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# ANALYSIS OF AN XFEM DISCRETIZATION FOR STOKES INTERFACE PROBLEMS

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**Abstract.** We consider a stationary Stokes interface problem. In the discretization the interface is not aligned with the triangulation. For the discretization we use the $P_1$ extended finite element space ($P_1$-XFEM) for the pressure and the standard conforming $P_2$ finite element space for the velocity. Since this pair is not necessarily LBB stable, a consistent stabilization term, known from the literature, is added. For the discrete bilinear form an inf-sup stability result is derived, which is uniform with respect to $h$ (mesh size parameter), the viscosity quotient $\mu_1/\mu_2$ and the position of the interface in the triangulation. Based on this, discretization error bounds are derived. An optimal preconditioner for the stiffness matrix corresponding to this pair $P_1$-XFE for pressure and $P_2$-FE for velocity is presented. The preconditioner has block diagonal form, with a multigrid preconditioner for the velocity block and a new Schur complement preconditioner. Optimality of this block preconditioner is proved. Results of numerical experiments illustrate properties of the discretization method and of a preconditioned MINRES solver.

**AMS subject classifications.** 65N15, 65N22, 65N30, 65F10

**Key words.** Stokes equations, interface problem, extended finite element space, preconditioning, Schur complement

## 1. Introduction.

In this paper we treat the following Stokes problem on a bounded connected Lipschitz domain $\Omega$ in $d$-dimensional Euclidean space ($d = 2, 3$): Find a velocity $u$ and a pressure $p$ such that

$$\begin{aligned} -\operatorname{div}(\mu(x)D(u)) + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with $D(u) := \nabla u + (\nabla u)^T$ and a *piecewise constant viscosity* $\mu = \mu_i > 0$ in $\Omega_i$. The subdomains $\Omega_1,\ \Omega_2$ are assumed to be Lipschitz domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. By $\Gamma$ we denote the interface between the subdomains, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. For a corresponding weak formulation we introduce the spaces $V := H_0^1(\Omega)^d$ and

$$L^2_\mu(\Omega) := \{\, p \in L^2(\Omega) \mid \int_\Omega \mu^{-1} p(x)\, dx = 0 \,\}. \tag{1.2}$$

The scaling with $\mu$ in the Gauge condition in (1.2) is convenient for obtaining estimates that are uniform w.r.t. the jump in the viscosity, cf. [13]. The variational problem reads as follows: given $f \in V'$ find $(u, p) \in V \times L^2_\mu(\Omega)$ such that

$$\left\{ \begin{aligned} \tfrac{1}{2}(\mu D(u), D(v))_{0,\Omega} - (\operatorname{div} v, p)_{0,\Omega} &= f(v) \quad && \text{for all } v \in V, \\ (\operatorname{div} u, q)_{0,\Omega} &= 0 && \text{for all } q \in L^2_\mu(\Omega). \end{aligned} \right. \tag{1.3}$$

Here $(\cdot,\cdot)_{0,\Omega}$ denotes the $L^2$ scalar product on $\Omega$. This is a well-posed weak formulation [6].

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An important motivation for considering this type of Stokes equations comes from two-phase incompressible flows. Often such problems are modeled by Navier-Stokes equations with discontinuous density and viscosity coefficients. The effect of interface tension can be taken into account by using a special localized force term at the interface [9]. If in such a setting one has highly viscous flows then the Stokes equations with discontinuous viscosity are a reasonable model problem for method developmemt and analysis. A well-known technique for capturing the unknown interface is based on the level set method, cf. [18, 3, 14] and the references therein. If the level set method is used, then typically in the discretization of the flow equations the interface is *not* aligned with the grid. This causes difficulties with respect to an accurate discretization of the flow variables. Recently, extended finite element techniques (XFEM; also called cut finite element methods) have been developed to obtain accurate finite element discretizations, cf. for example [5, 10, 9]. Concerning theoretical analysis of XFEM applied to such Stokes interface problems very little is known. In fact, the only paper with rigorous analysis of XFEM applied to Stokes interface problems we know of is [10]. In that paper a pair of XFEM spaces is considered, namely $P_1$-XFE for the pressure and iso$P_2$-XFE for the velocity. To obtain weak continuity of the velocity across the interface, a Nitsche method is used. In the bilinear form stabilization terms (controlling the jump in the normal gradient across faces between elements in a neighborhood of the interface) are added. For the discrete bilinear form an inf-sup stability result is derived, which is uniform with respect to $h$ (mesh size parameter), the viscosity quotient $\mu_1/\mu_2$ and the position of the interface in the triangulation. Based on this, an optimal discretization error bound is derived. Furthermore a uniform (w.r.t. the location of the interface) condition number bound for the stiffness matrix is derived.

In this paper we analyze an XFEM that differs from the one considered in [10]. In the discretization that we consider, the pressure variable is approximated in a conforming $P_1$-XFE space (as in [10]), but the velocity is approximated in the standard conforming $P_2$-FE space. In the discretization we use the same stabilization technique as in [10]. For the discrete bilinear form we derive an inf-sup stability result. Similar to [10], a key property of this result is that the stability constant is uniform with respect to $h$, the viscosity quotient $\mu_1/\mu_2$ and the position of the interface in the triangulation. Based on this result and interpolation error estimates, discretization error bounds are derived. Due to the use of the standard $P_2$-FE velocity space the error bound is not optimal if the normal derivative of the velocity is discontinous across the interface (which typically occurs if $\mu_1 \neq \mu_2$). However, the uniform stability result also holds if the $P_2$-FE velocity space is replaced by a larger conforming $P_2$-XFE space, cf. Remark 1 below. For this larger XFE velocity space improved error bounds hold. The reason why we consider the standard $P_2$-FE velocity space is that in real two-phase flow applications, with small viscosity jumps, the pair $P_1$-XFE for pressure and $P_2$-FE for the velocity has shown to work satisfactory [16, 4]. It turns out that the poor *asymptotic* approximation quality of the velocity in the $P_2$-FE space does not dominate the total error on realistic meshes. We will illustrate this in a numerical experiment in section 7.

Apart from the different spaces considered in this paper (compared to [10]) a further new key result is related to the linear algebra part. In [10] a condition number bound of the form $c(\mu_{\max}/\mu_{\min})^2h^{-2}$ is derived for the stiffness matrix. The issue of preconditioning is not treated in that paper. In this paper we derive an optimal preconditioner for the stiffness matrix corresponding to the pair $P_1$-XFE for pressure and $P_2$-FE for velocity. The preconditioner has block diagonal form, with a multigrid preconditioner for the velocity block and a new Schur complement preconditioner. Optimality of this block preconditioner is proved. Results of numerical experiments illustrate properties of the discretization method and of a preconditioned MINRES solver.

**2. The XFEM space of piecewise linears.** We assume a family of shape regular quasi-uniform triangulations consisting of simplices $\{\mathcal{T}_h\}_{h>0}$. The triangulations are *not* fitted to the interface $\Gamma$. To avoid technical details we make the following *generic intersection assumption*: if $\Gamma \cap T \neq \emptyset$ for a $T \in \mathcal{T}_h$, then $\text{meas}_{d-1}(\Gamma \cap \partial T) = 0$ holds. For example, if $d = 2$ this does not allow the case that $\Gamma \cap T$ coincides with an edge of $T$. We introduce the subdomains $\Omega_{i,h} := \{\, T \in \mathcal{T}_h \mid T \subset \Omega_i \text{ or } \text{meas}_{d-1}(T \cap \Gamma) > 0 \,\}$, $i = 1, 2$, and the corresponding standard linear finite element spaces

$$Q_{i,h} := \{\, v_h \in C(\Omega_{i,h}) \mid v_{h|T} \in \mathcal{P}_1 \quad \forall\, T \in \Omega_{i,h} \,\},\ i = 1, 2.$$

We use the same notation $\Omega_{i,h}$ for the set of tetrahedra as well as for the subdomain of $\Omega$ which is formed by these tetrahedra, as its meaning is clear from the context. For the stabilization procedure that is introduced below we need a further partitioning of $\Omega_{i,h}$. Define $\omega_{i,h} := \{\, T \in \Omega_{i,h} \mid \text{meas}_{d-1}(T \cap \Gamma) = 0 \,\}$, $i = 1, 2$ and $\mathcal{T}_h^\Gamma := \mathcal{T}_h \setminus (\omega_{1,h} \cup \omega_{2,h}) = \{\, T \in \mathcal{T}_h \mid \text{meas}_{d-1}(T \cap \Gamma) > 0 \,\}$. Note that $\mathcal{T}_h = \omega_{1,h} \cup \omega_{2,h} \cup \mathcal{T}_h^\Gamma$ holds and forms a disjoint union. Corresponding sets of faces (needed in the stabilization procedure) are given by

$$\mathcal{F}_i = \{\, F \subset \partial T \mid T \in \mathcal{T}_h^\Gamma, \quad F \not\subset \partial \Omega_{i,h} \,\}, \quad i = 1, 2,$$

and $\mathcal{F}_h := \mathcal{F}_1 \cup \mathcal{F}_2$. For each $F \in \mathcal{F}_h$ a fixed orientation of its normal is chosen and the unit normal with that orientation is denoted by $n_F$. These definitions are illustrated in Fig. 2.1.

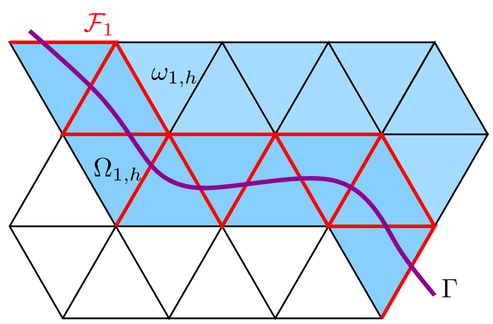



FIG. 2.1. *Set of faces $\mathcal{F}_1$ (in red) and subdomains $\omega_{1,h}$ (light-blue) and $\Omega_{1,h}$ (light- and darker blue triangles) for a 2D example.*

The generic intersection assumption implies that each $T \in \omega_{i,h}$, $i = 1, 2$, has at least one vertex in the interior of $\Omega_i$. A given $p_h = (p_{1,h}, p_{2,h}) \in Q_{1,h} \times Q_{2,h}$ has two values, $p_{1,h}(x)$ and $p_{2,h}(x)$, for $x \in \mathcal{T}_h^\Gamma$. We define a uni-valued function $p_h^\Gamma \in C(\Omega_1 \cup \Omega_2)$ by

$$p_h^\Gamma(x) = p_{i,h}(x) \quad \text{for} \quad x \in \Omega_i.$$

Using the generic intersection assumption we obtain that the mapping $p_h \mapsto p_h^\Gamma$ is bijective. On $Q_{1,h} \times Q_{2,h}$ we use a norm denoted by $\|p_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} := \|p_{1,h}\|^2_{0,\Omega_{1,h}} +$

$\|p_{2,h}\|^2_{0,\Omega_{2,h}}$. The XFEM space of piecewise linears is defined by

$$Q_h^\Gamma := (Q_{1,h} \times Q_{2,h})/\mathbb{R} = \{\, p_h \in Q_{1,h} \times Q_{2,h} \mid (\mu^{-1} p_h^\Gamma, 1)_{0,\Omega} = 0 \,\}. \tag{2.1}$$

Note that $\{\, p_h^\Gamma \mid p_h \in Q_h^\Gamma \,\}$ is a subspace of the pressure space $L^2_\mu(\Omega)$, cf. (1.2). In the analysis we need the following decomposition of this XFEM space into two orthogonal subspaces. We introduce the piecewise constant function

$$\bar{p}_\mu := \big(\mu_1 |\Omega_1|^{-1}, -\mu_2 |\Omega_2|^{-1}\big) \in Q_{1,h} \times Q_{2,h}. \tag{2.2}$$

