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Stabilisation of hyperbolic conservation laws using conservative finite–volume schemes

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***Abstract*—We discuss numerical stabilisation of dynamics governed by nonlinear hyperbolic conservation laws through feedback boundary conditions. Using a discrete Lyapunov function we prove exponential decay of the discrete solution to first–order finite volume schemes in conservative form. Decay rates are established for a large class of finite volume schemes including the Lax–Friedrichs scheme. Theoretical results are accompanied by computational results.**

***Index Terms*—stabilisation, finite–volume schemes, Lyapunov methods, boundary control, hyperbolic conservation laws**

I. Introduction

We are interested in the numerical analysis of feedback boundary control of nonlinear hyperbolic equations. Today, there exist a variety of analytical results by many authors on stability for general hyperbolic systems of conservation laws. We do not attempt to review all literature here and refer to [1]–[6] and the references therein for a more detailed discussion on analytical properties and existing results. The obtained results have many applications in engineering [7] including for example gas dynamics in pipes [8], water flow in canals [9]–[13] traffic flow [14], [15], supply chain [16] and electrical transmission lines [17], [18]. Also, more abstract results are available [19]. Recently, also aspects of switching systems and their control have been investigated [20]–[24]. Therein, a class of Lyapunov functions and suitable linearization has proven useful for analyzing feedback boundary control problems. For stability of the solution to the partial differential equation, the exponential decay of such a Lyapunov function has been established rigorously in a variety of cases, see for example [4], [25]–[27]. Typically, rather strong assumptions on the solution to the conservation (or balance) laws are required to establish decay rates. Therefore, the results are usually obtained using the equivalent non–conservative reformulation of the conservation law. Until now, only some of the theoretical results are accompanied by corresponding numerical analysis and we refer to [18], [28]–[30] for some recent discussion.

The focus of this paper is the stability analysis of numerical schemes. In a previous work [30] conditions on a numerical scheme for exponential decay of the discrete Lyapunov function have been established. However, therein the continuous approach has been extended to numerical discretizations of finite volume schemes. Therefore, the results are only applicable to numerical schemes in *non–conservative* form. For those schemes exponential decay of discrete L^2-Lyapunov functions has been established. However, non–conservative finite–volume schemes have very limited applications and they are not commonly used for simulation of hyperbolic problems [31]. In this paper we study general *conservative* first–order finite–volume methods in the context of stabilization. We establish convergence of the discrete L^2-Lyapunov function under general assumptions on first–order conservative finite volume schemes. The results are exemplified for the Lax–Friedrichs and Enquist–Osher scheme by computational results. So far, the discussion is limited to first–order schemes as well as scalar conservation laws. Further, we extend slightly the class of Lyapunov functions for the scalar by considering t entropy entropy–flux pairs. Those allow stabilization results without requiring linearization around steady–state as shown in the examples of Section II.

The paper is organized as follows. In Section II we present the motivation and some examples where stabilization is applied. Section III contains the main convergence theorem for general finite volume scheme and Section IV contains the accompanying numerical results. We close with an outlook to systems and high–order schemes in Section V.

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II. Motivation and Examples

In this section we present formal computations for stabilization of a nonlinear scalar hyperbolic equations. The results will then be used to rigorously prove stabilization results in Section III. The precise assumptions on arising functions are also given in the following section.

Consider a general nonlinear scalar conservation law (1) on $x \in [0,1], t \geq 0$ and for a possibly nonlinear flux $g : \mathbb{R} \to \mathbb{R}$.

$$\partial_t y(t,x) + \partial_x g(y(t,x)) = 0. \tag{1}$$

Clearly, any constant state y_0 is a solution to (1). Using suitable Lyapunov functions we stabilize y_0 through feedback boundary control. Consider a small perturbation $u = u(t,x)$ of a constant state y_0. Then, the perturbation fulfills the conservation law

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \; f(u) = g(u + y_0). \tag{2}$$

The feedback boundary condition $u(t,0) = \kappa\, u(t,1)$ can be prescribed for (2) provided that $f'(u) = g'(u+y_0) \geq 0$ for all u. For the scalar conservation law (2) we consider an entropy entropy–flux pairs (η, q), i.e., $q'(u) = \eta'(u) f'(u)$ and we have

$$\partial_t \eta(u(t,x)) + \partial_x q(u(t,x)) = 0 \tag{3}$$

for any smooth solution u to (2). According to [32, Section 6.2] the previous equality can be replaced by an inequality for convex entropies and solutions u to (2) enjoying discontinuities. Equation (3) holds as equality in case of sufficiently smooth solutions u and any smooth function η We additionally assume that $f'(u) \geq \nu,\ \eta \geq 0,\ \eta(0) = 0$, $sign(\eta'(u)) = sign(u)$. Then, we obtain $q(u) \geq \nu\eta(u) \geq 0$ according to Corollary 1. Also, we assume there exists $\kappa \in \mathbb{R}$ such that for all u :

$$q(\kappa\ u) - \frac{1}{e}q(u) \leq 0. \tag{4}$$

Formally, we establish exponential decay of the Lyapunov function

$$L(t) = \int_0^1 \exp(-x)\eta(u(t,x))dx \tag{5}$$

for any initial data $u(0,x) = u_0(x)$ and feedback boundary condition $u(t,0) = \kappa\ u(t,1)$. Indeed, a simple computation shows

$$\begin{aligned}\frac{d}{dt}L(t) &= -\int_0^1 \exp(-x)\partial_x q(u(t,x))dx \\ &= -\int_0^1 \exp(-x)q(u(t,x))dx \\ &\quad -\left(\frac{1}{e}q(u(t,1)) - q(u(t,0))\right) \\ &\leq -\nu\int_0^1 \exp(-x)\eta(u(t,x))dx = -\nu L(t).\end{aligned}$$

The previous inequality yields exponential decay of rate ν of the Lyapunov function. The formal result holds true for any sufficiently regular solution u and corresponding entropy entropy–flux pair (η, q). We present some examples.

