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ANALYSIS OF A DISCONTINUOUS GALERKIN METHOD APPLIED TO THE LEVEL SET EQUATION

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Abstract. In this paper we investigate an error analysis of the DG method in space and the Crank-Nicolson scheme in time applied to the level set equation.The exact solution is assumed to be sufficiently smooth. Under certain assumption on the underlying velocity field we proof an error bound of order $h^{k+\frac{1}{2}} + \Delta t^2$ for the error between the exact solution and the fully discrete solution in the L_2-norm , where h is the spatial gird size, Δt the time step size and k the polynomial degree.

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1. Introduction.

Level set methods [Set96, Set99, OF03] are very popular for numerically capturing moving surfaces or interfaces and used in many applications, e.g. in two-phase flow simulations [GR11, SAB+99, SSO94, SSH+07]. The surface or interface is implicitly given by the zero level of the continuous level set function ϕ. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a domain. We introduce the level set equation

$$\frac{\partial}{\partial t}\phi + \mathbf{u} \cdot \nabla\phi = 0 \text{for } t \geq 0, \quad x \in \Omega, \tag{1.1}$$

with a given velocity field $\mathbf{u}$ and with suitable initial conditions and boundary conditions on the inflow boundary. Due to the representation of the interface as the zero level of the level set function the level set method can handle topological changes of the interface easily. There are many different possibilities to solve the level set method numerically, e.g. finite differencing schemes or finite element methods. Often the method of lines is used. That means the level set equation is discretized in space first and then with respect to time. A popular finite element method that is used for the spatial discretization of the level set equation is the SUPG method. This method, that is based on continuous finite elements, is stable. In [Bur10] an analysis of the SUPG method combined with different time stepping schemes applied to the level set equation is presented. In case of the Crank-Nicolson scheme the discretization error is shown to be of order $h^{k+\frac{1}{2}} + \Delta t^2$, when polynomials of order k are used, h denotes the spatial grid size and Δt the time step size and the stabilization parameter δ is chosen to be of order h. In this analysis the error is measured in the norm $\|\cdot\|_{\mathbf{u}}^2 := \|\cdot\|_{L_2}^2 + \delta^2\|\mathbf{u}\cdot\nabla\cdot\|_{L_2}^2$.

Another popular finite element method that may be used for the spatial discretization of the level set equation is the Discontinuous Galerkin (DG) method with upwind flux [MRC06, PFP06, FK08]. In this paper we present an analysis of this DG method combined with the Crank-Nicolson scheme in time applied to the level set equation. The discretization error is estimated in the L_2-norm. The derivation of an error estimate in an $H^1(\Omega)$-norm is still an open problem. There are many papers on the analysis of DG methods. For instance, an analysis of the DG method for *stationary* convection diffusion reaction problems can be found in [AM09]. For a sufficiently

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smooth exact solution u and the DG-solution u_h the authors show the error bounds

$$|||u - u_h||| = C|u|_{\mathrm{H}^{k+1}(\Omega)} \begin{cases} h^{k+\frac{1}{2}} & \text{convection dominated} \\ h^k & \text{diffusion dominated} \\ h^{k+1} & \text{reaction dominated,} \end{cases}$$

where $|||\cdot|||$ includes among other terms a grid dependent $\mathrm{H}^1(\Omega)$-norm and an L_2-norm. The analysis in [AM09], however, does not apply to a pure advection problem as the presence of a strictly positive reaction coefficient is crucial. Furthermore, the classical DG method (with upwind numerical flux) for *stationary* first order advection-reaction problems is analyzed in [JP86]. The authors prove convergence of order $h^{k+\frac{1}{2}}$ in the norm $|||\cdot|||^2 := \|\cdot\|^2_{\mathrm{L}_2} + h\|\mathbf{u}\cdot\nabla\cdot\|^2_{\mathrm{L}_2} + \sum_{e\in\mathcal{F}} |\,[\![\cdot]\!]\,|^2_e$. The last summand denotes the integrals of the jump over all faces. Later in [BMS04] the same order of convergence is shown for a whole class of DG methods for the same stationary first order hyperbolic problem. The DG methods differ in the choice of the numerical flux function. The estimates, however, hold in a norm, that includes the L_2-norm and a semi-norm on the faces, but not an L_2-norm of the streamline gradient.
A special DG method for the *stationary* advection-reaction equation was analyzed in [BS07]. Here the estimates hold in a norm, that is similar to the norm in [JP86]. In particular the control over the streamline gradient is included. We are not aware of a complete analysis, i.e. treating discretization errors in *space* and *time*, of the DG method applied to a *transient* advection-reaction problem. There is some literature on analysis of transient problems. For instance, in [CHS90] the truncation error of the DG method for hyperbolic conservation laws of the form

$$\partial_t u + \mathrm{div}\mathbf{f}(u) = 0 \text{in } \Omega \times (0, T)$$

is shown to be of order $k + 1$.

A convergence result for the linear transport equation in 1D is presented in [CSJT98]. The spatial discretization error at the end time T can be shown to be of order $k + 1$, where k is the polynomial degree. The transport equation with constant velocity is solved on the interval $[0, 1]$ with periodic boundary conditions.
Although the level set equation is a linear transport equation, this result does not include the initial-boundary problem, that we consider in this paper, as the arguments used can not be generalized to higher dimensions and other boundary conditions than periodic boundary conditions. In [ZS04] and [ZS10] error estimates for Runge-Kutta DG methods applied to scalar conservation laws are presented. In [ZS04] the time discretization is a second order TVD Runge-Kutta method and in [ZS10] a third order explicit TVD Runge-Kutta method. An estimate of order $\mathcal{O}(h^{k+\frac{1}{2}} + \Delta t^m)$, $m = 1, 2$ is derived for a general monotone numerical flux function. If the upwind flux is used the order improves to $\mathcal{O}(h^{k+1} + \Delta t^m)$, $m = 1, 2$. The estimate relies on smooth exact solutions. The proofs are only given for scalar conservation laws in 1D while boundary conditions are not considered. Instead, it is assumed that the solution is periodic or compactly supported. The authors claim that the analysis can be extended to multidimensional conservation laws in case of the linear flux $f(\varphi) = \mathbf{u}\varphi$. This, however would still exclude the level set equation due to the assumption on the behavior of the solution at the boundary.

In this paper we present a complete analysis of the DG method in space and a time differencing scheme in time applied to the level set equation in arbitrary space dimensions and with boundary conditions.

In section 2 the level set equation, its spatial discretization by the classical DG method with upwind flux and its time discretization due to the Crank-Nicolson scheme are introduced. In section 3 the main result, the discretization error estimate, and its proof are presented. In section 4 numerical results are presented.

2. Problem setting. Consider the domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and a given velocity field $\mathbf{u}$ that is defined in Ω. Let $\mathbf{n}$ denote the outer unit normal on $\partial\Omega$. The inflow boundary is defined as

$$\partial\Omega_{in} := \{x \in \partial\Omega | \mathbf{u}(x) \cdot \mathbf{n}(x) < 0\}.$$

Boundary conditions for the level set equation need to be prescribed on $\partial\Omega_{in}$. In this paper we consider Dirichlet boundary conditions ϕ_D. Furthermore, initial conditions are needed. Let $\phi_0 : \Omega \to \mathbb{R}$ be (close to) a signed distance function. Then, the level set problem in strong formulation is

$$\begin{cases} \text{find } \phi \in \mathcal{C}^1([0,T];\mathcal{C}^1(\Omega)) & \text{such that} \\ \frac{\partial}{\partial t}\phi + \mathbf{u} \cdot \nabla\phi = 0 & \text{in } \Omega \times [0,T] \\ \phi(x,0) = \phi_0(x) & \text{in } \Omega \\ \phi(x,t) = \phi_D(x,t) & \text{on } \partial\Omega_{in} \times [0,T] \end{cases} \tag{2.1}$$

Let $\overline{\phi_D}$ be a smooth extension of ϕ_D into the whole domain Ω. By subtracting $\overline{\phi_D}$ problem (2.1) can be transformed into a problem of the form (2.2).

$$\begin{cases} \text{find } \Psi \in \mathcal{C}^1([0,T];\mathcal{C}^1(\Omega)) & \text{such that} \\ \frac{\partial}{\partial t}\Psi + \mathbf{u} \cdot \nabla\Psi = f & \text{in } \Omega \times [0,T] \\ \Psi(x,0) = \Psi_0(x) & \text{in } \Omega \\ \Psi(x,t) = 0 & \text{on } \partial\Omega_{in} \times [0,T], \end{cases} \tag{2.2}$$

where $f := -\frac{\partial}{\partial t}\overline{\phi_D} - \mathbf{u} \cdot \nabla\overline{\phi_D}$ and $\Psi_0 = \phi_0 - \overline{\phi_D}$. We will analyze the DG method for problem (2.2) in section 3. The corresponding DG method for problem (2.1) is derived in Appendix A. Furthermore, we prove that the two methods are equivalent in Appendix A. Thus, the error bound we derive for the DG method for problem (2.2) also holds for the DG method for problem (2.1). Throughout the analysis we need the following assumptions on the velocity field $\mathbf{u}$.

ASSUMPTION 1. *Let $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$ be a divergence-free, globally Lipschitz-continuous function that does not depend on time. Furthermore, $\boldsymbol{u} \in [W^{1,\infty}(\Omega)]^3$ has no closed curves and no stationary points, i.e. $|\boldsymbol{u}(x)| \neq 0$ for all $x \in \Omega$.*

ASSUMPTION 2. *Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular tetrahedral (triangular) triangulations of Ω We restrict to triangulations $\mathcal{T}_h$ that are quasi-uniform.* We introduce the notation $\mathbf{u_n} := \mathbf{u} \cdot \mathbf{n}$. We derive the DG method for problem (2.2). As the velocity field is divergence free, we can transform the level set equation into the conservation law

$$\frac{\partial}{\partial t}\Psi + \nabla \cdot (\mathbf{u}\Psi) = f \tag{2.3}$$

with initial and boundary conditions as above. To derive the classical DG discretization we multiply (2.3) by a differentiable test function v and integrate over $K \in \mathcal{T}_h$.

$$\int_K \frac{\partial}{\partial t}\Psi v \, dx + \int_K \nabla \cdot (\mathbf{u}\Psi) v \, dx = \int_K f v \, dx.$$

Integration by parts yields

$$\int_K \frac{\partial}{\partial t}\Psi v\,dx - \int_K \Psi \mathbf{u}\cdot\nabla v\,dx + \int_{\partial K} \Psi v \mathbf{u_n}\,ds = \int_K f v\,dx.$$

Define the finite dimensional trial and test space

$$\mathrm{V}_h^{k,DG} := \{v_h \,:\, v_h|_K \in \mathcal{P}_k(K), \forall\, K \in \mathcal{T}_h\}. \tag{2.4}$$

Substituting ϕ and v by their approximations in $\mathrm{V}_h^{k,DG}$ and summing over all $K \in \mathcal{T}_h$ leads to

$$\sum_{K\in\mathcal{T}_h} \int_K \frac{\partial}{\partial t}\Psi_h v_h\,dx - \int_K \Psi_h \mathbf{u}\cdot\nabla v_h\,dx + \int_{\partial K} \Psi_h v_h \mathbf{u_n}\,ds = \sum_{K\in\mathcal{T}_h}\int_K f v_h\,dx. \tag{2.5}$$