Using the one-dimensional subspace $M_0 := \mathrm{span}\{\bar{p}_\mu\} \subset Q_h^\Gamma$, the XFEM space is decomposed as $Q_h^\Gamma = M_0 \oplus M_0^\perp$, with $M_0^\perp := \{\, p_h \in Q_h^\Gamma \mid (p_h^\Gamma, \bar{p}_\mu^\Gamma)_{0,\Omega} = 0 \,\}$. We derive an elementary property:

LEMMA 2.1. *$p_h \in M_0^\perp$ has the property $(p_{i,h}, 1)_{0,\Omega_i} = 0$ for $i = 1, 2$.*

*Proof.* From $p_h \in Q_h^\Gamma$ it follows that $(\mu^{-1} p_h^\Gamma, 1)_{0,\Omega} = 0$, hence $\mu_1^{-1}(p_{1,h}, 1)_{0,\Omega_1} + \mu_2^{-1}(p_{2,h}, 1)_{0,\Omega_2} = 0$ holds. From $(p_h^\Gamma, \bar{p}_\mu^\Gamma)_{0,\Omega} = 0$ we get $\mu_1 |\Omega_1|^{-1}(p_{1,h}, 1)_{0,\Omega_1} - \mu_2 |\Omega_2|^{-1}(p_{2,h}, 1)_{0,\Omega_2} = 0$. These two relations imply $(p_{i,h}, 1)_{0,\Omega_i} = 0$ for $i = 1, 2$. □

ASSUMPTION 1. It is known that on the subdomain $\omega_{i,h}$, which is Lipschitz, the following inf-sup property (Necas-inequality) holds: there exists $c_N(\omega_{i,h}) > 0$ such that

$$\sup_{v \in H_0^1(\omega_{i,h})^d} \frac{(\mathrm{div}\, v, p)_{0,\omega_{i,h}}}{\|v\|_{1,\omega_{i,h}}} \geq c_N(\omega_{i,h}) \|p\|_{0,\omega_{i,h}} \quad \text{for all} \;\; p \in L^2(\omega_{i,h})/\mathbb{R}. \tag{2.3}$$

We *assume* that $\inf_{h>0} c_N(\omega_{i,h}) > 0$ holds, i.e. the inf-sup constants are uniformly bounded away from zero if $h \downarrow 0$.

We are not aware of a proof of such a uniform (w.r.t. the domain) inf-sup property. A heuristic argument which indicates that the assumption is plausible is the following: if a standard locally uniform simplex refinement strategy is used, then for $h \downarrow 0$ the domains $\omega_{i,h}$ converge "regularly" to $\Omega_i$.

For the stabilization we introduce the bilinear form

$$\begin{aligned} j(p_h, q_h) &:= \sum_{i=1}^{2} j_i(p_{i,h}, q_{i,h}), \quad p_h, q_h \in Q_{1,h} \times Q_{2,h}, \\ \text{with } j_i(p_{i,h}, q_{i,h}) &:= \mu_i^{-1} \sum_{F \in \mathcal{F}_i} h_F^3 ([\nabla p_{i,h} \cdot n_F], [\nabla q_{i,h} \cdot n_F])_{0,F}, \end{aligned} \tag{2.4}$$

which is also referred to as a ghost penalty term, cf. [2]. Here $[\nabla p_{i,h} \cdot n_F]$ denotes the jump of the normal component of the piecewise constant function $\nabla p_{i,h}$ across the face $F$. All constants used in the results below are *independent of $h$ and $\mu$, and of how the interface $\Gamma$ intersects the triangulation $\mathcal{T}_h$.*

LEMMA 2.2. *The following holds (Lemma 3.8 in [10]):*

$$\mu_i^{-1} \|p_{i,h}\|^2_{0,\Omega_{i,h}} \leq c\big(\mu_i^{-1} \|p_{i,h}\|^2_{0,\omega_{i,h}} + j_i(p_{i,h}, p_{i,h})\big) \quad \textit{for all} \;\; p_{i,h} \in Q_{i,h}, \;\; i = 1, 2.$$

*Proof.* Note that

$$\|p_{i,h}\|^2_{0,\Omega_{i,h}} = \|p_{i,h}\|^2_{0,\omega_{i,h}} + \sum_{T \in \Omega_{i,h} \setminus \omega_{i,h}} \|p_{i,h}\|^2_{0,T},$$

hence, we only have to treat $\|p_{i,h}\|_{0,T}$, $T \in \Omega_{i,h} \setminus \omega_{i,h}$. We write $p = p_{i,h}$, which is a piecewise linear function on $\Omega_{i,h}$. Take $T_0 = T \in \Omega_{i,h} \setminus \omega_{i,h}$, and $x \in T_0$. There is a sequence of simplices $T_1, \dots, T_k$ with faces $F_j = \bar{T}_j \cap \bar{T}_{j-1} \in \mathcal{F}_i$, $j = 1, \dots, k$, and $T_k \in \omega_{i,h}$. The number $k$ is uniformly bounded (often, $k = 1$ holds). The barycenter of $F_j$ is denoted by $m_j$. With an appropriate orientation of the jump operator $[\cdot]_F$ we have the relations

$$\sum_{j=1}^{k} [\nabla p]_{F_j} = \nabla p_{|T_k} - \nabla p_{|T_0},$$

$$\sum_{j=1}^{k} [\nabla p]_{F_j} \cdot m_j = p(m_1) - p(m_k) + \nabla p_{|T_k} \cdot m_k - \nabla p_{|T_0} \cdot m_1.$$

Using these, for $x \in T_0$ one obtains

$$p(x) = p(m_1) + \nabla p_{|T_0} \cdot (x - m_1) = p(m_k) + \nabla p_{|T_k} \cdot (x - m_k) + \sum_{j=1}^{k} [\nabla p]_{F_j} \cdot (m_j - x).$$

Because the tangential component of $\nabla p$ is continuous along the faces we have $[\nabla p]_{F_j} = [\nabla p \cdot n_{F_j}]_{F_j} n_{F_j}$. Using an inverse inequality $\|\nabla p\|_{0,T_k} \le c h_{F_k}^{-1} \|p\|_{0,T_k}$, the estimate $\|x - m_j\| \le c h_{F_j}$ and $|T_0| \sim |T_j|$, $j = 1, \dots, k$, we get

$$\begin{aligned}
\|p\|_{0,T_0}^2 &\le c\big(\|p\|_{0,T_k}^2 + \sum_{j=1}^{k} h_{F_j}^2 \frac{|T_0|}{|F_j|} \|[\nabla p \cdot n_{F_j}]\|_{0,F_j}^2\big) \\
&\le c\big(\|p\|_{0,T_k}^2 + \sum_{j=1}^{k} h_{F_j}^3 \|[\nabla p \cdot n_{F_j}]\|_{0,F_j}^2\big).
\end{aligned}$$

We sum over $T_0 = T \in \Omega_{i,h} \setminus \omega_{i,h}$ and use a finite overlap argument, resulting in

$$\begin{aligned}
\sum_{T \in \Omega_{i,h} \setminus \omega_{i,h}} \|p_{i,h}\|_{0,T}^2 &\le c\big(\|p_{i,h}\|_{0,\omega_{i,h}}^2 + \sum_{F \in \mathcal{F}_i} h_F^3 \|[\nabla p_{i,h} \cdot n_F]\|_{0,F}^2\big) \\
&\le c\big(\|p_{i,h}\|_{0,\omega_{i,h}}^2 + \mu_i j_i(p_{i,h}, p_{i,h})\big),
\end{aligned}$$

which completes the proof. □

**3. Discrete problem.** We introduce the usual bilinear forms

$$a(u, v) := \frac{1}{2} \int_{\Omega} \mu D(u) : D(v)\, dx, \quad b(v, p) = -(\operatorname{div} v, p)_{0,\Omega}.$$

with $D(v) := \nabla v + (\nabla v)^T$. For discretization of the pressure we use the XFEM space $Q_h^\Gamma$. Note that for $p_h \in Q_h^\Gamma$ we have $p_h^\Gamma \in L_\mu^2(\Omega)$. For the velocity discretization we use the standard conforming $P_2$-space

$$V_h := \{\, v_h \in C(\Omega)^d \mid v_{h|T} \in \mathcal{P}_2^d \ \ \forall\, T \in \mathcal{T}_h,\ v_{|\partial\Omega} = 0 \,\} \subset H_0^1(\Omega)^d.$$

The discretization of (1.3) that we consider is as follows: determine $(u_h, p_h) \in V_h \times Q_h^\Gamma$ such that

$$\begin{aligned}
&k\big((u_h, p_h), (v_h, q_h)\big) = f(v_h) \quad \text{for all } (v_h, q_h) \in V_h \times Q_h^\Gamma, \\
&k\big((u_h, p_h), (v_h, q_h)\big) := a(u_h, v_h) + b(v_h, p_h^\Gamma) - b(u_h, q_h^\Gamma) + \varepsilon_p j(p_h, q_h),
\end{aligned} \tag{3.1}$$

with a (sufficiently large) stabilization parameter $\varepsilon_p \ge 0$.

**4. Stability analysis.** In this section we derive a discrete inf-sup result for the bilinear form $k(\cdot,\cdot)$ w.r.t. the space $V_h \times Q_h^\Gamma$, cf. Theorem 4.4. Such a result also holds if we replace $V_h$ by a larger $H^1$-conforming space $\tilde{V}_h \supset V_h$, cf. Remark 1. The analysis is along the same lines as in [10, 13].
We will use the fact that the Taylor-Hood $P_2$-$P_1$ pair is uniformly stable on the subdomains $\omega_{i,h}$. To make this more precise, we need Assumption 1. Let $V_h(\omega_{i,h})$ be the space of continuous piecewise quadratics on $\omega_{i,h}$ that are zero on $\partial\omega_{i,h}$. Note that the domain $\omega_{i,h}$ varies with $h$. For proving the LBB stability of the $P_2$-$P_1$ pair on $\omega_{i,h}$ we use the approach as in [19]. For this one needs the inf-sup property for the pair $H_0^1(\omega_{i,h})^d \times L_0^2(\omega_{i,h})$ and a so-called weak inf-sup property for the $P_2$-$P_1$ on $\omega_{i,h}$. The latter is derived in [1] and uses only local properties on each simplex in the triangulation. Due to Assumption 1 the inf-sup constant $c(\omega_{i,h})$ in (2.3) is uniformly bounded away from zero. The analysis in [19, 1] thus yields that there exist constants $c_{N,i} > 0$, $i = 1, 2$, independent of $h$, such that

$$\sup_{v_h \in V_h(\omega_{i,h})} \frac{(\operatorname{div} v_h, q_h)_{0,\omega_{i,h}}}{\|v_h\|_{1,\omega_{i,h}}} \geq c_{N,i}\|q_h\|_{0,\omega_{i,h}} \ \forall\, q_h \in Q_{i,h} \ \text{ with } (q_h, 1)_{0,\omega_{i,h}} = 0. \quad (4.1)$$

In the next three lemmas we derive lower bounds for $\sup_{v_h \in V_h} \frac{b(v_h, p_h^\Gamma)}{\|v_h\|_1}$. We first consider $p_h \in M_0$ (Lemma 4.1), then $p_h \in M_0^\perp$ (Lemma 4.2), and then combine these results to,obtain an estimate for $p_h \in Q_h^\Gamma$ (Lemma 4.3).
*In the remainder we assume that Assumption 1 is fulfilled.*

LEMMA 4.1. *There exist $h_0 > 0$ and $c > 0$ such that for all $h \leq h_0$:*

$$\sup_{v_h \in V_h} \frac{b(v_h, p_h^\Gamma)}{\|\mu^{\frac{1}{2}}\nabla v_h\|_0} \geq c\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \quad \text{for all } p_h \in M_0.$$

