# High-Order Accurate, Fully Discrete Entropy Stable Schemes for Scalar Conservation Laws

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# HIGH-ORDER ACCURATE, FULLY DISCRETE ENTROPY STABLE SCHEMES FOR SCALAR CONSERVATION LAWS

H. ZAKERZADEH* AND U. S. FJORDHOLM†

**Abstract.** The recently developed TECNO schemes for hyperbolic conservation laws are designed to be high-order accurate and entropy stable, but are, as of yet, only semi-discrete. We perform an explicit discretization of the temporal derivative to obtain a fully discrete scheme, and derive a non-strict CFL condition that ensures global entropy stability. The scheme is tested in a series of numerical experiments.

## 1. Introduction.

We consider scalar conservation laws

$$
\begin{aligned}
u_t + \nabla \cdot f(u) &= 0 \\
u(x,0) &= u_0(x),
\end{aligned}
\tag{1.1}
$$

where the unknown $u = u(x,t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is the *conserved variable* and $f : \mathbb{R} \to \mathbb{R}^N$ is the *flux function*. The theory of well-posedness of scalar conservation laws is well-developed, and we give here a brief overview of its main features. Following characteristics of (1.1), it is seen that discontinuities will develop in finite time, regardless of the regularity of the initial data. Thus, (1.1) is interpreted in the distributional sense: we say that a function $u \in L^\infty(\mathbb{R}^N \times \mathbb{R})$ is a *weak solution* of (1.1) if it satisfies (1.1) in the sense of distributions. By the lack of uniqueness of weak solutions, an additional selection criterion must be enforced. We say that a weak solution $u$ is an *entropy solution* provided

$$
\eta(u)_t + \nabla \cdot q(u) \leqslant 0 \tag{1.2}
$$

in the sense of distributions for all entropy pairs $(\eta, q)$, that is, all pairs of functions $(\eta, q)$ such that $\eta$ is (strictly) convex and $q' = \eta' f'$. The entropy condition implies stability with respect to initial data, and in particular, uniqueness [7].

In numerical approximations of (1.1), it is natural to require that an analogue of the entropy condition (1.2) is satisfied. Such schemes are called *entropy stable*, and were first studied by Lax [8]. There, he established that the Lax-Friedrich's method is entropy stable, and as a corollary that if the method converges point-wise, then the limit is the unique entropy solution. This result was generalized by Harten, Hyman and Lax in 1976, who showed that all monotone schemes for scalar conservation laws are entropy stable [6]. Osher developed in his 1984 paper [10] the theory of E-schemes, of which monotone methods are a subset. E-schemes are designed precisely to be entropy stable for all entropy pairs, and since they are total variation diminishing, they converge to the entropy solution.

E-schemes, which enforce entropy stability for all entropy pairs, are at most first-order accurate [10]. Hence, to construct higher order accurate entropy stable schemes, we must restrict focus to a limited number of entropy pairs. This was done by Tadmor [14] for fully explicit first-order, and certain second-order, schemes. Fjordholm, Mishra and Tadmor [2] introduced the semi-discrete, arbitrarily high-order accurate, entropy stable *TECNO* schemes.

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The goal of the present paper is to derive CFL conditions which guarantee entropy stability for fully discrete TECNO schemes of arbitrary order of accuracy. Our result relies on a subtle balance between spatial diffusion and temporal anti-diffusion, and upon a conjecture on the ENO reconstruction first proposed in [1, Section 5.5] (see also Conjecture 2.4 below). The work presented here was initiated in the first author's M.Sc. thesis [15].

**2. Entropy stability for semi-discrete schemes.** In this section we review entropy stability for semi-discrete schemes. For notational simplicity we shall restrict ourselves to the one-dimensional conservation law (1.1) with $N = 1$ for the rest of the paper. For $\Delta x > 0$, let $x_j := j\Delta x$ and $x_{j+1/2} := \frac{x_j + x_{j+1}}{2}$, and define the computational cell $\mathcal{C}_j := [x_{j-1/2}, x_{j+1/2})$. The conservation law is discretized using a consistent and conservative scheme,

$$\frac{d}{dt}u_j + \frac{1}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right) = 0, \tag{2.1}$$

where the *numerical flux function* $F_{j+1/2} = F(u_{j-p+1}, \dots, u_{j+p})$ (for some $p \in \mathbb{N}$) is assumed to satisfy the consistency condition $F(u, \dots, u) = f(u)$ for all $u \in \mathbb{R}$. The unknown $u_j(t)$ is an approximation of $u(x_j, t)$ for finite difference schemes, and of $\frac{1}{\Delta x}\int_{\mathcal{C}_j} u(x,t)dx$ for finite volume schemes. For the time being we leave the time derivative undiscretized.

To ensure that the scheme converges towards the correct weak solution – if it at all converges – it must satisfy a discrete version of the entropy inequality (1.2). If $(\eta, q)$ is an entropy pair, then we say that the scheme (1.1) is *entropy stable* for $(\eta, q)$ provided that there is some numerical entropy flux $Q_{j+1/2} = Q(u_{j-p+1}, \dots, u_{j+p})$ such that the computed solution $u_j(t)$ satisfies

$$\frac{d}{dt}\eta(u_j) + \frac{1}{\Delta x}\left(Q_{j+1/2} - Q_{j-1/2}\right) \leqslant 0, \tag{2.2}$$

a discrete version of (1.2). If the computed solution is, say, compactly supported, then summing (2.2) over all $j \in \mathbb{Z}$ gives

$$\frac{d}{dt}\sum_j \eta(u_j)\Delta x \leqslant 0. \tag{2.3}$$

We say that the semi-discrete scheme (2.1) is *globally entropy stable* if (2.3) holds.

**2.1. Entropy conservative schemes.** Tadmor [12, 13] quantified the required amount of numerical diffusion needed for entropy stability for a single entropy pair $(\eta, q)$, through a comparison principle: a scheme is entropy stable precisely when it contains more diffusion than an *entropy conservative* scheme. A scheme

$$\frac{d}{dt}u_j + \frac{1}{\Delta x}\left(\tilde{F}_{j+1/2} - \tilde{F}_{j-1/2}\right) = 0 \tag{2.4}$$

is *entropy conservative* if computed solutions $u_j(t)$ satisfy the discrete entropy *equality*

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(\tilde{Q}_{j+1/2} - \tilde{Q}_{j-1/2}\right) = 0 \tag{2.5}$$

for some numerical entropy flux $\tilde{Q}_{j+1/2}$.

For a fixed entropy pair $(\eta, q)$, define the *entropy variables* as $v(u) := \eta'(u)$. Since $\eta$ is strictly convex, the mapping $u \mapsto v(u)$ is invertible, and we can write $u$ as a function of $v$. The *entropy potential* is defined as $\psi(v) := vf(u(v)) - q(u(v))$ via *Legendre transformation.* We will write $v_j = v(u_j)$ and $\psi_j = \psi(v_j)$, and denote the jump at a cell interface as $[\![v]\!]_{j+1/2} = v_{j+1} - v_i$, etc.

