# A Higher Order Finite Element Method for Partial Differential Equations on Surfaces

**Jörg Grande*  and  Arnold Reusken†**

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**Institut für Geometrie und Praktische Mathematik**
**RWTH Aachen**

**Templergraben 55, D-52056 Aachen, (Germany)**

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* Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: grande@igpm.rwth-aachen.de

† Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: reusken@igpm.rwth-aachen.de

# A HIGHER ORDER FINITE ELEMENT METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS ON SURFACES

JÖRG GRANDE* AND ARNOLD REUSKEN†

**Abstract.** A new higher order finite element method for elliptic partial differential equations on a stationary smooth surface $\Gamma$ is introduced and analyzed. We assume $\Gamma$ is characterized as the zero level of a level set function $\phi$ and *only a finite element approximation* $\phi_h$ *(of degree* $k \geq 1$*) of* $\phi$ *is known.* For the discretization of the partial differential equation, finite elements (of degree $m \geq 1$) on a piecewise *linear* approximation of $\Gamma$ are used. The discretization is lifted to $\Gamma_h$, which denotes the zero level of $\phi_h$, using a quasi-orthogonal coordinate system that is constructed by applying a gradient recovery technique to $\phi_h$.

A complete discretization error analysis is presented in which the error is split into a geometric error, a quadrature error, and a finite element approximation error. The main result is a $H^1(\Gamma)$-error bound of the form $c(h^m + h^{k+1})$. Results of numerical experiments illustrate the higher order convergence of this method.

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**1. Introduction.** In the past decade the study of numerical methods for PDEs on surfaces has been a rapidly growing research area. The development of finite element methods for solving elliptic equations on surfaces can be traced back to the paper [8], which considers a piecewise polygonal surface and uses a finite element space on a triangulation of this discrete surface. This approach has been further analyzed and extended in several directions, see, e.g., [9, 10] and the references therein. Another approach has been introduced in [4] and builds on the ideas of [2]. The method in that paper applies to cases in which the surface is given implicitly by some level set function and the key idea is to solve the partial differential equation on a narrow band around the surface. Unfitted finite element spaces on this narrow band are used for discretization. Another surface finite element method based on an outer (bulk) mesh has been introduced in [12] and further studied in [11, 6]. The main idea of this method is to use finite element spaces that are induced by triangulations of an outer domain to discretize the partial differential equation on the surface by considering *traces* of the bulk finite element space on the surface, instead of extending the PDE off the surface, as in [2, 4].

Most of the methods mentioned above have been studied both for stationary and evolving surfaces.

In all the papers we know of, except for [5], the discretization that is studied is based on piecewise *linear* finite elements. The paper [5] is the only one in which *higher order* finite element methods for partial differential equations on (stationary) surfaces are studied. We outline the key results of that paper. For a smooth bounded and connected surface $\Gamma \subset \mathbb{R}^3$ we consider the Laplace-Beltrami problem: for given $f \in L^2(\Gamma)$ with $\int_\Gamma f \, ds = 0$ determine $u \in H^1_*(\Gamma) := \{ u \in H^1(\Gamma) \mid \int_\Gamma u \, ds = 0 \}$ such

*Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: grande@igpm.rwth-aachen.de

†Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: reusken@igpm.rwth-aachen.de

that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds = \int_\Gamma f v \, ds \quad \text{for all} \ \ v \in H^1(\Gamma). \tag{1.1}$$

It is assumed that $\Gamma$ is represented as the *zero level of a smooth signed distance function* $d$. The exact surface is approximated by a quasi-uniform shape-regular polyhedral surface $\hat{\Gamma}_h$ having triangular faces, and with vertices on $\Gamma$. Based on the distance function $d$ a parametric mapping, consisting of piecewise polynomial mappings of degree $k$, is defined on $\hat{\Gamma}_h$, which results in a corresponding discrete surface $\hat{\Gamma}_h^k$. Using the same mapping a standard higher order finite element space on $\hat{\Gamma}_h$ is lifted to $\hat{\Gamma}_h^k$. This lifted space on $\hat{\Gamma}_h^k$ is used for the discretization of (1.1). An extensive error analysis of this method is presented in [5], resulting in optimal error bounds. For example, for the $H^1(\Gamma)$ error (where the discrete solution is lifted to $\Gamma$) a bound of the form $c(h^m + h^{k+1})$ is proved. Here $k$ is the degree of the polynomials used in the parametrization of $hat\Gamma_h^k$ and $m$ the degree of the polynomials in the finite element space on $\hat{\Gamma}_h$. We emphasize that in this method *explicit knowledge of the exact signed distance function to* $\Gamma$ *is an essential requirement.*

In many applications the exact signed distance function to the surface $\Gamma$ is not known. One often encounters situations in which $\Gamma$ is the zero level of a smooth level set function $\phi$ (not necessarily a signed distance function) and one *only has a finite element approximation of* $\phi$ *available.* This paper deals with the question: (how) can one develop a higher order finite element method in such a setting? We will present a constructive affirmative answer to this question.

We restrict ourselves to the model problem (1.1) with a stationary surface $\Gamma$. We assume $\Gamma$ to be sufficiently smooth. Our approach is fundamentally different from the one in [5], in the sense that we do not need the exact distance function $d$. Instead, we only(!) need a finite element approximation $\phi_h^k$ of a level set function $\phi$, which has $\Gamma$ as its zero level. The discrete level set function $\phi_h^k$ comes from a standard finite element space on a quasi-uniform triangulation of a bulk domain that contains $\Gamma$. In the error analysis we assume that $\phi_h^k$ satisfies an error bound of the form

$$\|\phi_h^k - \phi\|_{L^\infty(U)} + h\|\phi_h^k - \phi\|_{H^1_\infty(U)} \leq ch^{k+1}, \tag{1.2}$$

where $U$ is a (small) neighborhood of $\Gamma$ in $\mathbb{R}^3$. The zero level of $\phi_h^k$ is denoted by $\Gamma_h^k$. Note that for $k > 1$ , $\Gamma_h^k$ cannot be easily constructed. From (1.2) it follows that $\text{dist}(\Gamma, \Gamma_h^k) \leq ch^{k+1}$ holds. The method that we introduce is new and is built upon the following key ingredients:

- For $k = 1$ the function $\hat{\phi}_h := \phi_h^1$ is piecewise linear, hence its zero level is piecewise planar, consisting of quadrilaterals and triangles, and can easily be determined. The quadrilaterals are subdivided into triangles. The resulting triangulation is denoted by $\hat{\Gamma}_h$. This triangulation is in general *very* shape-*ir*regular. Nevertheless, the trace of an outer finite element space or a standard finite element space directly on $\hat{\Gamma}_h$ turns out to have optimal approximation properties [12, 13]. Such a finite element space on $\hat{\Gamma}_h$ is denoted by $\hat{S}_h$.
- We take $k > 1$. For the parametrization of $\Gamma_h^k$ we use a *quasi-normal field*, as introduced in [14]. Given $\phi_h^k$ we apply a *gradient recovery method* which results in a Lipschitz continuous vector field $n_h$ that is close to the normal field $n$ that corresponds to $\phi$. Using this quasi-normal field, there is a unique

decomposition $x = p_h(x) + d_h(x)n_h(p_h(x))$ for all $x$ in a neighborhood of $\Gamma_h^k$, with $p_h(x) \in \Gamma_h^k$ and $d_h \in \mathbb{R}$ an *approximate* signed distance function. It can be shown that $p_h : \hat{\Gamma}_h \to \Gamma_h^k$ is a *bijection*. This $p_h$ is used for the parametrization of $\Gamma_h^k$. For given $x \in \hat{\Gamma}_h$ its image $p_h(x) \in \Gamma_h^k$ can be determined (with high accuracy) using the known field $n_h$ and only few evaluations of $\phi_h^k$.

- Using the parametrization $p_h$ the finite element space $\hat{S}_h$ on $\hat{\Gamma}_h$ is lifted to $\Gamma_h^k$ and used for a Galerkin type discretization of (1.1), i.e. we take (1.1) with $\Gamma$ replaced by $\Gamma_h^k$, $H^1(\Gamma)$ replaced by the lifted finite element space, $f$ suitably extended, and instead of $\nabla_\Gamma$ we use the tangential gradient along $\Gamma_h^k$.
- Only evaluations of $p_h$ and $Dp_h$ can be computed. Hence, *quadrature* is needed. The finite element space is pulled back to $\hat{\Gamma}_h$, integrals over $\Gamma_h^k$ are transformed to integrals over $\hat{\Gamma}_h$ and quadrature is applied on triangles in $\hat{\Gamma}_h$. We then (only) need evaluations of $p_h$, $Dp_h$, and of exact normals on $\hat{\Gamma}_h$ and on $\Gamma_h^k$. The latter are easily determined using $\phi_h^k$.

The method is described more precisely in section 5. The implementation is discussed in section 14.

Apart from the new discretization method outlined above, the main contribution of this paper is an error analysis of this method. A key point related to this is the following. On each triangle $T$ of the "base" triangulation $\hat{\Gamma}_h$ the parametrization $p_h$ is *only Lipschitz*. This low regularity is due to the construction of the quasi-normal field $n_h$. Hence, the bilinear form pulled back to $\hat{\Gamma}_h$ consists of a sum of integrals of the form $\int_T G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h \, d\hat{s}_h$ with a function $G$ that has very low smoothness (not even continuous). Due to this the analysis of the quadrature error is not straightforward. This lack of smoothness is also an important reason why the analysis in this paper is (even) more technical than the one in [5]. The structure of the error analysis is outlined in section 6. As a main result, cf. Theorem 13.1, we prove an $H^1(\Gamma)$ error bound (where the discrete solution is lifted to $\Gamma$) of the form $c(h^m + h^{k+1})$. Here $m$ is the degree of the polynomials used in the finite element space $\hat{S}_h$.

**2. Preliminaries.** Let $\phi$ be a smooth function with a smooth, bounded and connected zero level set $\Gamma \subset \Omega \subset \mathbb{R}^3$, and let $\Omega_1 = \{x \in \Omega \mid \phi(x) \leq 0\}$ be the enclosed (compact) region. Furthermore $U$ is a (small) open subset of $\mathbb{R}^3$ with $\Gamma \subset U \subset \Omega$. This neighborhood is sufficiently small such that on $U$ we have a local coordinate system

$$x = p(x) + d(x)n(p(x)), \tag{2.1}$$

with $n$ the normal vector field on $\Gamma$ (pointing out of $\Omega_1$), $p : U \to \Gamma$ and $d$ the signed distance function to $\Gamma$ (negative in $\Omega_1$). For every $x \in U$ the normal field has the unique value $n(x) = n(p(x))$. We assume that $\|\nabla\phi(x)\| \geq c_0 > 0$ for all $x \in U$ holds.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular *quasi-uniform* tetrahedral triangulations on $\Omega$. Furthermore $V_h^k$ denotes a standard FE space on $\mathcal{T}_h$ consisting of continuous piecewise polynomial functions of degree $k$.

REMARK 1. Some of the assumptions introduced above are used to simplify the presentation and not essential for the applicability of the method or the validity of the error analysis. For example, $\mathbb{R}^3$ could easily be replaced by $\mathbb{R}^n$, with $n \geq 2$. Also the extension to a surface $\Gamma$ with a finite number of connected components is straightforward. Finally, with minor modifications the analysis also applies if $\{\mathcal{T}_h\}_{h>0}$ is a shape-regular (not necessarily quasi-uniform) family of triangulations. The assumption that $\Gamma$ is a smooth surface is an essential one.

Let $\phi_h^k \in V_h^k$ be an approximation of $\phi$ that satisfies

$$\|\phi_h^k - \phi\|_{L^\infty(U)} + h\|\phi_h^k - \phi\|_{H^1_\infty(U)} \le ch^{k+1}. \tag{2.2}$$

In the remainder we take a fixed value $k \ge 1$. To simplify notation, we write $\phi_h = \phi_h^k$. The *linear* finite element approximation $\phi_h^1$ plays a special role and is denoted by $\hat{\phi}_h$. The zero level sets of $\phi_h$ and $\hat{\phi}_h$ are denoted by $\Gamma_h$ and $\hat{\Gamma}_h$, respectively. The outward pointing normal fields on $\Gamma_h$ and $\hat{\Gamma}_h$ are denoted by $\bar{n}_h$ and $\hat{n}_h$, respectively. From (2.2) we obtain, cf. [5]:

$$\begin{aligned} &\mathrm{dist}(\Gamma, \hat{\Gamma}_h) \le ch^2, \quad \mathrm{dist}(\Gamma, \Gamma_h) \le ch^{k+1}, \\ &\|n - \hat{n}_h\|_{L^\infty(\Gamma_h)} \le ch, \quad \|n - \bar{n}_h\|_{L^\infty(\Gamma_h)} \le ch^k. \end{aligned} \tag{2.3}$$

The two eigenvalues of the Weingarten map $H := D^2 d \in \mathbb{R}^{3\times 3}$ corresponding to the eigenvectors orthogonal to $n(x)$ are denoted by $\kappa_i(x)$, $i = 1, 2$. These are related to the principal curvatures of $\Gamma$ by the formula $\kappa_i(x) = \kappa_i(p(x))/\big(1 + d(x)\kappa_i(p(x))\big)$. In [5] it is shown that for the surface measures $ds_h$ on $\Gamma_h$ and $ds$ on $\Gamma$ we have the relation $\mu_h(x)\, ds_h(x) = ds(p(x))$, with

$$\mu_h(x) = n(x)^T \bar{n}_h(x) \prod_{i=1}^{2} (1 - d(x)\kappa_i(x)), \quad x \in \Gamma_h. \tag{2.4}$$

Using this formula and the results in (2.3) one obtains:

$$\begin{aligned} \|\mu_h - 1\|_{L^\infty(\Gamma_h)} &\le c\|d\|_{L^\infty(\Gamma_h)} + c\|1 - n^T \bar{n}_h\|_{L^\infty(\Gamma_h)} \\ &= c\|d\|_{L^\infty(\Gamma_h)} + c\|n - \bar{n}_h\|^2_{L^\infty(\Gamma_h)} \le ch^{k+1}. \end{aligned} \tag{2.5}$$

Here and in the remainder, $c$ is used to denote different constants, which are all independent of $h$.

We need an $\mathcal{O}(h)$ neighborhood of $\Gamma_h$, denoted by $\Omega_{\Gamma_h}$, consisting of all tetrahedra with distance to $\Gamma_h$ smaller than $ch$, with a given $c > 0$. We assume that $h$ is sufficiently small such

$$\Gamma_h \subset \Omega_{\Gamma_h} \subset U \quad \text{and} \quad \hat{\Gamma}_h \subset \Omega_{\Gamma_h}$$

hold, cf. (2.3).

**3. Quasi-normal field.** In this section we define the notion of a quasi-normal field, as introduced in [14]. Such a quasi-normal field is constructed using a (simple) gradient recovery technique. Only this field, and not the gradient recovery technique, is then used in the finite element method further on.

A gradient recovery operator is a mapping $G_h : V_h^k \to (V_h^k)^3$, which has to satisfy certain reasonable approximation and stability conditions.

Assumption 3.1. *Let $I_h$ be the nodal interpolation in the finite element space $V_h^k$. We assume that for $\phi$ sufficiently smooth the gradient recovery method $G_h : V_h^k \to (V_h^k)^3$ satisfies:*

$$\|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \le ch^k, \tag{3.1}$$

$$\|G_h v_h\|_{L^\infty(U)} \le c\|v_h\|_{H^1_\infty(U^e)} \quad \textit{for all} \;\; v_h \in V_h^k. \tag{3.2}$$

*Here $U^e$ denotes the neighborhood $U$ enlarged with a suitable patch of surrounding elements.*

REMARK 2. In the literature gradient recovery techniques are known and often used in error estimators, cf. [1]. In such a setting one usually requires a power $k+1$, instead of $k$, in (3.1). In [14] the polynomial-preserving recovery (PPR) technique is considered. For the PPR technique, (3.2) and (3.1) with $k+1$ are shown to hold in two dimensions in [15]. To indicate that the conditions (3.1) and (3.2) are mild ones, as an example we describe a very simple gradient recovery technique satisfying Assumption 3.1. It is used in the experiments in section 14. The set of finite element nodes is denoted by $N_h$. To each finite element node $\xi \in N_h$ we assign the set $\mathcal{T}_\xi$ of all tetrahedra containing $\xi$. For $\xi \in U^e$ this $\mathcal{T}_\xi$ is chosen such that $T \in \mathcal{T}_\xi \Rightarrow T \subset U^e$. Let $n_\xi := |\mathcal{T}_\xi|$. The gradient recovery is defined by simple local averaging, namely $(Gv_h)(\xi) := \frac{1}{n_\xi}\sum_{T\in\mathcal{T}_\xi} \nabla v_{h|T_\xi}(\xi)$ for all $\xi$. Let $\phi_h = I_h\phi$ be the nodal interpolation of a smooth function $\phi$. From standard interpolation theory we get

$$\begin{aligned}\max_{\xi\in N_h\cap U^e} \|(G_h\phi_h)(\xi) - \nabla\phi(\xi)\| &= \max_{\xi\in N_h\cap U^e} \Big\| \frac{1}{n_\xi} \sum_{T\in\mathcal{T}_\xi} \Big(\nabla\phi_{h|T_\xi}(\xi) - \nabla\phi(\xi)\Big)\Big\| \\ &\le c \max_{T\in\mathcal{T}_h\cap U^e} \|\nabla\phi_h - \nabla\phi\|_{L^\infty(T)} \le ch^k\|\phi\|_{H^{k+1}(U^e)}.\end{aligned}$$

Hence, using $I_h(G_h\phi_h) = G_h\phi_h$ we get

$$\begin{aligned}&\|G_h\phi_h - \nabla\phi\|_{L^\infty(U)} \le \|I_h(G_h\phi_h - \nabla\phi)\|_{L^\infty(U)} + \|I_h(\nabla\phi) - \nabla\phi\|_{L^\infty(U)} \\ &\le c \max_{\xi\in N_h\cap U^e} \|(G_h\phi_h)(\xi) - \nabla\phi(\xi)\| + ch^k\|\phi\|_{H^{k+1}(U^e)} \le ch^k\|\phi\|_{H^{k+1}(U^e)},\end{aligned}$$

and thus the condition in (3.1) is satisfied. With similar arguments, using stability properties of $I_h$, one can verify that for this simple recovery operator condition (3.2) is satisfied, too. Properties of different gradient recovery techniques with respect to the construction of a quasi-normal field will be analyzed in a forthcoming paper. For the analysis in this paper it suffices to assume that we use a gradient recovery technique that has the properties given in Assumption 3.1.