- In the linear case $g(u) = a\ u$ with $a > 0$ we have $f'(u) \geq \nu = a$ and $\eta(u) = u^2$, and $q(u) = au^2$. For $\kappa^2 \leq \frac{1}{e}$ the assumptions on f, η and q as well as equation (4) is fulfilled. Obviously, we obtain a result already established e.g. in [4].
- Consider the stabilization of steady states in Burger's equation. The flux is $g(y) = \frac{1}{2}y^2$ and the state $y_0 > 0$. Then, $f'(u) = u + y_0 \geq \nu$ for $\nu = \frac{y_0}{2}$ and all $|u| \leq \frac{y_0}{2}$. For $\eta(u) = u^2$ we obtain $q(u) = \frac{2}{3}u^3 + y_0u^2$. Hence, for $\kappa^2 \leq \frac{1}{e}$ condition (4) is fulfilled. Note that $\eta(u) = u^{2k}$ for $k = 1, \ldots,$ is also possible choice within the Lyapunov function.
- Consider the stabilization of steady states $y_0 > 0$ in supply chain production models [33], [34]. A typical flux function is given by the M/M/1 queuing model with capacity one: $g(y) = \frac{y}{1+y}$. Hence, $f'(u) = \frac{1}{(1+y_0+u)^2} \geq \nu$ provided that $|u| \leq y_0$ and $\nu = \frac{1}{1+2y_0}$. Again, in equation (5) any choice $\eta(u) = u^{2k}$ for $k = 1, \ldots,$ yields exponential decay.

The previous examples already show that exponential stability is only expected under constraints on u. Due to Kruzkov Theorem [32, Theorem 6.2.3] bounds on $u(t,x)$ are imposed by bounds on the initial data u_0 even for weak solutions. We also recall that the Cauchy problem (2) with initial data $u_0(\cdot) \in C^k(\mathbb{R})$ has a unique solution $u(\cdot,\cdot) \in C^k([0,T] \times \mathbb{R})$ up to time T where $\frac{1}{T} = \inf_{y\in\mathbb{R}} f''(u_0(y))$. We also refer to [2], [4] for more results on classical solutions to initial and boundary value problems.

The previous results can be extended to Lyapunov functions $L(t) = \int_0^1 \exp(-x\ \mu)P(x)\nu(u(t,x))dx$ in a similar fashion as discussed in [4], [11], [17], [35], [36]. Further, exponential decay is also established in the following case using a similar formal computation. For $f'(u) \leq -\nu$ for all $u \in \mathcal{U}$ the Lyapunov function

$$\tilde{L}(t) = \int_0^1 \exp(x)\nu(u(t,x))dx$$

decays exponentially fast provided feedback boundary conditions of the type

$$u(1,t) = \kappa\ u(0,t)$$

and $q(u) - e\ q(\kappa u) \leq 0$ is fulfilled.

Corollary 1: Assume $f, \eta, q \in C^2(\mathbb{R};\mathbb{R})$ and $\eta(0) = 0,\ \eta \geq 0,\ sign(\eta'(u)) = sign(u)$. Let $q(u) = \int_0^u \eta'(s)f'(s)ds$. Further, assume there exists a set U such that $f'(u) \geq \nu > 0$ for all $u \in \mathcal{U}$. Then, $q(u) \geq \nu\ \eta(u)$ for all $u \in \mathcal{U}$.
The proof follows by simple integration.

III. Theoretical results

The main result is Theorem 3.1. We collect some assumptions on flux and entropy and entropy–flux pairs, respectively.

$$f, \eta, q \in C^2(\mathbb{R};\mathbb{R}),\ \eta(0) = 0,\ \eta \geq 0, \tag{6a}$$

$$sign(\eta'(u)) = sign(u),\ q'(u) = \eta'(u)f'(u). \tag{6b}$$

On the domain $x \in [0,1]$ we introduce an equidistant spatial grid $(x_i)_{i=1}^{N_x}$ with mesh size Δx. The grid points x_i are the cell centers of a finite volume scheme $x_i = (\Delta x)(i + \frac{1}{2})$ and N_x is such that $(N_x + \frac{1}{2})\Delta x = 1$. A general conservative finite–volume scheme [31] is given by

$$u_i^{n+1} - u_i^n = -\lambda\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) \tag{7}$$

for $i = 1, \ldots, N_x$, and $n = 0, \ldots, N_t$ and $\lambda = \frac{\Delta t}{\Delta x}$. The discrete boundary conditions are given by

$$u_0^n = \kappa_1 u^n_{N_x},\ \text{and}\ u^n_{N_x+1} = \kappa_N u_1^n. \tag{8}$$

and initial conditions for $i = 1, \ldots, N_x$ are denoted by

$$u_i^0 = u_{i,0}. \tag{9}$$

Here, u_i^n denotes the cell average of $u(\cdot,\cdot)$ at time $t_n = n\ \Delta t$, i.e.,

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t_n,\xi)d\xi.$$

In the first–order numerical finite volume schemes the reconstruction of the point values of u from the cell averages u_i^n is obtained through piecewise constant reconstruction. Therefore, we have that the numerical approximation to $u(t_n, x_i)$ is given by

$$u(t_n, x_i) \approx u_i^n.$$

The exponential stability of a general conservative finite–volume will be established under strong assumptions on the discrete solution $(u_i^n)_{i,n}$. Those are similar to [4] and assert a bound on the discrete C^2-norm of u_i^n. Here, we do not prove that the schemes conserves the bound but *assume* that the initial data $u_{i,0}$ is chosen such that $(u_i^n)_{i,n}$ fulfills bounds. The bounds are well–known for a C^2-solution $u(t,x)$ provided the time t is sufficiently small and the initial data $u_0(x)$ is sufficiently smooth. We assume there exists a bounded set $\mathcal{U} \subset \mathbb{R}$ with the following property: For any initial data $(u_{i,0})_i \in \mathcal{U}$ the solution $(u_i^n)_i^n$ obtained through equation (7), (8) and (9) fulfills for all $i = 1, \dots, N_x$ and $n = 0, \dots, N_t$:

$$u_i^n \in \mathcal{U},\ |\frac{u_{i-1}^n - u_i^n}{\Delta x}| \leq C_x, \tag{10a}$$

$$|\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}| \leq C_{xx}. \tag{10b}$$

Clearly, (10) are the discrete analog of the C^0, C^1 and C^2-norm of $u(t,x)$, respectively. The numerical flux $F_{i+\frac{1}{2}}^n = F(u_{i+1}^n, u_i^n)$ is expected to fulfill $F(u,u) = f(u)$. We assume that the numerical flux $F = F(x,y) : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is at least twice differentiable. Further, we assume that first and second derivatives of F on $\mathcal{U}$ are bounded.