THEOREM 2.1 (Tadmor [13]). *The scheme* (2.4) *is entropy conservative if*

$$[\![v]\!]_{j+1/2}\tilde{F}_{j+1/2} = [\![\psi]\!]_{j+1/2}.$$

*Furthermore, the scheme is second-order accurate.*

The existence of second-order accurate entropy conservative fluxes was shown in [13] by means of a phase space integral. Lefloch et al. in [9] showed that by a linear combination of entropy conservative fluxes over an extended stencil, a $2p$-th order of accuracy is achievable for any $p \in \mathbb{N}$. Specifically, they showed that if $\tilde{F}^{2p}_{j+1/2}$ is defined as

$$\tilde{F}^{2p}_{j+1/2} := \sum_{r=1}^{p} \alpha_{p,r} \sum_{s=0}^{r-1} \tilde{F}(v_{j-s}, v_{j-s+r}), \tag{2.6}$$

where the constants $\alpha_{p,r}$ satisfy

$$\sum_{r=1}^{p} r\alpha_{p,r} = 1, \qquad \sum_{r=1}^{p} r^{2s-1}\alpha_{p,r} = 0, \quad (s = 2, \ldots, p), \tag{2.7}$$

then the finite difference scheme with flux $\tilde{F}^{2p}$ is entropy conservative and $2p$-th order accurate.

**2.2. Entropy stable schemes.** Due to the lack of numerical diffusion, entropy conservative schemes will in general not converge to the entropy solution. To obtain consistency with the entropy condition, numerical diffusion must be added, as quantified in the following theorem.

THEOREM 2.2 (Tadmor [13]). *The scheme* (2.1) *is entropy stable if there are an entropy conservative flux* $\tilde{F}$ *and a number* $d_{\min} > 0$ *such that for each* $j$, $d_{j+1/2} \geqslant d_{\min}$ *and*

$$F_{j+1/2} = \tilde{F}_{j+1/2} - d_{j+1/2}[\![v]\!]_{j+1/2}. \tag{2.8}$$

Numerical schemes with flux of the form (2.8) will in general be only first-order accurate, due to the jump $[\![v]\!]_{j+1/2} \sim \Delta x$ in the numerical diffusion. We describe next a procedure to obtain arbitrarily high-order accurate entropy stable schemes.

Fix an integer $k > 1$ and choose $p := \lceil \frac{k}{2} \rceil$ so that $2p \geqslant k$. We perform a $k$-th order ENO (Essentially Non-Oscillatory) reconstruction of entropy variables $v_j$, resulting in a piecewise $(k-1)$-th order polynomial $v_j(x)$ for $x \in \mathcal{C}_j$ (see [5, 11]). Denoting the cell interface values $v_j^{\pm} := v_j(x_{j\pm1/2})$ and the cell interface jump $\langle\!\langle v \rangle\!\rangle_{j+1/2} := v_{j+1}^{-} - v_j^{+}$, we now define the numerical flux

$$F^k_{j+1/2} = \tilde{F}^{2p}_{j+1/2} - d_{j+1/2}\langle\!\langle v \rangle\!\rangle_{j+1/2}, \tag{2.9}$$

cf. (2.8). The resulting semi-discrete scheme is thus

$$\frac{d}{dt}u_j + \frac{1}{\Delta x}\left(\tilde{F}^{2p}_{j+1/2} - \tilde{F}^{2p}_{j-1/2}\right) = \frac{1}{\Delta x}\left(d_{j+1/2}\langle\!\langle v \rangle\!\rangle_{j+1/2} - d_{j-1/2}\langle\!\langle v \rangle\!\rangle_{j-1/2}\right). \tag{2.10}$$

We call this scheme the ($k$-th order) *TECNO scheme*. See [1, 2] for further details.

In [3] Fjordholm et al. showed that the ENO reconstruction procedure satisfies the so-called *sign property*:

$$\langle\!\langle v\rangle\!\rangle_{j+1/2}[\![v]\!]_{j+1/2} \geqslant 0 \qquad \forall j, \tag{2.11}$$

or in other words, the cell interface jump $v_{j+1}^- - v_j^+$ has the same sign as the cell mean value jump $v_{j+1} - v_j$. Moreover, there is a $C > 0$, only depending on $k$, such that

$$\frac{\langle\!\langle v\rangle\!\rangle_{j+1/2}}{[\![v]\!]_{j+1/2}} \leqslant C. \tag{2.12}$$

From the sign property (2.11) and the positivity of $d_{j+1/2}$, it follows that the TECNO scheme is entropy stable.

THEOREM 2.3 ([1, 2]). *The $k$-th order TECNO scheme is*

*(i) formally $k$-th order accurate*

*(ii) entropy stable; it satisfies*

$$\frac{d}{dt}\eta(u_j)+\frac{Q^k_{j+1/2}-Q^k_{j-1/2}}{\Delta x} = -\frac{d_{j+1/2}[\![v]\!]_{j+1/2}\langle\!\langle v\rangle\!\rangle_{j+1/2}+d_{j-1/2}[\![v]\!]_{j-1/2}\langle\!\langle v\rangle\!\rangle_{j-1/2}}{2\Delta x} \leqslant 0 \tag{2.13}$$

*for a numerical entropy flux function $Q^k_{j+1/2}$ consistent with $q$.*

Summing and integrating (2.13) over $j \in \mathbb{Z}$, $t \in [0, T]$, we find that

$$\sum_j \eta(u_j(T))\Delta x = \sum_j \eta(u_j(0))\Delta x - \int_0^T \sum_j d_{j+1/2}[\![v]\!]_{j+1/2}\langle\!\langle v\rangle\!\rangle_{j+1/2}\, dt.$$

From the ENO sign property (2.11) it then follows that $\sum_j \eta(u_j(T))\Delta x \leqslant \sum_j \eta(u_j(0))\Delta x$, and moreover,

$$\int_0^\infty \sum_j d_{j+1/2}[\![v]\!]_{j+1/2}\langle\!\langle v\rangle\!\rangle_{j+1/2}\, dt \leqslant \sum_j \eta(u_j(0))\Delta x. \tag{2.14}$$

**2.3. The ENO conjecture.** The estimate (2.12) gives an upper bound on the ENO reconstructed cell interface jump $v_{j+1}^- - v_j^+$ in terms of the jumps $v_{j+1} - v_j$. The opposite relation is *not* true in general, as $\langle\!\langle v\rangle\!\rangle_{j+1/2}$ can vanish where $v_{j+1}^- - v_j^+$ is non-zero. However, we pose the following conjecture (see [1, Chapter 5] for details).

CONJECTURE 2.4. *There is a $C > 0$, only depending on $k$, such that the $k$-th order ENO reconstruction $v_j^\pm$ of a compactly supported grid function $v_j$ satisfies*

$$\sum_j |[\![v]\!]_{j+1/2}|^{k+1} \leqslant C\|v\|_\infty^{k-1} \sum_j [\![v]\!]_{j+1/2}\langle\!\langle v\rangle\!\rangle_{j+1/2}. \tag{2.15}$$

Assuming that the computed solution is $L^\infty$-bounded, the a priori bound (2.14) combined with the ENO conjecture (2.15) implies that

$$\int_0^T \sum_j |[\![v]\!]_{j+1/2}|^{k+1}\, dt \leqslant C. \tag{2.16}$$

Using this bound on the spatial variation of the computed solution, it was shown in [1] that the $k$-th order (semi-discrete) TECNO scheme converges strongly to the entropy solution.

We will not proceed with a proof of convergence for the fully discrete TECNO scheme. Instead, we derive in the next section a CFL condition which ensures that a global time-discrete version of the essential entropy stability estimate (2.3) holds. This estimate will lead to an analogous time-discrete a priori bound (2.16) on the spatial variation.