*Proof.* It suffices to consider $p_h = \bar{p}_\mu$ as in (2.2). Define $\tilde{p} := \mu^{-1}\bar{p}_\mu = (|\Omega_1|^{-1}, -|\Omega_2|^{-1}) \in Q_{1,h} \times Q_{2,h}$. The relation $\|\mu^{-\frac{1}{2}}\bar{p}_\mu^\Gamma\|_{0,\Omega} = C(\mu,\Omega)^{\frac{1}{2}}\|\tilde{p}^\Gamma\|_{0,\Omega}$ holds, with

$$C(\mu,\Omega) = \frac{\mu_1|\Omega_1|^{-1} + \mu_2|\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} \geq \mu_{\max} \min_{i=1,2} \frac{|\Omega_i|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} = c\,\mu_{\max},$$

with $\mu_{\max} = \max\{\mu_1, \mu_2\}$. For $v_h \in V_h$ we have $0 = \int_\Omega \operatorname{div} v_h\, dx = \int_{\Omega_1} \operatorname{div} v_h\, dx + \int_{\Omega_2} \operatorname{div} v_h\, dx$, and using this one derives the relation

$$b(v_h, \bar{p}_\mu^\Gamma) = C(\mu,\Omega) b(v_h, \tilde{p}^\Gamma), \quad v_h \in V_h. \quad (4.2)$$

Let $q_h \in C(\Omega)$ be the continuous piecewise linear nodal interpolation of $\tilde{p}^\Gamma$. Then $\|q_h - \tilde{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$ holds. Define $\alpha = \frac{1}{|\Omega|}(q_h, 1)_{0,\Omega}$ and $q_h^* = q_h - \alpha$, hence, $(q_h^*, 1)_{0,\Omega} = 0$. Note that $|\alpha| = \frac{1}{|\Omega|}|(q_h, 1)_{0,\Omega}| = \frac{1}{|\Omega|}|(q_h - \tilde{p}^\Gamma, 1)_{0,\Omega}| \leq c\|q_h - \tilde{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$ holds. This implies $\|q_h^* - \tilde{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$. From the LBB stability of the standard $P_2$-$P_1$ Taylor-Hood pair on $\Omega$ it follows that there exists $\hat{v}_h \in V_h$ with $\|\hat{v}_h\|_1 = 1$ and $c > 0$ such that $b(\hat{v}_h, q_h^*) \geq c\|q_h^*\|_{0,\Omega}$ holds. Using this we obtain, with suitable constants $c > 0$:

$$\begin{aligned} b(\hat{v}_h, \tilde{p}^\Gamma) &\geq b(\hat{v}_h, q_h^*) - d^{\frac{1}{2}}\|\hat{v}_h\|_1\|q_h^* - \tilde{p}^\Gamma\|_{0,\Omega} \\ &\geq c\|q_h^*\|_{0,\Omega} - ch^{\frac{1}{2}} \geq c\|\tilde{p}^\Gamma\|_{0,\Omega} - ch^{\frac{1}{2}} \geq c\|\tilde{p}^\Gamma\|_{0,\Omega}, \end{aligned}$$

provided $h$ is sufficiently small. Combining this with the result in (4.2) yields

$$\begin{aligned} b(\hat{v}_h, \bar{p}_\mu^\Gamma) &= C(\mu,\Omega) b(v_h, \tilde{p}^\Gamma) \geq c\, C(\mu,\Omega) \|\tilde{p}^\Gamma\|_{0,\Omega} \\ &= c\, C(\mu,\Omega)^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega} \geq c \mu_{\max}^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega}. \end{aligned}$$

Finally note that $\|\mu^{-\frac{1}{2}} \bar{p}_\mu\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq (1+ch)\|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega} \leq c\|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega}$ and $\|\mu^{\frac{1}{2}} \nabla \hat{v}_h\|_0 \leq \mu_{\max}^{\frac{1}{2}} \|\hat{v}_h\|_1 = \mu_{\max}^{\frac{1}{2}}$ hold. ∎

LEMMA 4.2. *There exist $h_0 > 0$ and $c_1, c_2 > 0$ such that for all $h \leq h_0$:*

$$\sup_{v_h \in V_h(\omega_{1,h}\cup\omega_{2,h})} \frac{b(v_h, p_h^\Gamma)}{\|\mu^{\frac{1}{2}} \nabla v_h\|_0} \geq c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2 \frac{j(p_h,p_h)}{\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}}$$

*for all $p_h \in M_0^\perp \setminus \{0\}$, with $V_h(\omega_{1,h}\cup\omega_{2,h}) := \{\, v_h \in V_h \mid \operatorname{supp}(v_h) \subset \bar{\omega}_{1,h} \cup \bar{\omega}_{2,h} \,\}$.*

*Proof.* Take $p_h = (p_{1,h}, p_{2,h}) \in M_0^\perp$, $p_h \neq 0$. Define $\alpha_i = \frac{1}{|\omega_{i,h}|}(p_{i,h}, 1)_{0,\omega_{i,h}}$ and $p_{i,h}^* = p_{i,h} - \alpha_i$, hence, $(p_{i,h}^*, 1)_{0,\omega_{i,h}} = 0$. Using (4.1) it follows that there exist $\hat{v}_{i,h} \in V_h$ with $\operatorname{supp}(\hat{v}_{i,h}) \subset \bar{\omega}_{i,h}$, $\|\hat{v}_{i,h}\|_1 = \|p_{i,h}^*\|_{0,\omega_{i,h}}$, and a constant $c > 0$ such that $b(\hat{v}_{i,h}, p_{i,h}^*) \geq c\|p_{i,h}^*\|_{0,\omega_{i,h}}^2$ (with $p_{i,h}^*$ extended by zero outside $\Omega_{i,h}$). Using that $\hat{v}_{i,h} = 0$ on $\partial\omega_{i,h}$ and $p_{i,h} - p_{i,h}^* = \alpha_i$ is constant we get that $b(\hat{v}_{i,h}, p_{i,h}^*) = b(\hat{v}_{i,h}, p_{i,h})$ holds. Since $p_h^* = (p_{1,h}^*, p_{2,h}^*) \in Q_{1,h} \times Q_{2,h}$ we can apply Lemma 2.2 and thus get, with constant $c_1, c_2 > 0$:

$$\begin{aligned} b(\mu_i^{-1} \hat{v}_{i,h}, p_{i,h}) &= b(\mu_i^{-1} \hat{v}_{i,h}, p_{i,h}^*) \geq c \mu_i^{-1} \|p_{i,h}^*\|_{0,\omega_{i,h}}^2 \\ &\geq c_1 \mu_i^{-1} \|p_{i,h}^*\|_{0,\Omega_{i,h}}^2 - c_2 j(p_h^*, p_h^*). \end{aligned} \tag{4.3}$$

Since $j(p_h, p_h)$ depends only on $\nabla p_h$ we have $j(p_h^*, p_h^*) = j(p_h, p_h)$. From Lemma 2.1 we get $(p_{i,h}, 1)_{0,\Omega_i} = 0$. Using this we obtain

$$\begin{aligned} |\alpha_i| &= \frac{1}{|\omega_{i,h}|} |(p_{i,h}, 1)_{0,\omega_{i,h}}| = \frac{1}{|\omega_{i,h}|} \Big| \int_{\Omega_i \setminus \omega_{i,h}} p_{i,h}\, dx \Big| \\ &\leq \frac{1}{|\omega_{i,h}|} |\Omega_i \setminus \omega_{i,h}|^{\frac{1}{2}} \|p_{i,h}\|_{0,\Omega_i} \leq c h^{\frac{1}{2}} \|p_{i,h}\|_{0,\Omega_{i,h}}. \end{aligned} \tag{4.4}$$

Thus, for $h$ sufficiently small there exists $c > 0$ such that

$$\|p_{i,h}^*\|_{0,\Omega_{i,h}} \geq \|p_{i,h}\|_{0,\Omega_{i,h}} - c|\alpha_i| \geq \|p_{i,h}\|_{0,\Omega_{i,h}} (1 - c h^{\frac{1}{2}}) \geq c \|p_{i,h}\|_{0,\Omega_{i,h}}.$$

Using this in (4.3) we get

$$b(\mu_i^{-1} \hat{v}_{i,h}, p_{i,h}) \geq c_1 \|\mu_i^{-\frac{1}{2}} p_{i,h}\|_{0,\Omega_{i,h}}^2 - c_2 j(p_h, p_h),$$

and thus, with $\hat{v}_h := \mu_1^{-1} \hat{v}_{1,h} + \mu_2^{-1} \hat{v}_{2,h} \in V_h(\omega_{1,h} \cup \omega_{2,h})$:

$$\begin{aligned} b(\hat{v}_h, p_h^\Gamma) &= b(\mu_1^{-1} \hat{v}_{1,h}, p_{1,h}) + b(\mu_2^{-1} \hat{v}_{2,h}, p_{2,h}) \\ &\geq c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 - c_2 j(p_h, p_h). \end{aligned} \tag{4.5}$$

Using (4.4) we get $\|\hat{v}_{i,h}\|_1 = \|p_{i,h}^*\|_{0,\omega_{i,h}} \leq \|p_{i,h}\|_{0,\omega_{i,h}} + c|\alpha_i| \leq c\|p_{i,h}\|_{0,\Omega_{i,h}}$ and thus

$$\|\mu^{\frac{1}{2}} \nabla \hat{v}_h\|_0^2 = \sum_{i=1}^2 \mu_i^{-1} \|\nabla \hat{v}_{i,h}\|_0^2 \leq c \sum_{i=1}^2 \|\mu_i^{-\frac{1}{2}} p_{i,h}\|_{0,\Omega_{i,h}}^2 = c \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2$$

holds. Combining this with the estimate in (4.5) completes the proof. $\square$

LEMMA 4.3. *There exist $h_0 > 0$ and $c_1, c_2 > 0$ such that for all $h \leq h_0$:*

$$\sup_{v_h \in V_h} \frac{b(v_h, p_h^\Gamma)}{\|\mu^{\frac{1}{2}} \nabla v_h\|_0} \geq c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}} - c_2 \frac{j(p_h, p_h)}{\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}} \quad \forall\, p_h \in Q_h^\Gamma \setminus \{0\}.$$

*Proof.* Take $p_h = (p_{1,h}, p_{2,h}) \in Q_h^\Gamma \setminus \{0\}$. We use the decomposition $p_h = \bar{p}_h + \tilde{p}_h$, $\bar{p}_h \in M_0$, $\tilde{p}_h \in M_0^\perp$. From the lemmas above it follows that there exist $\bar{v}_h \in V_h$, $\tilde{v}_h \in V_h(\omega_{1,h} \cup \omega_{1,h})$, with $\|\mu^{\frac{1}{2}} \nabla \bar{v}_h\|_0 = \|\mu^{-\frac{1}{2}} \bar{p}_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}$, $\|\mu^{\frac{1}{2}} \nabla \tilde{v}_h\|_0 = \|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}$ such that

$$b(\bar{v}_h, \bar{p}_h^\Gamma) \geq c_1 \|\mu^{-\frac{1}{2}} \bar{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}}, \quad b(\tilde{v}_h, \tilde{p}_h^\Gamma) \geq c_2 \|\mu^{-\frac{1}{2}} \tilde{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} - c_3 j(\tilde{p}_h, \tilde{p}_h),$$

with $c_j > 0$, $j = 1, 2, 3$. Note that $\tilde{v}_h = 0$ on $\partial \omega_{i,h}$ and $\bar{p}_h^\Gamma$ is constant on $\omega_{i,h}$, hence $b(\tilde{v}_h, \bar{p}_h^\Gamma) = -\sum_{i=1}^2 (\operatorname{div} \tilde{v}_h, \bar{p}_h^\Gamma)_{0,\omega_{i,h}} = 0$ holds. Take $v_h := \bar{v}_h + \gamma \tilde{v}_h \in V_h$, with $\gamma > 0$. We then get

$$\begin{aligned} b(v_h, p_h^\Gamma) &= b(\bar{v}_h, \bar{p}_h^\Gamma) + \gamma b(\tilde{v}_h, \tilde{p}_h^\Gamma) + b(\bar{v}_h, \tilde{p}_h^\Gamma) \\ &\geq c_1 \|\mu^{-\frac{1}{2}} \bar{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} + \gamma c_2 \|\mu^{-\frac{1}{2}} \tilde{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} - \gamma c_3 j(\tilde{p}_h, \tilde{p}_h) + b(\bar{v}_h, \tilde{p}_h^\Gamma). \end{aligned}$$