$$F \in C^2(\mathbb{R}^2;\mathbb{R}),\ \sup_{u \in \mathcal{U}} \|DF(u,u)\| \leq C_{DF}, \tag{11a}$$

$$\sup_{u,v \in \mathcal{U}} \|D^2F(u,v)\| \leq C_{D^2F}. \tag{11b}$$

We collect the assumptions on η, q and f on the set U in equation (12).

$$f'(u) \geq \nu\ \forall u \in \mathcal{U},\ \sup_{u \in \mathcal{U}} \|q''(u)\| \leq C_{D^2q}, \tag{12a}$$

$$\sup_{u \in \mathcal{U}} \|f'(u)\| \leq C_{Df},\ \sup_{u \in \mathcal{U}} \|f''(u)\| \leq C_{D^2f}, \tag{12b}$$

$$\sup_{u \in \mathcal{U}} \|\eta'(u)\| \leq C_{D\eta},\ \sup_{u \in \mathcal{U}} \|\eta''(u)\| \leq C_{D^2\eta}. \tag{12c}$$

Theorem 3.1: Consider the conservative finite–volume scheme (7) and boundary conditions (8). Let $x_i = \Delta x\ (i + \frac{1}{2})$ for some $0 < \Delta x \leq 1$ and N_x, such that $(N_x + \frac{1}{2})\Delta x = 1$. Let $t_n = n\Delta t$. Assume there exists $\Delta t > 0$ and a bounded set $\mathcal{U} \subset \mathbb{R}$ such that the CFL condition

$$\sup_{u \in \mathcal{U}} |f'(u)|\lambda \leq 1$$

and (10) holds true. Assume the numerical flux F fulfills $F(u,u) = f(u)$ and (11). Further, the entropy–entropy-flux pair (η, q) fulfills assumption (6) and (12). Assume there exists κ_1 such that

$$e\ q(\kappa_1 u) - q(u) \leq 0, \forall u \in \mathcal{U}.$$

Define the discrete Lyapunov function at time $t^n = \Delta t\ n$ for $n = 0, \dots, N_t$ as

$$L^n := \Delta x \sum_{i=1}^{N_x} \exp(-x_i)\eta(u_i^n). \tag{13}$$

Then, there exists constants $C_1, C_2 > 0$ such that for all initial data $u_{i,0} \in \mathcal{U}$ and all times $t_n, n = 0, \dots,$

$$L^{n+1} \leq \exp(-\frac{e-1}{e}\nu\ t_{n+1})L^0 + C_1\Delta t + C_2 t_{n+1}(\Delta t)^3. \tag{14}$$

Proof: Since $F(u,u) = f(u)$ for all u, we have $F_y(u,u) = f'(u) - F_x(u,u)$. The difference in the numerical fluxes yields for some $\xi_x, \xi_y \in \mathcal{U}$

$$\begin{aligned} &F_{i-\frac{1}{2}}^n - F_{i+\frac{1}{2}}^n = (u_{i-1}^n - u_i^n)F_y(u_i^n, u_i^n) \\ &+(u_i^n - u_{i+1}^n)F_x(u_i^n, u_i^n) + \frac{1}{2}F_{yy}(u_i^n, \xi_y)(u_{i-1}^n - u_i^n)^2 \\ &-\frac{1}{2}F_{xx}(\xi_x, u_i^n)(u_{i+1}^n - u_i^n)^2 \\ &= (u_{i-1}^n - u_i^n)f'(u_i^n) - F_x(u_i^n, u_i^n)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \\ &+\frac{1}{2}F_{yy}(u_i^n, \xi_y)(u_{i-1}^n - u_i^n)^2 - \frac{1}{2}F_{xx}(\xi_x, u_i^n)(u_{i+1}^n - u_i^n)^2 \end{aligned}$$

Furthermore, we obtain for $\xi_\eta, \xi_q \in \mathcal{U}$

$$\begin{aligned} &\eta(u_i^{n+1}) - \eta(u_i^n) \\ &= -\lambda\eta'(u_i^n)(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) + \frac{1}{2}\eta''(\xi_\eta)(u_i^n - u_i^{n+1})^2 \\ &= \lambda(u_{i-1}^n - u_i^n)q'(u_i^n) + \tilde{R} \\ &= \lambda(q(u_{i-1}^n) - q(u_i^n)) + R, \end{aligned}$$

wherein R and $\tilde{R}$ are given by

$$\begin{aligned} R =& \tilde{R} - \frac{1}{2}\lambda q''(\xi_q)(u_i^n - u_{i-1}^n)^2 \\ \tilde{R} =& \frac{1}{2}\eta''(\xi_\eta)(u_i^n - u_i^{n+1})^2 \\ &+\lambda\eta'(u_i^n)F_x(u_i^n, u_i^n)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \\ &-\lambda\eta'(u_i^n)\frac{1}{2}F_{yy}(u_i^n, \xi_y)(u_{i-1}^n - u_i^n)^2 \\ &+\lambda\eta'(u_i^n)\frac{1}{2}F_{xx}(\xi_x, u_i^n)(u_{i+1}^n - u_i^n)^2 \end{aligned}$$

We estimate R by $R \leq (\Delta x)(\Delta t)C$ for a constant

$$\begin{aligned} C :=& \frac{1}{2}C_{D^2q}C_x + C_{xx}C_{DF}C_{D\eta} + C_{D\eta}C_{D^2F}C_x \\ &+\frac{1}{2}C_{D^2\eta}\lambda(C_{Df}C_x + C_{DF}C_{xx} + C_{D^2F}C_x)^2 \end{aligned}$$

The constant is obtained using the bounds given by assumption (10), (11), (12) and $\Delta x \leq 1$:

$$\begin{aligned} &\frac{\lambda}{2}q''(\xi_q)(u_i^n - u_{i-1}^n)^2 \leq (\Delta x)^2\frac{\lambda}{2}C_{D^2q}C_x, \\ &\lambda\eta'(u_i^n)F_x(u_i^n, u_i^n)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \\ &\leq (\Delta x)^2\lambda C_{D\eta}C_{DF}C_{xx}, \end{aligned}$$

and

$$\begin{aligned} &-\lambda\eta'(u_i^n)\frac{1}{2}F_{yy}(u_i^n, \xi_y)(u_{i-1}^n - u_i^n)^2 \\ &+\lambda\eta'(u_i^n)\frac{1}{2}F_{xx}(\xi_x, u_i^n)(u_{i+1}^n - u_i^n)^2 \\ &\leq (\Delta x)^2\lambda C_{D\eta}C_{D^2F}C_x, \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\eta''(\xi_\eta)(u_i^n-u_i^{n+1})^2\\
\le&\frac{1}{2}C_{D^2\eta}\lambda^2\Big((u_{i-1}^n-u_i^n)f'(u_i^n)\\
&-F_x(u_i^n,u_i^n)\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)\\
&+\frac{1}{2}F_{yy}(u_i^n,\xi_y)(u_{i-1}^n-u_i^n)^2\\
&-\frac{1}{2}F_{xx}(\xi_x,u_i^n)(u_{i+1}^n-u_i^n)^2\Big)^2\\
\le&\frac{1}{2}C_{D^2\eta}\lambda^2(\Delta x)^2\\
&\times(C_{Df}C_x+C_{DF}C_{xx}\Delta x+\Delta x C_{D^2F}C_x)^2
\end{aligned}$$