**3. Fully discrete schemes.** We proceed to discretize the time derivative in the semi-discrete scheme (2.1). This was carried out by Tadmor [14] for first-order accurate entropy stable schemes, which we write as

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{F_{j+1/2}^{n+1/2}-F_{j-1/2}^{n+1/2}}{\Delta x}=0. \tag{3.1a}$$

Here, $F_{j+1/2}^{n+1/2}$ will typically be in one of the forms

$$F_{j+1/2}^{n+1/2}=F\big(u_{j-p+1}^{n+1},\dots,u_{j+p}^{n+1}\big) \qquad \text{(implicit scheme)} \tag{3.1b}$$

$$F_{j+1/2}^{n+1/2}=F\big(u_{j-p+1}^{n},\dots,u_{j+p}^{n}\big) \qquad \text{(explicit scheme)} \tag{3.1c}$$

for some $2p$-point entropy stable numerical flux $F$.

Analogous to the semi-discrete formulation, we will say that the scheme (3.1a) is *locally entropy stable* if the computed solution satisfies

$$\frac{\eta(u_j^{n+1})-\eta(u_j^n)}{\Delta t}+\frac{Q_{j+1/2}^{n+1/2}-Q_{j-1/2}^{n+1/2}}{\Delta x}\leqslant 0 \tag{3.2}$$

for some numerical entropy flux $Q$. We say that (3.1a) is *globally entropy stable* if

$$\sum_j \eta(u_j^{n+1})\Delta x \leqslant \sum_j \eta(u_j^n)\Delta x. \tag{3.3}$$

These are discrete analogues of (2.2) and (2.3).

It was found in [14] that an implicit first-order time discretization (3.1a), (3.1b) gives an *unconditionally* locally entropy stable scheme: (3.2) holds for any choice of $\Delta t > 0$.

On the other hand, the *explicit* first-order discretization (3.1a), (3.1c) is locally entropy stable under a CFL condition $\frac{\Delta t}{\Delta x}\leqslant C = C(u^n)$ [14]. We wish to apply this stability analysis to the high-order accurate TECNO schemes (2.10).

**3.1. Implicit discretization.** The $k$-th order TECNO scheme with an implicit Euler discretization is the scheme (3.1a), (3.1b) with $F = F^k$, the TECNO flux (2.9). The proof of entropy stability of this scheme is a straightforward extension of [14, Example 7.1], but we include it here for the sake of completeness.

Defining $H(v) := \frac{du}{dv}(v) = (\eta''(u(v)))^{-1} > 0$ and $v_j(s) := v_j^n + s(v_j^{n+1} - v_j^n)$, we have

$$\begin{aligned}
\eta(u_j^{n+1})-\eta(u_j^n) &= \int_0^1 \frac{d}{ds}\eta(u(v_j(s)))\,ds = \int_0^1 v_j(s)H(v_j(s))(v_j^{n+1}-v_j^n)\,ds\\
&= v_j^{n+1}(u_j^{n+1}-u_j^n) - \left(v_j^{n+1}(u_j^{n+1}-u_j^n) - \int_0^1 v_j(s)H(v_j(s))(v_j^{n+1}-v_j^n)\,ds\right)\\
&= v_j^{n+1}(u_j^{n+1}-u_j^n) - \int_0^1 (1-s)H(v_j(s))(v_j^{n+1}-v_j^n)^2\,ds.
\end{aligned}$$

By construction (see [2, 13]), we have

$$v_j^{n+1}\frac{F_{j+1/2}^{n+1/2}-F_{j-1/2}^{n+1/2}}{\Delta x}=\frac{Q_{j+1/2}^{n+1/2}-Q_{j-1/2}^{n+1/2}}{\Delta x}+\mathcal{F}_j^{n+1},$$

where

$$\mathcal{F}_j^{n+1}:=-\frac{d_{j+1/2}^{n+1}[\![v^{n+1}]\!]_{j+1/2}\langle\!\langle v^{n+1}\rangle\!\rangle_{j+1/2}+d_{j-1/2}^{n+1}[\![v^{n+1}]\!]_{j-1/2}\langle\!\langle v^{n+1}\rangle\!\rangle_{j-1/2}}{2\Delta x}\leqslant 0 \tag{3.4}$$

is the entropy production from the spatial discretization (cf. (2.13)) and $Q_{j+1/2}^{n+1/2}=Q_{j+1/2}(u^{n+1})$ is some numerical entropy flux. Therefore,

$$\frac{\eta(u_j^{n+1})-\eta(u_j^n)}{\Delta t}+\frac{Q_{j+1/2}^{n+1/2}-Q_{j-1/2}^{n+1/2}}{\Delta x}=\mathcal{E}_j^{\mathrm{BE},n}+\mathcal{F}_j^{n+1},$$

where

$$\mathcal{E}_j^{\mathrm{BE},n}:=-\frac{1}{\Delta t}\int_0^1(1-s)H(v_j(s))(v_j^{n+1}-v_j^n)^2\;ds\leqslant 0.$$

is the entropy production from the time discretization.

Proposition 3.1. *The k-th order TECNO scheme with an implicit Euler discretization is unconditionally locally entropy stable.*

**3.2. Explicit discretization.** Unlike the implicit time discretization, an explicit time discretization may produce entropy locally. The analysis for first-order (in space) schemes relies on the fact that the spatial discretization always produces negative entropy of the order $-[\![u]\!]_{j+1/2}^2$, cf. (2.13). Thus, the time discretization may *produce entropy* of order $[\![u]\!]_{j+1/2}^2$, and still give entropy stability. For the high-order accurate TECNO scheme, however, the local numerical diffusion $d\langle\!\langle v\rangle\!\rangle$ (cf. (2.10)) might *vanish*, resulting in zero entropy diffusion from the spatial discretization. Therefore, we cannot hope for local entropy stability, unlike (3.2). Instead, we derive a CFL condition which ensures *global* entropy stability (3.3).

We consider the explicit discretization (3.1a), (3.1c) with $F=F^k$, the $k$-th order TECNO flux (2.9). Similar to the implicit discretization, we find that

$$\begin{aligned}\eta(u_j^{n+1})-\eta(u_j^n)&=\int_0^1\frac{d}{ds}\eta(u(v_j(s)))\;ds=\int_0^1 v_j(s)H(v_j(s))(v_j^{n+1}-v_j^n)\;ds\\&=v_j^n(u_j^{n+1}-u_j^n)+\left(\int_0^1 v_j(s)H(v_j(s))(v_j^{n+1}-v_j^n)\;ds-v_j^n(u_j^{n+1}-u_j^n)\right)\\&=v_j^n(u_j^{n+1}-u_j^n)+\int_0^1 sH(v_j(s))(v_j^{n+1}-v_j^n)^2\;ds.\end{aligned}$$

Thus, defining

$$\mathcal{E}_j^{\mathrm{FE},n}:=\frac{1}{\Delta t}\int_0^1 sH(v_j(s))(v_j^{n+1}-v_j^n)^2\;ds\geqslant 0,$$

we get

$$\frac{\eta(u_j^{n+1})-\eta(u_j^n)}{\Delta t}+\frac{Q_{j+1/2}^{n+1/2}-Q_{j-1/2}^{n+1/2}}{\Delta x}=\mathcal{E}_j^{\mathrm{FE},n}+\mathcal{F}_j^n, \tag{3.5}$$

where $\mathcal{F}_j^n$ is given by (3.4).