Given the gradient recovery operator $G_h$ we apply it to $\phi_h = \phi_h^k$ and define the *quasi-normal field*:

$$n_h(x) = \frac{(G_h\phi_h)(x)}{\|(G_h\phi_h)(x)\|}, \quad x \in \Omega_{\Gamma_h}. \tag{3.3}$$

Note that this field is only Lipschitz continuous; a main point in the analysis is that $n_h$ can be approximated by a smooth vector field (cf. Lemma 7.1). The result (3.5) in the following lemma explains why we call $n_h$ a "quasi-normal field". A proof of this lemma is given in [14] (where the power $k+1$ instead of $k$ is assumed in (3.1)). Since the lemma is of fundamental importance for the analysis in this paper, we include a proof. By $B(x;r)$ we denote the ball with center $x$ and radius $r$.

LEMMA 3.1. *Let Assumption 3.1 be satisfied. Let $r_x > 0$ (depending on $x$) be small enough such that $B(x,r_x) \subset U$ for all $x \in \Gamma_h$. There exist constants $c$ and $h_0 > 0$ such that for all $h \le h_0$ and all $x \in \Gamma_h$ the following holds:*

$$\|n_h(x) - n_h(y)\| \le c\|x-y\|, \quad \textit{for all } \ y \in B(x;r_x), \tag{3.4}$$

$$|\langle n_h(x), x-y\rangle| \le ch^k\|x-y\| + c\|x-y\|^2, \quad \textit{for all } \ y \in \Gamma_h \cap B(x;r_x). \tag{3.5}$$

*Proof.* Take $x \in \Gamma_h$ and $y \in B(x; r_x) \subset U$. From the definition of $n_h$ we get

$$\|n_h(x) - n_h(y)\| \leq 2\frac{\|(G_h\phi_h)(x) - (G_h\phi_h)(y)\|}{\|(G_h\phi_h)(x)\|}. \tag{3.6}$$

We write $G_h\phi_h = G_h(\phi_h - I_h\phi) + \big(G_h(I_h\phi) - \nabla\phi\big) + \nabla\phi$, and using (3.1), (3.2), (2.2), $\|\nabla\phi(x)\| \geq c_0 > 0$ and an interpolation bound we get

$$\begin{aligned} \|(G_h\phi_h)(x)\| &\geq \|\nabla\phi(x)\| - c\|\phi_h - I_h\phi\|_{H^1_\infty(U^e)} - c\|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \\ &\geq c_0 - ch^k \geq \frac{1}{2}c_0, \end{aligned} \tag{3.7}$$

provided $h$ is sufficiently small. The vector function $G_h v_h \in V_h^3$ is Lipschitz continuous and

$$\|(G_h\phi_h)(x) - (G_h\phi_h)(y)\| \leq \int_0^1 \|\nabla(G_h\phi_h)(x + t(x-y))\|\,dt\|x - y\| \tag{3.8}$$

holds. We write $z := x + t(x-y) \in B(x; r_x)$ and note that

$$\|\nabla(G_h\phi_h)(z)\| \leq \|\nabla\big(G_h\phi_h - I_h(\nabla\phi)\big)\|_{L^\infty(U)} + \|\nabla I_h(\nabla\phi)\|_{L^\infty(U)}.$$

Using an inverse inequality and the boundedness of $I_h$ in $H^1_\infty$, (3.1) and (3.2) we get

$$\begin{aligned} \|\nabla(G_h\phi_h)(z)\| &\leq ch^{-1}\|G_h\phi_h - I_h(\nabla\phi)\|_{L^\infty(U)} + c \\ &\leq ch^{-1}\|G_h(\phi_h - I_h\phi)\|_{L^\infty(U)} + ch^{-1}\|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \\ &\quad + ch^{-1}\|\nabla\phi - I_h(\nabla\phi)\|_{L^\infty(U)} + c \\ &\leq ch^{-1}\|\phi_h - I_h\phi\|_{H^1_\infty(U^e)} + ch^{k-1} + c \leq ch^{k-1} + c \leq c. \end{aligned} \tag{3.9}$$

Using this result in (3.8), in combination with (3.7) and (3.6) proves (3.4).

Now assume $y \in \Gamma_h$. The definition of $n_h$ and the lower bound in (3.7) yield

$$|\langle n_h(x), x - y\rangle| \leq \frac{2}{c_L}|\langle (G_h\phi_h)(x), x - y\rangle|. \tag{3.10}$$

Since $x, y \in \Gamma_h$ we have $0 = \phi_h(x) - \phi_h(y) = \int_0^1 \langle \nabla\phi_h(x + t(y-x)), x - y\rangle\,dt$, hence,

$$\begin{aligned} \langle (G_h\phi_h)(x), x - y\rangle &= \int_0^1 \langle (G_h\phi_h)(x) - (G_h\phi_h)(x + t(y-x)), x - y\rangle\,dt \\ &+ \int_0^1 \langle (G_h\phi_h)(x + t(y-x)) - \nabla\phi_h(x + t(y-x)), x - y\rangle\,dt. \end{aligned} \tag{3.11}$$

From the Lipschitz continuity estimate (3.8)-(3.9), with $y$ replaced by $x + t(y-x) \in B(x, r_x)$ we get

$$|\langle (G_h\phi_h)(x) - (G_h\phi_h)(x + t(y-x)), x - y\rangle| \leq c\|x - y\|^2. \tag{3.12}$$

For the second term on the right-hand side in (3.11) we get, using (3.1) and (3.2),

$$\begin{aligned} &|\langle (G_h\phi_h)(x + t(y-x)) - \nabla\phi_h(x + t(y-x)), x - y\rangle| \leq \|G_h\phi_h - \nabla\phi_h\|_{L^\infty(U)}\|x - y\| \\ &\leq \Big(\|G_h(\phi_h - I_h\phi)\|_{L^\infty(U)} + \|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} + \|\nabla\phi - \nabla\phi_h\|_{L^\infty(U)}\Big)\|x - y\| \\ &\leq c\big(\|\phi_h - I_h\phi\|_{H^1_\infty(U^e)} + h^k\big)\|x - y\| \leq ch^k\|x - y\|. \end{aligned}$$

Using this and (3.12) in (3.11) in combination with (3.10) proves (3.5). ∎

The quasi-normal field can be used to define a local coordinate system similar to (2.1). Given $n_h$ we define $F : \Gamma_h \times \mathbb{R} \to \mathbb{R}^3$, $F(z,t) := z + tn_h(z)$. In Lemma 3.1 and Theorem 3.2 in [14] it is proved that from (3.4) and (3.5) it follows that this mapping is a bijection between $\Gamma_h \times [-\epsilon, \epsilon]$ ($\epsilon > 0$, sufficiently small) and a (sufficiently small) neighborhood of $\Gamma_h$ in $\mathbb{R}^3$. This neighborhood is again denoted by $U$. Hence there is a unique decomposition

$$x = p_h(x) + d_h(x)n_h(p_h(x)), \quad x \in U, \tag{3.13}$$

with the *skew* projection $p_h : U \to \Gamma_h$ and $d_h$ an *approximate* signed distance function to $\Gamma_h$ , $|d_h(x)| = \|x - p_h(x)\|$. This decomposition resembles the one in (2.1). In the latter, however, one needs the exact level set function $\phi$ (to compute $n(p(x))$), whereas (3.13) is based on the quasi-normal field, which can be determined from the finite element approximation $\phi_h$. Furthermore, $d_h(x) = 0$ iff $x \in \Gamma_h$ holds, and we have the useful formula $d_h(x) = \langle x - p_h(x), n_h(p_h(x))\rangle$.

**4. Parametrization of $\mathbf{\Gamma_h}$.** We use $\hat{\Gamma}_h$ (the zero level of the piecewise linear function $\hat{\phi}_h$) and the quasi-normal field $n_h$ for a computable parametrization of $\Gamma_h$ (the zero level of the higher order finite element function $\phi_h$). From the assumptions above, it follows that

$$p_h|_{\hat{\Gamma}_h} : \hat{\Gamma}_h \to \Gamma_h \quad \text{is a bijection.}$$

Note that this bijection is (only) Lipschitz. The Lipschitz manifold $\hat{\Gamma}_h$ consists of triangles and convex quadrilaterals. Each quadrilateral is subdivided into two triangles. The resulting triangular triangulation of $\hat{\Gamma}_h$ is denoted by $\mathcal{F}_h$, i.e.,

$$\hat{\Gamma}_h = \cup\{T \mid T \in \mathcal{F}_h\}. \tag{4.1}$$

The family $\{\mathcal{F}_h\}_{h>0}$ may be quite shape-*irregular*, but this does not cause problems, cf. remark 3 below. The mapping $p_h$ is used for the parametrization of $\Gamma_h$. We need a transformation formula between integrals over $T \in \hat{\Gamma}_h$ and over $p_h(T)$, which is derived in the following lemma.

LEMMA 4.1. *For $T \in \mathcal{F}_h$, let $H \subset \mathbb{R}^3$ be the plane containing $T$, and let $\tilde{x} \mapsto U\tilde{x} + u$ be a parametrization $\mathbb{R}^2 \to H$ with an orthogonal matrix $U \in \mathbb{R}^{3\times 2}$. Then, for any measurable function $g\colon p_h(T) \to \mathbb{R}$ the transformation formula*

$$\int_{p_h(T)} g(y)\, d\sigma(y) = \int_T g(p_h(x))\, \hat{\mu}_h(x) d\sigma(x) \tag{4.2}$$

*holds, with $\hat{\mu}_h(x) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x)U)}$.*

*Proof.* Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be an injective Lipschitz-mapping, and let $T \subseteq \mathbb{R}^2$ be Lebesgue-measurable. We recall the transformation rule

$$\int_{F(T)} g(y)\, d\sigma(y) = \int_T g(F(x))\, \mu(x) d\sigma(x), \quad \mu(x) = \sqrt{\det(DF(x)^T DF(x))}.$$

We apply this formula to the parametrization $x = F(\tilde{x}) = U\tilde{x} + u$ of $H$. The surface measure on $H$ is

$$d\sigma(x) = \sqrt{\det(U^T U)}\, d\tilde{x} = d\tilde{x}, \tag{4.3}$$

because $U$ is orthogonal. We also apply this formula to the parametrization $y = F(x) := p_h(x) = p_h(U\tilde{x} + u)$. The surface measure on this set is

$$d\sigma(y) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x)U)}\, d\tilde{x} = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x)U)}\, d\sigma(x),$$

by the result in (4.3). ◻

**5. Finite element discretization.** We introduce the finite element discretization of the Laplace-Beltrami equation (1.1). Our method has some similarity with the one presented in [5], but an essential difference is that we (only) need the finite element approximations $\phi_h$ and $\hat{\phi}_h$ of $\phi$. From $\phi_h$ the quasi-normal field $n_h$ can be determined.

Let $\hat{S}_h$ be a finite element space of piecewise polynomials of degree $m \geq 1$ on the triangulation $\mathcal{F}_h$ of $\hat{\Gamma}_h$, cf. (4.1):

$$\hat{S}_h = \{\, \hat{v}_h \in C(\hat{\Gamma}_h) \mid \hat{v}_{h|T} \in P_m \quad \text{for all } T \in \mathcal{F}_h \,\}. \tag{5.1}$$

Remark 3. We briefly discuss two possible choices for the space $\hat{S}_h$. A first possibility is to use a trace space as introduced and analyzed in [12]. Such a space is constructed by taking the trace of a standard outer finite element space, e.g. the space $V_h^m$ used for the approximation of the level set function, cf. section 2. Its (optimal) approximation properties depend on the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$ not on the shape-regularity of the family $\{\mathcal{F}_h\}_{h>0}$.

A second possibility is to define standard polynomial spaces directly on the triangulation $\mathcal{F}_h$. Although this triangulation is in general very shape irregular, it has a maximal angle property in three dimensions: in [13] it is shown, that if in the construction of $\mathcal{F}_h$ the quadrilaterals are subdivided in two triangles in a suitable way, the maximal inner angles in the resulting triangulation are uniformly bounded away from $\pi$. Hence, standard finite element spaces on such a triangulation have optimal approximation quality, cf. [13] for more information.

We lift the space $\hat{S}_h$ to $\Gamma_h$ by using the bijection $p_h : \hat{\Gamma}_h \to \Gamma_h$:

$$S_h := \{\, v_h = \hat{v}_h \circ p_h^{-1} \mid \hat{v}_h \in \hat{S}_h \,\}. \tag{5.2}$$

For the discretization we need a (sufficiently accurate) extension of the data $f$ on $\Gamma$ to $\Gamma_h$. This extension is denoted by $f_h$ and is such that $\int_{\Gamma_h} f_h \, ds = 0$ holds.

Remark 4. One possible choice for the extension $f_h$ is $f_h = f^e - \frac{1}{|\Gamma_h|}\int_{\Gamma_h} f^e \, ds_h$, where $f^e$ denotes the constant extension along the exact normals on $\Gamma$. This, however, is not feasible, since in our setting it is not reasonable to assume that the normals to $\Gamma$ are known. Another possibility arises if we assume that $f$ is a (smooth) function that is defined in a neighborhood $U$ of $\Gamma$. As extension we may then take:

$$f_h(x) := f(x) - c_f \quad \text{for} \quad x \in \Gamma_h \quad \text{with } c_f := \frac{1}{|\Gamma_h|}\int_{\Gamma_h} f \, ds_h. \tag{5.3}$$

In the remainder we restrict to the latter choice of the extension. For $f$ we assume the smoothness property $f \in H^1_\infty(U)$.

The discrete problem is as follows: Determine $u_h \in S_h$ with $\int_{\Gamma_h} u_h \, ds_h = 0$ such that

$$\begin{aligned} &a(u_h, v_h) = l(v_h) \quad \text{for all } \ v_h \in S_h, \\ &a(u_h, v_h) := \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \, ds_h, \quad l(v_h) := \int_{\Gamma_h} f_h v_h \, ds_h. \end{aligned} \tag{5.4}$$

For the implementation of this method we pull the discretization back to $\hat{\Gamma}_h$ and apply quadrature on the triangulation $\mathcal{F}_h$ of $\hat{\Gamma}_h$. We first treat the pull back procedure. For this we derive a relation between the tangential gradient on $\Gamma_h$ and the tangential gradient on $\hat{\Gamma}_h$. For this we need several projectors, defined as follows, with $\hat{n}_h(x)$ the exact normal on $\hat{\Gamma}_h$:

$$\hat{\mathbf{Q}}(x) = \mathbf{I} - \frac{1}{\hat{\alpha}(x)}\hat{n}_h(x)\bar{n}_h(y)^T, \quad \hat{\alpha}(x) = \hat{n}_h(x)^T\bar{n}_h(y), \quad y = p_h(x), \quad x \in \hat{\Gamma}_h, \tag{5.5}$$

$$\hat{\mathbf{P}}(x) = \mathbf{I} - \hat{n}_h(x)\hat{n}_h(x)^T, \quad x \in \hat{\Gamma}_h, \tag{5.6}$$

$$\bar{\mathbf{P}}(y) := \mathbf{I} - \bar{n}_h(y)\bar{n}_h(y)^T, \quad y \in \Gamma_h, \tag{5.7}$$

Note that $\hat{\mathbf{Q}}(x)$, $x \in \hat{\Gamma}_h$, is an oblique projector which maps into the tangential space $\bar{n}_h(y)^\perp$. The following commutation relations hold:

$$\hat{\mathbf{Q}}(x)\bar{\mathbf{P}}(y) = \bar{\mathbf{P}}(y),\ \bar{\mathbf{P}}(y)\hat{\mathbf{Q}}(x) = \hat{\mathbf{Q}}(x),\ \hat{\mathbf{P}}(x)\hat{\mathbf{Q}}(x) = \hat{\mathbf{P}}(x),\ \hat{\mathbf{Q}}(x)\hat{\mathbf{P}}(x) = \hat{\mathbf{Q}}(x). \tag{5.8}$$

Lemma 5.1. *For $\hat{v}_h \in \hat{S}_h$, let $v_h = \hat{v}_h \circ (p_h|_{\hat{\Gamma}_h})^{-1} \in S_h$. For the tangential gradients the relations*

$$\begin{aligned} \nabla_{\hat{\Gamma}_h}\hat{v}_h(x) &= \hat{\mathbf{P}}(x)Dp_h(x)^T\nabla_{\Gamma_h}v_h(y) = W(x)\nabla_{\Gamma_h}v_h(y) \quad \text{with} \ \ y = p_h(x), \\ W(x) &:= \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x)Dp_h(x)^T, \end{aligned} \tag{5.9}$$

*hold for almost all $x \in \hat{\Gamma}_h$.*

*Proof.* As $p_h|_{\hat{\Gamma}_h} : \hat{\Gamma}_h \to \Gamma_h$ is a bijection, we have:

$$\hat{v}_h(x) = v_h(p_h(x)).$$

This relation and the ones below hold for almost all $x \in \hat{\Gamma}_h$. We apply the tangential gradient on $\hat{\Gamma}_h$ to both sides of the equation to obtain

$$\nabla_{\hat{\Gamma}_h}\hat{v}_h(x) = \hat{\mathbf{P}}(x)\nabla v_h(p_h(x)) = \hat{\mathbf{P}}(x)Dp_h(x)^T\nabla_{\Gamma_h}v_h(y).$$

This proves the first relation in (5.9). From

$$\begin{aligned} \hat{\mathbf{P}}(x)Dp_h(x)^T\nabla_{\Gamma_h}v_h(y) &= \hat{\mathbf{P}}(x)Dp_h(x)^T\bar{\mathbf{P}}(y)\nabla_{\Gamma_h}v_h(y), \\ \hat{\mathbf{P}}(x)Dp_h(x)^T\bar{\mathbf{P}}(y) &= W(x)\bar{\mathbf{P}}(y), \end{aligned}$$

we obtain the second relation in (5.9). ◻

From Lemma 9.1 below it follows that for $h$ sufficiently small the matrix $W$ is invertible. We assume that this condition on $h$ is satisfied, i.e. $W$ is invertible. We introduce the symmetric positive definite matrix function

$$T_h(x) := W(x)W(x)^T. \tag{5.10}$$

Using the transformation formulas in (4.2) and (5.9) we obtain the following "pulled back" *equivalent formulation of the discrete problem* (5.4): Determine $\hat{u}_h \in \hat{S}_h$ with $\int_{\hat{\Gamma}_h}\hat{u}_h\hat{\mu}_h\,d\hat{s}_h = 0$ such that

$$\begin{aligned} \int_{\hat{\Gamma}_h} G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h\,d\hat{s}_h &= \int_{\hat{\Gamma}_h}(f_h \circ p_h)\hat{v}_h\,\hat{\mu}_h\,d\hat{s}_h \quad \text{for all} \ \ \hat{v}_h \in \hat{S}_h, \\ G(\hat{x}) &:= T_h(\hat{x})^{-1}\hat{\mu}_h(\hat{x}), \quad \hat{x} \in \hat{\Gamma}_h. \end{aligned} \tag{5.11}$$

Clearly, for the implementation of this discretization we need quadrature.