The finite element functions Ψ_h and v_h are not uniquely defined on the faces $e \in \mathcal{F}^0$, where $\mathcal{F}^0$ denotes the set of interior faces of $\mathcal{T}_h$. To gain a uniquely defined discretization a numerical flux function has to be chosen. This numerical flux then substitutes the fluxes $f(\Psi_h)\cdot\mathbf{n} = \Psi_h\mathbf{u_n}$. In this paper we consider the upwind flux. Furthermore, we consider the DG method in the primal formulation. Let $\mathcal{F}^\partial$ denote the set of all faces located at the boundary $\partial\Omega$. Then the set of all faces is $\mathcal{F} := \mathcal{F}^0 \cup \mathcal{F}^\partial$. $\mathcal{F}^\partial$ can be further divided into $\mathcal{F}^{\partial_{in}}$, $\mathcal{F}^{\partial_{out}}$ and $\mathcal{F}^{\partial_0}$ according to the inflow boundary $\partial\Omega_{in}$, the outflow boundary $\partial\Omega_{out}$ and the part of the boundary $\partial\Omega_0$ where $\mathbf{u_n} = 0$. That means $\mathbf{u_n}$ is zero on all $e \in \mathcal{F}^{\partial_0}$ and thus, $\mathcal{F}^{\partial_0}$ will not be needed in the remainder.
For the interior faces $e \in \mathcal{F}^0$ there are two simplices K^+ and K^- sharing this face. Then the jump $[\![\cdot]\!]$ across e and the average $\{\cdot\}$ on e for a scalar function g and a vector valued function ψ are defined by

$$[\![g]\!] := g^+\mathbf{n}^+ + g^-\mathbf{n}^- \{g\} := \frac{1}{2}(g^+ + g^-)$$
$$[\![\psi]\!] := \psi^+\cdot\mathbf{n}^+ + \psi^-\cdot\mathbf{n}^- \{\psi\} := \frac{1}{2}(\psi^+ + \psi^-).$$

Furthermore, we need the scalar product $(\cdot,\cdot)_\mathcal{F} := \sum_{e\in\mathcal{F}}(\cdot,\cdot)_e$. The scalar products $(\cdot,\cdot)_{\mathcal{F}^{\partial_{out}}}$, $(\cdot,\cdot)_{\mathcal{F}^0}$ and $(\cdot,\cdot)_{\mathcal{F}^{\partial_{in}}}$ denote the summation over the corresponding subsets of $\mathcal{F}$. As a function in $\mathrm{V}_h^{k,DG}$ may be discontinuous across the interior faces, its gradient does not exist on $e \in \mathcal{F}^0$. However, it exists on every $K \in \mathcal{T}_h$. Thus, we use the L_2-scalar product $(\cdot,\cdot)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(\cdot,\cdot)_K$, the L_2-norm $\|\cdot\|_{\mathcal{T}_h} := \sqrt{(\cdot,\cdot)_{\mathcal{T}_h}}$ and the Sobolev-norm $\|\cdot\|_{\mathrm{H}^m(\mathcal{T}_h)} := \sqrt{\sum_{K\in\mathcal{T}_h}\|\cdot\|^2_{\mathrm{H}^{k+1}(K)}}$, $m \in \mathbb{N}$, for gradients of the discontinuous finite element functions.

We substitute the term $\sum_{K\in\mathcal{T}_h}\int_{\partial K}\Psi_h v_h \mathbf{u_n}\,ds$ in (2.5) by the sum over all faces. As Ψ_h is not single-valued on the faces we substitute the fluxes $\Psi_h\mathbf{u_n}$ by the upwind flux $\widehat{\mathbf{u}\Psi_h}^{\mathrm{uw}}$. Due to the inflow boundary conditions $\Psi = 0$ on $\partial\Omega_{in}$ we obtain

$$\begin{aligned}&(\frac{\partial}{\partial t}\Psi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\Psi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\Psi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}\Psi_h}^{\mathrm{uw}}, v_h)_{\mathcal{F}^0}\\ =&(f, v_h)_{\mathrm{L}_2(\Omega)},\end{aligned} \tag{2.6}$$

with

$$(\widehat{\mathbf{u}\psi}^{\mathrm{uw}}, v)_{\mathcal{F}^0} := \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![v]\!]\,\{\psi\} + \frac{1}{2}|\mathbf{u_n}|\,[\![v]\!]\cdot[\![\psi]\!]\,ds. \tag{2.7}$$

As $\Psi_h|_{\partial\Omega_{in}} = 0$ for all $t \in [0,T]$, we introduce the following space:

$$\mathrm{V}_{h,0}^{k,DG} := \{v_h \in \mathrm{V}_h^{k,DG} : v_h|_{\partial\Omega_{in}} = 0\}.$$

In the remainder we change the notation and use ϕ and ϕ_h also for the transformed variables Ψ and Ψ_h. We define the bilinear form $\mathbf{a}_h$ as follows

$$\begin{aligned} &\mathbf{a}_h : \mathrm{H}^1(\mathcal{T}_h) \times \mathrm{H}^1(\mathcal{T}_h) \to \mathbb{R}, \\ &\mathbf{a}_h(\varphi, v) = -(\varphi, \mathbf{u}\cdot\nabla v)_{\mathcal{T}_h} + (\varphi, v\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}\varphi}^{\mathrm{uw}}, v)_{\mathcal{F}^0} \end{aligned} \tag{2.8}$$

where $\mathrm{H}^1(\mathcal{T}_h) := \{\psi \in \mathrm{L}_2(\Omega); \psi|_T \in H^1(T), \forall T \in \mathcal{T}_h\}$ and

$$(\widehat{\mathbf{u}\varphi}^{\mathrm{uw}}, v)_{\mathcal{F}^0} := \sum_{e\in\mathcal{F}^0} \int_e \mathbf{u} \cdot [\![v]\!] \{\varphi\} + \frac{1}{2}|\mathbf{u_n}| [\![v]\!] \cdot [\![\varphi]\!] \, ds. \tag{2.9}$$

The semi-discrete level set DG method resulting form (2.2), then, reads:
find $\phi_h \in \mathcal{C}^1([0,T]; \mathrm{V}_{h,0}^{k,DG})$ such that

$$(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\phi_h, v_h) = (f, v_h)_{\mathrm{L}_2(\Omega)} \text{for all } v_h \in \mathrm{V}_h^{k,DG}. \tag{2.10}$$

Now, we apply the Crank-Nicolson scheme to discretize (2.10) in time. We introduce the time step size $\Delta t := T/N$, $N \in \mathbb{N}$, and $t^n := n\Delta t$, $n = 0, ..., N$. The fully discrete problem is given by the following scheme:

$$\begin{aligned} &\phi_h^0 = \phi_{0,h} \\ &\text{for} \quad n = 1, ..., N \quad \text{find} \quad \phi_h^n \in \mathrm{V}_{h,0}^{k,DG} \quad \text{such that for all } v_h \in \mathrm{V}_h^{k,DG} \\ &(\frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\frac{1}{2}(\phi_h^n + \phi_h^{n-1}), v_h) = (\frac{1}{2}(f(t^n) + f(t^{n-1})), v_h)_{\mathrm{L}_2(\Omega)}. \end{aligned} \tag{2.11}$$

Here, $\phi_{0,h}$ is an approximation in $\mathrm{V}_{h,0}^{k,DG}$ to ϕ_0. We will specify this approximation later. In the next section we derive error bounds for the scheme (2.11).

3. Error analysis. The analysis is based on ideas, that are used in [Bur10] for the analysis of the SUPG method applied to the level set equation and in [Tho97] for the analysis of FE methods for parabolic equations. We outline the main ingredients in the analysis below.

An inf-sup-condition with respect to a suitable norm is proven for the bilinear form $\mathbf{a}_h$. For the derivation of the inf-sup-condition the assumptions, that the velocity field is Lipschitz-continuous and has no closed curves and no stationary points, are crucial. This analysis is given in section 3.2. Using this result we show that in a suitable norm the projection error of the Ritz-projection related to $\mathbf{a}_h$ is of order $h^{k+\frac{1}{2}}$, cf. section 3.3. Then, in section 3.4, we consider the fully discrete system. As in the analysis for parabolic problems in [Tho97] the error between the fully discrete solution and the exact solution in the L_2-norm is decomposed into the projection error of the Ritz-projection and the difference between the Ritz-projection of the exact solution and the fully discrete solution. The latter is shown to be of order $h^{k+\frac{1}{2}} + \Delta t^2$. Here the assumption that the velocity field is time-independent is needed.

The proof of the inf-sup-condition (Theorem 3.11) is based on the construction of a suitable test function ψ_h^*. Similar arguments can be found in [BS07], where a stabilized DG method for the stationary advection-reaction equation is analyzed, in

[AM09], where DG methods for advection-diffusion-reaction problems are studied and in [BS11], where an analysis of the SUPG method applied to a transient convection-diffusion equation is presented. In our analysis the test function $\psi_h^* \in \mathrm{V}_h^{k,DG}$ is the sum of two functions, that are similar to those used in the papers mentioned above, i.e. $\psi_h^* := \delta_1 h \overline{\mathbf{u}} \cdot \nabla \varphi_h + \delta_2 \boldsymbol{\pi}_h(\chi \varphi_h)$, with a suitable choice for δ_1 and δ_2. Here, $\varphi_h \in \mathrm{V}_{h,0}^{k,DG}$, $\overline{\mathbf{u}}$ denotes the mean value of the velocity field $\mathbf{u}$, χ is a suitable smooth function, which will be introduced later, and $\boldsymbol{\pi}_h$ denotes the L_2-projection onto $\mathrm{V}_h^{k,DG}$. We show that for all $\varphi_h \in \mathrm{V}_{h,0}^{k,DG}$ and $\psi_h^* = \psi_h^*(\varphi_h)$ as above there are positive constants c_1 and $c_2 > 0$ such that

$$\mathbf{a}_h(\varphi_h, \psi_h^*) \geq c_1 |||\varphi_h|||^2 \quad \text{and} \quad |||\psi_h^*||| \leq c_2 |||\varphi_h||| \tag{3.1}$$

hold for a suitable norm $|||\cdot|||$. Thus, an inf-sup condition follows directly. The first summand, $h\overline{\mathbf{u}} \cdot \nabla \varphi_h$, is used in [BS07] together with the additional term $\kappa \varphi_h$ where κ is a suitable constant. $h\overline{\mathbf{u}} \cdot \nabla \varphi_h$ yields control over the L_2-norm of the streamline gradient. However, the test function as in [BS07] does not yield control over the L_2-norm of φ_h in the case of the level set equation due to the lack of the reaction term. Instead of adding $\kappa \varphi_h$, we add $\boldsymbol{\pi}_h(\chi \varphi_h)$ which yields the control over the L_2-norm of φ_h. The idea to construct such a test function was introduced in [JP86] and reused in [AM09].

The treatment of the time-dependent case is analog to the analysis of the SUPG method in [Bur10] and the approach in [Tho97].

3.1. Preliminaries. We introduce norms and semi-norms, that are related to the triangulation $\mathcal{T}_h$ and the bilinear form $\mathbf{a}_h$.