**3.3. Explicit discretization – the linear case.** For simplicity we shall first consider the linear advection equation

$$u_t + au_x = 0 \tag{3.6}$$

where $a$ is any real constant. Moreover, we let the entropy $\eta$ be given by $\eta(u) = u^2/2$. The $k$-th order TECNO scheme is then given by

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{a}{2}\sum_{i=1}^{p}\alpha_{p,i}(u_{j+i}^n - u_{j-i}^n) - d\langle\!\langle u^n\rangle\!\rangle_{j+1/2} + d\langle\!\langle u^n\rangle\!\rangle_{j-1/2}\right), \tag{3.7}$$

where we have put $d_{j+1/2} \equiv d > 0$ for some $d > 0$, to be determined.

We will denote

$$\|[\![u^n]\!]\|_{k+1} := \left(\sum_j |u_{j+1}^n - u_j^n|^{k+1}\right)^{1/(k+1)}.$$

and

$$\gamma = \gamma(u^n) := \frac{\|[\![u^n]\!]\|_{k+1}}{\|u^n\|_\infty}. \tag{3.8}$$

THEOREM 3.2. *Assume that Conjecture 2.4 is true, and that the computed solutions satisfy* $\max_{j,n}|u_j^n| \leqslant M$ *for some* $M > 0$. *Then there is a choice of* $d = d(u^n) > 0$ *and a* $C = C(u^n, k, p) > 0$ *such that if*

$$|a|\Delta t \leqslant C\gamma^{\frac{k-1}{2}}\Delta x^{1+\frac{k-1}{2(k+1)}} \tag{3.9}$$

*then the explicit $k$-th order TECNO scheme* (3.7) *is globally entropy stable, i.e. satisfies* (3.3). *In particular, a sufficient CFL condition is*

$$|a|\Delta t \leqslant \tilde{C}\Delta x^{1+\frac{k^2+k-2}{2(k+1)}}. \tag{3.10}$$

REMARK 3.3. *The proposed exponent in* (3.9), *denoted by* $\sigma$, *is shown for different choices of* $k$ *in Table 3.1.*

| Order of scheme $k$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| Exponent $\sigma$ | 1.000 | 1.1667 | 1.250 | 1.300 | 1.333 |

Table 3.1: The exponent appearing in the CFL condition (3.9), ensuring global entropy stability of fully-discrete TECNO scheme for linear equations.

*Proof.* We will estimate the sum $\sum_j \mathcal{E}_j^{\mathrm{FE},n} + \mathcal{F}_j^n$ and show that it is non-positive under the CFL condition. By our choice of $\eta$, we have $v(u) = u$ and $H(v) = 1$, whence $\mathcal{E}_j^{\mathrm{FE},n} = \frac{1}{2\Delta t}(u_j^{n+1} - u_j^n)^2$. We rewrite (3.7) as

$$\begin{aligned} u_j^{n+1} - u_j^n = &-\lambda\underbrace{\left(\frac{a}{2}\sum_{i=1}^{p}\alpha_{p,i}(u_{j+i}^n - u_j^n) - d\langle\!\langle u^n\rangle\!\rangle_{j+1/2}\right)}_{=:A_j} \\ &+\lambda\underbrace{\left(\frac{a}{2}\sum_{i=1}^{p}\alpha_{p,i}(u_{j-i}^n - u_j^n) - d\langle\!\langle u^n\rangle\!\rangle_{j-1/2}\right)}_{=:B_j}, \end{aligned}$$

where $\lambda := \frac{\Delta t}{\Delta x}$. Now, $(u_j^{n+1} - u_j^n)^2 \leqslant 2\lambda^2(A_j^2 + B_j^2)$. We reorder the above sums and use Jensen's inequality to get

$$\begin{aligned}
A_j^2 &= \left(\frac{a}{2}\sum_{i=1}^{p}\alpha_{p,i}(u_{j+i}^n - u_j^n) - d\langle\!\langle u^n\rangle\!\rangle_{j+1/2}\right)^2 \\
&= \left(\frac{a}{2}\sum_{m=0}^{p-1}\beta_m\left(u_{j+m+1}^n - u_{j+m}^n\right) - d\langle\!\langle u^n\rangle\!\rangle_{j+1/2}\right)^2 \\
&\leqslant \left(\theta\left(\frac{a}{2\theta}\sum_{m=0}^{p-1}\beta_m\left(u_{j+m+1}^n - u_{j+m}^n\right)\right)^2 + (1-\theta)\left(\frac{d}{1-\theta}\langle\!\langle u^n\rangle\!\rangle_{j+1/2}\right)^2\right) \\
&\leqslant \frac{a^2\beta}{4\theta}\sum_{m=0}^{p-1}|\beta_m|\left(u_{j+m+1}^n - u_{j+m}^n\right)^2 + \frac{d^2}{(1-\theta)}\langle\!\langle u^n\rangle\!\rangle_{j+1/2}^2.
\end{aligned}$$

where $\beta_m := \sum_{l=m+1}^{p}\alpha_{p,l}$, $\beta := \sum_{m=0}^{p-1}|\beta_m|$ and $\theta \in (0,1)$ is some number to be determined. An analogous estimate holds for $B_j^2$. Summing up, we find

$$\begin{aligned}
\sum_j \mathcal{E}_j^{\mathrm{FE},n}\Delta x &= \sum_j \frac{(u_j^{n+1} - u_j^n)^2}{2\Delta t}\Delta x \leqslant \lambda\sum_j A_j^2 + B_j^2 \\
&\leqslant \sum_j \frac{\lambda a^2\beta}{4\theta}\left(\sum_{m=0}^{p-1}|\beta_m|\left(u_{j+m+1}^n - u_{j+m}^n\right)^2 + \sum_{m=0}^{p-1}|\beta_m|\left(u_{j-m}^n - u_{j-m-1}^n\right)^2\right) \\
&\quad + \frac{\lambda d^2}{(1-\theta)}\sum_j\left(\langle\!\langle u^n\rangle\!\rangle_{j+1/2}^2 + \langle\!\langle u^n\rangle\!\rangle_{j-1/2}^2\right) \\
&= \frac{\lambda a^2\beta^2}{2\theta}\sum_j\left(u_{j+1}^n - u_j^n\right)^2 + \frac{2\lambda d^2}{(1-\theta)}\sum_j\langle\!\langle u^n\rangle\!\rangle_{j+1/2}^2.
\end{aligned}$$

Next,

$$\sum_j \mathcal{F}_j^n\Delta x = -\sum_j d[\![u^n]\!]_{j+1/2}\langle\!\langle u^n\rangle\!\rangle_{j+1/2} = -d\sum_j\left(u_{j+1}^n - u_j^n\right)\langle\!\langle u^n\rangle\!\rangle_{j+1/2}.$$

We now apply the estimates

$$\sum_j\left(u_{j+1}^n - u_j^n\right)^2 \leqslant C_1\Delta x^{-\frac{k-1}{k+1}}\|[\![u^n]\!]\|_{k+1}^2 \tag{3.11}$$

(Hölder's inequality),

$$\sum_j\langle\!\langle u^n\rangle\!\rangle_{j+1/2}^2 \leqslant C_2\sum_j\left(u_{j+1}^n - u_j^n\right)\langle\!\langle u^n\rangle\!\rangle_{j+1/2} \tag{3.12}$$

(the upper bound on ENO jumps, (2.12)) and

$$\sum_j\left(u_{j+1}^n - u_j^n\right)\langle\!\langle u^n\rangle\!\rangle_{j+1/2} \geqslant \frac{C_3}{\|u\|_\infty^{k-1}}\|[\![u^n]\!]\|_{k+1}^{k+1} \tag{3.13}$$