We introduce quadrature along the same lines as in [3]. Let $\tilde{T}$ be the unit triangle in $\mathbb{R}^2$, $T \in \mathcal{F}_h$ and $M_T : \tilde{T} \to T$ an affine mapping $M_T\tilde{x} = B_T\tilde{x} + b_T = x,\ \tilde{x} \in \tilde{T},\ x \in T$.

We consider a quadrature rule on $\tilde{T}$ of the form $Q_{\tilde{T}}(\tilde{\phi}) = \sum_{l=1}^{L} \tilde{\omega}_l \tilde{\phi}(\tilde{\xi}_l)$ with *strictly positive* weights $\omega_l$ and quadrature nodes $\tilde{\xi}_l \in \tilde{T}$. This induces a quadrature rule on $T$:

$$Q_T(\hat{\phi}) := \sum_{l=1}^{L} \hat{\omega}_l \hat{\phi}(\hat{\xi}_l), \quad \hat{\omega}_l = |T|\tilde{\omega}_l,\ \hat{\xi}_l = M_T(\tilde{\xi}_l). \tag{5.12}$$

Note that, although not explicit in the notation, $\hat{\omega}_l$, $\hat{\xi}_l$ depend on $T$. We apply quadrature to the discrete problem (5.11) as follows. First we consider the approximation of the bilinear form $a(u_h, v_h)$. Using the correspondence $v_h \circ p_h = \hat{v}_h$, we can represent $a(u_h, v_h)$ as follows, cf. (5.11):

$$\begin{aligned} a(u_h, v_h) &= \int_{\hat{\Gamma}_h} G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h \, d\hat{s}_h = \sum_{T\in\mathcal{F}_h} \int_T G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h \, d\hat{s}_h \qquad (5.13)\\ &= \sum_{T\in\mathcal{F}_h} \sum_{i,j=1}^{3} \int_T G_{ij}\partial_i^\Gamma \hat{u}_h \partial_j^\Gamma \hat{v}_h \, d\hat{s}_h, \end{aligned}$$

with $\partial_i^\Gamma$ the $i$th component of the vector $\nabla_{\hat{\Gamma}_h} = \hat{\mathbf{P}}\nabla$. Quadrature results in an approximate bilinear form, given by

$$a_h(u_h, v_h) = \sum_{T\in\mathcal{F}_h} Q_T(G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h). \tag{5.14}$$

For the right hand-side functional $l(v_h) = \int_{\Gamma_h} f_h v_h \, ds_h = \int_{\hat{\Gamma}_h} (f \circ p_h - c_f)\hat{v}_h\hat{\mu}_h \, d\hat{s}_h$ we have the approximation

$$\begin{aligned} l_h(v_h) &= \sum_{T\in\mathcal{F}_h} Q_T\big(f_h^q \hat{\mu}_h \hat{v}_h\big), \quad \text{for all } v_h \in S_h,\ \hat{v}_h = v_h \circ p_h, \\ f_h^q &:= f \circ p_h - c_f^q, \quad c_f^q := \frac{1}{A}\sum_{T\in\mathcal{F}_h} Q_T(f\circ p_h\, \hat{\mu}_h), \quad A := \sum_{T\in\mathcal{F}_h} Q_T(\hat{\mu}_h). \end{aligned} \tag{5.15}$$

The constant shift $c_f^q$ is taken such that the consistency condition $l_h(1) = 0$ is satisfied. The *final discrete problem*, i.e., after quadrature, *is as follows*: Determine $u_h^q \in S_h$ with $\sum_{T\in\mathcal{F}_h} Q_T(\hat{u}_h^q \hat{\mu}_h) = 0$ such that

$$a_h(u_h^q, v_h) = l_h(v_h) \quad \text{for all} \ \ v_h \in S_h. \tag{5.16}$$

Remark 5. In lemma 9.3 below we show

$$a_h(v_h, v_h) \geq \gamma \|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)} \quad \text{for all} \ \ v_h \in S_h$$

under some conditions on $Q_T$. Recall the Poincaré-inequality $\|v_h\|_{L^2(\Gamma_h)} \leq c\|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)}$ for all $v_h \in S_h$ with $\int_{\Gamma_h} v_h \, ds_h = 0$. This implies that the variational problem (5.16) has in $S_h$ a solution that is unique apart from a shift with the constant function on $\Gamma_h$. This shift is uniquely determined by the condition $\sum_{T\in\mathcal{F}_h} Q_T(\hat{u}_h^q \hat{\mu}_h) = 0$, which is a computable approximation of the standard condition $\int_{\Gamma_h} u_h^q \, ds_h = 0$. Hence, the final discrete problem has a unique solution.

**6. Outline of the analysis.** In the sections 7-12 we present an error analysis of the discrete problem (5.16). The analysis is rather technical and contains ingredients that are not standard in the literature. Therefore we outline the structure and main ideas of the analysis.

Central in the analysis is the Strang Lemma 9.4, in which the discretization error is bounded by three different error components, namely an approximation error, a geometric error and a quadrature error. In the sections 10–12 bounds for these three components are derived. Preliminaries for the analysis of these error components are derived in the sections 7 and 8. In these sections, properties of the quasi-normal field $n_h$ and the skew projection $p_h$, which is used for the parametrization of $\Gamma_h$, are derived. In the discrete problem (5.16), besides the skew projection $p_h$ its Jacobian $Dp_h$ plays a prominent role. Key estimates for this Jacobian are derived in Lemma 8.2. An important result in this lemma is that by using suitable projections the error bound of order $\mathcal{O}(h^k)$ in (8.8) can be improved to $\mathcal{O}(h^{k+1})$ in (8.9), (8.10).
In section 12 the error due to quadrature is analyzed. As far as we know, such quadrature errors have not been considered in other papers that treat error analyses of finite element methods for surface partial differential equations. The quadrature issue, however, is essential for the analysis of our method. The reason for this is that the discrete problem *before* quadrature (5.11) contains an integrand that is *not* smooth on the triangles $T \in \mathcal{F}_h$.
This non-smoothness is caused by the use of the quasi-normal field, which is only Lipschitz. Due to the nonsmooth integrand, standard analyses of quadrature errors as in e.g. [3], do not yield satisfactory bounds. The analysis of the quadrature error in section 12 is based on the following idea. Consider an integral $\int_T Gg\, d\hat{s}_h$, with a function $g$ that is smooth on $T$ and a function $G$ that is not necessarily smooth on $T$. Assume that $G^s$ is a smooth approximation of $G$. For the quadrature error we use the splitting

$$
\begin{aligned}
E_T(Gg) &:= \int_T Gg\, d\hat{s}_h - Q_T(Gg) \\
&= \int_T (G - G^s) g\, d\hat{s}_h + E_T(G^s g) + Q_T\big((G^s - G)g\big).
\end{aligned}
$$

The error terms $\int_T (G - G^s) g\, d\hat{s}_h$ and $Q_T((G^s - G)g)$ can be controlled by suitable bounds for $G - G^s$ (as in Corollary 12.3). Since $G^s g$ is smooth the term $E_T(G^s g)$ can be bounded using standard quadrature error analysis. A smooth approximation of the (matrix) function $G$ is derived and analyzed in section 12.1.
One further "nonstandard" ingredient is the following. As expected, in the analysis of the quadrature error we use the affine transformation between a triangle $T \in \mathcal{F}_h$ and the unit triangle $\tilde{T}$ in $\mathbb{R}^2$. We also need the usual relation between norms on $T$ and on $\tilde{T}$ as given in (12.1). In this estimate a Sobolev norm on $\tilde{T}$ is bounded by the corresponding norm on $T$. In our setting, due to the fact that the triangulation $\mathcal{F}_h$ is not shape-regular (inner angles are not uniformly bounded away from zero), an estimate in the other direction, i.e., bounding $|\hat{u}|_{H_p^n(T)}$ by $|\tilde{u}|_{H_p^n(\tilde{T})}$, does *not* hold. Fortunately, only the estimate (12.1) and not the one in the other direction is needed in our analysis.

In the analysis different (skew) projections play a key role. For these projections we use boldface notation. For the readers convenience we summarize these projections and the normal fields that are used:

$$n: U \to \mathbb{R}^3 \;\text{ (exact normal on } \Gamma), \quad n_h: \Omega_{\Gamma_h} \to \mathbb{R}^3 \;\text{ (quasi-normal field)},$$
$$\hat{n}_h: \hat{\Gamma}_h \to \mathbb{R}^3 \;\text{ (exact normal on } \hat{\Gamma}_h), \quad \bar{n}_h: \Gamma_h \to \mathbb{R}^3 \;\text{ (exact normal on } \Gamma_h),$$
$$\mathbf{P}(x) = \mathbf{I} - n(x)n(x)^T, \quad x \in U, \tag{6.1}$$
$$\hat{\mathbf{P}}(x) = \mathbf{I} - \hat{n}_h(x)\hat{n}_h(x)^T, \quad x \in \hat{\Gamma}_h, \tag{6.2}$$
$$\bar{\mathbf{P}}(y) := \mathbf{I} - \bar{n}_h(y)\bar{n}_h(y)^T, \quad y \in \Gamma_h. \tag{6.3}$$
$$\hat{\mathbf{Q}}(x) = \mathbf{I} - \frac{1}{\hat{\alpha}(x)}\hat{n}_h(x)\bar{n}_h(y)^T, \quad \hat{\alpha}(x) = \hat{n}_h(x)^T\bar{n}_h(y), \quad y = p_h(x), \quad x \in \hat{\Gamma}_h, \tag{6.4}$$
$$\mathbf{Q}(x) = \mathbf{I} - \frac{1}{\alpha(x)}n_h(x)\bar{n}_h(x)^T, \quad \alpha(x) := n_h(x)^T\bar{n}_h(x), \quad x \in \Gamma_h. \tag{6.5}$$

We use the following notation in many proofs below: For any $x \in \hat{\Gamma}_h$ (sometimes $x \in U$), we let

$$y := p_h(x) \in \Gamma_h, \quad z := p(y) = p \circ p_h(x) \in \Gamma, \quad \text{and} \quad \zeta := p(x) \in \Gamma.$$

**7. Properties of $n_h$ and $p_h$.** In this section we derive some properties of the quasi-normal field $n_h$ and the skew projection $p_h$ onto $\Gamma_h$ that we need in the analysis further on. We start with a lemma in which it is shown that $Dn_h$ is close to a smooth (matrix) function.

LEMMA 7.1. *The following holds for all sufficiently small $h$:*

$$\|n_h - \frac{\nabla\phi}{\|\nabla\phi\|}\|_{L^\infty(\Omega_{\Gamma_h})} \le ch^k, \tag{7.1}$$
$$\|n_h - n\|_{L^\infty(\Gamma_h)} \le ch^k, \tag{7.2}$$
$$\|Dn_h - D\big(\frac{\nabla\phi}{\|\nabla\phi\|}\big)\|_{L^\infty(\Omega_{\Gamma_h})} \le ch^{k-1}. \tag{7.3}$$

*Proof.* For the gradient recovery operator applied to the finite element approximation $\phi_h$ of the level set function $\phi$ we write $G_h = G_h\phi_h$. Using (3.1),(3.2),(2.2) and standard interpolation error results we get

$$\begin{aligned} \|G_h - \nabla\phi\|_{L^\infty(U)} &\le \|G_h(\phi_h - I_h\phi)\|_{L^\infty(U)} + \|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \\ &\le c\|\phi_h - I_h\phi\|_{H^1_\infty(U^e)} + ch^k \le ch^k. \end{aligned} \tag{7.4}$$

From this and $n_h = \|G_h\|^{-1}G_h$ the result in (7.1) follows. Using this result we get, for $x \in \Omega_{\Gamma_h}$:

$$\begin{aligned} \|n_h(x) - n(x)\| &\le ch^k + \|n(x) - \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|}\| \\ &= ch^k + \|\frac{\nabla\phi(p(x))}{\|\nabla\phi(p(x))\|} - \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|}\|. \end{aligned}$$

For $x \in \Gamma_h$ we have $\|x - p(x)\| \le ch^{k+1}$ and thus $\|\nabla\phi(x) - \nabla\phi(p(x))\| \le ch^{k+1}$. Hence we get the result (7.2). For the derivatives we have

$$\begin{aligned} Dn_h = D\big(\frac{G_h}{\|G_h\|}\big) &= \frac{1}{\|G_h\|}\big(I - \frac{1}{\|G_h\|^2}G_hG_h^T\big)DG_h \\ D\big(\frac{\nabla\phi}{\|\nabla\phi\|}\big) &= \frac{1}{\|\nabla\phi\|}\big(I - \frac{1}{\|\nabla\phi\|^2}\nabla\phi\nabla\phi^T\big)D^2\phi. \end{aligned} \tag{7.5}$$

Using an inverse inequality and the result in (7.4) we obtain

$$\begin{aligned}\|G_h-\nabla\phi\|_{H^1_\infty(U)} &\le \|G_h-I_h(\nabla\phi)\|_{H^1_\infty(U)}+\|I_h(\nabla\phi)-\nabla\phi\|_{H^1_\infty(U)}\\ &\le ch^{-1}\|G_h-I_h(\nabla\phi)\|_{L^\infty(U)}+ch^k\\ &\le ch^{-1}\big(\|G_h-\nabla\phi\|_{L^\infty(U)}+\|\nabla\phi-I_h(\nabla\phi)\|_{L^\infty(U)}\big)+ch^k\\ &\le ch^{k-1}.\end{aligned}$$

From this it follows that $\|DG_h-D^2\phi\|_{L^\infty(U)}\le ch^{k-1}$ holds. Using this and the result in (7.4) in combination with the formulas (7.5) proves the result in (7.3). ◻

The next lemma quantifies how well $p_h$ approximates $p$ and $d_h$ approximates $d$.

Lemma 7.2. *For $h$ sufficiently small the following holds:*

$$\|p_h-p\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1},\quad \|p\circ p_h-p\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1}, \tag{7.6}$$

$$\|d_h-d\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1}. \tag{7.7}$$

*Proof.* Take $x\in\Omega_{\Gamma_h}$ and $q\in\Gamma_h$, $\bar d\in\mathbb{R}$ such that $q=x-\bar d\,n(p(x))=x-\bar d\,n(q)$. Note that $p(x)=p(q)$ holds. Using $\mathrm{diam}(\Omega_{\Gamma_h})\le ch$ we get

$$|\bar d|=\|q-x\|\le\|q-p(q)\|+\|p(x)-x\|\le \mathrm{dist}(\Gamma_h,\Gamma)+|d(x)|\le ch. \tag{7.8}$$

In Lemma 4.1 in [14] is it shown that $d_h$ is uniformly Lipschitz on $U$, i.e. there is a constant $c$ such that $|d_h(z_1)-d_h(z_2)|\le c\|z_1-z_2\|$ for all $z_1,z_2\in U$. Since $q\in\Gamma_h$ we have $d_h(q)=0$. Using this we get

$$|d_h(x)|=|d_h(x)-d_h(q)|\le c\|x-q\|=c|\bar d|\le ch.$$

From $p_h(x)=x-d_h(x)n_h(p_h(x))$ we obtain $p_h(x)-q=\bar d\,n(q)-d_h(x)n_h(p_h(x))$, and thus

$$\|p_h(x)-q\|^2=\bar d\langle n(q),p_h(x)-q\rangle-d_h(x)\langle n_h(p_h(x)),p_h(x)-q\rangle.$$

For the first term we have, using (7.2) and (3.5),

$$\begin{aligned}|\langle n(q),p_h(x)-q\rangle| &\le |\langle n(q)-n_h(q),p_h(x)-q\rangle|+|\langle n_h(q),p_h(x)-q\rangle|\\ &\le ch^k\|p_h(x)-q\|+c\|p_h(x)-q\|^2.\end{aligned}$$

By (3.5), for the second term:

$$|\langle n_h(p_h(x)),p_h(x)-q\rangle|\le ch^k\|p_h(x)-q\|+c\|p_h(x)-q\|^2.$$

Hence,

$$\|p_h(x)-q\|^2\le ch^{k+1}\|p_h(x)-q\|+ch\|p_h(x)-q\|^2,$$

and thus, for $h$ sufficiently small, $\|p_h(x)-q\|\le ch^{k+1}$ holds. Using $\|q-p(x)\|=\|q-p(q)\|\le\mathrm{dist}(\Gamma_h,\Gamma)\le ch^{k+1}$ we thus get $\|p_h(x)-p(x)\|\le ch^{k+1}$, which proves the first estimate in (7.6). The second result in (7.6) follows from a triangle inequality:

$$\|p(p_h(x))-p(x)\|\le\|p(p_h(x))-p_h(x)\|+\|p_h(x)-p(x)\|\le\mathrm{dist}(\Gamma_h,\Gamma)+ch^{k+1}\le ch^{k+1}.$$

For the result in (7.7) we use the representation

$$\begin{aligned} d_h(x) - d(x) &= \langle x - p_h(x), n_h(p_h(x))\rangle - \langle x - p(x), n(p(x))\rangle \\ &= \langle p(x) - p_h(x), n(p(x))\rangle + \langle x - p_h(x), n_h(p_h(x)) - n(p_h(x))\rangle \\ &+ \langle x - p_h(x), n(p_h(x)) - n(p(x))\rangle. \end{aligned}$$