Combining the previous computations we estimate the time derivative of the discrete Lyapunov function (13).

$$\begin{aligned}
\frac{L^{n+1}-L^n}{\Delta t}=&\frac{1}{\lambda}\sum_{i=1}^{N_x}\exp(-x_i)\left(\eta(u_i^{n+1})-\eta(u_i^n)\right)\\
\le&\sum_{i=1}^{N_x}\exp(-x_i)\left(q(u_{i-1}^n)-q(u_i^n)\right)\\
&+\sum_{i=1}^{N_x}\exp(-x_i)(\Delta x)(\Delta t)C\\
\le&\sum_{i=1}^{N_x}q(u_i^n)\exp(-x_i)\left(\exp(-\Delta x)-1\right)\\
&+BC+(\Delta t)C
\end{aligned}$$

We obtain for the discrete boundary condition (8)

$$\begin{aligned}
BC=&\exp(-x_1)q(u_0^n)-q(u_{N_x}^n)\exp(-x_{N_x+1})\\
=&\exp(-\Delta x/2)\left(q(\kappa u_{N_x}^n)-\frac{1}{e}q(u_{N_x}^n)\right)\\
\le&0.
\end{aligned}$$

We estimate $\exp(-\Delta x)-1\le-\frac{e-1}{e}\Delta x$ for $\Delta x\le 1$. Using Corollary 1 we finally obtain

$$\begin{aligned}
L^{n+1}\le&L^n+\Delta t\sum_{i=1}^{N_x}q(u_i^n)\exp(-x_i)\left(\exp(-\Delta x)-1\right)\\
&+(\Delta t)^2C\\
\le&L^n-\Delta t\frac{e-1}{e}\nu\Delta x\sum_{i=1}^{N_x}\eta(u_i^n)\exp(-x_i)\\
&+(\Delta t)^2C\\
\le&(1-\Delta t\frac{e-1}{e}\nu)L^n+\sum_{i=0}^{n}(1-\frac{e-1}{e}\nu\Delta t)^iC(\Delta t)^2\\
\le&\exp(-\frac{e-1}{e}\nu\Delta t(n+1))L^0\\
&+C(\Delta t)^2\sum_{i=0}^{n}\exp(-\frac{e-1}{e}\nu\Delta t\ i).
\end{aligned}$$

The term $(\Delta t)\sum_{i=0}^{n}\exp(-\frac{e-1}{e}\nu\Delta t\ i)$ represents a first–order quadrature rule applied to

$$\begin{aligned}
&\int_0^{t_{n+1}}\exp(-\frac{e-1}{e}\nu s)ds\\
&=-\frac{e}{\nu(e-1)}\left(\exp(-\frac{e-1}{e}\nu t_{n+1})-1\right)\le 1.
\end{aligned}$$

The discretization error is bounded by $C_{quad}t_{n+1}(\Delta t)^2$ for some constant C_{quad}. Therefore, we obtain the estimate

$$L^{n+1}\le\exp(-\frac{e-1}{e}\nu t_{n+1})L^0+C\Delta t\left(1+C_{quad}t_{n+1}(\Delta t)^2\right).$$

■

We comment on the convergence properties for $n\to\infty$, $\Delta t\to 0$ and boundary conditions (8).

- A boundary condition at $x=0$ only is required according to (6), i.e., $f'(u)\ge 0$. For a general three–stencil scheme also a condition for u_{N_x+1} is imposed in equation (8). However, Theorem 3.1 shows that the decay is *independent* of the choice of k_N. This is coincides with the expected behavior of the continuous Lyapunov function.
- The constant C_1 and C_2 include in particular the constant C_{xx}. The later is related to the numerical diffusion of the solution u. This shows that the finite–volume scheme introduces additional numerical viscosity leading to a deterioration of the exponential decay rate. We refer to [31], [37] for more details on numerical viscosity of finite volume methods.
- Consider the limit $\Delta t\to 0, N_t\to\infty$ such that $T=N_t\Delta t$ remains a fixed arbitrary terminal time. Then, we obtain
$$L(T)=L^{N_t}\le\exp(-\frac{e-1}{e}\nu T)L(0),$$
corresponding to the expected result for the continuous result.
- In the general case the discrete Lyapunov function does not yield exponential decay for fixed Δt and $N_t\to\infty$. However, for fixed Δt and up to time $T=O(\frac{1}{(\Delta t)^2})$ we obtain for all n such that $n\Delta t\le T$:
$$L^n\le\exp(-\frac{e-1}{e}\nu t_n)L^0+O(\Delta t).$$
This implies that $L(0)$ decays exponentially up to a value of order $O(\Delta t)$. Hence, finer temporal meshes lead to a longer time horizon where we observe the exponential decay.

In the following we present some examples fulfilling the assertions of Theorem 3.1. The Enquist–Osher flux is given by

$$F_{i+\frac{1}{2}}^n=\frac{1}{2}\left(f(u_i^n)+f(u_{i+1}^n)\right)-\frac{1}{2}\int_{u_i^n}^{u_{i+1}^n}|f'(s)|ds.$$

Under assumption (6) and (12) the flux simplifies

$$F_{i+\frac{1}{2}}^n=f(u_i^n),$$

for the Enquist–Osher scheme. The Lax–Friedrichs flux is given by.

$$F^n_{i+\frac{1}{2}} = -\frac{1}{2\lambda}\left(u^n_{i+1} - u^n_i\right) + \frac{1}{2}\left(f(u^n_{i+1}) + f(u^n_i)\right),$$

In both cases (11) is fulfilled provided (6) and (12) holds.

Corollary 2: Assume (6), (10) and (12). Then, (11) is fulfilled for the the Lax–Friedrichs flux and the Enquist–Osher flux, respectively.