NOTATION 1. *Let* $\varphi \in \mathrm{H}^1(\mathcal{T}_h)$.

$$\|\varphi\|_{\mathcal{F}}^2 := \frac{1}{2} \sum_{e \in \mathcal{F}^0} \| |\boldsymbol{u_n}|^{\frac{1}{2}} [\![\varphi]\!] \|_e^2 + \frac{1}{2} \sum_{e \in \mathcal{F}^{\partial_{out}}} \| |\boldsymbol{u_n}|^{\frac{1}{2}} \varphi \|_e^2.$$

$$\|\varphi\|_{\boldsymbol{a}_h}^2 := \|\varphi\|_{\mathrm{L}_2(\Omega)}^2 + \|\varphi\|_{\mathcal{F}}^2.$$

$$|||\varphi|||^2 := h \|\boldsymbol{u} \cdot \nabla \varphi\|_{\mathcal{T}_h}^2 + \|\varphi\|_{\boldsymbol{a}_h}^2$$

Note that due to the definition of the jump

$$\| |\mathbf{u_n}| [\![\varphi_h]\!] \|_e = \| \mathbf{u} \cdot [\![\varphi_h]\!] \|_e \leq c \| |\mathbf{u_n}|^{\frac{1}{2}} [\![\varphi_h]\!] \|_e, \tag{3.2}$$

holds with $c = \| |\mathbf{u_n}|^{\frac{1}{2}} \|_\infty$ for every internal face $e \in \mathcal{F}^0$. Furthermore we will need inverse inequalities, trace inequalities and L_2-projection error estimates. For the finite element functions the standard inverse inequality and trace inequality hold due to Assumption 2. Consider an arbitrary element $K \in \mathcal{T}_h$. Then, the inverse inequalities

$$\|\nabla \varphi_h\|_{\mathrm{L}_2(K)} \leq C_K h_K^{-1} \|\varphi_h\|_{\mathrm{L}_2(K)} \quad \text{and} \quad \|\nabla \varphi_h\|_{\mathcal{T}_h} \leq C h^{-1} \|\varphi_h\|_{\mathrm{L}_2(\Omega)} \tag{3.3}$$

hold for all $\varphi_h \in \mathrm{V}_h^{k,DG}$ with constants C_k and C independent of h, e.g. [EG04]. Furthermore, the trace inequality

$$\|\varphi_h\|_{\mathrm{L}_2(\partial K)} \leq C_K h_K^{-\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(K)} \tag{3.4}$$

holds for all $\varphi_h \in \mathrm{V}_h^{k,DG}$ with a constant C_K independent of h [CL91]. Now consider an arbitrary face $e \in \mathcal{F}$ and $K^+ \in \mathcal{T}_h$ such that e is a face of K^+. Hence, there exists a constant C_{K^+} independent of h_{K^+} such that

$$\|\varphi_h^+\|_e \leq C_{K^+} h_{K^+}^{-\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(K^+)}$$

due to the trace inequality. Based on this observation we can derive trace inequalities for the jumps and averages on $e \in \mathcal{F}^0$ and for the face norm $\|\cdot\|_{\mathcal{F}}$.

LEMMA 3.1. *Let $\varphi_h, \psi_h \in \mathrm{V}_h^{k,DG}$. Let K^+, K^- be such that $\partial K^+ \cap \partial K^- = e \in \mathcal{F}^0$. Then,*

$$\|\{\varphi_h\}\|_e \leq Ch^{-\frac{1}{2}} \|\varphi_h\|_{K^+ \cup K^-} \tag{3.5}$$

$$\text{and} \quad \| [\![\varphi_h]\!] \|_e \leq Ch^{-\frac{1}{2}} \|\varphi_h\|_{K^+ \cup K^-}. \tag{3.6}$$

Furthermore,

$$\begin{rcases} \sum_{e\in\mathcal{F}^0} \|\{\psi_h\}\|_e^2 \\ \sum_{e\in\mathcal{F}^0} \| [\![\psi_h]\!] \|_e^2 \end{rcases} \leq Ch^{-1} \|\psi_h\|^2_{\mathrm{L}_2(\Omega)} \tag{3.7}$$

and

$$\|\varphi_h\|_{\mathcal{F}} \leq Ch^{-\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(\Omega)}. \tag{3.8}$$

A similar result can be found in [BS07], Lemma 2.2. We will need this version of the trace inequality rather than (3.4). Furthermore, it is helpful to use an alternative representation of $\mathbf{a}_h$. Let $\varphi, \psi \in \mathrm{H}^1(\mathcal{T}_h)$ with $\varphi = 0$ on $\partial\Omega_{in}$. By partially integrating the volume integral in (2.8) and applying the identity

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \varphi\psi \mathbf{u}_\mathbf{n}\, ds = \sum_{e\in\mathcal{F}^0} \int_e \mathbf{u}\cdot [\![\psi]\!] \{\varphi\} + \mathbf{u}\cdot [\![\varphi]\!] \{\psi\}\, ds + \sum_{e\in\mathcal{F}^{\partial_{out}}} \int_e \varphi\psi\mathbf{u}_\mathbf{n}\, ds \tag{3.9}$$

(e.g. see [AM09]) we obtain the representation

$$\mathbf{a}_h(\varphi,\psi) = \sum_{K\in\mathcal{T}_h} \int_K (\mathbf{u}\cdot\nabla\varphi)\psi\, dx + \sum_{e\in\mathcal{F}^0} \int_e -\mathbf{u}\cdot [\![\varphi]\!] \{\psi\} + \frac{1}{2}|\mathbf{u}_\mathbf{n}| [\![\varphi]\!] \cdot [\![\psi]\!]\, ds. \tag{3.10}$$

In a similar way one can show that

$$\mathbf{a}_h(\varphi,\varphi) = \|\varphi\|^2_{\mathcal{F}} \tag{3.11}$$

holds for every $\varphi \in \mathrm{H}^1(\mathcal{T}_h)$. To construct the first summand of the test function ψ_h^* the mean value $\overline{\mathbf{u}}$ of the velocity field and the following estimates are needed. Let $\overline{\mathbf{u}} \in \mathrm{V}_h^{0,DG}$ be the mean value of $\mathbf{u}$ on every $K \in \mathcal{T}_h$, i.e.

$$\overline{\mathbf{u}}|_K = \frac{1}{|K|}\int_K \mathbf{u}\, dx \quad \forall K \in \mathcal{T}_h.$$

Let L denote the Lipschitz-constant of $\mathbf{u}$. Then, for $K \in \mathcal{T}_h$ and an arbitrary $x \in K$ the estimate

$$\begin{aligned} \|\mathbf{u}(x) - \overline{\mathbf{u}}\|_2 =& \|\frac{1}{|K|}\int_K \mathbf{u}(x) - \mathbf{u}(y)\, dy\|_2 \\ \leq& \frac{1}{|K|}\int_K L\|x-y\|_2\, dy \leq \frac{1}{|K|} Lh_K \int_K 1\, dy = \qquad Lh_K \end{aligned}$$

follows due to the Lipschitz-continuity of $\mathbf{u}$. Thus,

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathrm{L}_\infty(K)} = \max_{x\in K} \|\mathbf{u}(x) - \overline{\mathbf{u}}\|_2 \leq Lh_K.$$

Summing over all $K \in \mathcal{T}_h$ yields

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathrm{L}_\infty(\Omega)} \leq Ch. \tag{3.12}$$

LEMMA 3.2. *For the velocity field $\boldsymbol{u}$ and its element-wise mean value $\overline{\boldsymbol{u}}$*

$$\|\overline{\boldsymbol{u}} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} \leq \|\boldsymbol{u} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} + c\|\varphi_h\|_{\mathrm{L}_2(\Omega)} \tag{3.13}$$

holds.

Proof. We add and subtract $\mathbf{u} \cdot \nabla\varphi_h$ and apply the triangle-inequality.

$$\|\overline{\mathbf{u}} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} \leq \|\mathbf{u} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} + \|(\overline{\mathbf{u}} - \mathbf{u}) \cdot \nabla\varphi_h\|_{\mathcal{T}_h}$$

Applying (3.12) and an inverse inequality yields

$$\|\overline{\mathbf{u}} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} \leq \|\mathbf{u} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} + Ch\|\nabla\varphi_h\|_{\mathcal{T}_h} \leq \|\mathbf{u} \cdot \nabla\varphi_h\|_{\mathcal{T}_h} + c\|\varphi_h\|_{\mathrm{L}_2(\Omega)}.$$

□

As the second summand is defined as the L_2-projection of $\chi\varphi_h$, we need the following approximation results. Let $\boldsymbol{\pi}_h$ denote the standard L_2-projection into $\mathrm{V}_h^{k,DG}$. Let $\psi \in \mathrm{H}^{k+1}(\Omega)$. Denote by $p_h := \psi - \boldsymbol{\pi}_h\psi$ the projection error. The standard error estimates

$$|p_h|_{\mathrm{H}^m(K)} \leq Ch_K^{k+1-m}|\psi|_{\mathrm{H}^{k+1}(K)}, \quad \forall\, \psi \in \mathrm{H}^{k+1}(K), \quad 0 \leq m \leq k \tag{3.14}$$

hold for any $K \in \mathcal{T}_h$ with a constant C independent of h. Also a trace inequality can be shown for $\psi \in \mathrm{H}^{k+1}(K)$, $K \in \mathcal{T}_h$ arbitrary, i.e.

$$\|p_h\|_{\mathrm{L}_2(\partial K)} \leq Ch_K^{k+\frac{1}{2}}|\psi|_{\mathrm{H}^{k+1}(K)} \tag{3.15}$$

holds with a constant C independent of h_K and ψ. Due to the trace inequality (3.15) and the estimates (3.14) we obtain the following estimates:

COROLLARY 3.3. *Let $e \in \mathcal{F}$. There exist constants C_e and C independent of h such that*

$$\|\,[\![p_h]\!]\,\|_e \leq C_e h^{k+\frac{1}{2}}|\psi|_{\mathrm{H}^{k+1}(U)}, \quad \|\{p_h\}\|_e \leq C_e h^{k+\frac{1}{2}}|\psi|_{\mathrm{H}^{k+1}(U)} \tag{3.16}$$

$$\text{and } \|p_h\|_{\mathcal{F}} \leq Ch^{k+\frac{1}{2}}|\psi|_{\mathrm{H}^{k+1}(\Omega)} \quad \text{for all } \psi \in \mathrm{H}^{k+1}(\Omega), \tag{3.17}$$

where $U = K^-(e) \cup K^+(e)$ (K^+ and K^- are the two elements sharing e) if e is an interior face and $U = K$ for the element K that has e as a face if $e \in \mathcal{F}^\partial$. Furthermore, there is a constant C independent of h such that

$$\|p_h\|_{\mathrm{H}^m(\mathcal{T}_h)} \leq Ch^{k+1-m}|\psi|_{\mathrm{H}^{k+1}(\Omega)}, \quad \forall\, \psi \in \mathrm{H}^{k+1}(\Omega), \quad 0 \leq m \leq k. \tag{3.18}$$

3.2. Proof of an inf-sup-property. First, we consider the first summand of ψ_h^*, i.e. $h\overline{\mathbf{u}} \cdot \nabla\varphi_h \in \mathrm{V}_{h,0}^{k,DG}$.

LEMMA 3.4. *Let $\varphi_h \in \mathrm{V}_{h,0}^{k,DG}$. Consider the function $h\overline{\boldsymbol{u}} \cdot \nabla\varphi_h \in \mathrm{V}_h^{k,DG}$. Then,*

$$\frac{1}{2}h\|\boldsymbol{u} \cdot \nabla\varphi_h\|_{\mathcal{T}_h}^2 \leq \boldsymbol{a}_h(\varphi_h, h\overline{\boldsymbol{u}} \cdot \nabla\varphi_h) + C(h\|\varphi_h\|_{\mathrm{L}_2(\Omega)}^2 + \|\varphi_h\|_{\mathcal{F}}^2) \tag{3.19}$$

holds with a constant C *that is independent of* h *and* φ_h.