The result in (7.7) follows from this if we use

$$\|p(x) - p_h(x)\| \le ch^{k+1}, \quad \|x - p_h(x)\| = |d_h(x)| \le ch,$$
$$\|n_h(p_h(x)) - n(p_h(x))\| \le ch^k, \quad \|n(p_h(x)) - n(p(x))\| \le ch^{k+1},$$

which completes the proof □

**8. Properties of the Jacobian $Dp_h$.** The Jacobian $Dp_h$ plays a key role in the discretization (5.11) (cf. definition of $W$ and $\hat{\mu}_h$). In this section we derive properties of this Jacobian that we need in our analysis. First we consider the Jacobian of the *exact* projection $p$ onto $\Gamma$ given in (2.1). Differentiating the relation (2.1) and using $n(x) = n(p(x))$ we get, for $x \in U$,

$$\big(\mathbf{I} + d(x)H(p(x))\big)Dp(x) = \mathbf{P}(x), \quad \mathbf{P}(x) = \mathbf{I} - n(x)n(x)^T, H(y) = Dn(y). \tag{8.1}$$

This formula has equivalent representations due to $\mathbf{P}(x) = \mathbf{P}(p(x))$ and $H(p(x)) = \mathbf{P}(x)H(p(x)) = H(p(x))\mathbf{P}(x)$. We derive a formula for $Dp_h$, cf. Lemma 8.1 below. It turns out that we need a skew projection as a substitute for the projection $\mathbf{P}$ in (8.1). This skew projection is as in (6.5):

$$\mathbf{Q}(x) = \mathbf{I} - \frac{1}{\alpha(x)} n_h(x)\bar{n}_h(x)^T, \quad \alpha(x) := n_h(x)^T \bar{n}_h(x), \quad x \in \Gamma_h.$$

The following relations hold, with $\bar{\mathbf{P}}$ as in (6.3):

$$\mathbf{Q}\bar{\mathbf{P}} = \bar{\mathbf{P}}, \quad \bar{\mathbf{P}}\mathbf{Q} = \mathbf{Q}. \tag{8.2}$$

LEMMA 8.1. *For a. e. $x \in U$, the following relations hold with $y = p_h(x) \in \Gamma_h$,*

$$\big(\mathbf{I} + d_h(x)\mathbf{Q}(y)Dn_h(y)\big)Dp_h(x) = \mathbf{Q}(y), \tag{8.3}$$
$$\mathbf{Q}(y)Dp_h(x) = Dp_h(x) = Dp_h(x)\mathbf{Q}(y). \tag{8.4}$$

*Proof.* Let $\bar{d}_h$ be the exact signed distance function to $\Gamma_h$. Differentiating $\bar{d}_h(p_h(x)) = 0$, which holds for a.e. $x \in U$, yields

$$\bar{n}_h(y)^T Dp_h(x) = 0. \tag{8.5}$$

Applying this to the differential of $p_h = id - d_h \cdot n_h \circ p_h$,

$$Dp_h = \mathbf{I} - d_h Dn_h(y)Dp_h - n_h(y)\nabla d_h^T, \tag{8.6}$$

yields $0 = \bar{n}_h(y)^T - d_h\bar{n}_h(y)^T Dn_h(y)Dp_h - \bar{n}_h(y)^T n_h(y)\nabla d_h^T$. This can be solved for $\nabla d_h^T$,

$$\nabla d_h^T = \frac{1}{\alpha}\left(\bar{n}_h(y)^T - d_h\bar{n}_h(y)^T Dn_h(y)Dp_h\right).$$

Inserting this into (8.6) and rearranging completes the proof of (8.3). The equation $\mathbf{Q}(y)Dp_h = Dp_h$ follows immediately from (8.5) and the definition of $\mathbf{Q}$. The equation $Dp_h(x) = Dp_h(x)\mathbf{Q}(y)$ follows from (8.3). □

Below, we frequently use that $\mathbf{I} + M$, $M \in \mathbb{R}^{n\times n}$, is invertible if $\rho(M) < 1$ and that

$$(\mathbf{I} + M)^{-1} = \mathbf{I} - (\mathbf{I} + M)^{-1}M, \quad M \in \mathbb{R}^{n\times n}, \quad \rho(M) < 1. \tag{8.7}$$

LEMMA 8.2. *For sufficiently small $h$, the following holds, with projections $\mathbf{P}$, $\hat{\mathbf{P}}$, $\bar{\mathbf{P}}$ defined in* (6.1),(6.2), (6.3)*:*

$$\|Dp - Dp_h\|_{L^\infty(\hat{\Gamma}_h)} \le ch^k, \tag{8.8}$$

$$\|\mathbf{P}\big(Dp - Dp_h\big)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}, \tag{8.9}$$

$$\|(\bar{\mathbf{P}} \circ p_h)\big(Dp - Dp_h\big)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}. \tag{8.10}$$

*Proof.* Let $x \in \hat{\Gamma}_h$ be arbitrary and $\zeta = p(x) \in \Gamma$. Let $\tilde{n} = \nabla\phi/\|\nabla\phi\|$ which is defined on $U$. As $\tilde{n} \equiv n$ on $\Gamma$, we have $p(x) = x - d(x)\tilde{n}(\zeta)$. Differentiating this relation we obtain the following representation for the Jacobian $Dp$:

$$Dp(x) = \big(\mathbf{I} + d(x)\mathbf{P}(\zeta)D\tilde{n}(\zeta)\big)^{-1}\mathbf{P}(\zeta) = (\mathbf{I} + B_1)^{-1}\mathbf{P}(\zeta), \quad B_1 = d(x)\mathbf{P}(\zeta)D\tilde{n}(\zeta).$$

From (8.3) we get, with $y = p_h(x) \in \Gamma_h$:

$$Dp_h(x) = \big(\mathbf{I}+d_h(x)\mathbf{Q}(y)Dn_h(y)\big)^{-1}\mathbf{Q}(y) = (\mathbf{I}+B_2)^{-1}\mathbf{Q}(y),\ B_2 = d_h(x)\mathbf{Q}(y)Dn_h(y).$$

Using (7.6) we get $\|\zeta - y\| \le ch^{k+1}$. Define $R_i := (\mathbf{I} + B_i)^{-1}B_i$, hence, by (8.7), $(\mathbf{I} + B_i)^{-1} = \mathbf{I} - R_i$. From $|d_h(x)| \le ch^2$, $|d(x)| \le ch^2$ and the definition of $B_i$ we get $\|R_i\| \le ch^2$. From the definitions we obtain

$$Dp(x) - Dp_h(x) = \mathbf{P}(\zeta) - \mathbf{Q}(y) + R_2\mathbf{Q}(y) - R_1\mathbf{P}(\zeta), \tag{8.11}$$

$$\begin{aligned}\mathbf{P}(\zeta) - \mathbf{Q}(y) &= \big(\frac{1}{\alpha(y)} - 1\big)n_h(y)\bar{n}_h(y)^T \\ &\quad + \big(n_h(y) - n(\zeta)\big)\bar{n}_h(y)^T + n(\zeta)\big(\bar{n}_h(y) - n(\zeta)\big)^T.\end{aligned} \tag{8.12}$$

Using (7.2) and (2.3) we get

$$\|n_h(y) - n(y)\| \le ch^k, \quad \|\bar{n}_h(y) - n(y)\| \le ch^k,$$

and combining this with $|\alpha(y) - 1| = \frac{1}{2}\|n_h(y) - \bar{n}_h(y)\|^2$, the smoothness of $n$ and $\|\zeta - y\| \le ch^{k+1}$, we obtain

$$\big|\frac{1}{\alpha(y)} - 1\big| \le ch^{2k}, \tag{8.13}$$

$$\|n_h(y) - n(\zeta)\| \le \|n_h(y) - n(y)\| + \|n(y) - n(\zeta)\| \le ch^k, \tag{8.14}$$

$$\|\bar{n}_h(y) - n(\zeta)\| \le \|\bar{n}_h(y) - n(y)\| + \|n(y) - n(\zeta)\| \le ch^k. \tag{8.15}$$

From these results and (8.12) we get

$$\|\mathbf{Q}(y) - \mathbf{P}(\zeta)\| \le ch^k. \tag{8.16}$$

Using the smoothness of $\tilde{n}$ and the result in (7.3) we obtain:

$$\|D\tilde{n}(\zeta) - Dn_h(y)\| \le \|D\tilde{n}(\zeta) - D\tilde{n}(y)\| + \|D\tilde{n}(y) - Dn_h(y)\| \le ch^{k-1}.$$

Combining this with $|d(x)| \le ch^2$ and $|d(x) - d_h(x)| \le ch^{k+1}$, cf. (7.7), yields

$$\begin{aligned} &\|R_1 - R_2\| = \|(\mathbf{I} + B_1)^{-1} - (\mathbf{I} + B_2)^{-1}\| \le c\|B_1 - B_2\| \qquad (8.17)\\ &\le c|d(x)|\|\mathbf{P}(\zeta) - \mathbf{Q}(y)\| + c|d(x)|\|D\tilde{n}(\zeta) - Dn_h(y)\| + c|d(x) - d_h(x)| \le ch^{k+1}. \end{aligned}$$

Combining this and (8.16) with the result in (8.11) proves the result in (8.8).

Let $\tilde{\mathbf{P}}$ denote either $\mathbf{P}(x)$ or $\bar{\mathbf{P}}(y)$. Using (8.11)-(8.12) we get

$$\begin{aligned} &\tilde{\mathbf{P}}\big(Dp(x) - Dp_h(x)\big)\hat{\mathbf{P}}(x) = M\hat{\mathbf{P}}(x) + R\hat{\mathbf{P}}(x),\\ &\qquad M := \tilde{\mathbf{P}}\Big(\big(n_h(y) - n(\zeta)\big)\bar{n}_h(y)^T - n(\zeta)\big(\bar{n}_h(y) - n(\zeta)\big)^T\Big)\hat{\mathbf{P}}(x), \qquad (8.18)\\ &\qquad R := \tilde{\mathbf{P}}\big(R_2\mathbf{Q}(y) - R_1\mathbf{P}(\zeta)\big) + \big(\frac{1}{\alpha(y)} - 1\big)\tilde{\mathbf{P}}n_h(y)\bar{n}_h(y)^T. \end{aligned}$$

Combining (8.16) and $\|R_i\| \le ch^2$ with (8.13), (8.17) we get $\|R\| \le ch^{k+1}$. We finally consider the term $M$. From

$$\|\hat{n}_h(x) - \bar{n}_h(y)\| \le \|\hat{n}_h(x) - n(x)\| + \|n(x) - n(y)\| + \|n(y) - \bar{n}_h(y)\| \le ch$$

we get

$$\|\bar{n}_h(y)^T\hat{\mathbf{P}}(x)\| = \|\hat{\mathbf{P}}(x)\big(\bar{n}_h(y) - \hat{n}_h(x)\big)\| \le ch. \qquad (8.19)$$

If $\tilde{\mathbf{P}} = \mathbf{P}(x)$ then $\tilde{\mathbf{P}}n(\zeta) = \tilde{\mathbf{P}}n(x) = 0$ holds. If $\tilde{\mathbf{P}} = \bar{\mathbf{P}}(y)$ we obtain from (8.15):

$$\|\tilde{\mathbf{P}}n(\zeta)\| = \|\bar{\mathbf{P}}(y)\big(n(\zeta) - \bar{n}_h(y)\big)\| \le \|n(\zeta) - \bar{n}_h(y)\| \le ch^k.$$

Combining these results we get

$$\|M\| \le \|n_h(y) - n(\zeta)\|\|\bar{n}_h(y)^T\hat{\mathbf{P}}(x)\| + \|\tilde{\mathbf{P}}n(\zeta)\|\|\bar{n}_h(y) - n(\zeta)\| \le ch^{k+1},$$

which completes the proof. □

The estimates in lemma 8.2 play a key role in the error analysis of our method. In Section 14 we give results of a numerical experiment which show that the bounds in the estimates are sharp. In particular, for obtaining the $h^{k+1}$ bounds the projections in the terms on the left hand-side are essential.

**9. Strang Lemma.** In this section we derive a Strang lemma. In the analysis we will need the constant extension of a function $w$ on $\Gamma$ along the normals $n$ to a function $w^e$ on $U$ given by

$$w^e(x) := w \circ p(x) \quad \text{for all } x \in U. \qquad (9.1)$$

We also use the *lift* of a function defined on $\Gamma_h$ (or on $\hat{\Gamma}_h$) to a function defined on $\Gamma$ along the normals $n$. More precisely, for a function $w$ defined on $\Gamma_h$ (or on $\hat{\Gamma}_h$), its lift $w^\ell$ to $\Gamma$ is given by

$$w^\ell \circ p(x) = w(x) \quad \text{for all } x \in \Gamma_h \text{ (or } x \in \hat{\Gamma}_h). \qquad (9.2)$$

The lifted finite element space is denoted by $S_h^\ell := \{ v_h^\ell \mid v_h \in S_h \}$. We need the following matrix function on $\Gamma$:

$$A_\Gamma(p(x)) = \frac{1}{\mu_h(x)} \mathbf{P}(x)[I - d(x)H(x)]\bar{\mathbf{P}}(x)[I - d(x)H(x)]\mathbf{P}(x), \quad x \in \Gamma_h, \tag{9.3}$$

with $\mu_h$ as in (2.4) and the projectors as in (6.1), (6.3). From [5, formula (2.14)] we have the integral identity

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \, ds_h = \int_\Gamma A_\Gamma \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell \, ds.$$

Using this we obtain that if $u_h$ solves (5.4), then the lifted function $u_h^\ell \in S_h^\ell$ satisfies

$$\int_\Gamma A_\Gamma \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h \, ds = \int_\Gamma \frac{1}{\mu_h^\ell} f_h^\ell v_h \, ds \quad \text{for all} \quad v_h \in S_h^\ell. \tag{9.4}$$

We also need the following estimates, which follow from the results (2.11), (2.12) in [5]:

$$\begin{aligned} \|\nabla_\gamma v^e\|_{L^2(\gamma)} &\le c\|\nabla_\Gamma v\|_{L^2(\Gamma)}, \quad v \in H^1(\Gamma), \gamma \in \{\hat{\Gamma}_h, \Gamma_h\}, \\ \|\nabla_\Gamma v^\ell\|_{L^2(\Gamma)} &\le c\|\nabla_\gamma v\|_{L^2(\gamma)}, \quad v \in H^1(\gamma), \gamma \in \{\hat{\Gamma}_h, \Gamma_h\}. \end{aligned} \tag{9.5}$$

We first derive ellipticity of the bilinear form $a_h(\cdot,\cdot)$ in (5.14). For this we need that the matrix $G$, cf. (5.11), is positive definite. To derive this result, in the next lemma we first consider the matrix $W$.

Lemma 9.1. *For $h$ sufficiently small the following holds:*

$$\|W - \mathbf{I}\|_{L^\infty(\hat{\Gamma}_h)} \le ch. \tag{9.6}$$

*Proof.* We recall the definition $W(x) = \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x) Dp_h(x)^T$. We drop the $x$-dependence in the notation. From (8.3) we get, due to $|d_h(x)| \le ch^2$, for $h$ sufficiently small, $Dp_h(x) = \big(\mathbf{I} + d_h(x)\mathbf{Q}(y)Dn_h(y)\big)^{-1}\mathbf{Q}(y) = \mathbf{Q}(y) + \mathcal{O}(h^2)$, with $y = p_h(x)$. Hence,

$$W = \mathbf{I} - \hat{\mathbf{Q}} + \hat{\mathbf{P}}\mathbf{Q}(y)^T + \mathcal{O}(h^2)$$

holds. Using (7.6) we get $\|x - y\| = \|x - p_h(x)\| \le \|x - p(x)\| + \|p(x) - p_h(x)\| \le \operatorname{dist}(\Gamma, \hat{\Gamma}_h) + ch^{k+1} \le ch^2$. Using this and the results in (2.3), we obtain

$$\|\hat{n}_h(x) - \bar{n}_h(y)\| \le \|\hat{n}_h(x) - n(x)\| + \|n(x) - n(y)\| + \|n(y) - \bar{n}_h(y)\| \le ch.$$

Hence,

$$\|\hat{\mathbf{P}}\bar{n}_h(y)\| = \|\hat{\mathbf{P}}(\bar{n}_h(y) - \hat{n}_h(x))\| \le \|\bar{n}_h(y) - \hat{n}_h(x)\| \le ch.$$

Using this, the definitions of the projections and $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}\hat{\mathbf{P}}$, cf. (5.8), yields

$$\begin{aligned} \|W - \mathbf{I}\| &\le \|(\mathbf{I} - \hat{\mathbf{Q}})\hat{\mathbf{P}}\| + \|\hat{\mathbf{P}}(\mathbf{Q}(y)^T - \mathbf{I})\| + ch^2 \\ &= \frac{1}{|\hat{\alpha}|}\|\hat{n}_h \bar{n}_h(y)^T \hat{\mathbf{P}}\| + \frac{1}{|\alpha(y)|}\|\hat{\mathbf{P}}\bar{n}_h(y) n_h(y)^T\| + ch^2 \\ &\le c\|\hat{\mathbf{P}}\bar{n}_h(y)\| + ch^2 \le ch, \end{aligned}$$

which completes the proof. □

COROLLARY 9.2. *For $h$ sufficiently small the matrix $G(x) = T_h(x)^{-1}\hat{\mu}_h(x)$, $x \in \hat{\Gamma}_h$, is uniformly symmetric positive definite, i.e. there exists a constant $\lambda_{\min}(G) > 0$ such that*

$$z^T G(x) z \geq \lambda_{\min}(G)\|z\|^2 \quad \textit{for almost all} \;\; x \in \hat{\Gamma}_h \quad \textit{and all} \;\; z \in \mathbb{R}^3.$$

*Proof.* From lemma 9.1 and (5.10), (5.11), we obtain for all sufficiently small $h$ and arbitrary $z \in \mathbb{R}^3$ that

$$z^T G(x) z = \hat{\mu}_h(x)\|W(x)^{-1}z\|^2 \geq \frac{\hat{\mu}_h(x)}{2}\|z\|^2, \quad x \in \hat{\Gamma}_h.$$

We recall the definition $\hat{\mu}_h(x) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x) U)}$. The matrix $U$ depends on the triangle $T \in \mathcal{F}_h$ and satisfies $\hat{\mathbf{P}}(x)U = U$, cf. Lemma 4.1. For $h$ sufficiently small we have $Dp_h(x) = \mathbf{Q}(y) + \mathcal{O}(h^2)$, $y = p_h(x)$. With $\zeta = p(x)$, we have $\mathbf{P}(\zeta) = \mathbf{P}(x)$. From (2.3) and $n(x)n(x)^T - \hat{n}_h(x)\hat{n}_h(x)^T = n(x)\big(n(x) - \hat{n}_h(x)\big)^T + \big(n(x) - \hat{n}_h(x)\big)\hat{n}_h(x)^T$ it follows that

$$\|\mathbf{P} - \hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \leq ch. \tag{9.7}$$

Using this and the result in (8.16), we get

$$\|\mathbf{Q}(y) - \hat{\mathbf{P}}(x)\| \leq \|\mathbf{Q}(y) - \mathbf{P}(\zeta)\| + \|\mathbf{P}(\zeta) - \mathbf{P}(x)\| + \|\mathbf{P}(x) - \hat{\mathbf{P}}(x)\| \leq ch.$$

This yields $Dp_h(x) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$ and consequently

$$U^T Dp_h(x)^T Dp_h(x) U = U^T \hat{\mathbf{P}}(x)\hat{\mathbf{P}}(x) U + \mathcal{O}(h) = U^T U + \mathcal{O}(h) = \mathbf{I} + \mathcal{O}(h).$$

Thus, for $h$ sufficiently small, we have that

$$\hat{\mu}_h(x) = 1 + \mathcal{O}(h), \quad x \in \hat{\Gamma}_h, \tag{9.8}$$

is uniformly (in $x$) bounded from below by a strictly positive constant. □

Using the result of the previous corollary we can derive ellipticity of the bilinear form $a_h(\cdot,\cdot)$:

LEMMA 9.3. *Assume that the quadrature rule $Q_{\tilde{T}}$ is exact for all polynomials of degree $2m-2$. There exists a constants $\gamma > 0$ and $h_0 > 0$ such that for all $h \leq h_0$*

$$a_h(v_h, v_h) \geq \gamma\|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)} \quad \textit{for all} \;\; v_h \in S_h. \tag{9.9}$$

*Proof.* Let $h$ be sufficiently small such that the matrix $G$ is uniformly positive definite, cf. Corollary 9.2. From this positive definiteness and the fact that the quadrature weights are strictly positive we get

$$Q_T(G\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h) \geq c_0 Q_T\big(\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|^2\big),$$

with $c_0 = \lambda_{\min}(G) > 0$, independent of $h$. Since the quadrature rule $Q_{\tilde{T}}$ on $\tilde{T}$ is exact for all polynomials of degree $2m-2$ and the mapping $M_T$ between $\tilde{T}$ and $T$

is affine this exactness property also holds for $Q_T$ on $T$. The functions $(\partial_i^\Gamma \hat{v}_h)^2$ are polynomials of degree $2m-2$ on $T$, and thus we have

$$a_h(v_h, v_h) \geq c_0 \sum_{T\in\mathcal{F}_h} Q_T\big(\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|^2\big) = c_0\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|^2_{L^2(\hat{\Gamma}_h)} \geq \gamma\|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)}$$

with $\gamma > 0$ due to (5.9) and (9.6). ◻

Based on this ellipticity property we apply standard arguments to derive the following variant of the Strang Lemma.