Since $\bar{p}_h$ is constant on $\Omega_{i,h}$ we have $j(\tilde{p}_h, \tilde{p}_h) = j(p_h, p_h)$. Furthermore:

$$\begin{aligned} |b(\bar{v}_h, \tilde{p}_h^\Gamma)| &\leq d^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla \bar{v}_h\|_0 \|\mu^{-\frac{1}{2}} \tilde{p}_h^\Gamma\|_{0,\Omega} \leq d^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} \bar{p}_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}} \|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}} \\ &\leq \frac{1}{2} c_1 \|\mu^{-\frac{1}{2}} \bar{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} + \frac{1}{2} d c_1^{-1} \|\mu^{-\frac{1}{2}} \tilde{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}}. \end{aligned}$$

For $\gamma = \frac{c_1^2 + d}{2 c_1 c_2}$ we thus get, with a suitable constant $c$:

$$b(v_h, p_h^\Gamma) \geq \frac{1}{2} c_1 \big( \|\mu^{-\frac{1}{2}} \bar{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} + \|\mu^{-\frac{1}{2}} \tilde{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} \big) - c j(p_h, p_h),$$

and combining this with $\|\mu^{-\frac{1}{2}} p_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} \leq 2 (\|\mu^{-\frac{1}{2}} \bar{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}} + \|\mu^{-\frac{1}{2}} \tilde{p}_h\|^2_{0,\Omega_{1,h} \cup \Omega_{2,h}})$ we obtain

$$\frac{b(v_h, p_h^\Gamma)}{\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}} \geq \frac{1}{4} c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}} - c \frac{j(p_h, p_h)}{\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}}. \tag{4.6}$$

From $0 = (\bar{p}_h^\Gamma, \tilde{p}_h^\Gamma)_{0,\Omega} = \sum_{i=1}^2 (\bar{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_i}$ we obtain

$$\begin{aligned} \Big| \sum_{i=1}^2 (\bar{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_{i,h}} \Big| = \Big| \sum_{i=1}^2 (\bar{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_{i,h} \setminus \Omega_i} \Big| &\leq c h^{\frac{1}{2}} \sum_{i=1}^2 \|\bar{p}_{i,h}\|_{0,\Omega_{i,h}} \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}} \\ &\leq c h^{\frac{1}{2}} \sum_{i=1}^2 \big( \|\bar{p}_{i,h}\|^2_{0,\Omega_{i,h}} + \|\tilde{p}_{i,h}\|^2_{0,\Omega_{i,h}} \big). \end{aligned}$$

Using this we get

$$
\begin{aligned}
&\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
&= \sum_{i=1}^2 \mu_i^{-1}\|\bar{p}_{i,h}+\tilde{p}_{i,h}\|^2_{0,\Omega_{i,h}} = \sum_{i=1}^2 \mu_i^{-1}\big(\|\bar{p}_{i,h}\|^2_{0,\Omega_{i,h}} + \|\tilde{p}_{i,h}\|^2_{0,\Omega_{i,h}} + 2(\bar{p}_{i,h},\tilde{p}_{i,h})_{0,\Omega_{i,h}}\big) \\
&\geq (1-ch^{\frac{1}{2}})\sum_{i=1}^2 \mu_i^{-1}\big(\|\bar{p}_{i,h}\|^2_{0,\Omega_{i,h}} + \|\tilde{p}_{i,h}\|^2_{0,\Omega_{i,h}}\big) \\
&= (1-ch^{\frac{1}{2}})\big(\|\mu^{-\frac{1}{2}}\bar{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|\mu^{-\frac{1}{2}}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}\big) \\
&= (1-ch^{\frac{1}{2}})\big(\|\mu^{\frac{1}{2}}\nabla\bar{v}_h\|_0^2 + \|\mu^{\frac{1}{2}}\nabla\tilde{v}_h\|_0^2\big).
\end{aligned}
$$

Hence, for $h$ sufficiently small there exists $c>0$ such that

$$
\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \geq c\big(\|\mu^{\frac{1}{2}}\nabla\bar{v}_h\|_0^2 + \|\mu^{\frac{1}{2}}\nabla\tilde{v}_h\|_0^2\big)^{\frac{1}{2}} \geq \frac{1}{2}\min\{1,\gamma^{-1}\}c\|\mu^{\frac{1}{2}}\nabla v_h\|_0,
$$

and combining this with (4.6) completes the proof. ∎

For the main result in the next theorem we introduce a mesh- and $\mu$-dependent norm on $V_h \times Q_h^\Gamma$:

$$
|||(u_h,p_h)|||_h^2 := \|\mu^{\frac{1}{2}}D(u_h)\|_0^2 + \|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + j(p_h,p_h). \tag{4.7}
$$

Recall that $\varepsilon_p$ is the stabilization parameter used in the discretization (3.1), and $D(u_h) = \nabla u_h + (\nabla u_h)^T$. From Korn's inequality it follows that this defines a norm on $V_h \times Q_h^\Gamma$.

THEOREM 4.4. *There exist constants $h_0>0$, $\epsilon_0>0$ and $c_s>0$ such that for all $h \leq h_0$, $\varepsilon_p \geq \epsilon_0$ the following holds:*

$$
\sup_{(v_h,q_h)\in V_h\times Q_h^\Gamma} \frac{k\big((u_h,p_h),(v_h,q_h)\big)}{|||(v_h,q_h)|||_h} \geq c_s|||(u_h,p_h)|||_h \quad \text{for all} \quad (u_h,p_h)\in V_h\times Q_h^\Gamma.
$$

*Proof.* Take $(u_h,p_h)\in V_h\times Q_h^\Gamma$. From Lemma 4.3 it follows that there exists, for $h_0>0$ sufficiently small, $w_h\in V_h$ with $\|\mu^{\frac{1}{2}}\nabla w_h\|_0 = \|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$ and

$$
b(-w_h,p_h) \geq c_1\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2 j(p_h,p_h).
$$

Take $(v_h,q_h) = (u_h-\alpha w_h, p_h)$, with $\alpha>0$. Note that $\|\mu^{\frac{1}{2}}D(v)\|_0 \leq c\|\mu^{\frac{1}{2}}\nabla v\|_0$ for $v\in H^1(\Omega)$ holds. We then obtain, with suitable strictly positive constants,

$$
\begin{aligned}
&k\big((u_h,p_h),(v_h,q_h)\big) \\
&= a(u_h,u_h) - \alpha a(u_h,w_h) + \alpha b(-w_h,p_h) + \varepsilon_p j(p_h,p_h) \\
&\geq \|\mu^{\frac{1}{2}}D(u_h)\|_0^2 - \tilde{c}\alpha\|\mu^{\frac{1}{2}}D(u_h)\|_0\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
&\quad + \alpha c_1\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (\varepsilon_p - \alpha c_2)j(p_h,p_h) \\
&\geq \frac{1}{2}\|\mu^{\frac{1}{2}}D(u_h)\|_0^2 + \alpha(c_1 - \frac{1}{2}\tilde{c}^2\alpha)\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (\varepsilon_p-\alpha c_2)j(p_h,p_h).
\end{aligned}
$$

We take $\alpha$ such that $c_1 - \frac{1}{2}\tilde{c}^2\alpha = \frac{1}{2}c_1$ holds, and $\varepsilon_p$ such that $\varepsilon_p - \alpha c_2 \geq 1$. Thus we obtain, with suitable $c>0$,

$$
k\big((u_h,p_h),(v_h,q_h)\big) \geq c|||(u_h,p_h)|||_h^2.
$$

Combining this with

$$\begin{aligned}
\|\!|(v_h,q_h)\|\!|_h^2 &= \|\mu^{\frac12}D(u_h-\alpha w_h)\|_0^2+\|\mu^{-\frac12}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2+j(p_h,p_h)\\
&\le 2\|\mu^{\frac12}D(u_h)\|_0^2+(c\alpha^2+1)\|\mu^{-\frac12}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2+j(p_h,p_h)\le c\|\!|(u_h,p_h)\|\!|_h^2
\end{aligned}$$

completes the proof. ∎

**5. Discretization error analysis.** We introduce the space $Q_{\mathrm{reg}} = H^2(\Omega_{1,h}) \times H^2(\Omega_{2,h})$. The norm in (4.7) is well-defined also for $(u,p) \in H^1(\Omega)^d \times Q_{\mathrm{reg}}$. Let $\mathcal{E}_i : H^2(\Omega_i) \to H^2(\Omega_{i,h})$ be a bounded extension operator. Hence, there is a constant $c$, independent of $h$, such that $\|\mathcal{E}_i p\|_{2,\Omega_{i,h}} \le c\|p\|_{2,\Omega_i}$ for all $p \in H^2(\Omega_i)$. For $p \in H^2(\Omega_1 \cup \Omega_2)$ we define $\mathcal{E}p := (\mathcal{E}_1 p_{|\Omega_1}, \mathcal{E}_2 p_{|\Omega_2}) \in Q_{\mathrm{reg}}$. Note that for such extensions the stabilization term vanishes: $j(\mathcal{E}p, q_h) = 0$ for all $p \in H^2(\Omega_1 \cup \Omega_2)$ and $q_h \in Q_h^\Gamma$, i.e., we have a *consistent stabilization*. Based on this observation we obtain the following Cea-estimate.

THEOREM 5.1. *Assume that the solution $(u,p)$ of* (1.3) *has the regularity property $p \in H^2(\Omega_1 \cup \Omega_2)$. Let $h_0 > 0$ and $\varepsilon_p$ be as in Theorem 4.4. Take $h \le h_0$ and let $(u_h,p_h) \in V_h \times Q_h^\Gamma$ be the solution of the discretization* (3.1). *There exists a constant $c > 0$, independent of $h$ and $\mu$ and of how the interface $\Gamma$ intersects the triangulation, such that*

$$\|\!|(u-u_h,\mathcal{E}p-p_h)\|\!|_h \le c \min_{(v_h,q_h)\in V_h\times Q_h^\Gamma} \|\!|(u-v_h,\mathcal{E}p-q_h)\|\!|_h.$$