Proof. We use the representation (3.10) of $\mathbf{a}_h$, which yields

$$\mathbf{a}_h(\varphi_h, h\overline{\mathbf{u}}\cdot\nabla\varphi_h) = h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\int_K \mathbf{u}\cdot\nabla\varphi_h h(\overline{\mathbf{u}}-\mathbf{u})\cdot\nabla\varphi_h\,dx$$
$$+\sum_{e\in\mathcal{F}^0}\int_e -\mathbf{u}\,[\![\varphi_h]\!]\,\{h\overline{\mathbf{u}}\cdot\nabla\varphi_h\} + \frac{1}{2}|\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\cdot[\![h\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\;ds.$$

Hence,

$$h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} = \mathbf{a}_h(\varphi_h, h\overline{\mathbf{u}}\cdot\nabla\varphi_h) + \sum_{K\in\mathcal{T}_h}\int_K \mathbf{u}\cdot\nabla\varphi_h h(\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\varphi_h\,dx$$
$$+\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\,[\![\varphi_h]\!]\,\{h\overline{\mathbf{u}}\cdot\nabla\varphi_h\} - \frac{1}{2}|\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\cdot[\![h\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\;ds.$$

Thus,

$$h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} \le \mathbf{a}_h(\varphi_h, h\overline{\mathbf{u}}\cdot\nabla\varphi_h) + \underbrace{\Big|\sum_{K\in\mathcal{T}_h}\int_K \mathbf{u}\cdot\nabla\varphi_h h(\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\varphi_h\,dx\Big|}_{=I_1}$$
$$+\underbrace{\Big|\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\,[\![\varphi_h]\!]\,\{h\overline{\mathbf{u}}\cdot\nabla\varphi_h\} - \frac{1}{2}|\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\cdot[\![h\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\;ds\Big|}_{I_2}. \tag{3.20}$$

We estimate I_j, $j = 1, 2$.
Due to the Cauchy-Schwarz inequality, estimate (3.12), Young's inequality with an arbitrary $\varepsilon_1 > 0$ we get

$$I_1 \le h\|\mathbf{u}\cdot\nabla\varphi_h\|_{\mathcal{T}_h}\|(\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\varphi_h\|_{\mathcal{T}_h} \overset{(3.12)}{\le} Ch^2\|\mathbf{u}\cdot\nabla\varphi_h\|_{\mathcal{T}_h}\|\nabla\varphi_h\|_{\mathcal{T}_h}$$
$$\le C\Big(\frac{1}{\varepsilon_1}h\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \varepsilon_1 h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h}\Big).$$

The Cauchy-Schwarz inequality, equation (3.2), Young's inequality with an arbitrary $\varepsilon_2 > 0$, equation (3.7) with $\psi_h = \overline{\mathbf{u}}\cdot\nabla\varphi_h$, and Lemma 3.2 are used to derive

$$I_2 \le \sum_{e\in\mathcal{F}^0}\left(\|\mathbf{u}\cdot[\![\varphi_h]\!]\|_e h\|\{\overline{\mathbf{u}}\cdot\nabla\varphi_h\}\|_e + \frac{1}{2}\||\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\|_e h\|[\![\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\|_e\right)$$
$$\le \sqrt{\sum_{e\in\mathcal{F}^0}\|\mathbf{u}\cdot[\![\varphi_h]\!]\|_e^2}\sqrt{\sum_{e\in\mathcal{F}^0}h^2\|\{\overline{\mathbf{u}}\cdot\nabla\varphi_h\}\|_e^2}$$
$$+\sqrt{\sum_{e\in\mathcal{F}^0}\||\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\|_e^2}\sqrt{\frac{1}{4}\sum_{e\in\mathcal{F}^0}h^2\|[\![\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\|_e^2}$$
$$\le \varepsilon_2\sum_{e\in\mathcal{F}^0}h^2\left(\|\{\overline{\mathbf{u}}\cdot\nabla\varphi_h\}\|_e^2 + \frac{1}{4}\|[\![\overline{\mathbf{u}}\cdot\nabla\varphi_h]\!]\|_e^2\right) + \frac{\tilde{C}}{\varepsilon_2}\sum_{e\in\mathcal{F}^0}\||\mathbf{u}_\mathbf{n}|^{\frac{1}{2}}\,[\![\varphi_h]\!]\|_e^2$$
$$\le \varepsilon_2 Ch\|\overline{\mathbf{u}}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} + \frac{\tilde{C}}{\varepsilon_2}\|\varphi_h\|^2_{\mathcal{F}} \le \varepsilon_2 Ch\left(\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} + c\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)}\right) + \frac{\tilde{C}}{\varepsilon_2}\|\varphi_h\|^2_{\mathcal{F}}.$$

Substituting these estimates in (3.20) and choosing $\varepsilon_1 = \frac{1}{4C(I_1)}$ and $\varepsilon_2 = \frac{1}{4C(I_2)}$ yields

$$\frac{1}{2}h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} \leq \mathbf{a}_h(\varphi_h, h\overline{\mathbf{u}}\cdot\nabla\varphi_h) + C(h\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \|\varphi_h\|^2_{\mathcal{F}}) \tag{3.21}$$

□

Furthermore, we will need the following result:
Lemma 3.5. *There exists a constant c independent of h such that*

$$|||h\overline{\boldsymbol{u}}\cdot\nabla\varphi_h||| \leq c|||\varphi_h|||, \quad \forall\, \varphi_h \in \mathrm{V}^{k,DG}_{h,0}. \tag{3.22}$$

Proof. Using the inverse inequality, (3.8) and Lemma 3.2 we get

$$\begin{aligned}|||h\overline{\mathbf{u}}\cdot\nabla\varphi_h|||^2 =& h^2\|\overline{\mathbf{u}}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} + h^3\|\mathbf{u}\cdot\nabla(\overline{\mathbf{u}}\cdot\nabla\varphi_h)\|^2_{\mathcal{T}_h} + h^2\|\overline{\mathbf{u}}\cdot\nabla\varphi_h\|^2_{\mathcal{F}}\\ \leq& h^2\left(\bar{c}^2\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h}\right) + C^2h\|\overline{\mathbf{u}}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h}\\ \leq& h^2\left(\bar{c}^2\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h}\right)\\ &+ C^2h\left(\tilde{c}^2\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h}\right) \leq c^2|||\varphi_h|||^2.\end{aligned}$$

□

Similar estimates can be shown for the second summand $\boldsymbol{\pi}_h(\chi\varphi_h)$. Before we prove these estimates we need the following results.
Lemma 3.6. *There exists a function $\eta \in W^{k+1,\infty}(\Omega)$ such that $\boldsymbol{u}\cdot\nabla\eta \geq C_u > 0$ in Ω.* The existence of such a function η was shown under assumptions that $\mathbf{u}$ has neither closed curves nor stationary points and $\mathbf{u} \in [W^{1,\infty}(\Omega)]^3$ in [AM09], Appendix A. Due to Assumption 1 these conditions hold.

Consider the function $\chi = e^{-\eta} \in W^{k+1,\infty}(\Omega)$. Due to the properties of η there exist positive constants χ_1, χ_2, χ_3 such that

$$\chi_1 \leq \chi \leq \chi_2 \quad |\nabla\chi| \leq \chi_3 \tag{3.23}$$

Note that for any $\varphi_h \in \mathrm{V}^{k,DG}_h$, $\chi\varphi_h \in \mathrm{H}^1(\mathcal{T}_h)$. Based on the estimate (3.18), (3.23) and the fact that $\chi \in W^{k+1,\infty}(\Omega)$ the following estimates can be shown: There exist positive constants C_1, C_2 and C_3 independent of h such that

$$\|\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\|_{\mathrm{L}_2(\Omega)} \leq C_1h\|\varphi_h\|_{\mathrm{L}_2(\Omega)} \quad \forall\,\varphi_h \in \mathrm{V}^{k,DG}_h \tag{3.24}$$

$$\|\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\|_{\mathrm{H}^1(\mathcal{T}_h)} \leq C_2\|\varphi_h\|_{\mathrm{L}_2(\Omega)} \quad \forall\,\varphi_h \in \mathrm{V}^{k,DG}_h \tag{3.25}$$

$$\left.\begin{cases}\sum_{e\in\mathcal{F}^0}\|\,[\![\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)]\!]\,\|_e\\ \sum_{e\in\mathcal{F}^0}\|\{\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\}\|_e\end{cases}\right. \leq C_3h^{\frac{1}{2}}\|\varphi_h\|_{\mathrm{L}_2(\Omega)} \quad \forall\,\varphi_h \in \mathrm{V}^{k,DG}_h. \tag{3.26}$$

Hence, also $\|\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\|_{\mathcal{F}} \leq C_3h^{\frac{1}{2}}\|\varphi_h\|_{\mathrm{L}_2(\Omega)}\,\forall\,\varphi_h \in \mathrm{V}^{k,DG}_h$ follows directly. A proof can be found in [AM09].
Lemma 3.7. *There exists a positive constant C independent of h such that*

$$\boldsymbol{a}_h(\varphi_h, \chi\varphi_h) \geq C\|\varphi_h\|^2_{\boldsymbol{a}_h} \quad \forall\,\varphi_h \in \mathrm{V}^{k,DG}_h. \tag{3.27}$$

Proof. Integration by parts, the identity $\frac{1}{2}\nabla(\varphi_h^2) = (\nabla\varphi_h)\varphi_h$ and the fact that φ_h vanishes on the inflow boundary yield:

$$\sum_{K\in\mathcal{T}_h} -\int_K \mathbf{u}\cdot\nabla(\chi\varphi_h)\varphi_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K -(\mathbf{u}\cdot\nabla\chi)\varphi_h^2 - \chi(\mathbf{u}\cdot\nabla\varphi_h)\varphi_h\,dx$$
$$= \sum_{K\in\mathcal{T}_h} -\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx - \frac{1}{2}\int_K \chi(\mathbf{u}\cdot\nabla\varphi_h^2)\,dx$$
$$= \sum_{K\in\mathcal{T}_h} -\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx + \frac{1}{2}\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx$$
$$- \frac{1}{2}\sum_{e\in\mathcal{F}^0}\int_e \chi\mathbf{u}\cdot [\![\varphi_h^2]\!]\;ds - \frac{1}{2}\sum_{e\in\mathcal{F}^{\partial_{out}}}\int_e \chi\varphi_h^2|\mathbf{u}_\mathbf{n}|\,ds$$
$$= -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx - \frac{1}{2}\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![\varphi_h^2]\!]\,\chi\,ds - \frac{1}{2}\sum_{e\in\mathcal{F}^{\partial_{out}}}\int_e \chi\varphi_h^2|\mathbf{u}_\mathbf{n}|\,ds.$$

