound exploration of human tissues. In all of these cases new phenomena can appear since reflections of waves at interfaces can lead to discontinuities even for linear equations.

The goal of the present work is to develop a numerical scheme for the propagation of acoustic waves in a heterogeneous medium in the context of the finite volume evolution Galerkin (FVEG) method. The FVEG method has been developed originally by Lukáčová and her coworkers, cf. e.g. [3, 4]. It is a predictor-corrector method combining the finite volume corrector step with the evolutionary predictor step. The corrector step approximates the fluxes by the midpoint rule in time and trapezoidal rule in space. At the space-time quadrature nodes, point values of the solution are predicted by a multidimensional approximate evolution operator. The latter is constructed using the theory of bicharacteristics under the assumption of spatially dependent Jacobian matrices. In the previous works of Lukáčová and others the evolution operators were derived for constant coefficient, locally linearised systems where bicharacteristics reduce to straight lines. An attempt to design a generalised FVEG scheme for linear hyperbolic systems with variable coefficients is done by Arun et al. in [1], where the methodology is demonstrated for a simple acoustic wave equation. The present work is a continuation along the lines of [1] to study wave propagation in complex media and to this end, we consider more general and practically relevant mathematical models.

Using a general version of the compatibility condition on a bicharacteristic curve [5], we derive an exact and (then use it to get) an approximate evolution operator. As shown in [1], in order to obtain a stable scheme, the coefficients of the heterogeneous medium must be approximated over a staggered grid that is centered at the integration points on cell interfaces. Our numerical experiments for wave propagation with continuous as well as discontinuous wave speeds, through smooth as well as non-smooth interfaces confirm robustness and reliability of the new FVEG scheme.

2. **Finite Volume Evolution Galerkin Method.** In this section we design an FVEG scheme for the numerical simulation of acoustic waves in a heterogeneous medium. In contrast to [1], the mathematical model used here for the propagation of acoustic waves is obtained by linearising the isentropic Euler equations or the elasticity equations; see [2] for a derivation. The system of equations reads

$$\partial_t U + \partial_x F_1(U) + \partial_y F_2(U) = 0, \tag{1}$$

with the vector of unknowns U and the flux-vectors $F_1(U)$ and $F_2(U)$ given as

$$U = \begin{pmatrix} \phi \\ \rho u \\ \rho v \end{pmatrix}, \ F_1(U) = \begin{pmatrix} u \\ K\phi \\ 0 \end{pmatrix}, \ F_2(U) = \begin{pmatrix} v \\ 0 \\ K\phi \end{pmatrix}. \tag{2}$$

Here, ϕ can be thought as the amplitude of a pressure wave and u, v are respectively the velocities in the x and y directions. The parameters $K(x, y)$ and $\rho(x, y)$ are respectively the bulk modulus and density and hence, are material dependent.

Let X be the vector-valued Sobolev space of solutions to (1) and let $E(\tau)\colon X \to X$ be the exact solution operator, i.e.

$$U(\cdot, t+\tau) = E(\tau)U(\cdot, t). \tag{3}$$

Let V_r be an approximation space of vector-valued piecewise polynomials of degree r and let us denote by U^n, the approximation to the exact solution $U(\cdot, t^n)$ in the space V_r. Since the exact solution is not always available, we suppose that an adequate approximate solution operator $E_\tau \colon V_r \to X$ is given. Let us also denote by $R \colon V_s \to V_r$, a suitable recovery operator, where $V_s \subset V_r$ is the space of vector-valued piecewise polynomials of degree s. Starting from U^n, the FVEG scheme can be recursively defined by

Definition 2.1.

$$U^{n+1} = U^n - \frac{1}{\Delta x}\int_0^{\Delta t} \delta_x F_1\left(U^{n+\frac{\tau}{\Delta t}}\right) d\tau - \frac{1}{\Delta y}\int_0^{\Delta t} \delta_y F_2\left(U^{n+\frac{\tau}{\Delta t}}\right) d\tau. \tag{4}$$

Here, δ_x and δ_y are finite difference operators, e.g. $\delta_x f(x) = f(x+h/2) - f(x-h/2)$ and $\delta_x F_1(U^{n+\tau/\Delta t})$ and $\delta_y F_2(U^{n+\tau/\Delta t})$ are respectively the flux differences in the x and y directions at time $t^n + \tau$. In order to evolve the these fluxes, the approximate evolution operator is used, i.e.

$$U^{n+\frac{\tau}{\Delta t}} = \sum \left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega} E_\tau R U^n d\sigma\right)\chi_{\partial\Omega}, \tag{5}$$

where χ is the characteristic function of the edge $\partial\Omega$ and summation is taken over all the computational cells.