Proof: Under assumption (6) and (12) the numerical flux F is given by

$$F^n_{i+\frac{1}{2}} = f(u^n_i),$$

for the Enquist–Osher scheme and

$$F^n_{i+\frac{1}{2}} = -\frac{1}{2\lambda}\left(u^n_{i+1} - u^n_i\right) + \frac{1}{2}\left(f(u^n_{i+1}) + f(u^n_i)\right),$$

for the Lax–Friedrichs schemes. Therefore, condition (11) is fulfilled. ■

Lemma 3.2 (Upwind discretization for linear flux): Let $f(u) = a\ u$ with $a > 0$. Then, $\eta(u) = u^2$ and $q(u) = au^2$ fulfill assumption (6).

Consider the conservative finite–volume scheme (7) with boundary conditions (8). Let $x_i = \Delta x\ (i + \frac{1}{2})$ for some $0 < \Delta x \leq 1$ and N_x, such that $(N_x + \frac{1}{2})\Delta x = 1$. Let $t_n = n\Delta t$ and assume

$$a\lambda = 1.$$

Let the numerical flux be given by the Enquist–Osher scheme, i.e.,

$$F^n_{i+\frac{1}{2}} = au^n_i.$$

Then, (11) holds with $C_{DF} = a$ and $C_{D^2F} = 0$.

Let $\kappa^2 \leq \frac{1}{e}$. Then, for the discrete Lyapunov function (13) the estimate (14) holds true with $C_1 = C_2 = 0$. Therefore, the discrete Lyapunov function decays exponentially fast.

Proof: For $\kappa^2 \leq \frac{1}{e}$ we note that $eq(\kappa u) - q(u) \leq 0$. Further, we note that for the Enquist–Osher (or Upwind) flux we have $F_x = F_{yy} = 0$ and therefore

$$F^n_{i-\frac{1}{2}} - F^n_{i+\frac{1}{2}} = (u^n_{i-1} - u^n_i)a.$$

Further, we note that

$$\begin{gathered}\eta(u^{n+1}_i) - \eta(u^n_i) = \\ -\lambda(2u^n_i)(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}) + (u^n_i - u^{n+1}_i)^2 \\ = \lambda(u^n_{i-1} - u^n_i)au^n_i + \tilde{R} \\ = \lambda(q(u^n_{i-1}) - q(u^n_i)) - \lambda a(u^n_i - u^n_{i-1})^2 + \tilde{R}\end{gathered}$$

where

$$\tilde{R} = (u^n_i - u^{n+1}_i)^2.$$

Hence, the constant C appearing in the proof of Theorem 3.1 is equal to zero and therefore also $C_1 = C_2 = 0$. ■

In the following we prove exponential convergence for a linear flux $f(u) = au$ and the Lax–Friedrichs scheme. We prove exponential stability of discrete Lyapunov function with $\eta(u) = u^2$. Lax–Friedrichs scheme is the prototype of a central scheme [38] and therefore the discretization is independent of the sign of a. Lemma 3.3 is a particular case of Theorem 3.1 that allows improve the obtained bounds using the properties of linear transport.

Lemma 3.3 (Lax–Friedrichs discretization for linear flux): Consider the linear transport equation (2) with $f(u) = a\ u$. Let $x_i = \Delta x(i + \frac{1}{2})$ for some $0 < \Delta x \leq 1$ and N_x such that $(N_x + \frac{1}{2})\Delta x = 1$. Let $t_n = n\Delta t$. Let $\lambda = \frac{\Delta t}{\Delta x}$ and assume Δt be such that the CFL condition

$$\frac{e}{e+1} \leq |a|\lambda \leq 1$$

holds. The Lax–Friedrichs discretization is given by

$$\begin{aligned}u^{n+1}_i - u^n_i &= -\lambda\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right), \\ F^n_{i+\frac{1}{2}} &= -\frac{1}{2\lambda}\left(u^n_{i+1} - u^n_i\right) + \frac{a}{2}\left(u^n_{i+1} + u^n_i\right),\end{aligned}$$

for $i = 0, \dots, N_x$ and $n = 0, \dots$. The initial condition is discretized as $u_{i,0} = u_{i,0}$. We consider boundary feedback control as

$$\begin{aligned}u^n_{N_x+1} &= \kappa_N u^n_1, \\ u^n_0 &= \kappa_1 u^n_{N_x}.\end{aligned}$$

Consider the discrete Lyapunov function

$$L^n = \Delta x \sum_{i=1}^{N_x} \exp(-\frac{a}{|a|}x)\left(u^n_i\right)^2. \tag{15}$$

Then, for any given $u_{i,0} \in \mathbb{R}$ the Lyapunov function decays exponentially fast to zero provided that $\kappa_1^2 \leq \frac{1}{e}$ and $\kappa_N^2 \leq \frac{1}{e}$. There exists

$$\nu := \frac{e-1}{2e\lambda}$$

such that

$$L^n \leq \exp(-\nu t_n)L^0. \tag{16}$$

Proof: Define

$$D := \frac{1}{2}\left(1 + a\lambda\right).$$

We have $0 \leq D \leq 1$ due to the CFL condition. A simple computation shows that the Lax–Friedrichs scheme can be rewritten as

$$u^{n+1}_i = u^n_{i+1}(1-D) + Du^n_{i-1}$$

and therefore

$$(u^{n+1}_i)^2 \leq (u^n_{i+1})^2(1-D) + D(u^n_{i-1})^2.$$

The discrete finite difference approximation is therefore estimated as follows:

$$\begin{aligned}
&\frac{L^{n+1}-L^n}{\Delta t}\\
=&\frac{1-D}{\lambda}\sum_{i=1}^{N_x}\exp(-\frac{a}{|a|}x_i)\left((u_{i+1}^n)^2-(u_i^n)^2\right)\\
&+\frac{D}{\lambda}\sum_{i=1}^{N_x}\exp(-\frac{a}{|a|}x_i)\left((u_{i-1}^n)^2-(u_i^n)^2\right)\\
=&\frac{1-D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\left(\exp(-\frac{a}{|a|}x_{i-1})-\exp(-\frac{a}{|a|}x_i)\right)\\
&+\frac{D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\left(\exp(-\frac{a}{|a|}x_{i+1})-\exp(-\frac{a}{|a|}x_i)\right)\\
&+\frac{1-D}{\lambda}\left(\exp(-\frac{a}{|a|}x_{N_x})(u_{N_x+1}^n)^2-\exp(-\frac{a}{|a|}x_0)(u_1^n)^2\right)\\
&+\frac{D}{\lambda}\left(\exp(-\frac{a}{|a|}x_1)(u_0^n)^2-\exp(-\frac{a}{|a|}x_{N_x+1})(u_{N_x}^n)^2\right).
\end{aligned}$$