*Proof.* For $A \in \mathbb{R}^{d\times d}$ we have $\mathrm{tr}(A)^2 = \frac14\,\mathrm{tr}(A+A^T)^2 \le \frac{d}{4}\,\mathrm{tr}\left((A+A^T)^2\right)$ and thus for $w \in C^1(\Omega)^d$ we get $|\mathrm{div}\, w|^2 = |\,\mathrm{tr}\,\nabla w|^2 \le \frac{d}{4}\,\mathrm{tr}\left((\nabla w+(\nabla w)^T)^2\right) = \frac{d}{4} D(w) : D(w)$. Hence, for $(w,q) \in H^1(\Omega)^d \times (Q_{\mathrm{reg}} + Q_h^\Gamma)$ the estimate

$$|b(w,q^\Gamma)| \le \|\mu^{\frac12}\mathrm{div}\, w\|_0\|\mu^{-\frac12}q^\Gamma\|_{0,\Omega} \le \frac12\sqrt{d}\|\mu^{\frac12}D(w)\|_0\|\mu^{-\frac12}q\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \tag{5.1}$$

holds. From this, the definition of the bilinear form $k(\cdot,\cdot)$ and the Cauchy-Schwarz inequality one obtains boundedness w.r.t. $\|\!|\cdot\|\!|_h$:

$$\left|k\big((w,r),(v,q)\big)\right| \le c\|\!|(w,r)\|\!|_h\|\!|(v,q)\|\!|_h \quad \forall\ (w,r),(v,q) \in H^1(\Omega)^d \times (Q_{\mathrm{reg}}+Q_h^\Gamma),$$

with $c$ depending only on $\varepsilon_p$ and $d$. For $p \in H^2(\Omega_1 \cup \Omega_2)$ we have $j(\mathcal{E}p,q_h) = 0$ for all $q_h \in Q_h^\Gamma$. Using this and the conformity property, i.e. $V_h \subset H_0^1(\Omega)^2$, $q_h^\Gamma \in L_\mu^2(\Omega)$ for $q_h \in Q_h^\Gamma$, we obtain consistency:

$$k\big((u,\mathcal{E}p),(v_h,q_h)\big) = k\big((u_h,p_h),(v_h,q_h)\big) \quad \text{for all } \ (v_h,q_h) \in V_h \times Q_h^\Gamma.$$

The proof is easily completed using the standard Cea-argument. ∎

REMARK 1. The results in Theorem 4.4 and 5.1 also hold if instead of $V_h$ one takes a larger velocity space $\tilde V_h \supset V_h$, which is conforming, i.e, $\tilde V_h \subset H_0^1(\Omega)^d$ holds. An obvious possibility is to extend the velocity space by additional basis functions to account for the kink of $u$ at the interface. In [11] a kink enrichment is presented, which leads to an XFEM space $\tilde V_h = V_h \oplus \mathrm{span}\{v_j \cdot \Psi^\Gamma \mid j \in \mathcal{J}_\Gamma\}$. Here $v_j$, $j \in \mathcal{J}_\Gamma$, denote basis functions with $\mathrm{supp}\, v_j \cap \Gamma \neq \emptyset$ and $\Psi^\Gamma$ is a special enrichment function with a kink at $\Gamma$, and which has a support only on tetrahedra cut by the interface. Theorem 4.4 also holds for the pair $\tilde{V}_h \times Q_h^\Gamma$. It is clear that $\tilde{V}_h$ has better approximation properties for functions with kinks than the standard space $V_h$, but it is not known whether an optimal approximation result $\inf_{v_h \in \tilde{V}_h} \|u - v_h\|_1 \leq ch^2$ holds for this space. The results on conditioning of the stiffness matrix, derived for the pair $V_h \times Q_h^\Gamma$ in the next section, do not hold for the pair $\tilde{V}_h \times Q_h^\Gamma$.

Bounds for the approximation error

$$\min_{(v_h,q_h)\in V_h\times Q_h^\Gamma} |||(u-v_h, \mathcal{E}p - q_h)|||_h^2 \tag{5.2}$$
$$= \min_{(v_h,q_h)\in V_h\times Q_h^\Gamma} \Big( \|\mu^{\frac12} D(u-v_h)\|_0^2 + \|\mu^{-\frac12}(\mathcal{E}p - q_h)\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) \Big)$$

can be derived using standard interpolation error bounds. We first consider the terms related to the pressure approximation.

Lemma 5.2. *There exists a constant $c$ such that for all $p \in H^2(\Omega_1 \cup \Omega_2)$ the following holds:*

$$\min_{q_h Q_h^\Gamma} \Big( \|\mu^{-\frac12}(\mathcal{E}p - q_h)\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) \Big) \leq ch^4 \|\mu^{-\frac12} p\|_{2,\Omega_1\cup\Omega_2}^2. \tag{5.3}$$

*Proof.* Take $p \in H^2(\Omega_1 \cup \Omega_2)$. For $\mathcal{E}p = (\hat{p}_1, \hat{p}_2) \in Q_{\text{reg}}$ let $I_h \hat{p}_i$ be the standard nodal interpolation on the vertices of $\Omega_{i,h}$. Hence,

$$\|\hat{p}_i - I_h \hat{p}_i\|_{\ell,\Omega_{i,h}} \leq ch^{2-\ell} \|\hat{p}_i\|_{2,\Omega_{i,h}} \leq ch^{2-\ell}\|p\|_{2,\Omega_i}, \quad \ell = 0,1, \tag{5.4}$$

holds. For $q = (q_1, q_2) \in Q_{\text{reg}} + Q_h^\Gamma$ and $F \in \mathcal{F}_i$, with $F = T_1 \cap T_2$ and $T_1, T_2 \in \Omega_{i,h}$, we have

$$\|[\nabla q_i \cdot n_F]\|_F^2 \leq \sum_{j=1}^2 \|\nabla q_i\|_{\partial T_j}^2 \leq c \sum_{j=1}^2 \big( h^{-1}\|\nabla q_i\|_{0,T_j}^2 + h\|\nabla^2 q_i\|_{0,T_j}^2 \big).$$

Using this we get

$$j(q,q) = \sum_{i=1}^2 \sum_{F\in\mathcal{F}_i} \mu_i^{-1} h_F^3 \|[\nabla q_i \cdot n_F]\|_F^2 \leq c \sum_{i=1}^2 \mu_i^{-1} \big( h^2 \|\nabla q_i\|_{0,\Omega_{i,h}}^2 + h^4 \sum_{T\in\Omega_{i,h}} \|\nabla^2 q_i\|_{0,T}^2 \big).$$

We take $q = \mathcal{E}p - q_h$, $q_h = (q_{1,h}, q_{2,h}) \in Q_h^\Gamma$, and noting that $\nabla^2 q_{i,h|T} = 0$ we thus obtain

$$\begin{aligned} j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) &\leq c \sum_{i=1}^2 \mu_i^{-1} \big( h^2 \|\nabla(\hat{p}_i - q_{i,h})\|_{0,\Omega_{i,h}}^2 + h^4 \|\nabla^2 \hat{p}_i\|_{0,\Omega_{i,h}}^2 \big) \\ &\leq ch^2 \sum_{i=1}^2 \mu_i^{-1} \|\nabla(\hat{p}_i - q_{i,h})\|_{0,\Omega_{i,h}}^2 + ch^4 \|\mu^{-\frac12} p\|_{2,\Omega_1\cup\Omega_2}^2. \end{aligned}$$

We take $q_h = (I_h \hat{p}_1, I_h \hat{p}_2)$, and using the interpolation error bounds in (5.4) we obtain the bound in (5.3). ∎

For the velocity term in (5.2) we obviously also have the optimal error bound

$$\min_{v_h \in V_h} \|\mu^{\frac12} D(u - v_h)\|_0^2 \leq c\mu_{\max} h^4 \|u\|_{3,\Omega} \quad \text{for} \quad u \in H^3(\Omega), \tag{5.5}$$

with $\mu_{\max} = \max\{\mu_1, \mu_2\}$. In our applications, however, we typically do not have the regularity property $u \in H^3(\Omega)$. The velocity $u$ is smooth in the interior of $\Omega_i$, but has a discontinuity in its first derivative across the interface $\Gamma$. Hence, globally, the best one can have is an asymptotic error bound of the form $\|u - v_h\|_1^2 \leq ch$. To improve on this one might use an XFEM velocity space, too, for example $\tilde{V}_h$ as explained in Remark 1. It turns out, however, that in many applications the poor velocity approximation using standard $P_2$ finite elements does not dominate the total error for realistic mesh sizes. This is illustrated by the numerical example in Section 7.3.
As far as we know, rigorous regularity results for the Stokes interface problem (1.1), e.g., $u \in H^2(\Omega_1 \cup \Omega_2),\ p \in H^1(\Omega_1 \cup \Omega_2)$ are not known in the literature.
Using a duality argument one can derive an $L^2$ error bound along the same lines as for the standard Stokes equation.

**6. Schur complement preconditioner.** We introduce a matrix-vector representation of the discrete problem (3.1). In $V_h$ we use the standard nodal basis denoted by $(\psi_j)_{1 \leq j \leq m}$, i.e.,

$$V_h \ni u_h = \sum_{j=1}^{m} x_j \psi_j. \tag{6.1}$$

The vector representation of $u_h$ is denoted by $\mathbf{x} = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$. In $Q_{i,h}$ we have a standard nodal basis denoted by $(\phi_{i,j})_{1 \leq j \leq n_i}$, $i = 1, 2$, i.e.,

$$Q_{1,h} \times Q_{2,h} \ni p_h = (p_{1,h}, p_{2,h}) = \Big( \sum_{j=1}^{n_1} y_{1,j} \phi_{1,j}, \sum_{j=1}^{n_2} y_{2,j} \phi_{2,j} \Big). \tag{6.2}$$

The vector representation of $p_h$ is denoted by $\mathbf{y} = (y_{1,1}, \ldots, y_{1,n_1}, y_{2,1}, \ldots, y_{2,n_2})^T \in \mathbb{R}^{n_1+n_2}$. Standard finite element theory yields that there are strictly positive constants $c_i$, independent of $h$, such that

$$c_1 h^d \|\mathbf{y}\|^2 \leq \|p_{1,h}\|_{0,\Omega_{1,h}}^2 + \|p_{2,h}\|_{0,\Omega_{2,h}}^2 = \|p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}^2 \leq c_2 h^d \|\mathbf{y}\|^2, \tag{6.3}$$

for all $p_h \in Q_{1,h} \times Q_{2,h}$. Here, $\|\cdot\|$ denotes the Euclidean vector norm. We use $\langle \cdot, \cdot \rangle$ to denote the Euclidean scalar product. The bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $j(\cdot, \cdot)$ have corresponding matrix representations, denoted by $A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{(n_1+n_2) \times m}, J \in \mathbb{R}^{(n_1+n_2) \times (n_1+n_2)}$, respectively. The following holds:

$$\begin{aligned}
a(u_h, u_h) &= \langle A\mathbf{x}, \mathbf{x} \rangle \quad \text{for all } u_h \in V_h, \\
b(u_h, p_h^\Gamma) &= \langle B\mathbf{x}, \mathbf{y} \rangle \quad \text{for all } u_h \in V_h,\ p_h \in Q_{1,h} \times Q_{2,h}, \\
j(p_h, p_h) &= \langle J\mathbf{y}, \mathbf{y} \rangle \quad \text{for all } p_h \in Q_{1,h} \times Q_{2,h}.
\end{aligned}$$

The matrix $A$ is symmetric positive definite. The matrix $J$ is symmetric positive semi-definite. Define $\mathbf{1} := (1, \ldots, 1)^T \in \mathbb{R}^{n_1+n_2}$. From $b(u_h, 1) = 0$ for all $u_h \in V_h$ and $j(1, q_h) = 0$ for all $q_h \in Q_{1,h} \times Q_{2,h}$ it follows that $B^T \mathbf{1} = J\mathbf{1} = 0$ holds.
Finally we introduce two mass matrices in the pressure space:

$$\begin{aligned}
M &= \text{blockdiag}(M_1, M_2), \quad (M_i)_{k,l} := (\mu_i^{-1} \phi_{i,k}, \phi_{i,l})_{0,\Omega_{i,h}}, \quad 1 \leq k, l \leq n_i, \quad i = 1, 2, \\
\hat{M} &= \text{blockdiag}(\hat{M}_1, \hat{M}_2), \quad (\hat{M}_i)_{k,l} := (\mu_i^{-1} \phi_{i,k}, \phi_{i,l})_{0,\Omega_i}, \quad 1 \leq k, l \leq n_i, \quad i = 1, 2.
\end{aligned}$$