In traditional predictor-corrector schemes like the two step Lax-Wendroff scheme, the predictor step is done by a multi-dimensional finite difference operator, for example the Lax-Friedrichs scheme. The FVEG scheme tries to replace the Lax-Friedrichs step by a more accurate evolution operator based on the theory of bicharacteristic curves, which is then approximated by quadrature; see [3] for details. The appealing element of the latter is that it systematically tries to take into account the infinitely many directions of wave propagation.

3. **Exact and Approximate Evolution Operators.**

Let us write the wave equation system in the primitive form:

$$\partial_t V + A_1 \partial_x V + A_2 \partial_y V = 0, \tag{6}$$

where

$$V = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & K(x,y) & 0 \\ \frac{1}{\rho(x,y)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ \frac{1}{\rho(x,y)} & 0 & 0 \end{pmatrix}. \tag{7}$$

We define the wavespeed c and the impedance Z via the relations $c := \sqrt{K/\rho}$ and $Z := \sqrt{K\rho}$.

We fix a point $P = (x, y, t^n + \tau)$ in space-time and consider the characteristic conoid of (6), passing through P and enveloped by the bicharacteristics given by

$$\frac{dx}{dt} = -c(x,y)\cos\theta, \; \frac{dy}{dt} = -c(x,y)\sin\theta, \; \frac{d\theta}{dt} = -\sin\theta\partial_x c + \cos\theta\partial_y c. \tag{8}$$

Here, $(\cos\theta, \sin\theta)$ is the unit normal to the wavefront, which is the section of the conoid by $t = \text{const}$ hyperplanes. We solve the system of equations (8) with the initial values $x(\omega, t+\tau) = x, \; y(\omega, t+\tau) = y$ and $\theta(\omega, t+\tau) = \omega \in [0, 2\pi]$. Let Q and

$\tilde{Q}$ be respectively arbitrary points on the wavefronts at $t = t^n$ and $t = \tilde{t} \in (t^n, t^n+\tau)$. Proceeding as in [1], we can derive the exact evolution operators

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix}(P) = \frac{1}{2\pi}\int_0^{2\pi} (p - Z\cos\theta u - Z\sin\theta v)(Q) \begin{pmatrix} 1 \\ \cos\omega \\ \sin\omega \end{pmatrix} d\omega$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} 1 \\ \frac{-2\cos\omega}{Z(P)} \\ \frac{-2\sin\omega}{Z(P)} \end{pmatrix} d\omega \int_{t^n}^{t^n+\tau} \left\{ u\frac{d}{dt}(Z\cos\theta) + v\frac{d}{dt}(Z\sin\theta) \right\}(\tilde{Q})d\tilde{t}$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} 1 \\ \frac{-2\cos\omega}{Z(P)} \\ \frac{-2\sin\omega}{Z(P)} \end{pmatrix} d\omega \int_{t^n}^{t^n+\tau} (ZS)(\tilde{Q})d\tilde{t}, \tag{9}$$

where

$$S := c\left\{\partial_x u \sin^2\theta - (\partial_y u + \partial_x v)\sin\theta\cos\theta + \partial_y v\cos^2\theta\right\}. \tag{10}$$

We begin the approximation to the operator (9) by applying the rectangular quadrature rule in time. The approximation of the last integral (involving the term S) is done exactly as in [1] and hence we do not elaborate them here. However, the approximation of the first two terms in (9) are done differently as outlined below. Let

$$I := (Z\cos\theta u)(Q) + \int_{t^n}^{t^n+\tau} u\frac{d}{dt}(Z\cos\theta)(\tilde{Q})d\tilde{t} \tag{11}$$

Using Taylor development for $Z(Q)\cos\theta$ in the first summand of (11) and rectangle rule for the time integral in the second summand yields

$$\begin{aligned} I &= u(Q)\left\{ Z(P)\cos\omega + (t^n - (t^n+\tau))\frac{d}{dt}(Z\cos\theta)(P)\right\} + \mathcal{O}(\tau^2) \\ &\quad + \tau u\frac{d}{dt}(Z\cos\theta)(Q) + \mathcal{O}(\tau^2). \\ &= u(Q)\left\{ Z(P)\cos\omega - \tau\frac{d}{dt}(Z\cos\theta)(P)\right\} + \tau u(Q)\frac{d}{dt}(Z\cos\theta)(P) + \mathcal{O}(\tau^2) \\ &= u(Q)Z(P)\cos\omega + \mathcal{O}(\tau^2). \end{aligned} \tag{12}$$