Thus, using the identity $\frac{1}{2}\mathbf{u}\cdot[\![\varphi_h^2]\!] = \mathbf{u}\cdot[\![\varphi_h]\!]\,\{\varphi_h\}$ and the continuity of χ we obtain

$$\mathbf{a}_h(\varphi_h, \chi\varphi_h)$$
$$= -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx - \frac{1}{2}\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![\varphi_h^2]\!]\,\chi\,ds - \frac{1}{2}\sum_{e\in\mathcal{F}^{\partial_{out}}}\int_e \chi\varphi_h^2|\mathbf{u}_\mathbf{n}|\,ds$$
$$+ \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![\varphi_h]\!]\,\{\varphi_h\}\chi + \frac{1}{2}|\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\cdot[\![\varphi_h]\!]\chi\,ds + \sum_{e\in\mathcal{F}^{\partial_{out}}}\int_e \chi\varphi_h^2|\mathbf{u}_\mathbf{n}|\,ds$$
$$= -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{u}\cdot\nabla\chi)\varphi_h^2\,dx + \frac{1}{2}\sum_{e\in\mathcal{F}^0}\int_e \chi|\mathbf{u}_\mathbf{n}|\,[\![\varphi_h]\!]\cdot[\![\varphi_h]\!]\;ds$$
$$+ \frac{1}{2}\sum_{e\in\mathcal{F}^{\partial_{out}}}\int_e \chi\varphi_h^2|\mathbf{u}_\mathbf{n}|\,ds.$$

Furthermore, $-\mathbf{u}\cdot\nabla\chi = -\mathbf{u}\cdot\nabla e^{-\eta} = (\mathbf{u}\cdot\nabla\eta)\chi \geq C_u\chi_1 > 0$. Hence,

$$\mathbf{a}_h(\varphi_h, \chi\varphi_h) \geq \frac{1}{2}C_u\chi_1\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \chi_1\|\varphi_h\|^2_{\mathcal{F}}$$

The result holds with $C := \max\{\frac{1}{2}C_u, 1\}\chi_1$. □

Lemma 3.8. *There exists a constant $\tilde{C}$ independent of h such that*

$$\boldsymbol{a}_h(\varphi_h, \chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)) \leq \tilde{C}h^{\frac{1}{2}}\|\varphi_h\|^2_{\boldsymbol{a}_h} \quad \forall\,\varphi_h \in \mathrm{V}_h^{k,DG}.$$

Proof. Let $\overline{\mathbf{u}}$ be the mean value of $\mathbf{u}$. As $\overline{\mathbf{u}}$ is piecewise constant, $\overline{\mathbf{u}}\cdot\nabla\varphi_h \in \mathrm{V}_h^{k,DG}$. Due to the definition of $\boldsymbol{\pi}_h$

$$\sum_{K\in\mathcal{T}_h}\int_K \overline{\mathbf{u}}\cdot\nabla\varphi_h(\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)) = 0 \tag{3.28}$$

holds. We use the alternative representation of $\mathbf{a}_h$ (3.10) and add a zero to obtain

$$\begin{aligned}
&\mathbf{a}_h(\varphi_h, \chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)) \\
&= \underbrace{\sum_{K\in\mathcal{T}_h} \int_K (\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\varphi_h(\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h))}_{=I} \\
&\quad + \underbrace{\sum_{e\in\mathcal{F}^0} \int_e -\mathbf{u}\cdot [\![\varphi_h]\!] \{\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\} + \frac{1}{2}|\mathbf{u_n}| [\![\varphi_h]\!] \cdot [\![\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)]\!] \, ds}_{=II}
\end{aligned}$$

Now we estimate I and II applying the estimates (3.24) and (3.26), an inverse inequality, (3.12) and (3.2).

$$\begin{aligned}
I \leq& \|(\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\varphi_h\|_{\mathcal{T}_h} \|\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h))\|_{\mathrm{L}_2(\Omega)} \\
\leq& C_{inv} h^{-1} \|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathrm{L}_\infty} \|\varphi_h\|_{\mathrm{L}_2(\Omega)} C_1 h \|\varphi_h\|_{\mathrm{L}_2(\Omega)} \leq Ch\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} \\
II \leq& \sum_{e\in\mathcal{F}^0} \|\mathbf{u}\cdot [\![\varphi_h]\!]\|_e \|\{\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\}\|_e + \frac{1}{2}\| |\mathbf{u_n}| [\![\varphi_h]\!]\|_e \| [\![\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)]\!]\|_e \\
\leq& \underbrace{\sqrt{\sum_{e\in\mathcal{F}^0} \|\mathbf{u}\cdot [\![\varphi_h]\!]\|_e^2}}_{\leq c\|\varphi_h\|_{\mathcal{F}}} \underbrace{\sqrt{\sum_{e\in\mathcal{F}^0} \|\{\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)\}\|_e^2}}_{\leq C_3 h^{\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(\Omega)}} \\
&+ \underbrace{\sqrt{\sum_{e\in\mathcal{F}^0} \| |\mathbf{u_n}| [\![\varphi_h]\!]\|_e^2}}_{\leq c\|\varphi_h\|_{\mathcal{F}}} \underbrace{\sqrt{\frac{1}{4}\sum_{e\in\mathcal{F}^0} \| [\![\chi\varphi_h - \boldsymbol{\pi}_h(\chi\varphi_h)]\!]\|_e^2}}_{\leq \frac{1}{2} C_3 h^{\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(\Omega)}} \\
\leq& c h^{\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(\Omega)} \|\varphi_h\|_{\mathcal{F}} \leq ch^{\frac{1}{2}} (\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + \|\varphi_h\|^2_{\mathcal{F}}) = ch^{\frac{1}{2}} \|\varphi_h\|^2_{\mathbf{a}_h}.
\end{aligned}$$

Summing the estimates for I and II and defining $\tilde{C} = C + c$ concludes the proof. □

LEMMA 3.9. *There exists a $h_0 > 0$ and a constant $\alpha = \alpha(h_0) > 0$ independent of h such that for all $h < h_0$*

$$\mathbf{a}_h(\varphi_h, \boldsymbol{\pi}_h(\chi\varphi_h)) \geq \alpha\|\varphi_h\|^2_{\mathbf{a}_h} \quad \forall\, \varphi_h \in \mathrm{V}_h^{k,DG} \tag{3.29}$$

$$\textit{and } |||\boldsymbol{\pi}_h(\chi\varphi_h)||| \leq c|||\varphi_h||| \quad \forall\, \varphi_h \in \mathrm{V}_h^{k,DG}. \tag{3.30}$$

Proof. Due to Lemma 3.7 and Lemma 3.8

$$\begin{aligned}
\mathbf{a}_h(\varphi_h, \boldsymbol{\pi}_h(\chi\varphi_h)) =& \mathbf{a}_h(\varphi_h, \boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h) + \mathbf{a}_h(\varphi_h, \chi\varphi_h) \\
\geq& -\tilde{C}h^{\frac{1}{2}}\|\varphi_h\|^2_{\mathbf{a}_h} + C\|\varphi_h\|^2_{\mathbf{a}_h} = (C - \tilde{C}h^{\frac{1}{2}})\|\varphi_h\|^2_{\mathbf{a}_h}.
\end{aligned}$$

Now choose $h_0 = \frac{1}{4}(\frac{C}{\tilde{C}})^2$. Then, $\alpha(h_0) = \frac{1}{2}C > 0$ and the first estimate holds for all $h < h_0$. To show (3.30) we apply the estimates (3.24) - (3.26) and note that
$\|\mathbf{u}\cdot\nabla(\chi\varphi_h)\|_{\mathcal{T}_h} \leq \|(\mathbf{u}\cdot\nabla\chi)\varphi_h\|_{\mathcal{T}_h} + \|\chi(\mathbf{u}\cdot\nabla\varphi_h)\|_{\mathcal{T}_h} \leq \chi_3\|\varphi_h\|_{\mathrm{L}_2(\Omega)} + \chi_2\|\mathbf{u}\cdot\nabla\varphi_h\|_{\mathcal{T}_h}$

due to (3.23). Then,

$$
\begin{aligned}
|||\boldsymbol{\pi}_h(\chi\varphi_h)||| \leq& |||\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h||| + |||\chi\varphi_h||| \\
=& \left(\|\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + h\|\mathbf{u}\cdot\nabla(\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h)\|^2_{\mathcal{T}_h} + \|\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h\|^2_{\mathcal{F}}\right)^{\frac{1}{2}} \\
&+ \left(\|\chi\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + h\|\mathbf{u}\cdot\nabla(\chi\varphi_h)\|^2_{\mathcal{T}_h} + \|\chi\varphi_h\|^2_{\mathcal{F}}\right)^{\frac{1}{2}} \\
=& \left(\|\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + h\|\mathbf{u}\cdot\nabla(\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h)\|^2_{\mathcal{T}_h} + \|\boldsymbol{\pi}_h(\chi\varphi_h) - \chi\varphi_h\|^2_{\mathcal{F}}\right)^{\frac{1}{2}} \\
&+ \left(\|\chi\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + h\|(\mathbf{u}\cdot\nabla\chi)\varphi_h + (\mathbf{u}\cdot\nabla\varphi_h)\chi\|^2_{\mathcal{T}_h} + \|\chi\varphi_h\|^2_{\mathcal{F}}\right)^{\frac{1}{2}} \\
\leq& \left(C_1^2h^2 + \|\mathbf{u}\|^2_{\mathrm{L}_\infty(\Omega)}C_2^2 + C_3^2h\right)^{\frac{1}{2}} \|\varphi_h\|_{\mathrm{L}_2(\Omega)} \\
&+ \left(\chi_2^2\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + 2h\chi_3^2\|\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + 2h\chi_2^2\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathcal{T}_h} + \chi_2^2\|\varphi_h\|^2_{\mathcal{F}}\right)^{\frac{1}{2}} \\
\leq& c|||\varphi_h|||.
\end{aligned}
$$

□

Now consider the function $\psi_h^* = \delta_1 h\overline{\mathbf{u}}\cdot\nabla\varphi_h + \delta_2\boldsymbol{\pi}_h(\chi\varphi_h)$.

LEMMA 3.10. *Let $\psi_h^* = \psi_h^*(\varphi_h) = \delta_1 h\overline{\boldsymbol{u}}\cdot\nabla\varphi_h + \delta_2\boldsymbol{\pi}_h\chi\varphi_h$, $\varphi_h \in \mathrm{V}_{h,0}^{k,DG}$. δ_1, δ_2 can be chosen independent of h such that for h small enough there are positive constants c_1 and c_2 independent of h such that*

$$\boldsymbol{a}_h(\varphi_h, \psi_h^*(\varphi_h)) \geq c_1|||\varphi_h|||^2 \quad \forall\,\varphi_h \in \mathrm{V}_{h,0}^{k,DG} \tag{3.31}$$

$$and |||\psi_h^*(\varphi_h)||| \leq c_2|||\varphi_h||| \quad \forall\,\varphi_h \in \mathrm{V}_{h,0}^{k,DG}. \tag{3.32}$$

Proof. To prove (3.31) we need (3.19) and (3.29).

$$
\begin{aligned}
\mathbf{a}_h(\varphi_h, \psi_h^*) =& \delta_1\mathbf{a}_h(\varphi_h, h\overline{\mathbf{u}}\cdot\nabla\varphi_h) + \delta_2\mathbf{a}_h(\varphi_h, \boldsymbol{\pi}_h(\chi\varphi_h)) \\
\geq& \delta_1\frac{1}{2}h\|\mathbf{u}\cdot\nabla\varphi_h\|^2_{\mathrm{L}_2(\Omega)} + (\delta_2\alpha - \delta_1 C)\|\varphi_h\|^2_{\mathbf{a}_h}
\end{aligned}
$$

Any choice for δ_1 and δ_2 such that $\delta_2\alpha - \delta_1 C > 0$ concludes the proof of (3.31). It is convenient to choose $\delta_1 = 2$ and $\delta_2 = \frac{1+2C}{\alpha}$. Then, $c_1 = 1$. With these choices for δ_1 and δ_2 (3.32) follows directly form (3.22) and (3.30). □

Now, we can proof the following theorem.