For these mass matrices we have the relations

$$\langle M\mathbf{y},\mathbf{y}\rangle = \|\mu_1^{-\frac{1}{2}}p_{1,h}\|^2_{0,\Omega_{1,h}} + \|\mu_2^{-\frac{1}{2}}p_{2,h}\|^2_{0,\Omega_{2,h}} = \|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}},$$
$$\langle \hat{M}\mathbf{y},\mathbf{y}\rangle = \|\mu_1^{-\frac{1}{2}}p_{1,h}\|^2_{0,\Omega_1} + \|\mu_2^{-\frac{1}{2}}p_{2,h}\|^2_{0,\Omega_2} = \sum_{i=1}^{2}\|\mu_i^{-\frac{1}{2}}p_h^\Gamma\|^2_{0,\Omega_i} = \|\mu^{-\frac{1}{2}}p_h^\Gamma\|^2_{0,\Omega}.$$

The matrix-vector representation of the discrete problem (3.1) is as follows. First note that $(\mu^{-1}p_h^\Gamma, 1)_{0,\Omega} = 0$ iff $\langle \hat{M}\mathbf{y},\mathbf{1}\rangle = 0$. The discrete problem is given by: determine $(\mathbf{x},\mathbf{y})$ with $\langle \hat{M}\mathbf{y},\mathbf{1}\rangle = 0$ such that

$$\begin{pmatrix} A & B^T \\ -B & \varepsilon_p J\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{b}\\ 0\end{pmatrix}, \quad b_j = (f,\psi_j)_{0,\Omega},\ 1\le j\le m.$$

For the iterative solution of this system it is convenient to use the following equivalent, symmetric formulation: determine $(\mathbf{x},\mathbf{y})$ with $\mathbf{y}\in \mathbf{1}^{\perp_{\hat{M}}}$ such that

$$K\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{b}\\ 0\end{pmatrix}, \quad K := \begin{pmatrix} A & B^T \\ B & -\varepsilon_p J\end{pmatrix}. \tag{6.4}$$

Note that $K$ has a one-dimensional kernel, spanned by $(0\ \mathbf{1})^T$. The Schur complement of $K$ is denoted by $S = BA^{-1}B^T + \varepsilon_p J$. We consider the block diagonal preconditioner,

$$Q = \begin{pmatrix} Q_A & 0 \\ 0 & Q_S\end{pmatrix}, \quad Q_S = \hat{M} + \varepsilon_p J, \quad Q_A \text{ symmetric positive definite.} \tag{6.5}$$

In our applications, cf. section 7, we use for $Q_A$ a symmetric multigrid iteration applied to $A$. The symmetric positive definite Schur complement preconditioner $Q_S = \hat{M} + \varepsilon_p J$ is analyzed in section 6.1.
When solving the linear system (6.4) we have to satisfy the consistency condition $\mathbf{y}\in\mathbf{1}^{\perp_{\hat{M}}}$. The following lemma shows that for a Krylov subspace method applied to the preconditioned matrix $Q^{-1}K$ this condition is automatically satisfied. We use the properties $J\mathbf{1}=0$, hence, $Q_S\mathbf{1} = \hat{M}\mathbf{1}$, i.e., $Q_S^{-1}\hat{M}\mathbf{1} = \mathbf{1}$.

LEMMA 6.1. *Define* $Y = \{(\mathbf{x}\ \mathbf{y})^T \in \mathbb{R}^{m+n_1+n_2} \mid \mathbf{y}\in\mathbf{1}^{\perp_{\hat{M}}}\}$. *Then* $Q^{-1}K : Y \to Y$ *is a bijection.*

*Proof.* The space $Y$ forms a direct sum with the kernel span$\{(0\ \mathbf{1})^T\}$ of the matrix $K$. Hence, the range of $K : Y \to \mathbb{R}^{m+n_1+n_2}$ has codimension 1. For $(\mathbf{x}\ \mathbf{y})^T$ define $(\tilde{\mathbf{x}}\ \tilde{\mathbf{y}})^T = Q^{-1}K(\mathbf{x}\ \mathbf{y})^T$. For $\tilde{\mathbf{y}}$ we have

$$\langle \hat{M}\tilde{\mathbf{y}},\mathbf{1}\rangle = \langle B\mathbf{x} - \varepsilon_p J\mathbf{y}, Q_S^{-1}\hat{M}\mathbf{1}\rangle = \langle B\mathbf{x} - \varepsilon_p J\mathbf{y},\mathbf{1}\rangle = \langle \mathbf{x}, B^T\mathbf{1}\rangle - \varepsilon_p\langle y, J\mathbf{1}\rangle = 0.$$

Hence, $(\tilde{\mathbf{x}}\ \tilde{\mathbf{y}})^T \in Y$ holds. □

As we will see in the next section, the matrices $M$ and $\hat{M}+\varepsilon_p J$ are spectrally equivalent. If we would use $Q_S = M$ as the Schur complement preconditioner, it is not clear how to satisfy the consistency condition $\mathbf{y}\in\mathbf{1}^{\perp_{\hat{M}}}$. This is the reason why besides the mass matrix $M$ we also need the mass matrix $\hat{M}$.

**6.1. Analysis of the preconditioner.** We analyze the quality of the block diagonal preconditioner $Q$ given in (6.5).

We start with a main result, which shows that the weighted mass matrix $M$ is uniformly spectrally equivalent to the Schur complement.

THEOREM 6.2. *Take $\varepsilon_p > 0$. There exist constants $c_1, c_2 > 0$, independent of $h$, $\mu$ and of how $\Gamma$ intersects the triangulation, such that with $S = BA^{-1}B^T + \varepsilon_p J$ we have:*

$$c_1\langle M\mathbf{y},\mathbf{y}\rangle \leq \langle S\mathbf{y},\mathbf{y}\rangle \leq c_2\langle M\mathbf{y},\mathbf{y}\rangle \quad \text{for all} \;\; \mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}. \tag{6.6}$$

*Proof.* Take $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$. We use the relation

$$\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}} = \max_{\mathbf{x}\in\mathbb{R}^m} \frac{\langle B\mathbf{x},\mathbf{y}\rangle}{\langle A\mathbf{x},\mathbf{x}\rangle^{\frac{1}{2}}} = \max_{u_h\in V_h} \frac{b(u_h,p_h^\Gamma)}{a(u_h,u_h)^{\frac{1}{2}}}. \tag{6.7}$$

We use the estimate (5.1) and thus get

$$\begin{aligned}\max_{u_h\in V_h} \frac{b(u_h,p_h^\Gamma)}{a(u_h,u_h)^{\frac{1}{2}}} &\leq c \max_{u_h\in V_h} \frac{\|\mu^{\frac{1}{2}}D(u_h)\|_{0,\Omega}\|\mu^{-\frac{1}{2}}p_h^\Gamma\|_{0,\Omega}}{\|\mu^{\frac{1}{2}}D(u_h)\|_{0,\Omega}} \\ &= c\|\mu^{-\frac{1}{2}}p_h^\Gamma\|_{0,\Omega} \leq c\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} = c\langle M\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}}.\end{aligned}$$

Hence $\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle \leq c\langle M\mathbf{y},\mathbf{y}\rangle$ holds. Using an inverse inequality we get

$$\begin{aligned}\langle J\mathbf{y},\mathbf{y}\rangle = j(p_h,p_h) &= \sum_{i=1}^2 \mu_i^{-1} \sum_{F\in\mathcal{F}_i} h_F^3\|[\nabla p_{i,h}\cdot n_F]\|_{0,F}^2 \\ &\leq c\sum_{i=1}^2 \mu_i^{-1} \sum_{T\in\Omega_{i,h}} h_T^3\|\nabla p_{i,h}\|_{0,\partial T}^2 \leq c\sum_{i=1}^2 \mu_i^{-1} \sum_{T\in\Omega_{i,h}} h_T^2\|\nabla p_{i,h}\|_{0,T}^2 \\ &\leq c\sum_{i=1}^2 \mu_i^{-1} \sum_{T\in\Omega_{i,h}} \|p_{i,h}\|_{0,T}^2 = c\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 = c\langle M\mathbf{y},\mathbf{y}\rangle.\end{aligned} \tag{6.8}$$

Hence,

$$\langle S\mathbf{y},\mathbf{y}\rangle = \langle (BA^{-1}B^T + \varepsilon_p J)\mathbf{y},\mathbf{y}\rangle \leq c(1+\varepsilon_p)\langle M\mathbf{y},\mathbf{y}\rangle \quad \text{for all} \;\; \varepsilon_p \geq 0,$$

holds, which proves the second inequality in (6.6).
Using (6.7) and Lemma 4.3 we get, suitable constants $c_1, c_2$,

$$\begin{aligned}\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}} &\geq c_1\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2\frac{j(p_h,p_h)}{\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}} \\ &= c_1\langle M\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}} - c_2\frac{\langle J\mathbf{y},\mathbf{y}\rangle}{\langle M\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}}}.\end{aligned}$$

This yields

$$\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}}\langle M\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}} + c_2\langle J\mathbf{y},\mathbf{y}\rangle \geq c_1\langle M\mathbf{y},\mathbf{y}\rangle.$$

Using $\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}}\langle M\mathbf{y},\mathbf{y}\rangle^{\frac{1}{2}} \leq \frac{1}{2}c_1^{-1}\langle BA^{-1}B^T\mathbf{y},\mathbf{y}\rangle + \frac{1}{2}c_1\langle M\mathbf{y},\mathbf{y}\rangle$ we thus get

$$\langle S\mathbf{y},\mathbf{y}\rangle \geq c_1^2 \min\big\{1, \frac{\varepsilon_p}{2c_1c_2}\big\}\langle M\mathbf{y},\mathbf{y}\rangle,$$

which proves the first inequality in (6.6). ∎

As can be seen from the proof, the constants $c_i$ in (6.6) depend on the value of the stabilization parameter $\varepsilon_p$.

As noted at the end of the previous section, in view of the consistency condition $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$, it is more convenient to use the matrix $Q_S = \hat{M} + \varepsilon_p J$ instead of $M$ as a preconditioner for the Schur complement $S$. In the next lemma we show that these two are uniformly spectrally equivalent.

LEMMA 6.3. *Take $\varepsilon_p > 0$. There exist constants $c_1, c_2 > 0$, independent of $h$, $\mu$ and of how $\Gamma$ intersects the triangulation, such that*

$$c_1\langle M\mathbf{y}, \mathbf{y}\rangle \leq \langle (\hat{M} + \varepsilon_p J)\mathbf{y}, \mathbf{y}\rangle \leq c_2\langle M\mathbf{y}, \mathbf{y}\rangle \quad \textit{for all} \ \ \mathbf{y} \in \mathbb{R}^{n_1+n_2}. \tag{6.9}$$

*Proof.* From $\|\mu^{-\frac{1}{2}} p_h^\Gamma\|_{0,\Omega} \leq \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$ we obtain $\langle \hat{M}\mathbf{y}, \mathbf{y}\rangle \leq \langle M\mathbf{y}, \mathbf{y}\rangle$. Combining this with the result in (6.8) proves the second inequality in (6.9). Using Lemma 2.2 we get,

$$\begin{aligned}\langle M\mathbf{y}, \mathbf{y}\rangle = \|\mu^{-\frac{1}{2}} p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} &\leq c\big(\|\mu^{-\frac{1}{2}} p_h^\Gamma\|^2_{0,\omega_{1,h}\cup\omega_{2,h}} + j(p_h, p_h)\big)\\ &\leq c\big(\|\mu^{-\frac{1}{2}} p_h^\Gamma\|^2_{0,\Omega} + \varepsilon_p j(p_h, p_h)\big) = c\big((\hat{M} + \varepsilon_p J)\mathbf{y}, \mathbf{y}\rangle,\end{aligned}$$

and thus the first inequality in (6.9) holds, too. ☐

The results above yield that the spectral condition number of $Q_S^{-1} S$ is uniformly bounded on $\mathbf{1}^{\perp_{\hat{M}}}$. Finally we show that linear systems with matrix $Q_S$ can be solved (approximately) with low computational costs. In [15] it is proved that for $\mu_1 = \mu_2 = 1$ the *diagonally scaled* matrix $\hat{D}^{-1}\hat{M}$, with $\hat{D} := \mathrm{diag}(\hat{M})$ is uniformly (w.r.t. $h$ and w.r.t. the position of the interface in the grid) well-conditioned. Due to the possibly small support of some extended basis functions, without the diagonal scaling the condition number of the mass matrix $\hat{M}$ is not uniformly bounded. Here, we have to study the conditioning of $Q_S = \hat{M} + \varepsilon_p J$. We benefit from the stabilizing term $\varepsilon_p J$, and a conditioning result is easily obtained, as shown in the following lemma.