(the ENO conjecture (2.15)). Assuming for the moment that

$$d - C_2 \frac{2\lambda d^2}{(1-\theta)} \geqslant 0, \tag{3.14}$$

we get

$$\begin{aligned}
&\sum_j \left(\mathcal{E}_j^{\mathrm{FE},n} + \mathcal{F}_j^n\right)\Delta x \\
&\leqslant \frac{\lambda a^2\beta^2}{2\theta}\sum_j \left(u_{j+1}^n - u_j^n\right)^2 + \frac{2\lambda d^2}{(1-\theta)}\sum_j \langle\!\langle u^n\rangle\!\rangle_{j+1/2}^2 - d\sum_j \left(u_{j+1}^n - u_j^n\right)\langle\!\langle u^n\rangle\!\rangle_{j+1/2} \\
&\leqslant C_1 \Delta x^{-\frac{k-1}{k+1}} \frac{\lambda a^2\beta^2}{2\theta} \|[\![u^n]\!]\|_{k+1}^2 - \left(d - C_2\frac{2\lambda d^2}{(1-\theta)}\right)\sum_j \left(u_{j+1}^n - u_j^n\right)\langle\!\langle u^n\rangle\!\rangle_{j+1/2} \\
&\leqslant C_1 \Delta x^{-\frac{k-1}{k+1}} \frac{\lambda a^2\beta^2}{2\theta} \|[\![u^n]\!]\|_{k+1}^2 - \left(d - C_2\frac{2\lambda d^2}{(1-\theta)}\right)\frac{C_3}{\|u\|_\infty^{k-1}}\|[\![u^n]\!]\|_{k+1}^{k+1} \\
&= \|[\![u^n]\!]\|_{k+1}^2\left(\Delta t\left(C_1\frac{a^2\beta^2}{2\theta\Delta x^{\frac{2k}{k+1}}} + C_2C_3\frac{2d^2}{(1-\theta)\Delta x}\gamma^{k-1}\right) - dC_3\gamma^{k-1}\right),
\end{aligned}$$

where $\gamma$ is defined by (3.8). Choosing $\theta$ such that the coefficient of $\Delta t$ is minimized, we get

$$\sum_j \left(\mathcal{E}_j^{\mathrm{FE},n} + \mathcal{F}_j^n\right)\Delta x \leqslant \|[\![u^n]\!]\|_{k+1}^2\left(2\Delta t\left(\frac{|a|\beta\sqrt{C_1}}{2\Delta x^{\frac{k}{k+1}}} + \frac{d\gamma^{\frac{k-1}{2}}\sqrt{C_2C_3}}{\Delta x^{\frac{1}{2}}}\right)^2 - dC_3\gamma^{k-1}\right).$$

Thus, we require

$$\Delta t \leqslant \frac{C_3 d\gamma^{k-1}}{2\left(\frac{|a|\beta\sqrt{C_1}}{2\Delta x^{\frac{k}{k+1}}} + \frac{d\gamma^{\frac{k-1}{2}}\sqrt{C_2C_3}}{\Delta x^{\frac{1}{2}}}\right)^2}.$$

The right-hand side is maximized when

$$d = \frac{|a|\beta}{2}\sqrt{\frac{C_1}{C_2C_3}}\left(\frac{\Delta x^{\frac{-1}{k+1}}}{\gamma}\right)^{\frac{k-1}{2}},$$

giving the optimal CFL condition

$$\Delta t \leqslant \sqrt{\frac{C_3}{C_1C_2}}\frac{\gamma^{\frac{k-1}{2}}\Delta x^{\frac{3k+1}{2(k+1)}}}{4|a|\beta}. \tag{3.15}$$

Note that $\gamma \gtrsim \Delta x$, regardless of the value of $u^n$. Therefore, a sufficient CFL condition is

$$\Delta t \lesssim \frac{\Delta x^{\frac{k(k+3)}{2(k+1)}}}{|a|}.$$

Finally, we verify that the condition (3.14) is satisfied when $\theta$ and $d$ are chosen as above. Indeed, with these choices of $\theta$ and $d$, (3.14) can be rewritten as

$$\Delta t \leqslant \sqrt{\frac{C_3}{C_1 C_2}} \frac{\gamma^{\frac{k-1}{2}} \Delta x^{\frac{3k+1}{2(k+1)}}}{2|a|\beta},$$

which is twice the right-hand side of (3.15). □

Remark 3.4. *The CFL condition derived in [15] requires $\sigma = k$ which is much more restrictive than Theorem 3.2.*

**3.4. Extension to higher order time integration.** The generalization of the preceding analysis to higher order time integration is straightforward, provided so-called Strong Stability Preserving (SSP) Runge-Kutta time integration is applied (see [4] and the references therein). If we write the explicit Euler time-stepping (3.7) in the abstract formulation

$$u^{n+1} = u^n + \Delta t L(u^n), \tag{3.16}$$

then a $k$-th order accurate SSP Runge-Kutta method is of the form

$$u^{(0)} := u^n, \qquad u^{(i)} := \sum_{l=0}^{i-1} \delta_{il} u^{(l)} + \beta_{il} \Delta t L(u^{(l)}) \quad (i = 1, \ldots, k), \qquad u^{n+1} := u^{(k)}, \tag{3.17}$$

where $\delta_{il} > 0$, $\beta_{il} \geqslant 0$, and $\sum_{l=0}^{i-1} \delta_{il} = 1$ for consistency. Each intermediate step can be written as a convex combination of explicit Euler steps with step sizes $\frac{\beta_{il}}{\delta_{il}} \Delta t$:

$$u^{(i)} = \sum_{l=0}^{i-1} \delta_{il} \left( u^{(l)} + \Big(\frac{\beta_{il}}{\delta_{il}} \Delta t\Big) L(u^{(l)}) \right).$$

Thus, if the forward Euler method (3.16) is globally entropy stable with respect to an entropy pair $(\eta, q)$, then convexity of the entropy function gives

$$\begin{aligned}
\sum_j \eta\big(u_j^{(i)}\big) &\leqslant \sum_j \sum_{l=0}^{i-1} \delta_{il} \eta \left( u_j^{(l)} + \left(\frac{\beta_{il}}{\delta_{il}} \Delta t\right) L(u^{(l)})_j \right) \\
&\leqslant \sum_j \sum_{l=0}^{i-1} \delta_{il} \left( \sum_j \eta\big(u_j^{(l)}\big) \right) = \sum_{l=0}^{i-1} \delta_{il} \sum_j \eta\big(u_j^{(l)}\big)
\end{aligned}$$

for every $i = 1, \ldots, k$. Hence, iterating over $i = 1, \ldots, k$, we find that

$$\sum_j \eta\big(u_j^{(k)}\big) \leqslant \ldots \leqslant \sum_j \eta\big(u_j^{(0)}\big),$$

and so the method is globally entropy stable.

Theorem 3.5. *Under the same assumptions as in Theorem 3.2, there is a choice of $d = d(u^n) > 0$ and a $C = C(u^n, k, p) > 0$ such that if*

$$|a| \Delta t \leqslant C \left( \frac{\|[\![u^n]\!]\|_{k+1}}{\|u\|_\infty} \right)^{\frac{k-1}{2}} \Delta x^{1 + \frac{k-1}{2(k+1)}},$$

*then the explicit $k$-th order TECNO scheme using SSP time-stepping* (3.17), (3.7) *is globally entropy stable, i.e. satisfies* (3.3).