THEOREM 3.11. *For h small enough there exists a constant $C > 0$ independent of h such that the following inf-sup-condition holds:*

$$C|||\varphi_h||| \leq \sup_{\psi_h\in\mathrm{V}_h^{k,DG}} \frac{\boldsymbol{a}_h(\varphi_h, \psi_h)}{|||\psi_h|||} \quad \forall\,\varphi_h \in \mathrm{V}_{h,0}^{k,DG}. \tag{3.33}$$

Proof. Let $\psi_h^* = 2h\overline{\mathbf{u}}\cdot\nabla\varphi_h + \frac{1+2C}{\alpha}\boldsymbol{\pi}_h\chi\varphi_h$ be the test function defined in the proof of Lemma 3.10. Then, Lemma 3.10 holds with $c_1 = 1$. Define $c := c_2$, where c_2 is the constant in (3.32). We obtain

$$|||\varphi_h||| \leq \frac{\mathbf{a}_h(\varphi_h, \psi_h^*)}{|||\varphi_h|||} \leq c\frac{\mathbf{a}_h^*(\varphi_h, \psi_h^*)}{|||\psi_h^*|||} \leq c \sup_{\psi_h\in\mathrm{V}_h^{k,DG}} \frac{\mathbf{a}_h(\varphi_h, \psi_h)}{|||\psi_h|||}$$

We conclude by setting $C = c^{-1}$. □

3.3. Error bound for the Ritz-projection. In this section we analyze the fully discrete problem. The error between the exact solution and the fully discrete solution is decomposed into a projection error and the error between the projection of the exact solution and the fully discrete solution. This difference is a function in $\mathrm{V}_h^{k,DG}$. Therefore we refer to this error as the discrete error. The projection we need is the Ritz-projection defined in Definition 3.12 below. We derive an error bound for the projection error of the Ritz-projection and the discrete error in suitable norms. Applying these results to the fully discrete scheme we obtain an error bound of order $h^{k+\frac{1}{2}}+\Delta t^2$. The idea of this approach is common in analyses of finite element methods for parabolic problems. A similar derivation for the standard Galerkin method with backward Euler or Crank-Nicolson in time can be found in [Tho97], Chap. 1.

DEFINITION 3.12. *Let $\psi \in \mathrm{H}^1(\Omega)$. The Ritz-projection $\boldsymbol{\pi}_h^{DG}\psi \in \mathrm{V}_h^{k,DG}$ is defined by*

$$\boldsymbol{a}_h(\boldsymbol{\pi}_h^{DG}\psi - \psi, v_h) = 0 \quad \forall\, v_h \in \mathrm{V}_h^{k,DG}. \tag{3.34}$$

THEOREM 3.13. *Let $\psi \in \mathrm{H}^{k+1}(\Omega)$ with $\psi = 0$ on $\partial\Omega_{in}$. Then, the following holds with a constant C independent of ψ and h:*

$$|||\boldsymbol{\pi}_h^{DG}\psi - \psi||| \le Ch^{k+\frac{1}{2}}\|\psi\|_{\mathrm{H}^{k+1}(\Omega)} \tag{3.35}$$

Proof. Denote with $\boldsymbol{\pi}_h$ the L_2-projection to $\mathrm{V}_h^{k,DG}$. Then,

$$|||\boldsymbol{\pi}_h^{DG}\psi - \psi||| \le \underbrace{|||\boldsymbol{\pi}_h^{DG}\psi - \boldsymbol{\pi}_h\psi|||}_{:=|||e_h|||} + \underbrace{|||\psi - \boldsymbol{\pi}_h\psi|||}_{:=|||p_h|||} \tag{3.36}$$

First, we consider the L_2-projection error. Due to Corollary 3.3 we obtain

$$\begin{aligned} |||p_h|||^2 =& \|p_h\|_{\mathrm{L}_2(\Omega)}^2 + h\|\mathbf{u}\cdot\nabla p_h\|_{\mathcal{T}_h}^2 + \|p_h\|_{\mathcal{F}}^2 \\ \le& \tilde{C}h^{2k+1}|\psi|_{\mathrm{H}^{k+1}(\Omega)}^2\,(h + c + \tilde{c}) \le C^2h^{2k+1}|\psi|_{\mathrm{H}^{k+1}(\Omega)}^2 \end{aligned}$$

Thus,

$$|||p_h||| \le Ch^{k+\frac{1}{2}}|\psi|_{\mathrm{H}^{k+1}(\Omega)} \le Ch^{k+\frac{1}{2}}\|\psi\|_{\mathrm{H}^{k+1}(\Omega)}.$$

Now consider e_h. According to Theorem 3.11 the inf-sup-condition (3.33) holds for every $\psi_h \in \mathrm{V}_{h,0}^{k,DG}$. Note that $e_h = \boldsymbol{\pi}_h^{DG}\psi - \boldsymbol{\pi}_h\psi \in \mathrm{V}_{h,0}^{k,DG}$ as $e_h = 0$ on $\partial\Omega_{in}$, because $\psi = 0$ on $\partial\Omega_{in}$. Thus, we can apply Theorem 3.11 to e_h and get

$$|||e_h||| \le c \sup_{v_h \in \mathrm{V}_h^{k,DG}} \frac{\mathbf{a}_h(e_h, v_h)}{|||v_h|||}.$$

Furthermore,

$$\mathbf{a}_h(e_h, v_h) = \mathbf{a}_h(\boldsymbol{\pi}_h^{DG}\psi - \boldsymbol{\pi}_h\psi, v_h) \overset{\text{def}}{=} \mathbf{a}_h(\psi - \boldsymbol{\pi}_h\psi, v_h) = \mathbf{a}_h(p_h, v_h).$$

We will show that

$$\mathbf{a}_h(p_h, v_h) \le Ch^{k+\frac{1}{2}}\|\psi\|_{\mathrm{H}^{k+1}(\Omega)}|||v_h|||. \tag{3.37}$$

This implies

$$|||e_h||| \leq Ch^{k+\frac{1}{2}} \|\psi\|_{\mathrm{H}^{k+1}(\Omega)},$$

which concludes the proof.
Now we prove (3.37). As $p_h = 0$ on the inflow boundary we obtain

$$\mathbf{a}_h(p_h, v_h) = -(p_h, \mathbf{u} \cdot \nabla v_h)_{\mathcal{T}_h} + \sum_{e \in \mathcal{F}^0} \int_e \mathbf{u} \cdot [\![v_h]\!] \{p_h\} + \frac{1}{2}|\mathbf{u_n}| [\![v_h]\!] \cdot [\![p_h]\!] \, ds.$$

Due to the Cauchy-Schwarz inequality and Corollary 3.3

$$|(p_h, \mathbf{u} \cdot \nabla v_h)_{\mathcal{T}_h}| \leq Ch^{k+1} |\psi|_{\mathrm{H}^{k+1}(\Omega)} \|\mathbf{u} \cdot \nabla v_h\|_{\mathcal{T}_h}.$$

Furthermore, with the Cauchy-Schwarz inequality, equality (3.2) and the estimates in (3.16) we obtain

$$\begin{aligned}
&|\sum_{e \in \mathcal{F}^0} \int_e \mathbf{u} \cdot [\![v_h]\!] \{p_h\} + \frac{1}{2}|\mathbf{u_n}| [\![v_h]\!] \cdot [\![p_h]\!] \, ds| \\
&\leq \sum_{e \in \mathcal{F}^0} \|\mathbf{u} \cdot [\![v_h]\!]\|_e \|\{p_h\}\|_e + \frac{1}{2} \||\mathbf{u_n}| [\![v_h]\!]\|_e \|[\![p_h]\!]\|_e \\
&\leq c \sqrt{\sum_{e \in \mathcal{F}^0} \||\mathbf{u_n}|^{\frac{1}{2}} [\![v_h]\!]\|_e^2} \left(\sqrt{\sum_{e \in \mathcal{F}^0} \|\{p_h\}\|_e^2} + \sqrt{\sum_{e \in \mathcal{F}^0} \|[\![p_h]\!]\|_e^2} \right) \\
&\leq Ch^{k+\frac{1}{2}} |\psi|_{\mathrm{H}^{k+1}(\Omega)} \|v_h\|_{\mathcal{F}}.
\end{aligned}$$

Combining these estimates, we obtain

$$\mathbf{a}_h(p_h, v_h) \leq Ch^{k+\frac{1}{2}} \|\psi\|_{\mathrm{H}^{k+1}(\Omega)} |||v_h|||.$$

Hence, (3.37) holds. □

3.4. Error analysis for the fully discrete problem. Now, let ϕ denote the exact solution of problem (2.2) and let ϕ_h denote the exact solution in time of the system of ODEs that results from the spatial discretization by means of the DG method. Let $\phi_h^0 = \phi_{0,h} = \boldsymbol{\pi}_h^{DG} \phi_0$ be the approximation of ϕ_0 in $\mathrm{V}_{h,0}^{k,DG}$. Consider the time steps $t^n = n\Delta t$, where Δt is the step size. $t^N = T$ is the final time. Hence, $\phi(t^n)$ denotes the exact solution at t^n and $\phi_h(t^n)$ denotes the semi-discrete solution at time t^n. Accordingly, ϕ_h^n denotes the fully discrete solution at the n-th time step. The fully discrete scheme for solving the level set equation with DG in space and Crank-Nicolson in time is given by (2.11). The next estimate will be applied to the discrete error $\theta^n := \phi_h^n - \boldsymbol{\pi}_h^{DG} \phi(t^n)$. Similar derivations are presented in [Tho97], Theorem 1.6, p.15 and [Bur10].

Lemma 3.14. *For a given initial function $\psi^0 \in \mathrm{V}_h^{k,DG}$ and given source terms $\{g^n\}_{n=0}^N$, $g^n \in \mathrm{L}_2(\Omega)$, let $\{\psi^n\}_{n=0}^N$, $\psi^n \in \mathrm{V}_h^{k,DG}$, be the sequence that satisfies*

$$(\frac{\psi^n - \psi^{n-1}}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} + \boldsymbol{a}_h(\frac{\psi^n + \psi^{n-1}}{2}, v_h) = (g^n, v_h)_{\mathrm{L}_2(\Omega)} \tag{3.38}$$

for all $v_h \in \mathrm{V}_h^{k,DG}$, $n = 1, ..., N$. Then,

$$\|\psi^n\|_{\mathrm{L}_2(\Omega)} \leq \|\psi^0\|_{\mathrm{L}_2(\Omega)} + \Delta t \sum_{m=1}^{n} \|g^m\|_{\mathrm{L}_2(\Omega)} \quad \textit{for all } n = 1, ..., N \tag{3.39}$$

Proof. Note that

$$\begin{aligned}
&\|\psi^n\|^2_{L_2(\Omega)} - \|\psi^{n-1}\|^2_{L_2(\Omega)} + \frac{\Delta t}{2}\|\psi^n + \psi^{n-1}\|^2_{\mathcal{F}} \\
=&\Delta t(\frac{\psi^n - \psi^{n-1}}{\Delta t}, \psi^n + \psi^{n-1})_{L_2(\Omega)} + \Delta t \mathbf{a}_h(\frac{\psi^n + \psi^{n-1}}{2}, \psi^n + \psi^{n-1}) \\
=&\Delta t(g^n, \psi^n + \psi^{n-1})_{L_2(\Omega)}
\end{aligned}$$