THEOREM 9.4. *Assume $h$ is sufficiently small such that $a_h(\cdot,\cdot)$ has the ellipticity property* (9.3). *Define the data extension error $\tilde{E}_f := \|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)}$. For the solution $u_h^q$ of* (5.16) *the following error bound holds:*

$$\begin{aligned}
&\|\nabla_\Gamma(u-(u_h^q)^\ell)\|_{L^2(\Gamma)} \\
&\leq c \min_{v_h\in S_h}\Big[\|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + \|(I-A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h}v_h\|_{L^2(\Gamma_h)} \\
&+ \sup_{w_h\in S_h/\mathbb{R}} \frac{a(v_h,w_h)-a_h(v_h,w_h)}{\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}}\Big] + c\sup_{w_h\in S_h/\mathbb{R}}\frac{l(w_h)-l_h(w_h)}{\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}} + \tilde{E}_f.
\end{aligned} \tag{9.10}$$

*Proof.* Take an arbitrary $v_h \in S_h$. We start with a triangle inequality and (9.5):

$$\begin{aligned}
\|\nabla_\Gamma(u-(u_h^q)^\ell)\|_{L^2(\Gamma)} &\leq \|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + \|\nabla_\Gamma(v_h^\ell-(u_h^q)^\ell)\|_{L^2(\Gamma)} \\
&\leq \|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + c\|\nabla_{\Gamma_h}(v_h-u_h^q)\|_{L^2(\Gamma_h)}.
\end{aligned}$$

We derive a bound for $\|\nabla_{\Gamma_h}e_h\|_{L^2(\Gamma_h)}$, $e_h := u_h^q - v_h$. Let $c_1$ be a constant such that $\tilde{e}_h := e_h + c_1$ satisfies $\int_{\Gamma_h}\tilde{e}_h\, ds_h = 0$. For arbitrary constants $c$, there holds $\nabla_{\Gamma_h}c \equiv 0$. In particular, by (5.14), we get the consistency property $a_h(c,\tilde{e}_h) = 0$. Using this, (9.9), and the definition of the discrete problems (5.4), (5.16), we obtain

$$\begin{aligned}
&\|\nabla_{\Gamma_h}e_h\|^2_{L^2(\Gamma_h)} = \|\nabla_{\Gamma_h}\tilde{e}_h\|^2_{L^2(\Gamma_h)} \leq \gamma^{-1}a_h(e_h,\tilde{e}_h) \\
&= \gamma^{-1}\big(l_h(\tilde{e}_h) - l(\tilde{e}_h) + a(u_h,\tilde{e}_h) - a_h(v_h,\tilde{e}_h)\big) \\
&= \gamma^{-1}\big(a(u_h-v_h,e_h) + a(v_h,\tilde{e}_h) - a_h(v_h,\tilde{e}_h) + l_h(\tilde{e}_h) - l(\tilde{e}_h)\big).
\end{aligned} \tag{9.11}$$

We will derive the bound

$$\begin{aligned}
a(u_h-v_h,e_h) \leq c\big(&\|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + \|(I-A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h}v_h\|_{L^2(\Gamma_h)} \\
&+ \tilde{E}_f\big)\|\nabla_{\Gamma_h}e_h\|_{L^2(\Gamma_h)},
\end{aligned} \tag{9.12}$$

and combination of this with the relation (9.11) and the triangle inequality above proves the result (9.10). For the derivation of (9.12) we note, cf. (9.4), that for all $w_h \in S_h^\ell$ we have

$$\begin{aligned}
\int_\Gamma A_\Gamma\nabla_\Gamma u_h^\ell\cdot\nabla_\Gamma w_h\, ds &= \int_\Gamma \frac{1}{\mu_h^\ell}f_h^\ell w_h\, ds = \int_\Gamma\big(\frac{1}{\mu_h^\ell}f_h^\ell - f\big)w_h\, ds + \int_\Gamma f w_h\, ds \\
&= \int_\Gamma\big(\frac{1}{\mu_h^\ell}f_h^\ell - f\big)w_h\, ds + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma w_h\, ds.
\end{aligned}$$

Let $\bar{c}$ be a constant that is chosen below and $\bar{e}_h := e_h - \bar{c}$. From the previous equation,

$$
\begin{aligned}
a(u_h - v_h, e_h) &= a(u_h - v_h, \bar{e}_h) = \int_{\Gamma_h} \nabla_{\Gamma_h}(u_h - v_h)\cdot \nabla_{\Gamma_h}\bar{e}_h \, ds_h \\
&= \int_\Gamma A_\Gamma \nabla_\Gamma (u_h^\ell - v_h^\ell)\cdot \nabla_\Gamma \bar{e}_h^\ell \, ds \qquad (9.13)\\
&= \int_\Gamma \nabla_\Gamma (u - v_h^\ell)\cdot \nabla_\Gamma e_h^\ell \, ds + \int_\Gamma \big(\frac{1}{\mu_h^\ell} f_h^\ell - f\big)\bar{e}_h^\ell \, ds + \int_\Gamma (I - A_\Gamma)\mathbf{P}\nabla_\Gamma v_h^\ell \cdot \nabla_\Gamma e_h^\ell \, ds.
\end{aligned}
$$

holds. Now $\bar{c}$ is chosen as $\bar{c} := \frac{1}{|\Gamma|}\int_{\Gamma_h} \mu_h e_h \, ds_h$ such that we have

$$\int_\Gamma \bar{e}_h^\ell \, ds = \int_\Gamma e_h^\ell \, ds - \int_\Gamma \bar{c}\, ds = \int_{\Gamma_h} \mu_h e_h \, ds_h - |\Gamma|\bar{c} = 0.$$

Hence, the Poincare inequality $\|\bar{e}_h^\ell\|_{L^2(\Gamma)} \le c\|\nabla_\Gamma e_h^\ell\|_{L^2(\Gamma)}$ holds. Using this, the Cauchy-Schwarz inequality and $\|\nabla_\Gamma e_h^\ell\|_{L^2(\Gamma)} \le c\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}$, cf. (9.5), in (9.13), we get the estimate (9.12). □

In the total error there are three different components, namely a *geometric error* (approximation of $\Gamma$ by $\Gamma_h$), an *approximation error* (results from using the finite element space) and a *quadrature error*. In section 10 we study the first term on the right hand-side in (9.10), which quantifies the approximation error. The second term and the fifth term are related to the geometric error and are analyzed in section 11. The third and fourth term on the right hand-side in (9.10) arise from the quadrature and are treated in section 12.

**10. Approximation error.** For the analysis of the approximation error we assume the following approximation quality of the finite element space $\hat{S}_h$ on $\hat{\Gamma}_h$, cf. (5.1): there are $m \ge 1$ and an interpolation operator $I_h : H^{m+1}(\Gamma) \to \hat{S}_h$ such that for $s = 0, \ldots, m$:

$$\sum_{T \in \mathcal{F}_h} \|w^e - I_h w\|_{H^s(T)}^2 \le c h^{2(m+1-s)} \|w\|_{H^{m+1}(\Gamma)}^2 \quad \text{for all } w \in H^{m+1}(\Gamma). \qquad (10.1)$$

Such an approximation property holds for the two possible choices for $\hat{S}_h$ mentioned in Remark 3. The estimate (10.1) for $s = 1$ implies

$$\|\nabla_{\hat{\Gamma}_h}(w^e - I_h w)\|_{L^2(\hat{\Gamma}_h)} \le c h^m \|w\|_{H^{m+1}(\Gamma)} \quad \text{for all } \; w \in H^{m+1}(\Gamma). \qquad (10.2)$$

In the analysis we use the spaces $\hat{S}_h$ (on $\hat{\Gamma}_h$), cf. (5.1), $S_h$ (on $\Gamma_h$), cf. (5.2), and the lifted space $S_h^\ell$ (on $\Gamma$), cf. (9.2). The analysis requires smoothness of the solution of (1.1),

Assumption 10.1. *The solution $u$ of* (1.1) *satisfies $u \in H^{m+1}(\Gamma) \cap H^2_\infty(\Gamma)$.*

In the analysis below we use the following test function $u_{h,*} \in S_h$ to prove an upper bound for the minimum over $v_h \in S_h$ in (9.10):

$$u_{h,*} := \hat{u}_{h,*} \circ p_h^{-1} \in S_h, \quad \hat{u}_{h,*} := I_h u \in \hat{S}_h. \qquad (10.3)$$

Theorem 10.1. *Let $m \ge 1$ be such that* (10.2) *and assumption 10.1 are fulfilled. For $h$ sufficiently small the following holds, with $u_{h,*}$ as in* (10.3)*:*

$$\min_{v_h \in S_h} \|\nabla_\Gamma (u - v_h^\ell)\|_{L^2(\Gamma)} \le \|\nabla_\Gamma (u - u_{h,*}^\ell)\|_{L^2(\Gamma)} \le c h^m \|u\|_{H^{m+1}(\Gamma)} + c h^{k+1} \|u\|_{H^2_\infty(\Gamma)}$$

*Proof.* The test functions in (10.3) satisfy

$$u_{h,*} \circ p_h(x) = \hat{u}_{h,*}(x), \quad u_{h,*}^\ell \circ p(y) = u_{h,*}(y), \quad y := p_h(x) \in \Gamma_h,\ x \in \hat{\Gamma}_h,$$

cf. (5.2), (9.2). Using $\hat{u}_* := u^e \in H^1(\hat{\Gamma}_h)$, we define $u_* \in H^1(\Gamma_h)$, and $\tilde{u} \in H^1(\Gamma)$ by

$$u_* \circ p_h(x) = \hat{u}_*(x), \quad \tilde{u} \circ p(y) = u_*(y), \quad y := p_h(x) \in \Gamma_h,\ x \in \hat{\Gamma}_h.$$

Note that $\tilde{u} = u_*^\ell$ on $\Gamma$ holds. From (5.9) it follows that $\|\nabla_{\Gamma_h}\big(\hat{v} \circ p_h^{-1}\big)\|_{L^2(\Gamma_h)} \leq c\|\nabla_{\hat{\Gamma}_h}\hat{v}\|_{L^2(\hat{\Gamma}_h)}$ for all $\hat{v} \in H^1(\hat{\Gamma}_h)$ holds. Using this and (9.5) we get

$$\begin{aligned}\|\nabla_\Gamma(\tilde{u} - u_{h,*}^\ell)\|_{L^2(\Gamma)} &= \|\nabla_\Gamma(u_* - u_{h,*})^\ell\|_{L^2(\Gamma)} \leq c\|\nabla_{\Gamma_h}(u_* - u_{h,*})\|_{L^2(\Gamma_h)} \\ &\leq c\|\nabla_{\hat{\Gamma}_h}(u^e - \hat{u}_{h,*})\|_{L^2(\hat{\Gamma}_h)}.\end{aligned}$$

Hence, with the triangle inequality and (10.2) we obtain

$$\begin{aligned}\|\nabla_\Gamma(u - u_{h,*}^\ell)\|_{L^2(\Gamma)} &\leq c\|\nabla_{\hat{\Gamma}_h}(u^e - \hat{u}_{h,*})\|_{L^2(\hat{\Gamma}_h)} + \|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)} \\ &\leq ch^m\|u\|_{H^{m+1}(\Gamma)} + \|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)}.\end{aligned} \tag{10.4}$$

We derive a bound for the term $\|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)}$. Let $x \in \hat{\Gamma}_h$ be arbitrary, $y = p_h(x) \in \Gamma_h$, $z = p(y) \in \Gamma$, and $\zeta = p(x) \in \Gamma$. From (7.6) it follows that

$$\|y - \zeta\| \leq ch^{k+1}, \quad \|z - \zeta\| \leq ch^{k+1}, \quad \|x - z\| \leq ch^2. \tag{10.5}$$

Our starting point is the identity $\tilde{u}(z) = u(\zeta)$, which holds a. e. on $\hat{\Gamma}_h$ by definition of $\tilde{u}$. Taking the tangential gradient on $\hat{\Gamma}_h$ yields

$$\hat{\mathbf{P}}(x)Dp_h(x)^T Dp(y)^T \nabla_\Gamma \tilde{u}(z) = \hat{\mathbf{P}}(x)Dp(x)^T \nabla_\Gamma u(\zeta). \tag{10.6}$$

From the smoothness of $u$ and $\|z - \zeta\| \leq ch^{k+1}$, we get $\nabla_\Gamma u(\zeta) = \nabla_\Gamma u(z) + r_0$, with $\|r_0\| \leq ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}$. We insert this into (10.6) and rearrange the terms to obtain

$$\begin{aligned}&\hat{\mathbf{P}}(x)Dp_h(x)^T Dp(y)^T \nabla_\Gamma\big(\tilde{u}(z) - u(z)\big) \\ &= \hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T Dp(y)^T\big)\mathbf{P}(z)\nabla_\Gamma u(z) + \hat{\mathbf{P}}(x)Dp_h(x)^T r_0 =: r_1\end{aligned} \tag{10.7}$$

For the matrix in first term on the right hand-side in (10.7) we have

$$\begin{aligned}&\hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T Dp(y)^T\big)\mathbf{P}(z) \\ &= \hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T\big)\mathbf{P}(z) + \hat{\mathbf{P}}(x)Dp_h(x)^T(\mathbf{I} - Dp(y)^T\big)\mathbf{P}(z) =: A_0 + A_1.\end{aligned}$$

Using $\|\mathbf{P}(z) - \mathbf{P}(x)\| = \|\mathbf{P}(z) - \mathbf{P}(\zeta)\| \leq c\|z - \zeta\| \leq ch^{k+1}$ and (8.9) we obtain $\|A_0\| \leq ch^{k+1}$. Differentiating the relation $p(y) = y - d(y)n(p(y))$ one obtains $Dp(y) = \big(\mathbf{I} + d(y)H(z)\big)^{-1}\mathbf{P}(z)$, with $H(z) = Dn(z)$. From (7.7) one obtains $|d(y)| \leq ch^{k+1}$, and this yields

$$Dp(y) = \mathbf{P}(z) + \mathcal{O}(h^{k+1}). \tag{10.8}$$

Thus we get $\|A_1\| \leq c\|\mathbf{P}(z)(\mathbf{I} - Dp(y))\| \leq ch^{k+1}$. Using the bounds for $A_0$, $A_1$ in (10.7) we get

$$\begin{aligned}A\nabla_\Gamma\big(\tilde{u}(z) - u(z)\big) &= r_1, \quad \|r_1\| \leq ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}, \\ A &:= \hat{\mathbf{P}}(x)Dp_h(x)^T Dp(y)^T\mathbf{P}(z).\end{aligned} \tag{10.9}$$

We now analyze the matrix $A$. From (8.3) and $|d_h(x)| \le ch^2$ we get

$$Dp_h(x) = \mathbf{Q}(y) + \mathcal{O}(h^2). \tag{10.10}$$

Combining this with (10.8) yields $A = (\hat{\mathbf{P}}(x)\mathbf{Q}(y)^T + E_1)\mathbf{P}(z)$, with $\|E_1\| \le ch^2$.