We estimate the terms independently. Since $\kappa_N^2, \kappa_1^2 \leq \frac{1}{e}$ we have $\kappa_N^2 \leq \exp(\frac{a}{|a|}(x_{N_x}-x_0))$ and $\kappa_1^2 \leq \exp(-\frac{a}{|a|})$. Hence,

$$\begin{aligned}
&\frac{1-D}{\lambda}\left(\exp(-\frac{a}{|a|}x_{N_x})\kappa_N^2(u_1^n)^2-\exp(-\frac{a}{|a|}x_0)(u_1^n)^2\right)\\
&+\frac{D}{\lambda}\left(\exp(-\frac{a}{|a|}x_1)\kappa_1^2(u_{N_x}^n)^2-\exp(-\frac{a}{|a|}x_{N_x+1})(u_{N_x}^n)^2\right)\\
&\leq 0,
\end{aligned}$$

Further, we simplify

$$\begin{aligned}
&\frac{1-D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\left(\exp(-\frac{a}{|a|}x_{i-1})-\exp(-\frac{a}{|a|}x_i)\right)\\
&+\frac{D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\left(\exp(-\frac{a}{|a|}x_{i+1})-\exp(-\frac{a}{|a|}x_i)\right)\\
=&\frac{1-D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\exp(-\frac{a}{|a|}x_i)\left(\exp(\frac{a}{|a|}\Delta x)-1\right)\\
&+\frac{D}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\exp(-\frac{a}{|a|}x_i)\left(\exp(-\frac{a}{|a|}\Delta x)-1\right)\\
=&\frac{\mathcal{X}}{\lambda}\sum_{i=1}^{N_x}(u_i^n)^2\exp(-\frac{a}{|a|}x_i)\\
=&\frac{\mathcal{X}}{\Delta x\lambda}L^n,
\end{aligned}$$

for

$$\begin{aligned}
\mathcal{X}=&(1-D)\left(\exp(\frac{a}{|a|}\Delta x)-1\right)\\
&+D\left(\exp(-\frac{a}{|a|}\Delta x)-1\right).
\end{aligned}$$

We have $\exp(\Delta x)-1 \leq (e-1)\Delta x$ and $\exp(-\Delta x)-1 \leq -\frac{e-1}{e}\Delta x$ for $0<\Delta x\leq 1$. According to the CFL condition we have $\frac{e}{e+1} \leq |a|\lambda < 1$. If $a>0$ then $2D = 1 + a\lambda \geq 1+\frac{e}{e+1}$ and $2(1-D) \leq 1-\frac{e}{e+1}$. If $a<0$, then $2(1-D)\geq 1+\frac{e}{e+1}$ and $2D \leq 1-\frac{e}{e+1}$. Hence, if $a>0$ we estimate

$$\begin{aligned}
\mathcal{X}\leq&\frac{(e-1)\Delta x}{2}(1-\frac{e}{e+1})-(1+\frac{e}{e+1})\frac{\Delta x(e-1)}{2e}\\
=&\quad -\frac{e-1}{2e}\Delta x+R.
\end{aligned}$$

For $a<0$ we have

$$\begin{aligned}
\mathcal{X}\leq&-\frac{\Delta x(e-1)}{2e}(1+\frac{e}{e+1})+\frac{(e-1)\Delta x}{2}(1-\frac{e}{e+1})\\
=&\quad -\frac{e-1}{2e}\Delta x+R.
\end{aligned}$$

In both cases we have R given by

$$R=\frac{(e-1)\Delta x}{2}(1-\frac{e}{e+1})-\frac{e}{e+1}\frac{\Delta x(e-1)}{2e}=0.$$

Therefore,

$$\frac{L^{n+1}-L^n}{\Delta t}\leq -\frac{e-1}{2e\lambda}L^n=-\nu L^n.$$

The previous inequality yields the exponential decay (16) for any initial data $u_{i,0}$. ■

For any fixed a the equation linear transport equation and does require only one boundary conditions for well–posedness. Due to the central scheme we have to describe two conditions for a well–posed problem. Consider for example the case $a>0$. The condition on the damping required to proceed in the previous proof reads $\kappa_N^2 \leq \exp(a(x_{N_x}-x_0))$. This does not necessarily introduce a damping and shows that the influence of the right boundary condition is very weak in the discrete scheme. This is consistent with the expected behavior of the continuous equation.

The proof of Lemma 3.3 simplifies in case of Upwind or Enquist–Osher fluxes. For this particular case a similar result has already been established in [30] and therefore it is omitted here.

IV. Computational results

We present computational results for Theorem 3.1 and Lemma 3.3. If not stated otherwise we use $N_x = 100$ equi–distant spatial grid points on $[0,1]$. As in Theorem 3.1 we have the cell–centers given by $x_i = \Delta x(i+\frac{1}{2})$. The temporal stepsize Δt is chosen to fulfill the CFL condition. We chose the feedback control $\kappa_1^2 = \kappa_N^2 = \frac{1}{e}$ corresponding to the lowest possible damping of the data. If not stated otherwise we consider the stationary state $y_0 = 1$. As initial perturbation we chose

$$u_{i,0} = c_{y_0}\sin(2\pi x_i)$$

for a constant c_{y_0}. The constant c_{y_0} is chosen such that $f'(u) \geq \nu$ for all $u = u_{i,0}$. We consider $f(u) = a(u+y_0)$, $f(u) = (u+y_0)^2$ (Burgers) and $f(u) = \frac{u+y_0}{1+y_0+u}$ (supply chain). We present results for a Lyapunov function with $\eta(u) = u^2$ and $\eta(u) = u^4$, respectively.