As $\frac{\Delta t}{2}\|\psi^n + \psi^{n-1}\|^2_{\mathcal{F}}$ is positive, the estimate

$$\|\psi^n\|^2_{L_2(\Omega)} - \|\psi^{n-1}\|^2_{L_2(\Omega)} \leq \Delta t\|g^n\|_{L_2(\Omega)}(\|\psi^n\|_{L_2(\Omega)} + \|\psi^{n-1}\|_{L_2(\Omega)})$$

holds due to the Cauchy-Schwarz inequality. Dividing through the common factor $(\|\psi^n\|_{L_2(\Omega)} + \|\psi^{n-1}\|_{L_2(\Omega)})$ and adding $\|\psi^{n-1}\|_{L_2(\Omega)}$ yields

$$\|\psi^n\|_{L_2(\Omega)} \leq \|\psi^{n-1}\|_{L_2(\Omega)} + \Delta t\|g^n\|_{L_2(\Omega)}$$

Estimate (3.39) follows via mathematical induction. □

THEOREM 3.15. *For $N \in \mathbb{N}$ let $\{\phi_h^n\}_{n=0}^N$, be the solution of* (2.11) *and ϕ the exact solution of* (2.2)*. Assume that ϕ is sufficiently smooth. Then,*

$$\|\phi_h^n - \phi(t^n)\|_{L_2(\Omega)} \leq C(h^{k+\frac{1}{2}} + \Delta t^2). \tag{3.40}$$

Proof. From

$$\phi_h^n - \phi(t^n) = \underbrace{\phi_h^n - \boldsymbol{\pi}_h^{DG}\phi(t^n)}_{:=\theta^n} + \underbrace{\boldsymbol{\pi}_h^{DG}\phi(t^n) - \phi(t^n)}_{:=\eta^n}$$

we obtain

$$\|\phi_h^n - \phi(t^n)\|_{L_2(\Omega)} \leq \|\theta^n\|_{L_2(\Omega)} + \|\eta^n\|_{L_2(\Omega)}$$

The estimate for $\|\eta^n\|_{L_2(\Omega)}$ is given by Theorem 3.13 as $\|\cdot\|_{L_2(\Omega)} \leq |||\cdot|||$. Hence, $\|\eta^n\|_{L_2(\Omega)} \leq Ch^{k+\frac{1}{2}}$.
To estimate $\|\theta^n\|_{L_2(\Omega)}$ we first derive a sequence $\{\omega^n\}_{n=1}^N$ such that

$$(\frac{\theta^n - \theta^{n-1}}{\Delta t}, v_h)_{L_2(\Omega)} + \mathbf{a}_h(\frac{\theta^n + \theta^{n-1}}{2}, v_h) = (\omega^n, v_h)_{L_2(\Omega)} \quad \forall v_h \in V_h^{k,DG},\ n = 1, ..., N$$

holds. Furthermore, we use the decomposition $\omega^n = \omega_1^n + \omega_2^n + \omega_3^n$ and derive estimates for each summand ω_j^n, $j = 1, 2, 3$ in terms of time-derivatives of the exact solution ϕ. Analog approaches can be found in [Tho97], Chap. 1 and [Bur10].

$$\begin{aligned}
&(\frac{\theta^n - \theta^{n-1}}{\Delta t}, v_h)_{L_2(\Omega)} + \mathbf{a}_h(\frac{\theta^n + \theta^{n-1}}{2}, v_h) \\
\overset{\text{Def. of } \boldsymbol{\pi}_h^{DG}}{=}&(-\frac{\boldsymbol{\pi}_h^{DG}\phi(t^n) - \boldsymbol{\pi}_h^{DG}\phi(t^{n-1})}{\Delta t} + \frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}, v_h)_{L_2(\Omega)} \\
&+ \mathbf{a}_h(\frac{\phi_h^n + \phi_h^{n-1}}{2} - \frac{\phi(t^n) + \phi(t^{n-1})}{2}, v_h)
\end{aligned}$$

By adding and subtracting $\frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t}$, we obtain

$$\begin{aligned}
&(\frac{\theta^n-\theta^{n-1}}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\frac{\theta^n+\theta^{n-1}}{2}, v_h) \\
&= (-\frac{\boldsymbol{\pi}_h^{DG}\phi(t^n) - \boldsymbol{\pi}_h^{DG}\phi(t^{n-1})}{\Delta t} + \frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} \\
&+ (\frac{\phi_h^n - \phi_h^{n-1}}{\Delta t} - \frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} \\
&+ \mathbf{a}_h(\frac{\phi_h^n + \phi_h^{n-1}}{2} - \frac{\phi(t^n)+\phi(t^{n-1})}{2}, v_h).
\end{aligned}$$

Now we use the consistency of the method, i.e. for the exact solution ϕ

$$\begin{aligned}
&(\frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\frac{\phi_h^n + \phi_h^{n-1}}{2}, v_h) = (\frac{1}{2}(f(t^n)+f(t^{n-1})), v_h)_{\mathrm{L}_2(\Omega)} \\
&= (f(t^{n-\frac{1}{2}}), v_h)_{\mathrm{L}_2(\Omega)} = (\frac{\partial}{\partial t}\phi(t^{n-\frac{1}{2}}), v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\phi(t^{n-\frac{1}{2}}), v_h)
\end{aligned}$$

holds for all $v_h \in \mathrm{V}_h^{k,DG}$. Note that ϕ is continuous. Thus, the jumps are zero and the inner boundary integrals vanish. We use the alternative representation of $\mathbf{a}_h$ (3.10) to verify that $\mathbf{a}_h(\phi, v_h) = (\mathbf{u}\cdot\nabla\phi, v_h)_{\mathrm{L}_2(\Omega)}$. Applying this result yields

$$\begin{aligned}
&(\frac{\theta^n-\theta^{n-1}}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} + \mathbf{a}_h(\frac{\theta^n+\theta^{n-1}}{2}, v_h) \\
&(-\frac{\boldsymbol{\pi}_h^{DG}\phi(t^n) - \boldsymbol{\pi}_h^{DG}\phi(t^{n-1})}{\Delta t} + \frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} \\
&+ (\frac{\partial}{\partial t}\phi(t^{n-\frac{1}{2}}) - \frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t}, v_h)_{\mathrm{L}_2(\Omega)} \\
&+ \mathbf{a}_h(\phi(t^{n-\frac{1}{2}}) - \frac{\phi(t^n)+\phi(t^{n-1})}{2}, v_h).
\end{aligned}$$

Thus,

$$\begin{aligned}
\omega_1^n =& -(\boldsymbol{\pi}_h^{DG} - \mathrm{id})\frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t} \\
\omega_2^n =& -(\frac{\phi(t^n)-\phi(t^{n-1})}{\Delta t} - \frac{\partial}{\partial t}\phi(t^{n+\frac{1}{2}})) \\
\omega_3^n =& \mathbf{u}\cdot\nabla\phi(t^{n-\frac{1}{2}}) - \frac{1}{2}\mathbf{u}\cdot\nabla\phi(t^n) - \frac{1}{2}\mathbf{u}\cdot\nabla\phi(t^{n-1}).
\end{aligned}$$

Now, we can apply Lemma 3.14, i.e.

$$\begin{aligned}
\|\theta^n\|_{\mathrm{L}_2(\Omega)} \leq& \|\theta^0\|_{\mathrm{L}_2(\Omega)} + \Delta t \sum_{m=1}^{n} \|\omega^m\|_{\mathrm{L}_2(\Omega)} \\
\leq& \Delta t \sum_{m=1}^{n} (\|\omega_1^m\|_{\mathrm{L}_2(\Omega)} + \|\omega_2^m\|_{\mathrm{L}_2(\Omega)} + \|\omega_3^m\|_{\mathrm{L}_2(\Omega)})
\end{aligned} \tag{3.41}$$

as $\theta^0 = 0$ due to the definition of ϕ_h^0. We derive estimates for $\Delta t \sum_{m=1}^{n} \|\omega_i^m\|_{\mathrm{L}_2(\Omega)}$, $i = 1, 2, 3$.

$$\begin{aligned}
\Delta t\|\omega_1^m\|_{\mathrm{L}_2(\Omega)} =& \|(\boldsymbol{\pi}_h^{DG} - \mathrm{id})(\phi(t^m) - \phi(t^{m-1}))\|_{\mathrm{L}_2(\Omega)} \\
\leq& Ch^{k+\frac{1}{2}}\|\phi(t^m) - \phi(t^{m-1})\|_{\mathrm{H}^{k+1}(\Omega)}
\end{aligned}$$

due to Theorem 3.13. Furthermore,

$$\phi(t^m) - \phi(t^{m-1}) = \int_{t^{m-1}}^{t^m} \frac{\partial}{\partial t}\phi(s)\, ds$$

Thus,

$$\Delta t \sum_{m=1}^{n} \|\omega_1^m\|_{\mathrm{L}_2(\Omega)} \leq ch^{k+\frac{1}{2}} \int_{t^0}^{t^n} \|\frac{\partial}{\partial t}\phi(s)\|_{\mathrm{H}^{k+1}(\Omega)}\, ds \leq ch^{k+\frac{1}{2}}$$

Define

$$g_2(t) := \frac{1}{2}((t - t^{m-1})^2 \chi_{[t^{m-1}, t^{m-\frac{1}{2}}]} + (t - t^m)^2 \chi_{[t^{n-\frac{1}{2}}, t^m]})$$

Then,

$$\omega_2^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} g_2(s) \frac{\partial^3}{\partial t^3}\phi(s)\, ds$$

As $|g_2(t)| \leq \frac{1}{4}\Delta t^2$, we get

$$\Delta t \sum_{m=1}^{n} \|\omega_2^m\|_{\mathrm{L}_2(\Omega)} \leq \frac{1}{4}\Delta t^2 \int_{t^{m-1}}^{t^m} \|\frac{\partial^3}{\partial t^3}\phi(s)\|_{\mathrm{L}_2(\Omega)}\, ds \leq C\Delta t^2$$

To estimate ω_3^m define $g_3(t) := \frac{1}{2}(-(t - t^{m-1})\chi_{[t^{m-1}, t^{m-\frac{1}{2}}]} + (t - t^m)\chi_{[t^{m-\frac{1}{2}}, t^m]})$.
As the velocity field does not depend on time, we can switch the time-derivative and the operator $\mathbf{u} \cdot \nabla$. Thus,

$$\begin{aligned}\|\omega_3^m\|_{\mathrm{L}_2(\Omega)} &= \| \int_{t^m - 1}^{t^m} g_3(s)\mathbf{u} \cdot \nabla \frac{\partial^2}{\partial t^2}\phi(s)\, ds\|_{\mathrm{L}_2(\Omega)} \\ &\leq \int_{t^m - 1}^{t^m} \|g_3(s)\mathbf{u} \cdot \nabla \frac{\partial^2}{\partial t^2}\phi(s)\|_{\mathrm{L}_2(\Omega)}\, ds\end{aligned}$$

As $|g_3(t)| \leq \Delta t$, we get

$$\Delta t \|\omega_3^m\|_{\mathrm{L}_2(\Omega)} \leq \Delta t^2 \int_{t^m - 1}^{t^m} \|\mathbf{u} \cdot \nabla \frac{\partial^2}{\partial t^2}\phi(s)\|_{\mathrm{L}_2(\Omega)}\, ds$$

And thus,

$$\Delta t \sum_{m=1}^{n} \|\omega_3^m\|_{\mathrm{L}_2(\Omega)} \leq \Delta t^2 \int_{t^0}^{t^n} \|\mathbf{u} \cdot \nabla \frac{\partial^2}{\partial t^2}\phi(s)\|_{\mathrm{L}_2(\Omega)}\, ds \leq C\Delta t^2$$

Combining these results yields

$$\Delta t \sum_{m=1}^{n} \|\omega^n\|_{\mathrm{L}_2(\Omega)} \leq ch^{k+\frac{1}{2}} + C\Delta t^2$$

and, thus, (3.40). □

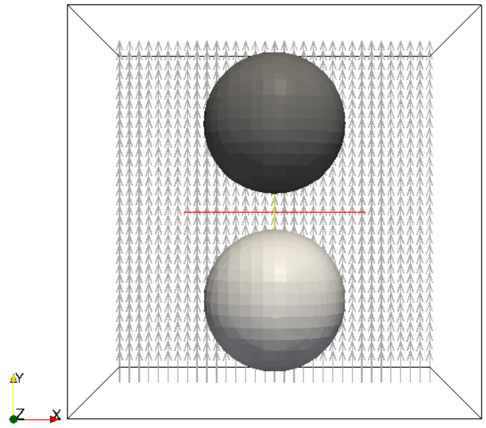

Fig. 4.1: initial (light gray) and final (dark gray) position of interface

4. Numerical Results. The DG method (A.1) in space and the Crank-Nicolson scheme in time is applied to the original level set equation (2.1) to solve a test problem.