**4. Nonlinear equations.** Next, we generalize the approach from the previous section to general, nonlinear scalar conservation laws (1.1). Thus, consider the explicitly discretized TECNO scheme (3.1a), (3.1c) with the numerical flux $F^k$ given by (2.9). The key observation is that the $2p$-th order entropy conservative flux (2.6) is Lipschitz continuous whenever the underlying entropy conservative flux $\tilde{F}$ is Lipschitz, in the sense that

(4.1)
$$\left|\tilde{F}^{2p}(u_{j-p},\dots,u_{j+p-1}) - \tilde{F}^{2p}(u_{j-p+1},\dots,u_{j+p})\right| \leqslant C_{\mathrm{Lip}} \sum_{m=-p}^{p-1} |u_{j+m+1} - u_{j+m}|.$$

As in the linear case, we will let the entropy be given by $\eta(u) = u^2/2$, which gives entropy variable $v(u) := \eta'(u) = u$.

THEOREM 4.1. *Assume that the TECNO diffusion constant $d_{j+1/2}$ in* (2.9) *satisfies $d_{\max} \geqslant d_{j+1/2} \geqslant d_{\min} > 0$ for all $j$. Then, under the same assumptions as in Theorem 3.2, there are constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 > 0$, depending on the flux function $f$, order of the scheme $k$ and the diffusion of the numerical flux such that if*

$$\Delta t \leqslant \frac{\mathcal{C}_1 \gamma^{k-1}}{\left(\mathcal{C}_2 \Delta x^{-\frac{k}{k+1}} + \mathcal{C}_3 \gamma^{\frac{k-1}{2}} \Delta x^{-\frac{1}{2}}\right)^2} \tag{4.2}$$

*with $\gamma$ defined in* (3.8)*, then the explicit $k$-th order TECNO scheme is globally entropy stable, i.e. satisfies* (3.3).

*In particular, there is a $\mathcal{C}_4 > 0$ and a choice of $d_{\max} > 0$ such that the CFL condition*

$$\Delta t \leqslant \mathcal{C}_4 \frac{d_{\min}}{d_{\max}} \gamma^{\frac{k-1}{2}} \Delta x^{1+\frac{k-1}{2(k+1)}} \tag{4.3}$$

*for any $d_{\min} \leqslant d_{\max}$ , implies global entropy stability.*

*Proof.* From the Lipschitz condition (4.1) we find that

$$\begin{aligned}
\left|u_j^{n+1} - u_j^n\right| &= \lambda \left|\tilde{F}^{2p}_{j+1/2} - \tilde{F}^{2p}_{j-1/2} - d_{j+1/2}\langle\!\langle u^n \rangle\!\rangle_{j+1/2} + d_{j-1/2}\langle\!\langle u^n \rangle\!\rangle_{j-1/2}\right| \\
&\leqslant \lambda \left( C_{\mathrm{Lip}} \sum_{m=-p}^{p-1} |u_{j+m+1} - u_{j+m}| + d_{j+1/2}|\langle\!\langle u^n \rangle\!\rangle_{j+1/2}| + d_{j-1/2}|\langle\!\langle u^n \rangle\!\rangle_{j-1/2}| \right).
\end{aligned}$$

Through an argument very similar to the linear case, the elementary inequality $(a_1 + \dots + a_m)^2 \leqslant m(a_1^2 + \dots + a_m^2)$, and the estimates (3.11), (3.12) and (3.13) imply that

$$\begin{aligned}
\sum_j \left(\mathcal{E}_j^{\mathrm{FE},n} + \mathcal{F}_j^n\right) \Delta x \leqslant\ & \frac{\lambda}{2\theta}(2p-1)^2 C_1 C_{\mathrm{Lip}}^2 \|[\![u^n]\!]\|_{k+1}^2 \Delta x^{-\frac{k-1}{k+1}} \\
& - \left(d_{\min} - \frac{2\lambda C_2 d_{\max}^2}{(1-\theta)}\right) \sum_j [\![u^n]\!]_{j+1/2} \langle\!\langle u^n \rangle\!\rangle_{j+1/2},
\end{aligned}$$

where $d_{\min}$ and $d_{\max}$ are the minimum-positive and maximum characteristic speeds of the flux derivative in Lax-Friedrichs or Local Lax-Friedrichs sense (see [1] for different choices of $d$). Now if we claim that—analogously to (3.14)— we have

$$d_{\min} - \frac{2\lambda C_2 d_{\max}^2}{(1-\theta)} \geqslant 0. \tag{4.4}$$

It follows that

$$\sum_j (\mathcal{E}_j^{\mathrm{FE},n} + \mathcal{F}_j^n)\Delta x$$
$$\leqslant \|[\![u^n]\!]\|_{k+1}^2 \left( \Delta t \left( C_1 \frac{(2p-1)^2 C_{\mathrm{Lip}}^2 C_1}{2\theta \Delta x^{\frac{2k}{k+1}}} + C_2 C_3 \frac{2d_{max}^2}{(1-\theta)\Delta x} \gamma^{k-1} \right) - d_{\min} C_3 \gamma^{k-1} \right).$$

Again choosing $\theta$ such that the coefficient of $\Delta t$ is minimized, we get the CFL condition

$$\Delta t \leqslant \frac{d_{\min} C_3 \gamma^{k-1}}{2\left( \frac{(2p-1)C_{\mathrm{Lip}}\sqrt{C_1}}{2\Delta x^{\frac{k}{k+1}}} + \frac{\sqrt{C_2C_3}d_{\max}}{\Delta x^{1/2}} \gamma^{\frac{k-1}{2}} \right)^2}, \tag{4.5}$$

which is (4.2).

If we choose $d_{\min}$ such that $d_{\min} = \epsilon d_{\max} = \epsilon d$ for some $\epsilon \in [0,1]$ and $d > 0$, then (4.5) is minimized when

$$d = \sqrt{\frac{C_1}{C_2C_3}} \frac{(2p-1)C_{\mathrm{Lip}}}{2\epsilon} \left( \frac{\Delta x^{\frac{-1}{k+1}}}{\gamma} \right)^{\frac{k-1}{2}},$$

whence (4.5) gets

$$\Delta t \leqslant \sqrt{\frac{C_3}{C_1C_2}} \frac{\epsilon}{4(2p-1)C_{\mathrm{Lip}}} \gamma^{\frac{k-1}{2}} \Delta x^{1+\frac{k-1}{2(k+1)}}. \tag{4.6}$$

With our choice of $\theta$, the condition (4.4) becomes

$$\Delta t \leqslant \frac{d_{\min}\Delta x}{2C_2 d_{\max}^2 \left( 1 + \sqrt{\frac{C_1}{C_2C_3}} \frac{(2p-1)C_{\mathrm{Lip}}}{d_{\max}} \frac{\Delta x^{-\frac{k-1}{2(k+1)}}}{\gamma^{\frac{k-1}{2}}} \right)}. \tag{4.7}$$

Finally, we verify the assumption (4.4). For general $0 < d_{\min} \leqslant d_{\max}$, the estimate $\gamma \gtrsim \Delta x$ shows that the asymptotic behavior of (4.5) is $\Delta t \lesssim \Delta x^{\frac{k^2+2k-1}{k+1}}$, while for (4.7) it is $\Delta t \lesssim \Delta x^{\frac{5k-1}{2(k+1)}}$, which is less restrictive than the former. Hence, the CFL condition (4.2) implies that our assumption (4.4) indeed holds. □

REMARK 4.2. *The exponent appearing in the worst-case scenario* $\Delta t \lesssim \Delta x^{\frac{k^2+2k-1}{k+1}}$ *of* (4.2)*, as derived in the above proof, is presented in Table 4.1. On the other hand, the exponent in the optimal CFL condition* (4.3) *is identical to that of the linear case, cf. Table 3.1.*

| Order of scheme $k$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| Exponent $\sigma$ | 1.00 | 2.33 | 3.50 | 4.60 | 5.66 |

Table 4.1: The exponent $\sigma$ proposed by Theorem 4.1, $\Delta t \leqslant C\Delta x^{\sigma}$, for global entropy stability of the fully-discrete TECNO scheme, nonlinear equations.