We consider the matrix $\hat{\mathbf{P}}(x)\mathbf{Q}(y)^T$. Using (8.19) we get

$$\|\hat{\mathbf{P}}(x)(\mathbf{Q}(y)^T - \mathbf{I})\| \le \frac{1}{|\alpha(y)|}\|\hat{\mathbf{P}}(x)\bar{n}_h(y)\| \le ch.$$

Using $\|\mathbf{P}(z) - \mathbf{P}(\zeta)\| \le ch^{k+1}$, $\mathbf{P}(\zeta) = \mathbf{P}(x)$, (9.7) gives $\|\hat{\mathbf{P}}(x) - \mathbf{P}(z)\| \le ch$. Hence,

$$\|\hat{\mathbf{P}}(x)\mathbf{Q}(y)^T - \mathbf{P}(z)\| \le \|\hat{\mathbf{P}}(x)(\mathbf{Q}(y)^T - \mathbf{I})\| + \|\hat{\mathbf{P}}(x) - \mathbf{P}(z)\| \le ch.$$

From this we get $A = (\mathbf{I} + E_2)\mathbf{P}(z)$, with $\|E_2\| \le ch$. Using this in (10.9) we get, for $h$ sufficiently small,

$$\|\nabla_\Gamma\big(\tilde{u}(z) - u(z)\big)\| = \|(I + E_2)^{-1}r_1\| \le c\|r_1\| \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)},$$

which implies $\|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)} \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}$. Combining this with the result in (10.4) completes the proof. ∎

**11. Geometric error.** We study the terms $\|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h}v_h\|_{L^2(\Gamma_h)}$ and $\tilde{E}_f := \|f - \frac{1}{\mu_h^\ell}f_h^\ell\|_{L^2(\Gamma)}$ that occur in the Strang Lemma, cf. (9.10).

THEOREM 11.1. *Let $u_{h,*} \in S_h$ be as in (10.3). For $h$ sufficiently small the following estimates hold:*

$$\|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h}u_{h,*}\|_{L^2(\Gamma_h)} \le ch^{k+1}\|u\|_{H^2(\Gamma)}, \tag{11.1}$$

$$\|f - \frac{1}{\mu_h^\ell}f_h^\ell\|_{L^2(\Gamma)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}. \tag{11.2}$$

*Proof.* Using $\|d\|_{L^\infty(\Gamma_h)} \le ch^{k+1}$, $\|\frac{1}{\mu_h} - 1\|_{L^\infty(\Gamma_h)} \le ch^{k+1}$, cf. (2.3) and (2.5), in (9.3) yields $A_\Gamma(p(x)) = \mathbf{P}(x)\bar{\mathbf{P}}(x)\mathbf{P}(x) + \mathcal{O}(h^{k+1})$, $x \in \Gamma_h$. From $\mathbf{P}(p(x)) = \mathbf{P}(x)$ and the identity $\mathbf{P}\bar{\mathbf{P}}\mathbf{P} - \mathbf{P} = \mathbf{P}(\mathbf{P} - \bar{\mathbf{P}})(\bar{\mathbf{P}} - \mathbf{P})\mathbf{P}$ we obtain

$$\begin{aligned}\|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)} &\le c\|\mathbf{P}\bar{\mathbf{P}}\mathbf{P} - \mathbf{P}\|_{L^\infty(\Gamma_h)} + ch^{k+1}\\ &\le c\|\mathbf{P} - \bar{\mathbf{P}}\|^2_{L^\infty(\Gamma_h)} + ch^{k+1} \le ch^{k+1}.\end{aligned} \tag{11.3}$$

The last estimate above follows from (2.3). Using (9.5), (5.9) and (10.2) we get

$$\begin{aligned}\|\nabla_{\Gamma_h}u_{h,*}\|_{L^2(\Gamma_h)} &\le c\|\nabla_{\hat{\Gamma}_h}I_hu\|_{L^2(\hat{\Gamma}_h)} \le c\big(\|\nabla_{\hat{\Gamma}_h}u^e\|_{L^2(\hat{\Gamma}_h)} + \|u\|_{H^2(\Gamma)}\big)\\ &\le c\big(\|\nabla_\Gamma u\|_{L^2(\Gamma)} + \|u\|_{H^2(\Gamma)}\big) \le c\|u\|_{H^2(\Gamma)}.\end{aligned} \tag{11.4}$$

Combination of (11.3) and (11.4) yields the proof of (11.1).

We consider the data extension $f_h(x) = f(x) - c_f$, $x \in \Gamma_h$ with $c_f$ as in (5.3). For the constant $c_f$ we get, using the smoothness of $f$, $\mathrm{dist}(\Gamma, \Gamma_h) \le ch^{k+1}$, (2.5), and $\int_\Gamma f\,ds = 0$:

$$\begin{aligned}|c_f| &\le \frac{1}{|\Gamma_h|}\left|\int_{\Gamma_h}(f - f\circ p)\,ds_h\right| + \frac{1}{|\Gamma_h|}\left|\int_{\Gamma_h}f\circ p\,ds_h\right|\\ &\le c\|\nabla f\|_{L^\infty(U)}h^{k+1} + \frac{1}{|\Gamma_h|}\left|\int_\Gamma f\frac{1}{\mu_h^\ell}\,ds\right|\\ &\le c\|\nabla f\|_{L^\infty(U)}h^{k+1} + c\|\frac{1}{\mu_h^\ell} - 1\|_{L^\infty(\Gamma)}\|f\|_{L^1(\Gamma)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}.\end{aligned} \tag{11.5}$$

With this we obtain

$$
\begin{aligned}
&\|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)} \le c\|f - f_h^\ell\|_{L^2(\Gamma)} + ch^{k+1}\|f\|_{L^2(\Gamma)} \\
&\le c\|f^e - f_h\|_{L^2(\Gamma_h)} + ch^{k+1}\|f\|_{L^2(\Gamma)} \le c\|f^e - f\|_{L^2(\Gamma_h)} + c|c_f| + ch^{k+1}\|f\|_{L^2(\Gamma)} \\
&\le c\|\nabla f\|_{L^\infty(U)}\|x - p(x)\|_{L^\infty(\Gamma_h)} + ch^{k+1}\|f\|_{H^1_\infty(U)} \le ch^{k+1}\|f\|_{H^1_\infty(U)},
\end{aligned}
$$

which proves the result in (11.2). ∎

**12. Quadrature error.** In this section we analyze the quadrature error, i.e., we derive bounds for the third and fourth term in the Strang Lemma. Recall from section 5 the affine mapping from the unit reference triangle $\tilde{T}$ to $T \in \mathcal{F}_h$, given by $x = M_T\tilde{x} = B_T\tilde{x} + b_T$, $\tilde{x} \in \tilde{T}$, $x \in T$. Note that $B_T \in \mathbb{R}^{3\times 2}$. Furthermore $\|B_T\| \le ch$ holds. Correspondence of functions on $\tilde{T}$ and $T$ is given by $\tilde{u}(\tilde{x}) = \hat{u}(B_T\tilde{x} + b_T) = \hat{u}(x)$. Note that for $n \in \mathbb{N}$, $\tilde{u} \in C^n(\bar{\tilde{T}})$ and $\xi_i \in \mathbb{R}^2$, $1 \le i \le n$ we have $D^n\tilde{u}(\tilde{x})(\xi_1, \ldots \xi_n) = D^n\hat{u}(x)(B_T\xi_1, \ldots, B_T\xi_n) = D_T^n\hat{u}(\hat{x})(B_T\xi_1, \ldots, B_T\xi_n)$ (where $D_T$ denotes the tangential derivative along $T$), and thus as in Theorem 15.1 in [3] we obtain, for $n \in \mathbb{N}$, $p \in [1, \infty]$,

$$
|\tilde{u}|_{H_p^n(\tilde{T})} \le c\|B_T\|^n|T|^{-1/p}|\hat{u}|_{H_p^n(T)} \le ch^n|T|^{-1/p}|\hat{u}|_{H_p^n(T)} \quad \text{for } \tilde{u} \in H_p^n(\tilde{T}). \tag{12.1}
$$

In the seminorm $|\hat{u}|_{H_p^n(T)}$ only the derivatives of order $n$ are involved and these derivatives are the *tangential* ones along the triangle $T$. We note that an estimate in the other direction, i.e. bounding derivatives of $\hat{u}$ by those of $\tilde{u}$, causes problems, because the triangle $\hat{T}$ may have arbitrary small angles. Thus the smallest singular value of $B_T$ cannot be bounded from below by $ch$ with a uniform (w.r.t. $T$ and $h$) constant $c > 0$.

The quadrature error for the quadrature rule (5.12) is defined by

$$
E_{\tilde{T}}(\tilde{\phi}) = \int_{\tilde{T}} \tilde{\phi}\, d\tilde{x} - Q_{\tilde{T}}(\tilde{\phi}), \quad E_T(\hat{\phi}) = \int_T \hat{\phi}\, d\hat{x} - Q_T(\hat{\phi}). \tag{12.2}
$$

Note that $E_T(\hat{\phi}) = |T|E_{\tilde{T}}(\tilde{\phi})$ holds.

**12.1. Smooth approximation of $G$.** In the bilinear form $a_h(u_h, v_h)$ the quadrature rule $Q_T$ is applied to the function $G\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h$. On each triangle $T \in \hat{\Gamma}_h$ the vector functions $\nabla_{\hat{\Gamma}_h}\hat{u}_h$ and $\nabla_{\hat{\Gamma}_h}\hat{v}_h$ are polynomials and thus have $C^\infty$ smoothness. The matrix $G = G(x) = T_h(x)^{-1}\hat{\mu}_h(x)$, $x \in T$, however, contains derivatives of the function $p_h$, which is *only Lipschitz*. Hence, $G$ is not even continuous. In this section we show that, on $T$, this matrix function can be approximated with accuracy $\mathcal{O}(h^{k+1})$ by a smooth matrix function, denoted by $G^s$. The components $T_h^{-1} = (WW^T)^{-1}$ and $\hat{\mu}_h$ of $G$ are treated in the lemmas 12.1 and 12.2 below.

Recall the definition $W(x) = \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x)Dp_h(x)^T$, cf. Lemma 5.1. From (5.8), (8.2), (8.4), and the definition of $W$, we get for almost all $x \in \hat{\Gamma}_h$:

$$
\begin{aligned}
\hat{\mathbf{P}}(x)W(x) &= \hat{\mathbf{P}}(x)Dp_h(x)^T = \hat{\mathbf{P}}(x)Dp_h(x)^T\mathbf{Q}(y)^T \\
&= \hat{\mathbf{P}}(x)Dp_h(x)^T\bar{\mathbf{P}}(y) = W(x)\bar{\mathbf{P}}(y), \quad y = p_h(x).
\end{aligned} \tag{12.3}
$$

This implies the commutator relations (note that $W$ is invertible, cf. (9.6)):

$$
\begin{aligned}
W(x)W(x)^T\hat{\mathbf{P}}(x) &= \hat{\mathbf{P}}(x)W(x)W(x)^T, \\
\big(W(x)W(x)^T\big)^{-1}\hat{\mathbf{P}}(x) &= \hat{\mathbf{P}}(x)\big(W(x)W(x)^T\big)^{-1}.
\end{aligned} \tag{12.4}
$$

The oblique projector $\hat{\mathbf{Q}}(x)$, $x \in \hat{\Gamma}_h$, is approximated by:

$$\hat{\mathbf{Q}}^s(x) := \mathbf{I} - \frac{1}{\hat{\alpha}^s(x)} \hat{n}_h(x) n(x)^T, \quad \text{with } \hat{\alpha}^s(x) = \hat{n}_h(x)^T n(x). \tag{12.5}$$

Note that $\hat{n}_h$ is constant on $T \in \hat{\Gamma}_h$ and the normal $n$ is smooth (depending only on the smoothness of $\Gamma$). Hence, $\hat{\mathbf{Q}}^s(x)$ is a piecewise smooth matrix function. Similar to (5.8), the following commutation relations hold for $x \in \hat{\Gamma}_h$:

$$\begin{aligned} \hat{\mathbf{Q}}^s(x)\mathbf{P}(x) = \mathbf{P}(x), \quad \mathbf{P}(x)\hat{\mathbf{Q}}^s(x) = \hat{\mathbf{Q}}^s(x), \\ \hat{\mathbf{P}}(x)\hat{\mathbf{Q}}^s(x) = \hat{\mathbf{P}}(x), \quad \hat{\mathbf{Q}}^s(x)\hat{\mathbf{P}}(x) = \hat{\mathbf{Q}}^s(x). \end{aligned} \tag{12.6}$$

We use $\hat{\mathbf{Q}}^s$ to define a piecewise smooth approximation of $W(x)$:

$$W^s(x) := \mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x) Dp(x)^T, \quad x \in \hat{\Gamma}_h. \tag{12.7}$$

Note that $\hat{\mathbf{P}}(x)$ is constant on $T \in \mathcal{F}_h$ and $Dp(x)$ is a smooth matrix function (depending only on the smoothness of $\Gamma$), hence, $W^s(x)$ is smooth for $x \in T$. Using (2.3) and (8.1), we get (for $h$ sufficiently small) $W^s(x) = \mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x)\mathbf{P}(x) + \mathcal{O}(h^2)$. An elementary computation yields $-\hat{\mathbf{Q}}^s + \hat{\mathbf{P}}\mathbf{P} = \hat{n}_h(\hat{\alpha}^s n - \hat{n}_h)^T - \frac{1}{\hat{\alpha}^s}(\hat{\alpha}^s n - \hat{n}_h)n^T$. Using $|\hat{\alpha}^s - 1| = \frac{1}{2}\|\hat{n}_h - n\|^2 \leq ch^2$ we thus get

$$W^s(x) = \mathbf{I} + \mathcal{O}(h), \quad x \in \hat{\Gamma}_h. \tag{12.8}$$

In particular, for $h$ sufficiently small, $W^s$ is invertible. Similar to (12.3) and (12.4), we obtain for almost all $x \in \hat{\Gamma}_h$:

$$\hat{\mathbf{P}}(x) W^s(x) = \hat{\mathbf{P}}(x) Dp(x)^T = W^s(x)\mathbf{P}(x), \tag{12.9}$$

and this yields the commutation relations

$$\begin{aligned} W^s(x)W^s(x)^T\hat{\mathbf{P}}(x) = \hat{\mathbf{P}}(x)W^s(x)W^s(x)^T, \\ \big(W^s(x)W^s(x)^T\big)^{-1}\hat{\mathbf{P}}(x) = \hat{\mathbf{P}}(x)\big(W^s(x)W^s(x)^T\big)^{-1}. \end{aligned} \tag{12.10}$$

Lemma 12.1. *For $h$ sufficiently small the following holds:*

$$\left\| \hat{\mathbf{P}} \left( \left(WW^T\right)^{-1} - \left(W^s {W^s}^T\right)^{-1} \right) \hat{\mathbf{P}} \right\|_{L^\infty(\hat{\Gamma}_h)} \leq ch^{k+1}. \tag{12.11}$$

*Proof.* We suppress the argument $x$ in the notation below. We use the matrix identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ and the commutator relations (12.4), (12.10) to compute

$$\begin{aligned} &\hat{\mathbf{P}} \left( (WW^T)^{-1} - (W^s {W^s}^T)^{-1} \right) \hat{\mathbf{P}} \\ &= \hat{\mathbf{P}}(WW^T)^{-1}\big(W^s{W^s}^T - WW^T\big)(W^s{W^s}^T)^{-1}\hat{\mathbf{P}} \\ &= (WW^T)^{-1}\hat{\mathbf{P}}\big(W^s{W^s}^T - WW^T\big)\hat{\mathbf{P}}(W^s{W^s}^T)^{-1}. \end{aligned} \tag{12.12}$$

From (9.6) and (12.8) we obtain

$$\|(WW^T)^{-1}\|_{L^\infty(\hat{\Gamma}_h)} \leq c, \quad \|(W^s{W^s}^T)^{-1}\|_{L^\infty(\hat{\Gamma}_h)} \leq c. \tag{12.13}$$

The relations (12.3), (12.9) yield

$$\hat{\mathbf{P}}(W^s W^{s\,T} - WW^T)\hat{\mathbf{P}} = \hat{\mathbf{P}}\Big((Dp - Dp_h)^T Dp + Dp_h^T(Dp - Dp_h)\Big)\hat{\mathbf{P}}$$
$$= \hat{\mathbf{P}}\Big((Dp - Dp_h)^T \mathbf{P} Dp + Dp_h^T \bar{\mathbf{P}}(y)(Dp - Dp_h)\Big)\hat{\mathbf{P}},$$

where in the last equality we used $Dp(x) = \mathbf{P}(x)Dp(x)$, $Dp_h(x) = \bar{\mathbf{P}}(y)Dp_h(x)$ with $y = p_h(x)$, cf. (8.1), (8.2), (8.4). Using the estimates in (8.9) and (8.10) results in

$$\|\hat{\mathbf{P}}(W^s W^{s\,T} - WW^T)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1},$$

and combining this with the results in (12.12) and (12.13) completes the proof. ∎

For the estimate in (12.11) to hold the use of the projections $\hat{\mathbf{P}}$ on the left hand-side is essential. Without these projections, only the asymptotically worse upper bound $ch^k$ holds. This difference in the upper bounds is directly related to the different bounds in (8.8) and (8.9).

For given $T \in \mathcal{F}_h$ let $U = U_T$ be the orthogonal matrix from Lemma 4.1. We introduce

$$\hat{\mu}^s(x) := \sqrt{\det(U^T Dp(x)^T Dp(x) U)}, \quad x \in \hat{\Gamma}_h. \tag{12.14}$$

Since $U$ is constant on each $T$ and $Dp(x)$ is a smooth matrix function (depending only on the smoothness of $\Gamma$) we have that $\hat{\mu}^s$ is a smooth matrix function on each $T \in \mathcal{F}_h$.