The computational domain is the unit cube $[0,1]^3$. The initial interface $\Gamma(0)$ is the sphere $\partial B_r(m(0))$ with the radius $r = 0.2$ and center $m(0) = (0.5, 0.25, 0.5)$. The initial level set function ϕ_0 is the signed distance function to $\Gamma(0)$ and the boundary conditions are $\phi(x,t) = \phi_0(x)$ on $\partial\Omega_{in}$. The velocity field is the constant translation in x_2-direction $\mathbf{u} = (0, 0.2, 0)$. We compute the movement of the level set function until $T = 2.5$. Thus, the exact solution at T is given by the signed distance function to $\Gamma(T) = \partial B_r(m(T))$ with $m(T) = (0.5, 0.75, 0.5)$ in $U(T) = \Omega \setminus \Omega_{\text{in}}(T)$ and ϕ_0 in $\Omega_{\text{in}}(T)$. $\Omega_{\text{in}}(T)$ denotes the subset of Ω where the boundary conditions influence the solution, i.e. $[0,1] \times [0, 0.5] \times [0.1]$.

For the spatial discretization the cube Ω is divided into 8^3 sub-cubes and each sub-cube in 6 tetrahedra. This gird is regularly refined and $l = 0, 1, 2, 3$ gives the level of refinement. The initial numerical level set function $\phi_{0,h}$ is the L_2-projection of the signed distance function.

The theoretical estimate for the discretization error holds only for a sufficiently smooth exact solution. As in this example the exact solution is not globally smooth, we compute the discretization error in a subset $U(T)$ of Ω, where it is smooth.

The discretization error is computed in the L_2-norm $\|\cdot\|_{\mathcal{T}_h}$. The discretization error at $t = 0$ is the L_2-projection error, and thus, of order three for quadratic finite elements. We compute the discretization error at $t = 0$ in a subset $U(0) \subset \Omega$ which has the same structure and size as $U(T)$ and covers a part of $\phi_{0,h}$ that is comparable to the part of $\phi_h(T)$ that is covered by $U(T)$. Hence we can compare the discretization error at $t = T$ to the discretization error at $t = 0$. Table 4.1 shows the discretization error at $t = 0$ and $t = T$ for different levels of refinement. The time step size is chosen small enough such that only the spatial error is observed. The initial error is almost of order 3 which reflects the L_2-projection error as the polynomial degree is 2. At $t = T$ the discretization error is approximately 5 times bigger.The order is 3 again. Although only an order of 2.5 is obtained theoretically, in practice the optimal order is observed often. In this case this might be due to the simple, constant velocity field and the convex shape of the initial interface. To examine the order of the time

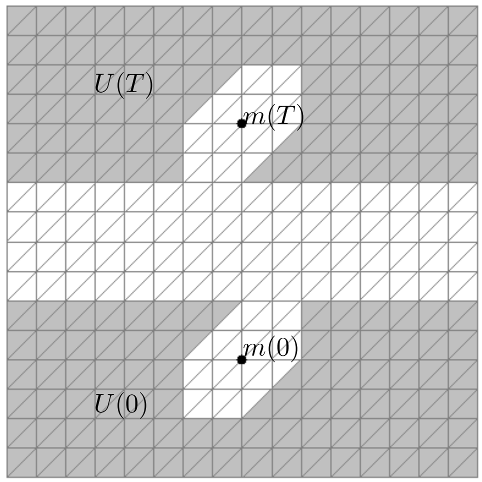

Fig. 4.2: 2D sketch of $U(0)$ and $U(T)$ for the grid of level 1

	$t=0$		$t=T$	
level	error	order	error	order
0	3.9114 e-5	-	2.0653 e-4	-
1	5.5861 e-6	2.81	2.1782 e-5	3.25
2	7.0731 e-7	2.98	2.6894 e-6	3.02
3	8.8908 e-8	2.99	e-	

Table 4.1: Discretization error $\|\phi_h - \phi\|_{L_2(U(t))}$

integration we fix the spatial resolution to $h(2) = \frac{1}{32}$. We halve the time step size Δt several times until Δt is smaller than $h^{1\frac{1}{4}}$. Table 4.2 shows the discretization error at $t = T$. We observe the order Δt^2 as long as Δt is bigger than $1,19 \cdot h^{1\frac{1}{4}}$. For smaller Δt the discretization error converges to the corresponding value in Table 4.1, where a very small time step size was used.

Δt	error	order
$\frac{1}{4}$	7.6183 e-4	-
$\frac{1}{8}$	1.8917 e-4	2.01
$\frac{1}{16}$	4.5304 e-5	2.06
$\frac{1}{32}$	1.1548 e-5	1.97
$\frac{1}{64}$	3.8799 e-6	1.57
$\frac{1}{128}$	2.7737 e-6	0.48

Table 4.2: Discretization error $\|\phi_h - \phi\|_{L_2(U(T))}$, $h = \frac{1}{32}$

Appendix A. Appendix. Analog to the derivation of the DG method for (2.2), one can derive a DG method for problem (2.1). This DG discretization of the level set equation is already known in the literature, e.g. it was used in [MRC06, PFP06,

FK08]. With the notation introduced above this DG-level set method is:
find $\phi_h \in \mathcal{C}^1([0,T];\mathrm{V}_h^{k,DG})$ such that for all $v_h \in \mathrm{V}_h^{k,DG}$

$$(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\phi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\phi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}\phi_h}^{\mathrm{uw}}, v_h)_{\mathcal{F}^0} = -(\phi_{D,h}, v_h)_{\mathcal{F}^{\partial_{in}}}, \tag{A.1}$$

where ϕ_h is *not* zero on the inflow boundary, but fulfills $\phi_h|_{\partial\Omega_{in}} = \phi_{D,h}$. We proof that the two DG methods based on (2.1) and (2.2), respectively, are equivalent. Let $\overline{\phi_{D,h}} \in \mathrm{V}_h^{k,DG}$ be a continuous approximation of the extension $\overline{\phi_D}$. For instance, take the nodal interpolation, where the values are $\phi_{D,h}$ in the nodes, that lie on the inflow boundary, and zero in the remaining nodes. Consider the DG method derived of the transformed problem:

$$\begin{aligned}
&(\frac{\partial}{\partial t}\Psi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\Psi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\Psi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}\Psi_h}^{\mathrm{uw}}, v_h)_{\mathcal{F}^0}\\
=&-(\frac{\partial}{\partial t}\overline{\phi_{D,h}} + \mathbf{u}\cdot\nabla\overline{\phi_{D,h}}, v_h)_{\mathrm{L}_2(\Omega)}\\
&\phi_h := \Psi_h + \overline{\phi_{D,h}}
\end{aligned}$$

Now, we substitute Ψ_h by $\phi_h - \overline{\phi_{D,h}}$, use the definition of the upwind flux and integrate the divergence term in the right hand side by parts:

$$\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h - \overline{\phi_{D,h}}, v_h)_{\mathrm{L}_2(\Omega)} - (\phi_h - \overline{\phi_{D,h}}, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h}\\
&+(\phi_h - \overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}(\phi_h - \overline{\phi_{D,h}})}^{\mathrm{uw}}, v_h)_{\mathcal{F}^0}\\
=&(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\phi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\phi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}}\\
&-(\frac{\partial}{\partial t}\overline{\phi_{D,h}}, v_h)_{\mathrm{L}_2(\Omega)} + (\overline{\phi_{D,h}}, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h}\\
&-(\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![v_h]\!]\,\{\phi_h - \overline{\phi_{D,h}}\} + \frac{1}{2}|\mathbf{u_n}|\,[\![v_h]\!]\cdot[\![\phi_h - \overline{\phi_{D,h}}]\!]\;ds\\
=&-(\frac{\partial}{\partial t}\overline{\phi_{D,h}} + \mathbf{u}\cdot\nabla\overline{\phi_{D,h}}, v_h)_{\mathrm{L}_2(\Omega)}\\
=&-(\frac{\partial}{\partial t}\overline{\phi_{D,h}}, v_h)_{\mathrm{L}_2(\Omega)} + (\overline{\phi_{D,h}}, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} - \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![\overline{\phi_{D,h}}v_h]\!]\;ds\\
&-(\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} - (\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{in}}}
\end{aligned}$$

The volume integrals and the integrals over the outflow boundary for $\overline{\phi_{D,h}}$ cancel, thus we obtain:

$$\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\phi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\phi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}}\\
&+\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![v_h]\!]\,\{\phi_h - \overline{\phi_{D,h}}\} + \frac{1}{2}|\mathbf{u_n}|\,[\![v_h]\!]\cdot[\![\phi_h - \overline{\phi_{D,h}}]\!]\;ds\\
=&-\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![\overline{\phi_{D,h}}v_h]\!]\;ds - (\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{in}}}.
\end{aligned}$$

As $\overline{\phi_{D,h}}$ is continuous and the jump and the average are linear, the method can be reduced to

$$
\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathcal{T}_h} - (\phi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\phi_h, v_h \mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} \\
&+ \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot [\![v_h]\!]\,\{\phi_h\} + \frac{1}{2}|\mathbf{u_n}|\,[\![v_h]\!]\cdot[\![\phi_h]\!]\,ds - \sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![v_h]\!]\,\overline{\phi_{D,h}}\,ds \\
&= -\sum_{e\in\mathcal{F}^0}\int_e \mathbf{u}\cdot[\![v_h]\!]\,\overline{\phi_{D,h}}\,ds - (\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{in}}},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, v_h)_{\mathrm{L}_2(\Omega)} - (\phi_h, \mathbf{u}\cdot\nabla v_h)_{\mathcal{T}_h} + (\phi_h, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{out}}} + (\widehat{\mathbf{u}\phi_h}^{\mathrm{uw}}, v_h)_{\mathcal{F}^0} \\
&= -(\overline{\phi_{D,h}}, v_h\mathbf{u_n})_{\mathcal{F}^{\partial_{in}}}.
\end{aligned}
$$

Due to the definition of the upwind flux this is exactly the DG discretization based on the original formulation of the level set equation.

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