REMARK 4.3. *[15] includes an asymptotic analysis for nonlinear case that proposes $\sigma = k$, the same as in the linear case (cf. Remark 3.4), before the discontinuity appears.*

| Gridpoints \ $k$ | FE | | | SSP RK2 | | |
|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 2 | 3 | 4 |
| 20 | 0.7468 | 0.9441 | 0.9780 | 0.7051 | 0.9326 | 0.9746 |
| 40 | 0.9102 | 0.9856 | 0.9970 | 0.8909 | 0.9825 | 0.9964 |
| 80 | 0.9770 | 0.9964 | 0.9995 | 0.9692 | 0.9956 | 0.9995 |

Table 5.1: The growth factor for different orders and mesh sizes at $T = 2$. Computed with the TECNO scheme using forward Euler time integration and second-order Runge-Kutta scheme applied to linear advection.

**5. Numerical experiments.** In the present section we present numerical experiments demonstrating global entropy stability for both linear and nonlinear conservation laws. The experiments are done on a periodic domain $\Omega := [-1, 1)$ for TECNO schemes of order $k = 2$, $k = 3$ and $k = 4$, using the CFL conditions proposed by Theorem 3.2 and Theorem 4.1. We consider the linear advection equation (3.6) and Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{5.1}$$

and in each case we use the initial data as

$$\begin{aligned} u_0(x) &= \sin(\pi x) && \text{(linear advection)} \\ u_0(x) &= \frac{1}{2}\big(\sin(\pi x) + 1\big) && \text{(Burger's equation)}, \end{aligned}$$

when the latter develops a shock at $t = \frac{2}{\pi} \approx 0.636$. For the square entropy function $\eta(u) = u^2/2$, global entropy stability is equivalent to stability of the solution in the 2-norm. We therefore display the *growth factor*

$$\mathcal{G}_T := \frac{\|u(T)\|_2}{\|u(0)\|_2},$$

for the numerical solution over time.

**5.1. Linear Advection.** Consider the linear advection equation (3.6) with $a = 1$. As the equation is linear, entropy should be constant in time, i.e. $\|u(T)\|_2 = \|u(0)\|_2$ for all $T > 0$. We use Theorem 3.2 to calculate the time step for global entropy stability. We set $C_1 = |\Omega|^{\frac{k-1}{k+1}} = 2^{\frac{k-1}{k+1}}$ and $C_3 = \frac{1}{2^{k-1}}$ as proposed in [1]. $C_2$ is the ENO upper bound constant found in [3].

Table 5.1 shows the growth factor at time $T = 2$ with various combinations of mesh size and order of accuracy. As well as forward Euler time integration, we show results for the second-order SSP Runge-Kutta (Heun's method). Higher (than second) order accurate SSP RK methods and forth-order non-SSP RK give very similar results. The growth factor $\mathcal{G}_T$ over time is shown in Figure 5.2. It can be seen that the more spatial accuracy the scheme has, the less entropy is dissipated. On the other hand, using SSP Runge-Kutta schemes dissipates entropy more than forward Euler. As Figure 5.2 shows, this dissipation decreases as the mesh is refined.

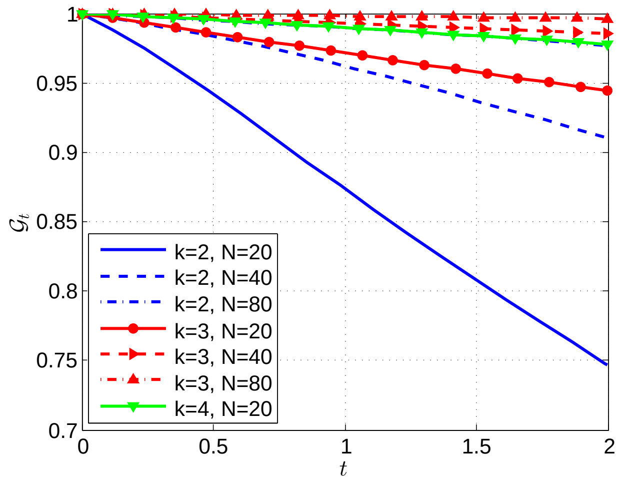


Fig. 5.1: The growth factor over time for different orders and mesh sizes, computed with the TECNO scheme using forward Euler time integration, linear advection.

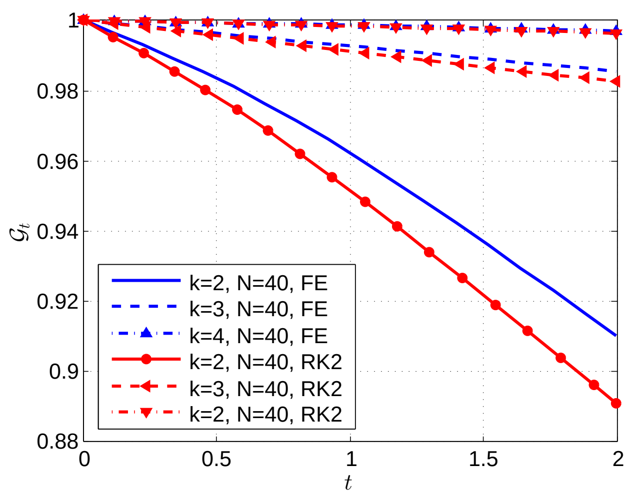


Fig. 5.2: The growth factor over time, computed with the TECNO scheme using forward Euler time integration and SSP RK2, linear advection.

**5.2. Burgers' equation.** As the prototypical example of a nonlinear scalar conservation laws, we consider Burgers' equation (5.1). As the CFL condition depends on the Lipschitz constant of the entropy conservative flux $\tilde{F}$, we estimate this as follows.

PROPOSITION 5.1. *Consider Burgers' equation* (5.1) *and suppose that the data* $u_j$ *is* $L^\infty$*-bounded,* $\max_j |u_j| \leqslant M$. *Then the high-order entropy conservative flux* (2.6) *based on the second-order entropy conservative flux*

$$\tilde{F}_{j+1/2} := \frac{u_j^2 + u_{j+1}u_j + u_{j+1}^2}{6}$$

*is Lipschitz-continuous with constant*

$$C_{\text{Lip}} = \frac{M}{2}\sum_{r=1}^{p} |\alpha_{p,r}| r.$$

*Proof.* It is straightforward to show that

$$|\tilde{F}(u_j, u_{j+1}) - \tilde{F}(u_{j-1}, u_j)| \leqslant \frac{M}{2}(|u_j - u_{j-1}| + |u_{j+1} - u_j|).$$

Thus,

$$\begin{aligned} |\tilde{F}_{j+1/2} - \tilde{F}_{j-1/2}| &\leqslant \sum_{r=1}^{p} |\alpha_{p,r}| \left| \tilde{F}(u_j, u_{j+r}) - \tilde{F}(u_{j-r}, u_j) \right| \\ &\leqslant \frac{M}{2}\sum_{r=1}^{p} |\alpha_{p,r}| r \big(|u_{j-p+1} - u_{j-p}| + \cdots + |u_{j+p} - u_{j+p-1}|\big). \end{aligned}$$

□

The CFL condition (4.2), valid for general diffusion coefficients $d_{j+1/2}$, is too restrictive for practical computations. Instead we select $d_{\min} = d_{\max}$ according to the optimal choice suggested in Theorem 4.1. Figure 5.3 shows how the growth factor $\mathcal{G}_T$ decreases with time. It can be seen that higher order time integration dissipated more entropy, similar to the results for the linear equation, although the difference is small. Also Figure 5.3 suggests that higher order schemes have more entropy dissipation. Note that in this case, due to shock formation, the entropy should be dissipated in the exact (entropy) solution.