LEMMA 6.4. *Take $\varepsilon_p > 0$. Define $D := \mathrm{diag}(\hat{M} + \varepsilon_p J)$. There exist constants $c_1, c_2 > 0$, independent of $h$, $\mu$ and of how $\Gamma$ intersects the triangulation, such that*

$$c_1\langle D\mathbf{y}, \mathbf{y}\rangle \leq \langle (\hat{M} + \varepsilon_p J)\mathbf{y}, \mathbf{y}\rangle \leq c_2\langle D\mathbf{y}, \mathbf{y}\rangle \quad \textit{for all} \ \ \mathbf{y} \in \mathbb{R}^{n_1+n_2}. \tag{6.10}$$

*Proof.* By $A \sim B$ we denote uniform spectral equivalence of the s.p.d. matrices $A$ and $B$. Define $D_M := \mathrm{diag}(M)$. If in (6.9) for $\mathbf{y}$ we take the standard basis vectors we obtain $D \sim D_M$. From the definition of $M$ and the result in (6.3) it follows that $D_M \sim M$. Thus we get $D \sim M$. Using (6.9) we conclude $D \sim \hat{M} + \varepsilon_p J$. ☐

Now we apply a standard analysis as in e.g. [17, 12], to derive results on the spectrum of the preconditioned matrix $Q^{-1}K$. From the results in Theorem 6.2 and Lemma 6.3 it follows that there are constants $\gamma_S > 0$ and $\Gamma_S$, independent of $h, \mu$ and of how the interface intersects the triangulation, such that for the Schur complement preconditioner $Q_S$ as in (6.5), with a fixed $\varepsilon_p > 0$, we have the following spectral equivalence:

$$\gamma_S\langle Q_S\mathbf{y}, \mathbf{y}\rangle \leq \langle S\mathbf{y}, \mathbf{y}\rangle \leq \Gamma_S\langle Q_S\mathbf{y}, \mathbf{y}\rangle \quad \text{for all} \ \ \mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}. \tag{6.11}$$

For $Q_A$ we take a symmetric multigrid preconditioner. Thus there exists $\gamma_A > 0$ independent of $h$ and of how the interface intersects the triangulation such that

$$\gamma_A\langle Q_A\mathbf{x}, \mathbf{x}\rangle \leq \langle A\mathbf{x}, \mathbf{x}\rangle \leq \langle Q_A\mathbf{x}, \mathbf{x}\rangle \quad \text{for all} \ \ \mathbf{x} \in \mathbb{R}^m. \tag{6.12}$$

In the upper bound in (6.12) we have a constant 1, because the iteration matrix of a symmetric multigrid method for the diffusion equation is positive definite. The spectral constant $\gamma_A$ may depend on the quotient $\mu_1/\mu_2$.

COROLLARY 6.5. *Using* (6.11), (6.12) *and the analysis in [12] (Lemma 5.14) it follows that all nonzero eigenvalues of $Q^{-1}K$ lie in the union of the intervals*

$$[\gamma_A, 1] \cup \big[\frac{1}{2}(\gamma_A + \sqrt{\gamma_A^2 + 4\gamma_A\gamma_S})\,,\ \frac{1}{2}(1+\sqrt{1+4\Gamma_S})\big] \\ \cup \big[\frac{1}{2}(1-\sqrt{(1+4\Gamma_S})\,,\ \frac{1}{2}(\gamma_A - \sqrt{\gamma_A^2 + 4\gamma_A\gamma_S})\big].$$

This shows that $Q$ is an optimal preconditioner for $K$. Systems with the Schur complement preconditioner $Q_S$ can be solved (approximately) with acceptable computational costs, cf. Lemma 6.4.

## 7. Numerical experiments.

**7.1. The sliver experiment.** In our first experiment we want to investigate the influence of the parameter $\varepsilon_p$ on the stability of the resulting discretizations. To this end, we introduce the so-called sliver experiment. Praxis has shown that in unstabilized discretizations the stability problems seem to arise from those XFEM functions which have a tiny support. In this experiment we deliberately create such functions and repeatedly shrink their support. We choose a uniform grid of the domain $\Omega = (-1,1)^3$, consisting of $4 \times 4 \times 4$ equally sized cubes. Each of these cubes is then sub-divided into six tetrahedra. In order to shrink the support of XFEM functions we define a sequence of planar interfaces $(\Gamma_k)_{k\in\mathbb{N}_0}$ approaching the $x$-$y$-plane:

$$\Gamma_k := \big\{\mathbf{x} \mid \mathbf{x} = (x, y, 0.1 \cdot 2^{-k}),\ x, y \in (-1,1)\big\}. \tag{7.1}$$

We take $\mu_1 = \mu_2 = 1$. As a measure of stability, we want to estimate

$$\inf_{(u_h,p_h)\in V_h\times Q_h^\Gamma} \sup_{(v_h,q_h)\in V_h\times Q_h^\Gamma} \frac{k\big((u_h,p_h),(v_h,q_h)\big)}{\|(u_h,p_h)\|\|(v_h,q_h)\|}, \quad \text{where } \|(u_h,p_h)\|^2 = \|u_h\|_1^2 + \|p_h\|_0^2. \tag{7.2}$$

Using Lemma 6.3 and the coercivity of the bilinear form $a$, it can be shown that this can be estimated by the smallest non-zero eigenvalue of the following matrix:

$$\begin{pmatrix} A + M_v & 0 \\ 0 & \hat{M} + \varepsilon J \end{pmatrix}^{-1} \begin{pmatrix} A & B^T \\ -B & \varepsilon_p J \end{pmatrix}, \tag{7.3}$$

where $M_v$ is mass matrix in the velocity space. We denote this smallest non-zero eigenvalue by $C_{\text{stab}}$.

Figure 7.1 shows the values of $C_{\text{stab}}$ for the two choices $\varepsilon_p = 10^{-5}$ and $\varepsilon_p = 1$. Even though there are five orders of magnitude between them, the stability results are almost identical. In fact, for $\varepsilon_p = 10^{-5}$ the results even slightly improve with increasing $k$. This is also true for choices of $\varepsilon_p$ in between those values, which we did not plot here for the sake of a better visualization. For the unstabilized discretization, we remark that due to numerical instabilities the computed values for $C_{\text{stab}}$ might be inaccurate. However, the value of $C_{\text{stab}}$ seems to deteriorate approximately as $\mathcal{O}(\delta^3)$, where $\delta$ is the distance of the interface to the $x$-$y$-plane. It appears that already "tiny amounts" of the stabilization suffice to restore the method's stability. Furthermore, variation of the parameter $\varepsilon_p$ seems to have a very mild influence on the stability of the method.

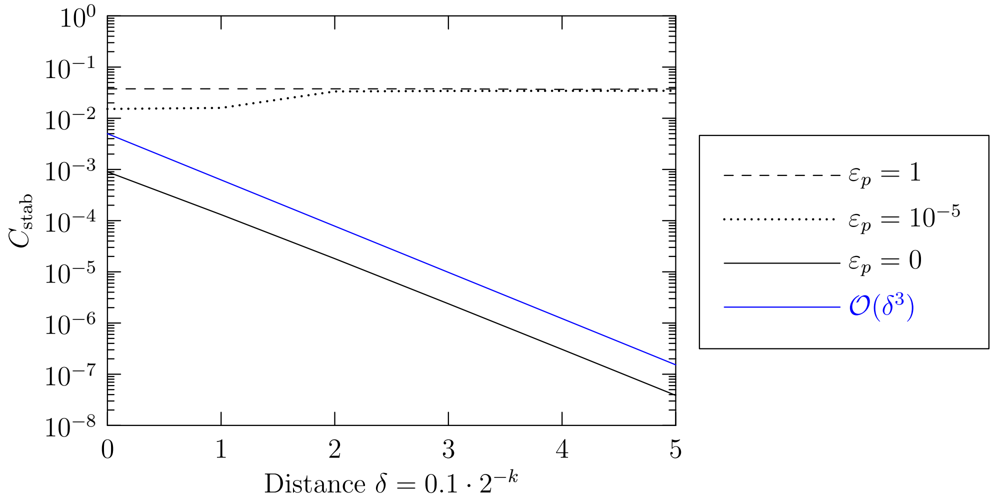



FIG. 7.1. *Values of the stability constant $C_{\text{stab}}$ for different values of $\varepsilon_p$ as the interface's distance to the x-y-plane is decreased.*

**7.2. Experiments with a smooth velocity solution.** In this section we want to investigate the convergence properties of the method. To this end, we prescribe Dirichlet boundary conditions and an external force $f = f_\Omega + \hat{f}_\Gamma$, $\hat{f}_\Gamma(v) := \sigma \int_\Gamma v \cdot \mathbf{n}\, ds$ with $\sigma := 10$ for $v \in V$, such that the analytical solution is:

$$u(x,y,z) = \alpha(r)\, e^{-r^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad \text{where } r = \sqrt{x^2+y^2+z^2},$$

$$\alpha(r) = \begin{cases} \mu_1^{-1} & \text{for } r < r_\Gamma, \\ \mu_2^{-1} + (\mu_1^{-1} - \mu_2^{-1}) e^{r^2 - r_\Gamma^2} & \text{for } r \geq r_\Gamma, \end{cases} \tag{7.4}$$

$$p(x,y,z) = x^3 + \begin{cases} \sigma & x \in \Omega_1, \\ 0 & \text{else}, \end{cases}$$

where the domain is $\Omega := (-1,1)^3$ and $\Omega_1 := S_{2/3}$ the sphere of radius $r_\Gamma := 2/3$ around the origin. Note that the function $\alpha(r)$ is continuous and has a kink at $r = r_\Gamma$ in case of non-matching viscosities $\mu_i$. Note also that the velocity vectors are tangential to the interface, i.e., $u \cdot n_\Gamma = 0$ with $n_\Gamma$ the outer normal to $\Omega_1$, which is necessary for the assumption of a stationary interface.

For simplicity, as a first test case we choose $\mu_1 = \mu_2 = 1$, while a more realistic setting will be examined in Section 7.3. For this choice we have $\alpha \equiv 1$ and thus the functions $u, p$ can be ideally approximated by the ansatz spaces while not being a part of them. For the discretization of $\hat{f}_\Gamma$ we use $\hat{f}_{\Gamma_h}(v_h) := \sigma \int_{\Gamma_h} v_n \cdot \mathbf{n}_h\, ds$, which is second-order accurate. Here $\Gamma_h$ is a piece-wise planar approximation to $\Gamma$ with $\text{dist}(x, \Gamma) \leq ch^2$ for all $x \in \Gamma_h$, cf. [8].