The terms involving v are treated analogously. Using these approximations together with the approximation of the source term integrals as in [1] yields the approximate evolution operators, e.g. for pressure,

$$\begin{aligned} p(P) =& \frac{1}{2\pi}\int_0^{2\pi} [p - Z(P)(u\cos\omega + v\sin\omega)]d\omega \\ & - \frac{1}{2\pi}\sum_j \Big[Z(\omega)(-u\sin\omega + v\cos\omega)\Big]_{\omega_j^-}^{\omega_j^+} \\ & - \frac{1}{2\pi}\int_0^{2\pi} \Big[u\frac{(dZ(\omega)\sin\omega)}{d\omega} - v\frac{d(Z(\omega)\cos\omega)}{d\omega}\Big]d\omega. \end{aligned} \tag{13}$$

The expressions for u and v are analogous; see also [1, 3, 4] for more details on the use of the evolution operators in a finite volume framework, leading to the FVEG scheme.

4. **Numerical Case Studies.** In this section we demonstrate the performance of our scheme for smooth and non-smooth data. The scheme is implemented as follows: In order to avoid the overlap of the discontinuities along the integration paths in (13), both c and Z are stored at the centres of a staggered grid, whereas p, u and v are stored in the physical grid; see also [1]. We use a piecewise linear reconstruction for Z, p, u and v on their respective grids with the minmod limiter to limit overshoots and undershoots of the linearly recovered approximations. The rectangle rule is used for spatial integral of flux (5) on the edges and for all the numerical experiments performed, the CFL number is set to be 0.5.

4.1. **Order of Convergence.** Our first goal is to demonstrate the second order convergence of the scheme by computing the experimental order of convergence (EOC). To this end, we choose a smooth coefficients and initial data

$$\rho(x, y) = K(x, y) = 1 + \frac{1}{4}\left(\sin(4\pi x) + \cos(4\pi y)\right),$$
$$p(x, y, 0) = \sin(2\pi x) + \cos(2\pi y),\ u(x, y, 0) = v(x, y, 0) = 0.$$

The computational domain $[0, 1] \times [0, 1]$ is successively divided into $10 \times 10, 20 \times 20, \ldots, 320 \times 320$ mesh cells and the final time is set to $t = 1.0$. The boundary conditions are periodic everywhere. Since an exact solution of this initial value problem is not available, the numerical solution obtained an $N \times N$ grid is compared to the one obtained on a $2N \times 2N$ grid. The errors in p, u and v and the corresponding EOCs obtained in the L^1 norm is shown in table 1. The table clearly shows the second order convergence of the scheme.

N	error of p	EOC	error of u	EOC	error of v	EOC
10	8.52E-02	-	7.54E-02	-	5.54E-02	-
20	3.96E-02	1.11	2.57E-02	1.55	1.55E-02	1.84
40	1.70E-02	1.22	5.93E-03	2.12	4.60E-03	1.75
80	3.53E-03	2.27	1.37E-03	2.11	1.24E-03	1.89
160	6.35E-04	2.47	3.05E-04	2.17	3.00E-04	2.05
320	1.32E-04	2.27	7.30E-05	2.06	7.37E-05	2.02

TABLE 1. Wave propagation in a medium with smoothly varying density and bulk modulus: EOCs for p, u and v measured in the L^1-norm.

4.2. **Wave Propagation in a Heterogeneous Layered Medium.** This test problem is motivated an anlogous study in [1] and the problem models the propagation of a pressure pulse through a heterogeneous layered medium with a single interface. The density and bulk modulus are initialised as

$$(\rho(x, y), K(x, y)) = \begin{cases} (1, 1), & \text{if } x \leq 0.5, \\ (4, 2), & \text{otherwise.} \end{cases}$$

The initial data read

$$p(x, y, 0) = \begin{cases} 1 + 0.5(\cos(\pi r/0.1) - 1), & \text{if } r \leq 0.1, \\ 0, & \text{otherwise,} \end{cases}$$
$$u(x, y, 0) = 0 = v(x, y, 0),$$

where r denotes the distance $r = \sqrt{(x-0.25)^2 + (y-0.4)^2}$. The computational domain is $[-0.95, 1.05] \times [-0.8, 1.6]$ and the boundary conditions are absorbing via simple extrapolation of the variables on all sides. The contours of p, u and v at times $t = 0.2, 0.4, 0.6$ and 0.8 are plotted in Figure 1. In the figure we clearly notice a