LEMMA 12.2. *For $h$ sufficiently small the following holds:*

$$\|\hat{\mu}_h - \hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}. \tag{12.15}$$

*Proof.* Let $x \in \hat{\Gamma}_h$, $y = p_h(x)$, $\zeta = p(x)$, and $U \in \mathbb{R}^{3\times 2}$ be the orthogonal matrix from Lemma 4.1. Note that $\hat{\mathbf{P}}(x)U = U$ holds. We define $A = Dp(x)U \in \mathbb{R}^{3\times 2}$ and $B = Dp_h(x)U \in \mathbb{R}^{3\times 2}$, hence $\hat{\mu}^s(x)^2 - \hat{\mu}_h(x)^2 = \det(A^T A) - \det(B^T B)$ holds. Using $Dp(x) = \mathbf{P}(x) + \mathcal{O}(h^2) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$, cf. (9.7), and $Dp_h(x) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$, we get with $\hat{\mathbf{P}}(x)U = U$ that

$$A^T A = \mathbf{I} + \mathcal{O}(h), \quad B^T B = \mathbf{I} + \mathcal{O}(h) \tag{12.16}$$

hold. Let $M(t) := A^T A + t(B^T B - A^T A)$ be a matrix valued function, $t \in [0, 1]$. Due to (12.16) we have, for $h$ sufficiently small,

$$|\det M(t) - 1| \le \frac{1}{2}, \quad t \in [0, 1]. \tag{12.17}$$

Hence, $M(t)$ is invertible and has a uniformly bounded condition number. We apply the mean value theorem to the scalar valued function $f(t) := \det M(t)$, $t \in [0, 1]$: There exists $s \in (0, 1)$ with

$$\det B^T B - \det A^T A = f(1) - f(0) = f'(s) = f(s)\,\mathrm{tr}\big(M(s)^{-1}(B^T B - A^T A)\big).$$

With this we obtain

$$\big|\det B^T B - \det A^T A\big| \le 3|\det M(s)|\,\|M(s)^{-1}\|\|B^T B - A^T A\| \le c\,\big\|B^T B - A^T A\big\|\,.$$

Using $\hat{\mathbf{P}}U = U$, $Dp_h(x) = \bar{\mathbf{P}}(y)Dp_h(x)$, $Dp(x) = \mathbf{P}(x)Dp(x)$ and the estimates in Lemma 8.2 we obtain

$$\begin{aligned}\|A^TA - B^TB\| &\leq \|(B-A)^TB\| + \|A^T(B-A)\| \\ &= \|U^T\hat{\mathbf{P}}(Dp_h - Dp)^T\bar{\mathbf{P}}(y)Dp_hU\| + \|U^TDp^T\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}U\| \\ &\leq c\big(\|\hat{\mathbf{P}}(Dp_h - Dp)^T\bar{\mathbf{P}}(y)\| + \|\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}\|\big) \leq ch^{k+1}.\end{aligned}$$

Both $\hat{\mu}^s(x)$ and $\hat{\mu}_h(x)$ are strictly positive and uniformly bounded away from 0 for $h$ sufficiently small due to (12.16). Hence, we obtain

$$|\hat{\mu}^s(x) - \hat{\mu}_h(x)| = \frac{|\hat{\mu}^s(x)^2 - \hat{\mu}_h(x)^2|}{|\hat{\mu}^s(x) + \hat{\mu}_h(x)|} \leq c\left|\det B^TB - \det A^TA\right| \leq ch^{k+1}.$$

□

The results in the Lemmas 12.1, 12.2, induce a piecewise (on $T \in \mathcal{F}_h$) smooth approximation of the matrix $G(x) = \hat{\mu}(x)\big(W(x)W(x)^T\big)^{-1}$.

Corollary 12.3. *Define*

$$G^s(x) = \hat{\mu}^s(x)\big(W^s(x)W^s(x)^T\big)^{-1}, \quad x \in T \in \mathcal{F}_h.$$

*Take $m \in \mathbb{N}$, $m \geq 1$. Provided $\Gamma$ is sufficiently smooth, the entries of the matrix $G^s$ have the smoothness property $G^s_{ij} \in H^m_\infty(T)$, $1 \leq i,j \leq 3$, for all $T \in \mathcal{F}_h$. Furthermore the estimates*

$$\max_{T\in\mathcal{F}_h} \|G^s\|_{H^m_\infty(T)} \leq c, \tag{12.18}$$

$$\max_{T\in\mathcal{F}_h} \left\|\hat{\mathbf{P}}(G - G^s)\hat{\mathbf{P}}\right\|_{L^\infty(T)} \leq ch^{k+1}, \tag{12.19}$$

*hold, with constants $c$ independent of $h$.*

*Proof.* From (2.1) it is clear that the smoothness of $p$ and $n$, in a small neighborhood of $\Gamma$, depends only on the smoothness of $\Gamma$. On $T \in \mathcal{F}_h$ the normal $\hat{n}_h$ is constant, hence the smoothness of (the entries of) the matrix $W^s(x)^{-1} = \big(\mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x)Dp(x)^T\big)^{-1}$ depends only on the smoothness of the matrix $Dp$ and of the vector field $n$. Similarly, on $T \in \mathcal{F}_h$ the orthogonal matrix $U = U_T$ is constant and thus the smoothness of $\hat{\mu}^s$ depends only on the smoothness of the matrix $Dp$. On $T \in \mathcal{F}_h$, (higher) derivatives of $G^s_{ij}$ can be estimated by bounds that depend only on bounds for (higher) derivatives of $p$ and $n$. If $\Gamma$ is sufficiently smooth, these bounds are uniform w.r.t. $T \in \mathcal{F}_h$. From these observations it follows that for all entries of the matrix $G^s$, we have $G^s_{ij} \in H^m_\infty(T)$ for all $T \in \mathcal{F}_h$ and that the result (12.18) holds. The result (12.19) directly follows from (12.15) and Lemma 12.1. □

**12.2. Bound on the quadrature error in the bilinear form.** We derive a bound for the term $\sup_{w_h\in S_h/\mathbb{R}} \frac{a(v_h,w_h)-a_h(v_h,w_h)}{\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}}$ in the Strang Lemma, where we take $v_h = u_{h,*}$ as in Theorem 10.1. The technique used in the error analysis below is very similar to the one used in the analysis of the quadrature error in [3].

Theorem 12.4. *Assume that the quadrature rule $Q_{\hat{T}}$ is exact for all polynomials of degree $2m-2$ and that* (10.1) *and assumption 10.1 hold. For $u_{h,*}$ from* (10.3)*, the following holds for $h$ sufficiently small:*

$$\sup_{w_h\in S_h/\mathbb{R}} \frac{a(u_{h,*},w_h) - a_h(u_{h,*},w_h)}{\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}} \leq ch^m\|u\|_{H^{m+1}(\Gamma)} + ch^{k+1}\|u\|_{H^2(\Gamma)}.$$

*Proof.* We write $v_h$, $\hat{v}_h$ for $u_{h,*}$ and $\hat{u}_{h,*}$. Note that

$$\int_{\hat{\Gamma}_h} G\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{w}_h \, d\hat{s}_h = \int_{\hat{\Gamma}_h} \hat{\mathbf{P}}G\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{w}_h \, d\hat{s}_h$$

holds, and similarly with $G$ replaced by $G^s$. We use the splitting

$$\begin{aligned} a(v_h, w_h) - a_h(v_h, w_h) &= \int_{\hat{\Gamma}_h} \hat{\mathbf{P}}(G - G^s)\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{w}_h \, d\hat{s}_h \\ &+ \sum_{T\in\mathcal{F}_h} E_T(G^s\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{w}_h) + \sum_{T\in\mathcal{F}_h} Q_T\big(\hat{\mathbf{P}}(G^s - G)\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{w}_h\big) \\ &=: A + B + C, \end{aligned} \tag{12.20}$$

cf. (12.2), (5.13), (5.14). For the first term in (12.20) we get, using (12.19) and $\|\hat{v}_h\|_{H^1(\hat{\Gamma}_h)} = \|I_h u\|_{H^1(\hat{\Gamma}_h)} \le \|u^e\|_{H^1(\hat{\Gamma}_h)} + ch\|u\|_{H^2(\Gamma)} \le c\|u\|_{H^2(\Gamma)}$ , cf. (10.1) and (9.5),

$$|A| \le ch^{k+1}\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^2(\hat{\Gamma}_h)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \le ch^{k+1}\|u\|_{H^2(\Gamma)}\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}. \tag{12.21}$$

For the third term, we use the positivity of the quadrature weights to obtain

$$|C| \le \sum_{T\in\mathcal{F}_h} \|\hat{\mathbf{P}}(G^s - G)\hat{\mathbf{P}}\|_{L^\infty(T)}\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^\infty(T)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^\infty(T)}\, Q_T(1).$$

Clearly, $Q_T(1) = |T|$. We use the local estimate $\sqrt{|T|}\|f\|_{L^\infty(T)} \le c\|f\|_{L^2(T)}$ which is valid for finite element functions $f$ on arbitrarily shaped triangles. We again apply (12.19) and combine this with a Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |C| &\le ch^{k+1}\sum_{T\in\mathcal{F}_h} \|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^2(T)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(T)} \\ &\le ch^{k+1}\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^2(\hat{\Gamma}_h)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \le ch^{k+1}\|u\|_{H^2(\Gamma)}\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}. \end{aligned} \tag{12.22}$$

In the second term in (12.20) we have smooth integrands $G^s_{ij}$, $\partial^\Gamma_i\hat{v}_h$, $\partial^\Gamma_j\hat{w}_h$, on each $T \in \mathcal{F}_h$. The latter two are polynomials of degree $m-1$. For the derivation of a bound we can apply an analysis as in [3]. The result (28.16) in [3] states:

$$|E_{\tilde{T}}(\tilde{a}\tilde{v}\tilde{w})| \le c\Big(\sum_{j=0}^{m-1}|\tilde{a}|_{H^{m-j}_\infty(\tilde{T})}|\tilde{v}|_{H^j(\tilde{T})}\Big)\|\tilde{w}\|_{L^2(\tilde{T})}$$

for all $\tilde{a} \in H^m_\infty(\tilde{T})$, $\tilde{v} \in P_{m-1}(\tilde{T})$, $\tilde{w} \in P_{m-1}(\tilde{T})$. With the result in (12.1) (note that $H^j(T) := H^j_2(T)$) and using $E_T(\hat{\phi}) = |T|E_{\tilde{T}}(\tilde{\phi})$ we get

$$\begin{aligned} |E_T(avw)| &\le ch^m_T\Big(\sum_{j=0}^{m-1}|a|_{H^{m-j}_\infty(T)}|v|_{H^j(T)}\Big)\|w\|_{L^2(T)} \\ &\le ch^m_T\|a\|_{H^m_\infty(T)}\|v\|_{H^{m-1}(T)}\|w\|_{L^2(T)} \end{aligned}$$

for all $a \in H^m_\infty(T),\ v \in P_{m-1}(T),\ w \in P_{m-1}(T)$. For the second term in (12.20), we take $a = G^s_{ij}$, $v = \partial^\Gamma_i \hat{v}_h$, $w = \partial^\Gamma_j \hat{w}_h$, and using (12.18) we get

$$
\begin{aligned}
|B| &= \Big| \sum_{T\in\mathcal{F}_h} \sum_{i,j=1}^{3} E_T(G^s_{ij}\partial^\Gamma_i \hat{v}_h \partial^\Gamma_j \hat{w}_h)\Big| \le ch^m \sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|_{H^m(T)} \|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(T)} \\
&\le ch^m \Big(\sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|^2_{H^m(T)}\Big)^{\frac{1}{2}} \|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \\
&\le ch^m \|u\|_{H^{m+1}(\Gamma)} \|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}.
\end{aligned} \tag{12.23}
$$

In the last inequality we used $\sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|^2_{H^m(T)} = \sum_{T\in\mathcal{F}_h} \|I_h u\|^2_{H^m(T)} \le c\|u\|^2_{H^{m+1}(\Gamma)}$, which follows from (10.1). Combining the bounds (12.21), (12.22), (12.23) with the splitting (12.20) completes the proof. ◻

**12.3. Bound on the quadrature error in the right hand-side functional.** We finally analyze the term $\sup_{w_h\in S_h/\mathbb{R}} \frac{l(w_h)-l_h(w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}}$ in the Strang Lemma. The idea of the analysis is the same as used above: We replace the Lipschitz functions $f_h \circ p_h\, \hat{\mu}_h$ and $f^q_h \circ p_h\, \hat{\mu}_h$ by piecewise smooth ones and split the error into three terms. As $f_h$ and $f^q_h$ only differ by a constant, the estimates are very similar. For the approximation of $\hat{\mu}_h$ we use $\hat{\mu}^s$ defined in (12.14), with error bound as in (12.15). For the approximation of $f \circ p_h$ we introduce $f^s := f \circ p\, (= f^e)$. Using the result in (7.6) we obtain,

$$
\|f^s(x) - f\circ p_h\|_{L^\infty(\hat{\Gamma}_h)} \le \|\nabla f\|_{L^\infty(U)}\, \|p - p_h\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}. \tag{12.24}
$$

Similar to the proof of Corollary 12.3, the smoothness of $f^s$ on $\hat{\Gamma}_h$ only depends on the smoothness of $f$ and $p$ and on piecewise constant quantities pertaining to $\hat{\Gamma}_h$. Thus, we have

$$
\|f^s\|_{H^m_\infty(\hat{\Gamma}_h)} \le c\, \|f\|_{H^m_\infty(\Gamma)}\,. \tag{12.25}
$$

Recall the definition $f_h(x) = f(x) - c_f$, $c_f = \frac{1}{|\Gamma_h|}\int_{\Gamma_h} f\, ds_h$, cf. (5.3). Using the bound for $c_f$ given in (11.5) and the results in (12.24), (12.15) we get

$$
\|(f_h\circ p_h)\hat{\mu}_h - f^s\hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}. \tag{12.26}
$$

THEOREM 12.5. *Assume that the quadrature rule $Q_{\tilde{T}}$ is exact for all polynomials of degree $2m-2$. Furthermore, assume that $f \in H^1_\infty(U) \cap H^m_\infty(\Gamma)$. The following holds:*

$$
\sup_{w_h\in S_h/\mathbb{R}} \frac{l(w_h)-l_h(w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}} \le ch^m\|f\|_{H^m_\infty(\Gamma)} + ch^{k+1}\|f\|_{H^1_\infty(U)}. \tag{12.27}
$$

*Proof.* Take an arbitrary $w_h \in S_h/\mathbb{R}$ and recall the Poincaré inequality

$$
\|w_h\|_{L^2(\Gamma_h)} \le c\, \|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}\,. \tag{12.28}
$$

We use the error splitting

$$
\begin{aligned}
l(w_h) - l_h(w_h) &= \int_{\hat{\Gamma}_h} \big((f_h\circ p_h)\hat{\mu}_h - f^s\hat{\mu}^s\big)\hat{w}_h\, d\hat{s}_h \\
&\quad + \sum_{T\in\mathcal{F}_h} E_T(f^s\hat{\mu}^s\hat{w}_h) \\
&\quad + \sum_{T\in\mathcal{F}_h} Q_T\big((f^s\hat{\mu}^s - f^q_h\hat{\mu}_h)\hat{w}_h\big).
\end{aligned} \tag{12.29}
$$

For the first term we obtain, using (12.26), (9.8), and (12.28):

$$\begin{aligned}\Big|\int_{\hat\Gamma_h}\big((f_h\circ p_h)\hat\mu_h - f^s\hat\mu^s\big)\hat w_h\,d\hat s_h\Big| &\le ch^{k+1}\|f\|_{H^1_\infty(U)}\|\hat w_h\|_{L^2(\hat\Gamma_h)}\\ \le ch^{k+1}\|f\|_{H^1_\infty(U)}\|w_h\|_{L^2(\Gamma_h)} &\le ch^{k+1}\|f\|_{H^1_\infty(U)}\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}.\end{aligned} \tag{12.30}$$

For the analysis of the second term, with the smooth integrand $f^s\hat\mu^s\hat w_h$, we use a result from [3]. From Theorem 28.3 in [3], we have

$$|E_{\tilde T}(\tilde a\tilde v)| \le c\left(|\tilde a|_{H^m_\infty(\tilde T)}|\tilde v|_{L^2(\tilde T)} + \big(|\tilde a|_{H^{m-1}_\infty(\tilde T)} + |\tilde a|_{H^m_\infty(\tilde T)}\big)|\tilde v|_{H^1(\tilde T)}\right) \tag{12.31}$$

for all $\tilde a \in H^m_\infty(\tilde T)$, $\tilde v \in P_m(\tilde T)$. Using this with $\tilde a = \tilde f^s\tilde\mu^s$, $\tilde v = \tilde w_h$, (12.1), $\sup_{T\in\mathcal F_h}\|\hat\mu^s\|_{H^m_\infty(T)} \le c$, and $E_T(\phi) = |T|E_{\tilde T}(\tilde\phi)$, we get

$$\begin{aligned}|E_T(f^s\hat\mu^s\hat w_h)| &= |T|\left|E_{\tilde T}(\tilde f^s\tilde\mu^s\tilde w_h)\right|\\ &\le c\,|T|\left(|\tilde f^s\tilde\mu^s|_{H^m_\infty(\tilde T)}|\tilde w_h|_{L^2(\tilde T)} + \big(|\tilde f^s\tilde\mu^s|_{H^{m-1}_\infty(\tilde T)} + |\tilde f^s\tilde\mu^s|_{H^m_\infty(\tilde T)}\big)|\tilde w_h|_{H^1(\tilde T)}\right)\\ &\le ch^m|T|^{\frac12}\left(|f^s\hat\mu^s|_{H^m_\infty(T)}|\hat w_h|_{L^2(T)} + \big(|f^s\hat\mu^s|_{H^{m-1}_\infty(T)} + |f^s\hat\mu^s|_{H^m_\infty(T)}\big)|\hat w_h|_{H^1(T)}\right)\\ &\le ch^m|T|^{\frac12}\|f^s\|_{H^m_\infty(T)}\|\hat w_h\|_{H^1(T)}.\end{aligned} \tag{12.32}$$

Summation over $T$ and (12.25) yield

$$\begin{aligned}\Big|\sum_{T\in\mathcal F_h} E_T(f^s\hat\mu^s\hat w_h)\Big| &\le ch^m\sum_{T\in\mathcal F_h}|T|^{\frac12}\|f^s\|_{H^m_\infty(T)}\|\hat w_h\|_{H^1(T)}\\ &\le ch^m\|f\|_{H^m_\infty(\Gamma)}\|\hat w_h\|_{H^1(\hat\Gamma_h)}.\end{aligned} \tag{12.33}$$

Using $\|\hat w_h\|_{H^1(\hat\Gamma_h)} \le c\|w_h\|_{H^1(\Gamma_h)}$ (which follows from (5.9), (9.6), and (9.8)) and the Poincaré inequality (12.28) yields

$$\Big|\sum_{T\in\mathcal F_h} E_T(f^s\hat\mu^s\hat w_h)\Big| \le ch^m\|f\|_{H^m_\infty(\Gamma)}\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}. \tag{12.34}$$

Finally, we treat the third term in (12.29). Using

$$\sum_{T\in\mathcal F_h} Q_T(|\hat w_h|) \le c\sum_{T\in\mathcal F_h}|T|\|\hat w_h\|_{L^\infty(T)} \le c\|\hat w_h\|_{L^2(\hat\Gamma_h)} \le c\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)},$$

we get

$$\begin{aligned}\Big|\sum_{T\in\mathcal F_h} Q_T\big((f^s\hat\mu^s - f^q_h\hat\mu_h)\hat w_h\big)\Big| &\le \|f^s\hat\mu^s - f^q_h\hat\mu_h\|_{L^\infty(\hat\Gamma_h)}\sum_{T\in\mathcal F_h} Q_T(|\hat w_h|)\\ &\le c\,\|f^s\hat\mu^s - f^q_h\hat\mu_h\|_{L^\infty(\hat\Gamma_h)}\,\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}.\end{aligned} \tag{12.35}$$