We also compare the time step for the linear and non-linear cases as shown in Figure 5.4. After the formation of a shock, there is a slight increase in $\Delta t$; cf. Remark 4.3.

**6. Conclusion and future work.** We have derived CFL conditions that ensure the global entropy stability of an explicit temporal discretization of the high-order accurate TECNO schemes. Using SSP Runge-Kutta methods the schemes are high-order accurate in time as well as space. By a judicious choice of the diffusion constant $d_{j+1/2}$ appearing in the TECNO scheme, we obtain a CFL condition of the form $\Delta t \leqslant C\Delta x^\sigma$ for a $\sigma \in [1, 2)$, cf. (3.9) in the linear case and (4.2) in the nonlinear case. This should be contrasted with the CFL condition $\Delta t \leqslant C\Delta x$ for monotone first-order schemes, and $\Delta t \leqslant C\Delta x^2$ for convection-diffusion equations.

A natural next step would be to carry out, for the fully discrete schemes, the convergence analysis performed in [1] for semi-discrete schemes. This entails deriving an a priori bound on the spatial variation of the solution, a discrete version of the bound (2.16).

Further suggestions for future work include sharpen the CFL condition, for instance by doing a more careful, local choice of the diffusion coefficient $d_{j+1/2}$, and to generalize our analysis to arbitrary entropy functions and to systems of conservation laws. Finally, a proof (or counterexample) of the ENO conjecture would be of great interest.

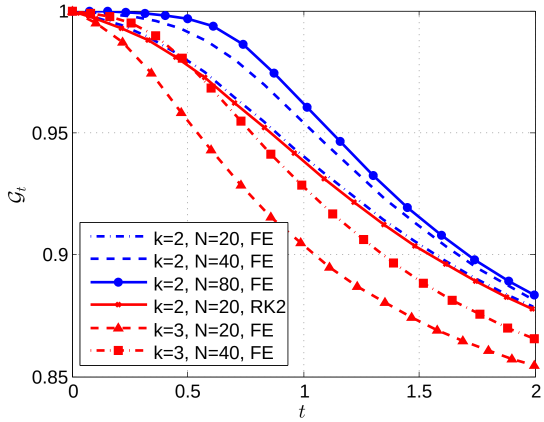


Fig. 5.3: Comparison of growth factor for different time integration methods, mesh sizes and orders for fully discrete TECNO scheme, applied to Burgers' equation.

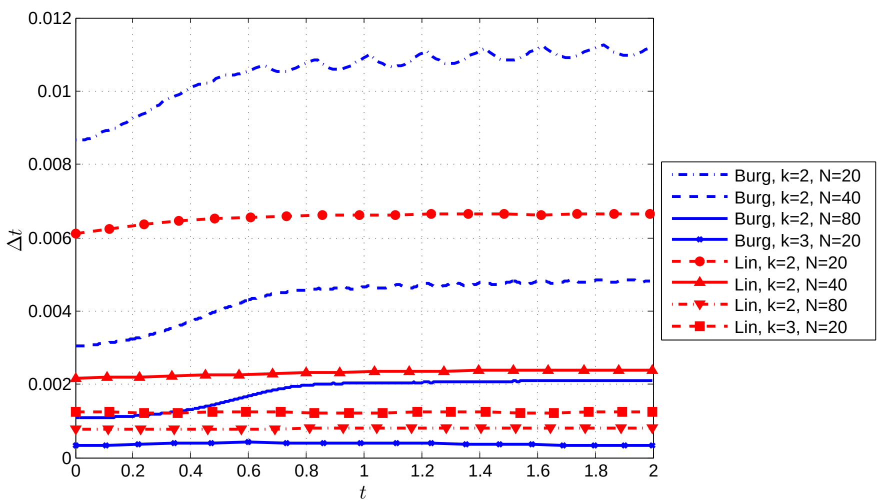


Fig. 5.4: Comparison of required $\Delta t$ for global entropy stability of fully discrete TECNO scheme with forward Euler time integration.

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## REFERENCES

[1] U. S. Fjordholm, *High-order accurate entropy stable numerical schemes for hyperbolic conservation laws*, PhD thesis, ETH Zürich, 2013. No. 21025.

[2] U. S. Fjordholm, S. Mishra, and E. Tadmor, *Arbitrarily high order accurate entropy stable essentially non-oscillatory schemes for systems of conservation laws*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 544–573.

[3] ———, *ENO reconstruction and ENO interpolation are stable*, Foundations of Computational

Mathematics, (2012).

[4] S. Gottlieb and C.-W. Shu, *Total variation diminishing Runge-Kutta schemes*, Mathematics of Computation, 67 (1998), pp. 73–85.

[5] A. Harten, B. Engquist, S. Osher, and S. R. Chakravarthy, *Uniformly high order accurate essentially non-oscillatory schemes, III*, J. Comput. Phys., 71 (1987), pp. 231–303.

[6] A. Harten, J. M. Hyman, P. D. Lax, and B. Keyfitz, *On finite-difference approximations and entropy conditions for shocks*, Communications on Pure and Applied Mathematics, 29 (1976), pp. 297–322.

[7] S. N. Kruzkov, *First order quasilinear equations in several independent variables*, Math USSR SB, 10 (1970), pp. 217–243.

[8] P. D. Lax, *Shock waves and entropy*, Contributions to nonlinear functional analysis, (1971), pp. 603–634.

[9] P. G. LeFloch, J. M. Mercier, and C. Rohde, *Fully discrete, entropy conservative schemes of arbitrary order*, SIAM Journal on Numerical Analysis, 40 (2003), pp. pp. 1968–1992.

[10] S. Osher, *Riemann solvers, the entropy condition, and difference approximations*, SIAM Journal on Numerical Analysis, 21 (1984), pp. pp. 217–235.

[11] C.-W. Shu, *High order ENO and WENO schemes for computational fluid dynamics*, in High-Order Methods for Computational Physics, T. J. Barth and H. Deconinck, eds., vol. 9 of Lecture Notes in Computational Science and Engineering, Springer, 1999, pp. 439–582.

[12] E. Tadmor, *Numerical viscosity and the entropy condition for conservative difference schemes*, Mathematics of Computation, 43 (1984), pp. pp. 369–381.

[13] ———, *The numerical viscosity of entropy stable schemes for systems of conservation laws. I.*, Mathematics of Computation, 49 (1987), pp. 91–103.

[14] ———, *Entropy stability theory for difference approximations of nonlinear conservation laws and related time dependent problems*, Acta Numerica, 12 (2003), pp. 451–512.

[15] H. Zakerzadeh, *Entropy stable essentially non–oscillatory schemes*, Master's thesis, RWTH Aachen, 2013.