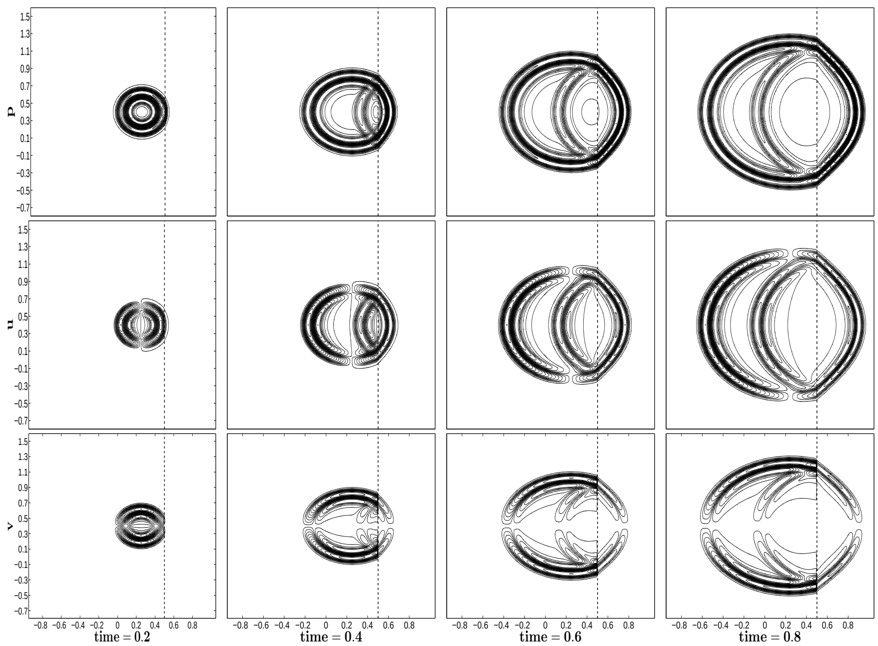

FIGURE 1. Waves passing through the interface of a layered medium.

good resolution of the circular waves, which confirms the genuine multidimensional behaviour of the FVEG scheme. There are no spurious oscillations at the interface and the deformation of the wave due to the change in the medium is captured very well. Due to the jump in the impedances of the media, a part of the wave is reflected backwards as seen in the plots at $t = 0.4$ onwards.

4.3. **Waves passing through a wavy interface.** In this test we simulate the waves passing through a complex, wavy interface, which is not aligned to the grid. The initial values of p, u and v are same as in the previous problem. The material parameters K and ρ are initialised as

$$K(x,y) = 1, \;\; \rho(x,y) = \begin{cases} 1, & \text{if } x \le 0.5\cos(2\pi(y-0.4)) + 0.4, \\ 4, & \text{otherwise.} \end{cases}$$

The computational domain is $[-0.95, 1.2] \times [-0.675, 1.475]$ and the boundary conditions are absorbing everywhere. The isolines of the solutions at times $t = 0.2, 0.4, 0.6$ and 1.0 are depicted in Figure 2. As in the previous problem we observe both the reflection and transmission of the waves at the interface. However, due to the wavy geometry of the material interface, a complex flow pattern of the reflected waves can be observed at the interface.

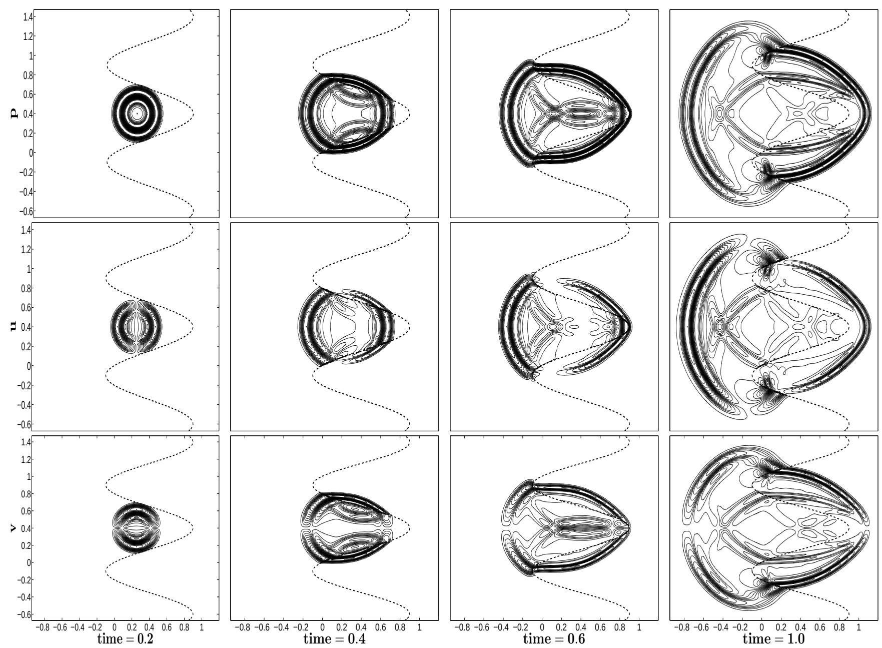

FIGURE 2. Waves passing through a wavy interface of a layered medium.