We derive a bound for the term $\|f^s\hat\mu^s - f^q_h\hat\mu_h\|_{L^\infty(\hat\Gamma_h)}$. Recall the definitions

$$f^q_h = f\circ p_h - c^q_f, \quad c^q_f = \frac{1}{A}\sum_{T\in\mathcal F_h} Q_T(f\circ p_h\hat\mu_h), \quad A = \sum_{T\in\mathcal F_h} Q_T(\hat\mu_h)$$

from (5.15). Using the results in (12.24) and (12.15), we get

$$\begin{aligned}\|f^s\hat{\mu}^s - f_h^q\hat{\mu}_h\|_{L^\infty(\hat{\Gamma}_h)} &\le \|f^s\hat{\mu}^s - f\circ p_h\hat{\mu}_h\|_{L^\infty(\hat{\Gamma}_h)} + |c_f^q|\ \|\hat{\mu}_h\|_{L^\infty(\hat{\Gamma}_h)} \\ &\le ch^{k+1}\|f\|_{H^1_\infty(U)} + c\,|c_f^q|.\end{aligned} \tag{12.36}$$

Using $\int_{\hat{\Gamma}_h}(f_h\circ p_h)\hat{\mu}_h\,d\hat{s}_h = \int_{\Gamma_h} f_h\,ds_h = 0$, we obtain the splitting

$$\begin{aligned}\sum_{T\in\mathcal{F}_h} Q_T(f\circ p_h\hat{\mu}_h) &= \sum_{T\in\mathcal{F}_h} Q_T(f\circ p_h\hat{\mu}_h - f^s\hat{\mu}^s) + \sum_{T\in\mathcal{F}_h} E_T(f^s\hat{\mu}^s) \\ &\quad + \int_{\hat{\Gamma}_h}(f^s\hat{\mu}^s - f_h\circ p_h\hat{\mu}_h)\,d\hat{s}_h =: I + II + III.\end{aligned} \tag{12.37}$$

Due to (12.24) and (12.15), for the first term in (12.37) we get

$$|I| \le \|f^s\hat{\mu}^s - f\circ p_h\hat{\mu}_h\|_{L^\infty(\hat{\Gamma}_h)} \sum_{T\in\mathcal{F}_h} Q_T(1) \le ch^{k+1}\,\|f\|_{H^1_\infty(U)}\,.$$

We insert $\hat{w}_h = 1$ into (12.33). The second term in (12.37) can be bounded using this and (12.25):

$$|II| = \Big|\sum_{T\in\mathcal{F}_h} E_T(f^s\hat{\mu}^s)\Big| \le ch^m\|f\|_{H^m_\infty(\Gamma)}.$$

Furthermore, using the first inequality in (12.30) with $\hat{w}_h = 1$, we get

$$|III| = \Big|\int_{\hat{\Gamma}_h}(f_h\circ p_h)\hat{\mu}_h - f^s\hat{\mu}^s\,d\hat{s}_h\Big| \le ch^{k+1}\|f\|_{H^1_\infty(U)}.$$

From (9.8) it follows that $\|\hat{\mu}_h - 1\|_{L^\infty(\hat{\Gamma}_h)} \le ch$ and thus, for $h$ sufficiently small we have $|A|^{-1} \le c$. Using this and the bounds for $I$, $II$, and $III$ in (12.37), we get

$$|c_f^q| \le ch^{k+1}\|f\|_{H^1_\infty(U)} + ch^m\|f\|_{H^m_\infty(\Gamma)}.$$

Combining this with the results in (12.35) and (12.36) we finally obtain the bound for the third term:

$$\Big|\sum_{T\in\mathcal{F}_h} Q_T\big((f^s\hat{\mu}^s - f_h^q\hat{\mu}_h)\hat{w}_h\big)\Big| \le \big(ch^{k+1}\|f\|_{H^1_\infty(U)} + ch^m\|f\|_{H^m_\infty(\Gamma)}\big)\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}.$$

Combining this bound with the estimates in (12.30) and (12.34) completes the proof. ∎

**13. Main theorem.** Combining the results derived in the previous sections with the Strang Lemma we obtain a discretization error bound. This main result and the key assumptions are summarized in the following theorem. We assume that $\Gamma$ is sufficiently smooth, but do not specify the required smoothness.

THEOREM 13.1. *Assume that the finite element level set function $\phi_h^k$ satisfies* (2.2)*. For the construction of a quasi-normal field we apply a gradient recovery method to $\phi_h^k$ that satisfies Assumption 3.1. On $\hat{\Gamma}_h$ (zero level of $\phi_h^1$) we use a finite element space $\hat{S}_h$ that has the approximation property* (10.1)*, with $m \ge 1$. Assume that $f \in H^m_\infty(\Gamma)\cap H^1_\infty(U)$ and that the solution $u$ of* (1.1) *has regularity $u \in H^{m+1}(\Gamma)\cap H^2_\infty(\Gamma)$. We consider the discrete problem* (5.16) *with data extension $f_h$ as in* (5.3) *and with* *a quadrature rule $Q_T$ that is exact for all polynomials of degree $2m-2$. Then there exist constants $h_0 > 0$ and $c$ such that for all $h \le h_0$ the error in the solution $u_h^q$ of* (5.16) *is bounded by*

$$\begin{aligned}&\big\|\nabla_\Gamma\big(u-(u_h^q)^\ell\big)\big\|_{L^2(\Gamma)}\\&\le ch^m\big(\|u\|_{H^{m+1}(\Gamma)}+\|f\|_{H^m_\infty(\Gamma)}\big)+ch^{k+1}\big(\|u\|_{H^2_\infty(\Gamma)}+\|f\|_{H^1_\infty(U)}\big).\end{aligned}$$

**14. Numerical experiment.** As surface we take the unit sphere $\Gamma = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ embedded in $\Omega = (-2, 2)^3$. This surface is characterized as the zero-level of the level set function $\phi(x) := \|x\| - 1$. A family $\{\mathcal{T}_l\}_{l\ge0}$ of tetrahedral triangulations of $\Omega$ is used. We triangulate $\Omega$ starting with a uniform subdivision into 48 tetrahedra with mesh size $h_0 = \sqrt{3}$. Then, an adaptive red-green refinement-algorithm (implemented in the software package DROPS [7]) is applied; in each refinement step the tetrahedra that contain the (approximate) surface are refined such that on level $l = 1, 2, \ldots$ there holds $h_T \le \sqrt{3}\, 2^{-l}$ in a small neighborhood of $\Gamma$. The family $\{\mathcal{T}_l\}_{l\ge0}$ is consistent and quasi-uniform (in a neighborhood of the interface). As piecewise linear approximation of $\phi$ we take $\hat\phi_h := \phi_h^1 := I^1(\phi)$ where $I^1$ is the standard nodal interpolation operator on $\mathcal{T}_l$ for piecewise linear finite elements. The piecewise linear interface is given by $\hat\Gamma_h := \{\, x \in \Omega \mid \hat\phi_h(x) = 0 \,\}$. For the approximation of $\phi$ two choices are considered below. A piecewise quadratic approximation of $\phi$ is taken as $\phi_h := \phi_h^2 := I^2(\phi)$ where $I^2$ is the standard nodal interpolation operator on $\mathcal{T}_l$ for piecewise quadratic finite elements. The higher order interface is given by $\Gamma_h := \{\, x \in \Omega \mid \phi_h(x) = 0 \,\}$. This choice satisfies (2.2) for $k = 2$. We also consider the choice $\phi_h := \hat\phi_h$, hence $\Gamma_h = \hat\Gamma_h$, which satisfies (2.2) with $k = 1$. The point in taking $\Gamma_h = \hat\Gamma_h$ is to show the dependence on both $m$ and $k$ of the bound in Theorem 13.1.

For the case $\phi_h = \phi_h^2$ ($k = 2$) the quasi normal field $n_h$ is a vector-valued, continuous, piecewise quadratic finite element function. It is computed as described in remark 2. The skew projection $y = p_h(x) \in \Gamma_h$, $x \in U$, is computed as in [14, Sec. 5]. Given $x$ and $y$, one can compute $n_h(y)$, $Dn_h(y)$, and $d_h(x) = \langle x - y, n_h(y)\rangle$. Furthermore, the exact normal on $\Gamma_h$ can be determined from $\bar n_h(y) = \|\nabla\phi_h(y)\|^{-1}\nabla\phi_h(y)$. Hence, the Jacobian $Dp_h(x)$ can be computed using the relation (8.3), and $\hat\mu_h$ can be computed as in lemma 4.1.

**Experiment 1.** We perform an experiment to show that the estimates in lemma 8.2 and lemma 12.2 are sharp. These estimates are crucial in the error analysis for bounding the errors resulting from variational crimes by $\mathcal{O}(h^{k+1})$ instead of $\mathcal{O}(h^k)$ terms. For the unit sphere, one computes $Dp(x) = \|x\|^{-1}(\mathbf{I} - xx^T/\|x\|^2)$. For given $x \in \hat\Gamma_h$, the Jacobian $Dp_h(x)$ can be determined as explained above. The corresponding error $\|Dp_h - Dp\|_{L^\infty(\hat\Gamma_h)}$ is approximated by taking the maximum of $\|Dp_h(x) - Dp(x)\|$ over the vertices $x$ of all triangles $T \in \mathcal{F}_h$. In Table 14.1 this error is given. The results show the $\mathcal{O}(h^2)$ behavior as proven in (8.8).

The projected error $\|\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}\|_{L^\infty(\hat\Gamma_h)}$ (approximated in the same way as explained above) is also given in Table 14.1 and shows a $\mathcal{O}(h^3)$ behavior as proven in (8.9). Finally, in Table 14.2, we give the error quantity $\|\hat\mu_h - \hat\mu^s\|_{L^\infty(\hat\Gamma_h)}$ which has an $\mathcal{O}(h^3)$ behavior, as proven in lemma 12.2.

**Experiment 2.** We apply the discretization method to the Laplace-Beltrami equation (1.1), with two different surfaces $\Gamma$ (as used in [12]). As a first example

| level $l$ | $\|Dp_h - Dp\|_{L^\infty(\hat{\Gamma}_h)}$ | factor | $\|\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)}$ | factor |
|---|---|---|---|---|
| 1 | 0.09229 | – | 0.02021 | – |
| 2 | 0.02704 | 3.4 | 3.584e-3 | 5.6 |
| 3 | 7.004e-3 | 3.9 | 4.645e-4 | 7.7 |
| 4 | 1.722e-3 | 4.1 | 6.837e-5 | 6.8 |
| 5 | 4.579e-4 | 3.8 | 8.919e-6 | 7.7 |
| 6 | 1.148e-4 | 4.0 | 1.141e-6 | 7.8 |

TABLE 14.1
*Error of the (projected) Jacobian.*

| level $l$ | $\|\hat{\mu}_h - \hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)}$ | factor |
|---|---|---|
| 1 | 0.02880 | – |
| 2 | 4.851e-3 | 5.9 |
| 3 | 6.441e-4 | 7.5 |
| 4 | 9.422e-05 | 6.8 |
| 5 | 1.239e-05 | 7.6 |
| 6 | 1.585e-06 | 7.8 |

TABLE 14.2
*Error of the functional determinant.*

we take the unit sphere $\Gamma = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ embedded in $\Omega = (-2, 2)^3$. The right-hand side $f$ is such that the solution is given by

$$u(x) = \frac{12}{\|x\|^3}\left(3x_1^2 x_2 - x_2^3\right), \quad x = (x_1, x_2, x_3) \in \Omega.$$

The function $u$ is an eigenfunction of the Laplace-Beltrami operator, $-\Delta_\Gamma u = 12u =: f$. The right-hand side $f$ satisfies the compatibility condition $\int_\Gamma f\,ds = 0$, likewise does $u$. Note that $u$ and $f$ are constant along normals of $\Gamma$, that is $u \equiv u^e$, $f \equiv f^e$.

The triangulations $\mathcal{T}_l$ and $\hat{\phi}_h$ are the same as explained above. For the finite element space $\hat{S}_h$, cf. (5.1), we use the *trace of the outer piecewise quadratic finite element space*, as explained in Remark 3. Thus, we have $m = 2$ in (10.1). For $\phi_h = \phi_h^k$ we consider the choices with $k \in \{1, 2\}$ as explained above.

We outline the approach used for the evaluation of the bilinear form $a_h$ in (5.14) and the right-hand side $l_h$ in (5.15). For the quadrature rule $Q_T$, cf. (5.12), we use a fifth order accurate quadrature rule with positive weights on the reference triangle. Given $x \in \hat{\Gamma}_h$, the functional determinant $\hat{\mu}_h(x)$ can be evaluated as described above. With these data, $l_h$ can be computed as in (5.15). For $a_h$, lemma 5.1 and (5.10) are used to obtain an expression for $G$ in (5.14). The computed solution $u_h^q$ is normalized such that $\sum_{T\in\mathcal{F}_h} Q_T(\hat{u}_h^q \circ p_h\, \hat{\mu}_h) = 0$. The discrete problem is solved with a standard CG method with symmetric Gauss-Seidel preconditioner to a relative tolerance of $10^{-7}$.

We start with the case $k = 1$ where $\Gamma_h = \hat{\Gamma}_h$. From theorem 13.1, we know that, for $m = 2$, $k = 1$, the $H^1(\Gamma_h)$-error is bounded by $ch^2 + ch^2 = \mathcal{O}(h^2)$. This can clearly be observed in Table 14.3. Since the geometric errors are of the order $\mathcal{O}(h^2)$ we expect that the $L^2(\Gamma_h)$-norm of the discretization error is dominated by this term and does not scale better than $\mathcal{O}(h^2)$. The $L^2(\Gamma_h)$-norm discretization error is also

| level $l$ | $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ | factor | $\|\nabla_{\hat{\Gamma}_h}(u^e - u_h^q)\|_{L^2(\Gamma_h)}$ | factor |
|---|---|---|---|---|
| 1 | 0.1431 | – | 0.6911 | – |
| 2 | 0.03239 | 4.4 | 0.1636 | 4.2 |
| 3 | 7.986e-3 | 4.1 | 0.04219 | 3.9 |
| 4 | 1.968e-3 | 4.1 | 0.01054 | 4.0 |
| 5 | 4.935e-4 | 4.0 | 2.689e-3 | 3.9 |
| 6 | 1.230e-4 | 4.0 | 6.685e-4 | 4.0 |

TABLE 14.3
*Sphere, $m = 2$, $k = 1$: Discretization errors and error reduction.*

given in table 14.4 and clearly scales like $\mathcal{O}(h^2)$.
The number of iterations needed on level $l = 1, 2, \ldots, 6$, is 21, 40, 68, 147, 272, 588.

| level $l$ | $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ | factor | $\|\nabla_{\Gamma_h}(u^e - u_h^q)\|_{L^2(\Gamma_h)}$ | factor |
|---|---|---|---|---|
| 1 | 0.03910 | – | 0.5615 | – |
| 2 | 4.541e-3 | 8.6 | 0.1319 | 4.3 |
| 3 | 6.197e-4 | 7.3 | 0.03452 | 3.8 |
| 4 | 7.772e-5 | 8.0 | 8.647e-3 | 4.0 |
| 5 | 1.006e-5 | 7.7 | 2.224e-3 | 3.9 |
| 6 | 1.243e-6 | 8.1 | 5.444e-4 | 4.1 |

TABLE 14.4
*Sphere, $m = 2$, $k = 2$: Discretization errors and error reduction.*

We finally considr the case $k = m = 2$, i.e. a *higher order* approximation. From theorem 13.1, we know that $H^1(\Gamma_h)$-error is bounded by $ch^2 + ch^3 = \mathcal{O}(h^2)$. This order can be observed in Table 14.4. Since the geometric errors are of the order $\mathcal{O}(h^3)$ we expect, cf. the analysis in [5], that the $L^2(\Gamma_h)$-norm of the discretization error is of the order $\mathcal{O}(h^3)$. The $L^2(\Gamma_h)$-norm discretization error is given in table 14.4. These results clearly show that this error indeed scales like $\mathcal{O}(h^3)$. Hence our method, based on piecewise quadratics both for the surface approximation and for the Galerkin discretization of the Laplace-Beltrami equation, has third order convergence.
The number of CG iterations needed on level $l = 1, 2, \ldots, 6$, is 21, 39, 68, 147, 272, 588, which is almost identical to the previous experiment.

As a second example we take a torus instead of the unit sphere. Let

$$\phi(x) = \sqrt{x_3^2 + \left(\sqrt{x_1^2 + x_2^2} - R\right)^2} - r, \quad \Gamma = \{x \in \Omega \mid \phi(x) = 0\}.$$

We take $R = 1$ and $r = 0.6$. In the coordinate system $(\rho, \varphi, \theta)$ with

$$x = R \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} + \rho \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$

the $\rho$-direction is normal to $\Gamma$, $\frac{\partial x}{\partial \rho} \perp \Gamma$ for $x \in \Gamma$. Thus, the following solution $u$ and corresponding right-hand side $f$ are constant in normal direction:

$$
\begin{aligned}
u(x) &= \sin(3\varphi)\cos(3\theta+\varphi),\\
f(x) &= r^{-2}(9\sin(3\varphi)\cos(3\theta+\varphi))\\
&\quad - (R + r\cos(\theta)^{-2}(-10\sin(3\varphi)\cos(3\theta+\varphi) - 6\cos(3\varphi)\sin(3\theta+\varphi))\\
&\quad - (r(R + r\cos(\theta))^{-1}(3\sin(\theta)\sin(3\varphi)\sin(3\theta+\varphi)).
\end{aligned}
\tag{14.1}
$$

Both $u$ and $f$ satisfy the zero mean compatibility condition. In the discretization

| level $l$ | $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ | factor |
|---|---|---|
| 1 | 0.4567 | – |
| 2 | 0.05133 | 8.9 |
| 3 | 0.006674 | 7.7 |
| 4 | 9.053e-4 | 7.4 |
| 5 | 1.194e-4 | 7.6 |
| 6 | 1.468e-5 | 8.1 |

TABLE 14.5
*Torus: Discretization errors and error reduction.*

all components are the same as in the example with the unit sphere above. We only present the results for $k = m = 2$. The $L^2(\Gamma_h)$-norm discretization errors are given in table 14.5. Again we observe the expected $\mathcal{O}(h^3)$ behavior.

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