In Figure 1 we present results corresponding to Theorem 3.1. We use the Enquist–Osher scheme for feedback stabilization of Burgers' equation. We present the values L^n for the Lyapunov function given by (13), as well as, the theoretical bound $\exp(-\frac{e-1}{e}\nu t)L^0$ and the size of the temporal grid Δt. The later indicates the threshold in the estimate (14). All plots are in logarithmic scale. In Figure 1 we present results for $\eta(u) = u^2$ and for $\eta(u) = u^4$. We observe that the decay is stronger than the theoretical bound suggests. Also, we observe the strong decay even for values of order smaller than $O(\Delta t)$. This shows that the presented estimate (14) is not a sharp bound. The decay is similar for both Lyapunov functions. In Figure 2 we use the same scheme, feedback condition and Lyapunov functions but consider the flux function of the supply chain model. As expected a similar behavior as for Burger's example is observed. The actual decay of the Lyapunov function is bounded by the theoretical bound and the temporal grid. As in the previous computational results we observe that the bound is not sharp. The results for the Lax–Friedrichs scheme are similar and we present only results for the stabilization of Burger's equation. In Figure 3 we show the exponential decay for a Lyapunov function with $\eta(u) = u^2$ and $\eta(u) = u^4$, respectively. The decay in the computed values of the Lyapunov function is stronger compared with the Enquist–Osher scheme. A possible reason could be the larger numerical viscosity of the Lax–Friedrichs scheme compared with the Enquist–Osher scheme.

Finally, we present results for the Lax–Friedrichs scheme and linear transport with both positive and negative transport. We chose $\lambda|a| = 1$ and the lowest possible feedback damping $\kappa_1^2 = \kappa_N^2 = \frac{1}{e}$. We set $\eta(u) = u^2$. In Figure 4 we present the decay of the Lyapunov function as well as the theoretical bounds for linear transport with $a < 0$ and $a > 0$, respectively. We observe that the actual decay is stronger than the computed numerical bound. The Lax–Friedrichs scheme as a symmetric flux with respect to the sign of a. As expected the behavior of Lyapunov function is therefore similar for $a < 0$ and $a > 0$, respectively. The numerical decay rates are summarized in Table I.

TABLE I

DECAY OF THE DISCRETE LYAPUNOV FUNCTION FOR DIFFERENT NUMERICAL GRIDS N_x. WE CONSIDER A LINEAR FLUX AND THE LAX–FRIEDRICHS SCHEME. IN THE MIDDLE WE COMPUTE THE NUMERICAL DECAY RATE AS $-\ln\left(\frac{L(t)}{L(0)}\right)\frac{1}{T}$. THE LAST COLUMN CORRESPONDS TO THE THEORETICAL RATE OBTAINED IN LEMMA 3.3.

N_x	Numerical rate	Theor. rate
25	2.42	0.35
50	1.77	0.35
100	1.42	0.35
200	1.24	0.35

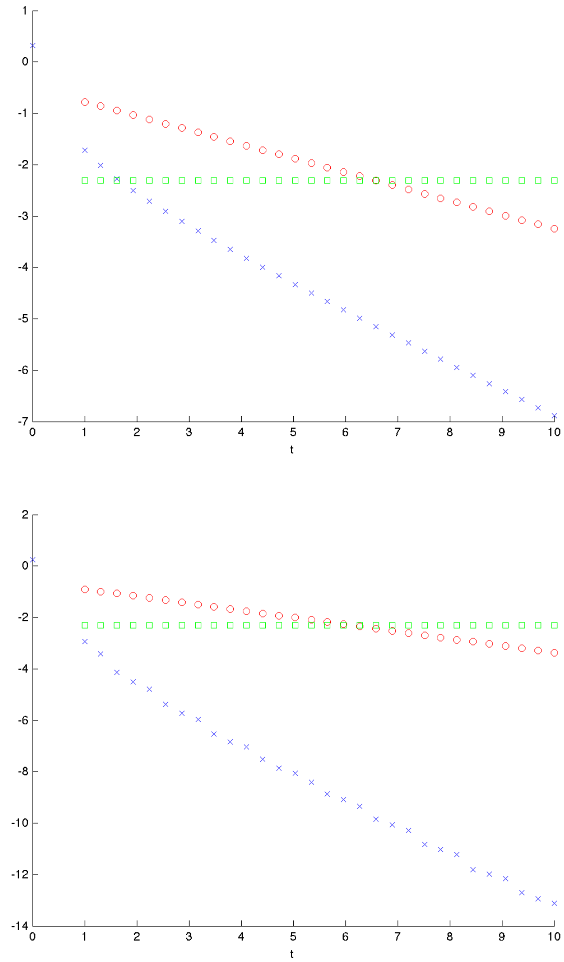

Fig. 1. Decay of the discrete Lyapunov function in logarithmic scale (blue crosses). Feedback stabilization of the state $y_0 = 1$ with low damping $\kappa^2 = \frac{1}{e}$ and dynamics governed by the Burgers equation. Numerical flux is the Enquist–Osher flux. In red dots the theoretical decay rate $-\frac{e-1}{e}\nu$ is shown. In green the size of the temporal grid Δt is shown. Left picture is the Lyapunov function for $\eta(u) = u^2$ and to the right for $\eta(u) = u^4$.

V. CONCLUSION

We present stabilization results for nonlinear conservation laws using feedback boundary control. In the discrete case stabilization is proven for general finite volume schemes. The result is obtained under assumptions on uniform bounds on entropy, entropy–flux and discrete C^2–norm of the solution u. Some assumptions are relaxed depending on the numerical scheme. Convergence of Lax–Friedrichs scheme in case of linear transport is also established. Numerical results illustrate the expected behavior.

In view of the existing theoretical results on stabilization of systems of conservation laws, it would be desirable to have similar results as Theorem 3.1 for systems. However, the theoretical proof [4] follows by linearization and diagonalization of the hyperbolic system. However, most even the simplest available finite–volume schemes do not use this technique but rely on a central approximation as for example the Lax–Friedrichs scheme. A possible remedy might be the use of a suitable entropy function, however, in the case of systems those might not exist.

Nowadays, high–order schemes are common and important in the discretization of hyperbolic equations. An extension of the previous results to high–order schemes is therefore desirable. However, the current estimate relies on Taylor expansion of the numerical flux up to second–order. This restricts strongly the use of limiters in the high–order scheme and therefore it is expected that the current approach does not transfer directly to high–order discretizations.