4.4. **Wave Propagation through a Nonsmooth Interface.** This test taken from the reference [2]. The setup consists of a planar square wave pressure pulse passing through a heterogeneous medium with piecewise constant density and bulk modulus. The density and bulk modulus have the initial values

$$\rho(x,y) = 1.0,\ K(x,y) = \begin{cases} 0.25 & \text{if } x > 0 \text{ and } y < 0.55x, \\ 1.0 & \text{otherwise.} \end{cases}$$

The initial data read

$$v(x,y,0) = 0,\ p(x,y,0) = u(x,y,0) = \begin{cases} 1 & \text{if } -0.35 < x < -0.2, \\ 0 & \text{otherwise.} \end{cases}$$

We apply periodic boundary conditions and the simulations are performed for $t = 0.4, 0.6$ and 1.0. The isolines of the pressure obtained on a 100×100 mesh are plotted in Figure 3 and for the sake of comparison we also plot the pressure obtained on finer mesh of 400×400 cells. The results clearly show the reflection and transmission of waves at the interface. After passing through interface, a part of the waves get reflected off due to the ramp-like geometry of the interface. It has to be noted that both reflected and transmitted waves are oblique to the grid and the genuinely multidimensional FVEG scheme resolve these waves without any grid alignment effect or spurious oscillations.

REFERENCES

[1] K. R. Arun, M. Kraft, M. Lukáčová-Medviďová, and P. Prasad. Finite volume evolution Galerkin method for hyperbolic conservation laws with spatially varying flux functions. *J. Comput. Phys.*, 228:565–590, 2009.

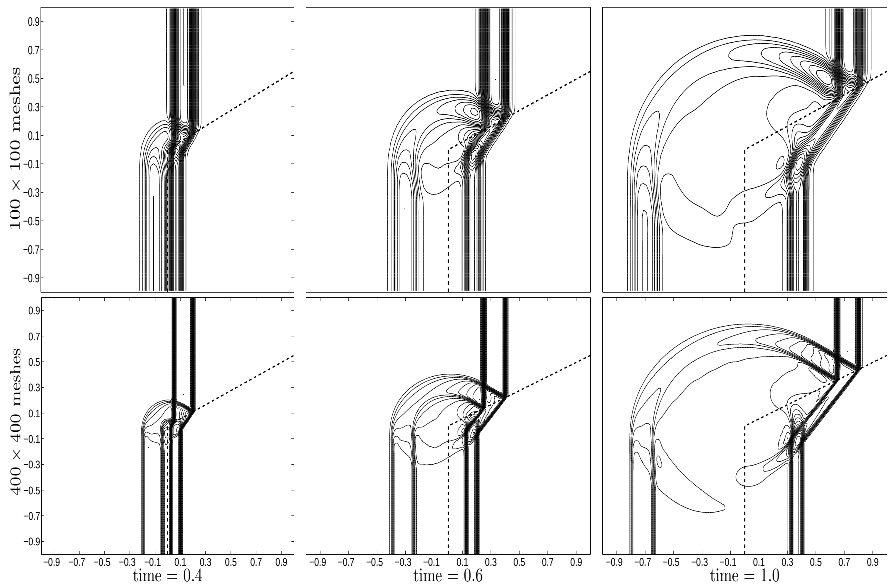

FIGURE 3. Waves passing through a non-smooth interface of a layered medium.

[2] R. J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, 2002.

[3] M. Lukáčová-Medviďová, K. W. Morton, and G. Warnecke. Finite volume evolution Galerkin (FVEG) methods for hyperbolic problems. *SIAM J. Sci. Comput.*, 26:1–30, 2004.

[4] M. Lukáčová-Medviďová, J. Saibertová, and G. Warnecke. Finite volume evolution Galerkin methods for nonlinear hyperbolic systems. *J. Comput. Phys.*, 183:533–562, 2002.

[5] P. Prasad. *Nonlinear Hyperbolic Waves in Multi-dimensions*. Chapman and Hall/CRC, London, 2001.

Received xxxx 20xx; revised xxxx 20xx.

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