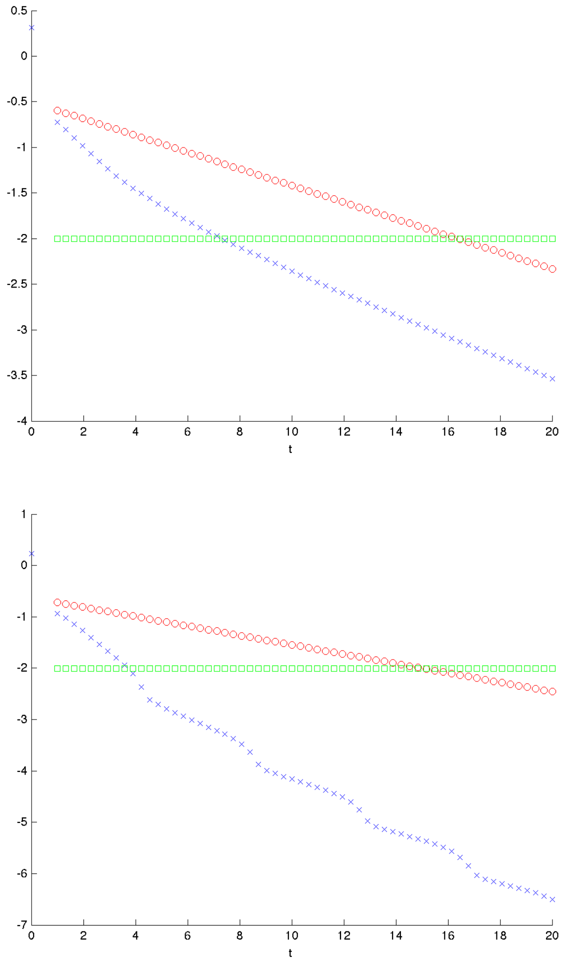

Fig. 2. Decay of the discrete Lyapunov function in logarithmic scale (blue crosses). Feedback stabilization of the state $y_0 = 1$ with low damping $\kappa^2 = \frac{1}{e}$ and dynamics governed by the supply chain flux. Numerical flux is the Enquist–Osher flux. In red dots the theoretical decay rate $-\frac{e-1}{e}\nu$ is shown. In green the size of the temporal grid Δt is shown. Left picture is the Lyapunov function for $\eta(u) = u^2$ and to the right for $\eta(u) = u^4$.

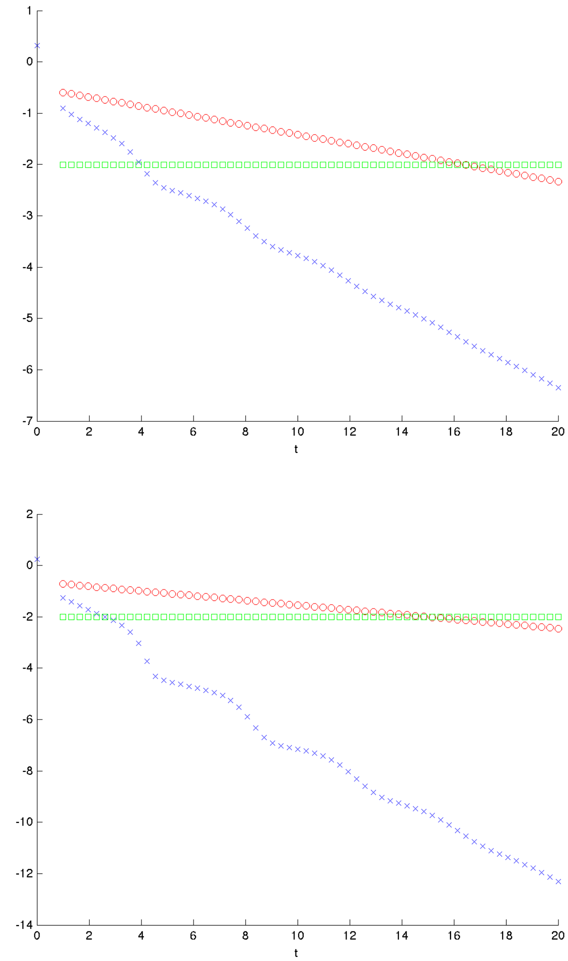

Fig. 3. Decay of the discrete Lyapunov function in logarithmic scale (blue crosses). Feedback stabilization of the state $y_0 = 1$ with low damping $\kappa^2 = \frac{1}{e}$ and dynamics governed by Burger's equation. Numerical flux is the Lax–Friedrichs flux. In red dots the theoretical rate $-\frac{e-1}{e}\nu$ is shown. In green the size of the temporal grid Δt is shown. Left picture is the Lyapunov function for $\eta(u) = u^2$ and to the right for $\eta(u) = u^4$.

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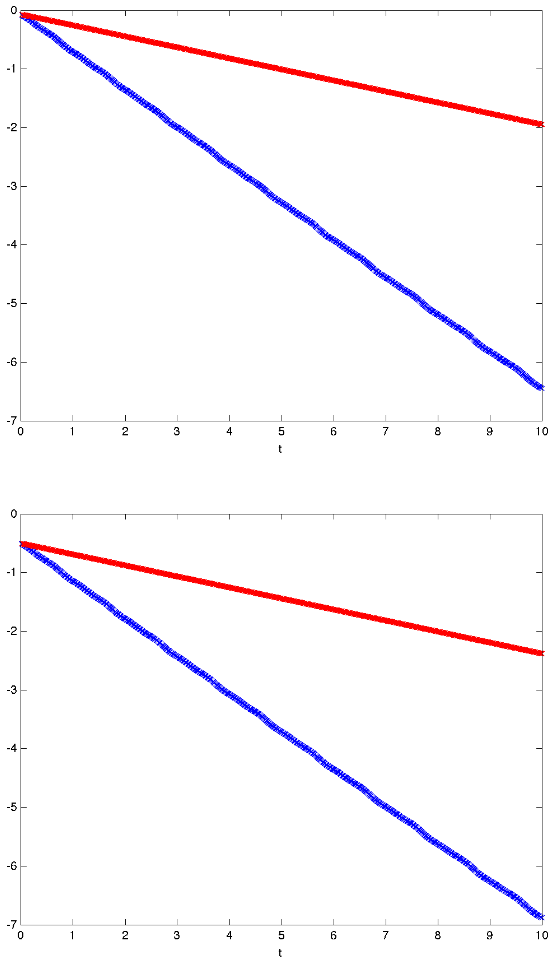

Fig. 4. Decay of the discrete Lyapunov function in logarithmic scale (blue crosses). Feedback stabilization of the state $y_0 = 1$ with low damping $\kappa^2 = \frac{1}{e}$ and linear flux $f(u) = a(u + y_0)$. Numerical flux is the Lax–Friedrichs flux. In red dots the theoretical decay rate is shown. The Lyapunov function is computed using $\eta(u) = u^2$. Left part corresponds to $a < 0$ and right part to $a > 0$.

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