In a first step, we want to investigate the sensitivity of the discretization error with respect to $\varepsilon_p$. We therefore choose a fixed grid of $16 \times 16 \times 16$ cubes which are each subsequently subdivided into six tetrahedra. Afterwards we change the value of $\varepsilon_p$ and compute the discretization error. For the solution of the linear system of equations a preconditioned MINRES method was used, with the preconditioners

| $\varepsilon_p$ | $\|e_p^\Gamma\|_0$ | $\|e_u\|_1$ | Iterations |
|---|---|---|---|
| 0 | $1.82 \cdot 10^{-2}$ | $4.90 \cdot 10^{-3}$ | $> 1000$ |
| $10^{-5}$ | $9.64 \cdot 10^{-3}$ | $4.87 \cdot 10^{-3}$ | 97 |
| $10^{-3}$ | $9.49 \cdot 10^{-3}$ | $4.86 \cdot 10^{-3}$ | 96 |
| $10^{-1}$ | $9.76 \cdot 10^{-3}$ | $4.97 \cdot 10^{-3}$ | 102 |
| 1 | $1.33 \cdot 10^{-2}$ | $6.41 \cdot 10^{-3}$ | 95 |
| 10 | $3.01 \cdot 10^{-2}$ | $1.43 \cdot 10^{-2}$ | 102 |
| $10^3$ | $9.34 \cdot 10^{-2}$ | $4.61 \cdot 10^{-2}$ | 97 |

TABLE 7.1
*Discretization errors and iteration counts for various values of* $\varepsilon_p$. *For* $\varepsilon_p = 0$ *the residual did not fall below* $10^{-6}$.

defined as in the previous section. The MINRES iteration was stopped when the residual fell below the threshold of $10^{-9}$.

Table 7.1 shows the resulting iteration counts and discretization errors for various values of $\varepsilon_p$. One can clearly see that for $\varepsilon_p < 1$, its magnitude is virtually insignificant for the properties of the resulting discretization. Note that the error in the pressure variable is only about half of the corresponding value for the unstabilized discretization. For $\varepsilon_p = 1$ the errors increase only very slightly. Also note that for $\varepsilon_p = 0$ the MINRES iteration did not converge to the target residual due to the poor stability properties. For all other choices of $\varepsilon_p$, the introduced preconditioners show to be effective with iteration counts around 100. We can therefore conclude that $\varepsilon_p$ only has a very mild influence on the properties of the resulting discretization.

After having established the discretization's small sensitivity with respect to $\varepsilon_p$, we want to inspect the convergence behavior with respect to $h$. To this end we choose a fixed value $\varepsilon_p = 1$ and start with a uniform grid of $4 \times 4 \times 4$ cubes which are subsequently each divided into six tetrahedra. We then perform uniform mesh refinements and look at the influence on the discretization error and the iteration counts.

Figure 7.2 shows the discretization errors for the different refinement levels. As predicted by the analysis, we have a second order convergence behavior. The iteration counts varied between values of 95 and 102, confirming the optimal behavior of the preconditioners introduced. Note that for the second order convergence behavior, the use of the $P_1$-XFE space for the pressure is essential. If one uses the standard $P_1$-FE space instead, the rate of convergence drops to $\mathcal{O}(h^{\frac{1}{2}})$, cf. [9].

**7.3. Experiments with more realistic parameter settings.** In two-phase flows the pressure jump at the interface is induced by surface tension. To incorporate the effect of surface tension we consider the same test case as in Section 7.2, but we replace the artificial surface force $\hat{f}_\Gamma$ by the surface tension force $f_\Gamma(v) = \int_\Gamma \tau\kappa v \cdot \mathbf{n}\, ds$. Here $\tau > 0$ is a constant surface tension coefficient and $\kappa(x)$ denotes the local curvature of $\Gamma$.

Choosing $\tau = \frac{10}{3}$ we have $\tau\kappa = \tau\frac{2}{r_\Gamma} = 10 = \sigma$, thus for the continuous setting both surface forces coincide, i.e., $f_\Gamma = \hat{f}_\Gamma$. This, however, does *not* hold for the discrete case, i.e., $f_{\Gamma_h} \neq \hat{f}_{\Gamma_h}$, which is due to the fact that for $f_{\Gamma_h}$ the curvature has to be evaluated from the *approximate* interface $\Gamma_h$. For the discretization $f_{\Gamma_h}$ of the surface tension force we use a Laplace-Beltrami technique described in [8] and

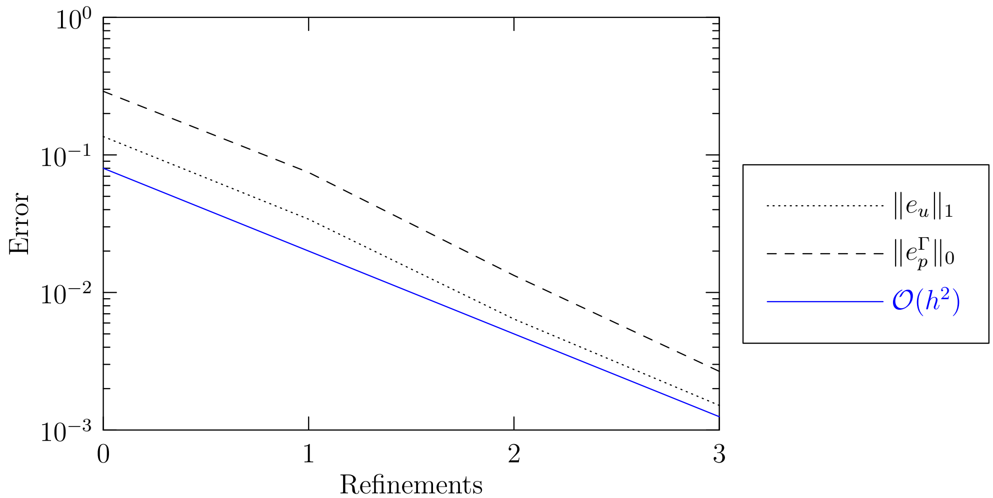



FIG. 7.2. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ using an artificial surface force term $\hat{f}_\Gamma$.*

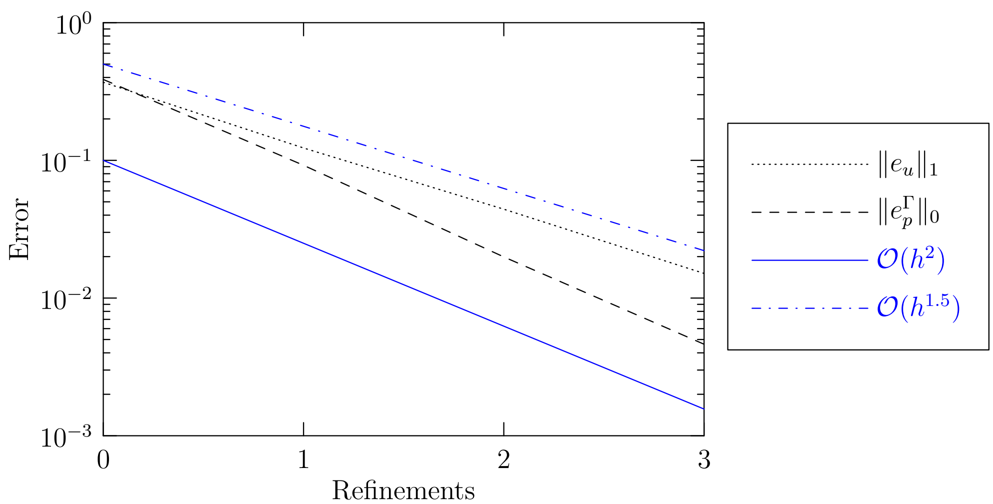
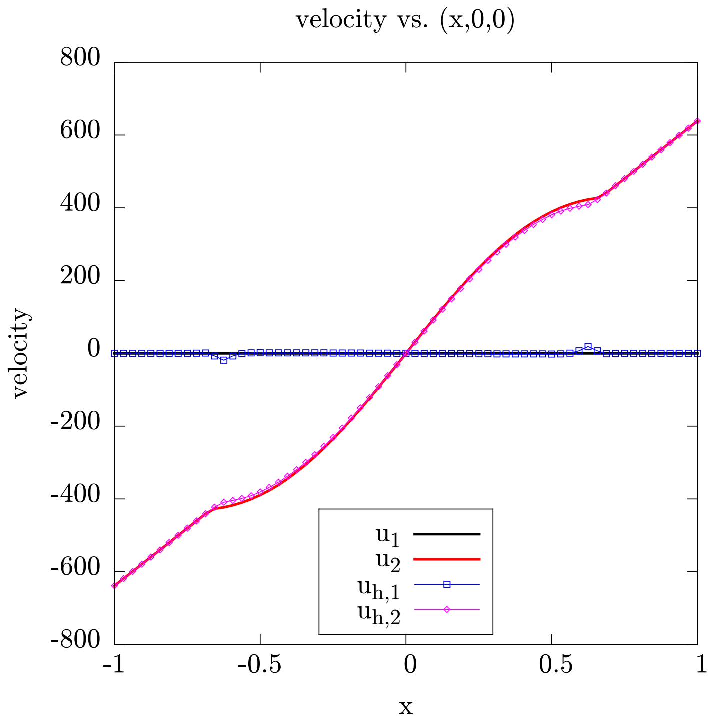



FIG. 7.3. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ using a surface tension force term $f_\Gamma$.*

analyzed in [8, 7] which has a discretization order of 1.5. Due to the first Strang lemma the same convergence order is expected for the sum of the velocity error (in $\|\cdot\|_1$) and pressure error (in $\|\cdot\|_0$). We take $\varepsilon_p = 1$ and apply grid refinement as in the previous experiment. The error plot given in Figure 7.3 shows that the velocity error has an $\mathcal{O}(h^{\frac{3}{2}})$ behavior and the pressure error converges with second order.

Finally, we consider an experiment which mimics a two-phase flow water/air system with non-matching viscosities. The solution is chosen as in (7.4) with $\mu_1 = 10^{-3}$, $\mu_2 = 10^{-1}$ and $\tau = 700$. These values for viscosity and surface tension coefficient $\tau$ correspond to the dimensionless formulation of the two-phase Stokes equations for an air bubble with radius $\frac{2}{3}\,mm$ in ambient water, assuming a characteristic length



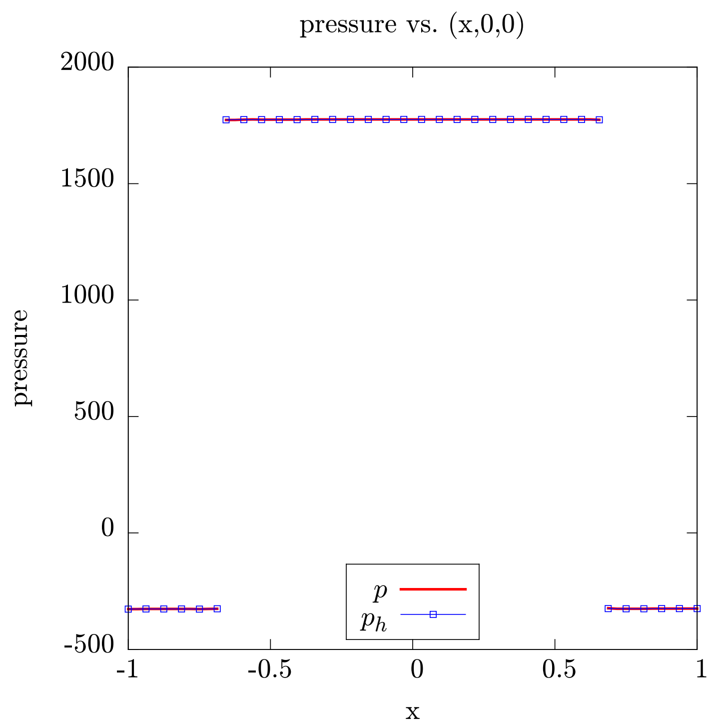




FIG. 7.4. *Exact and discrete velocity and pressure solutions along the x-axis for $\varepsilon_p = 1$ on grid refinement level 3 and a realistic parameter setting corresponding to an air bubble in water.*

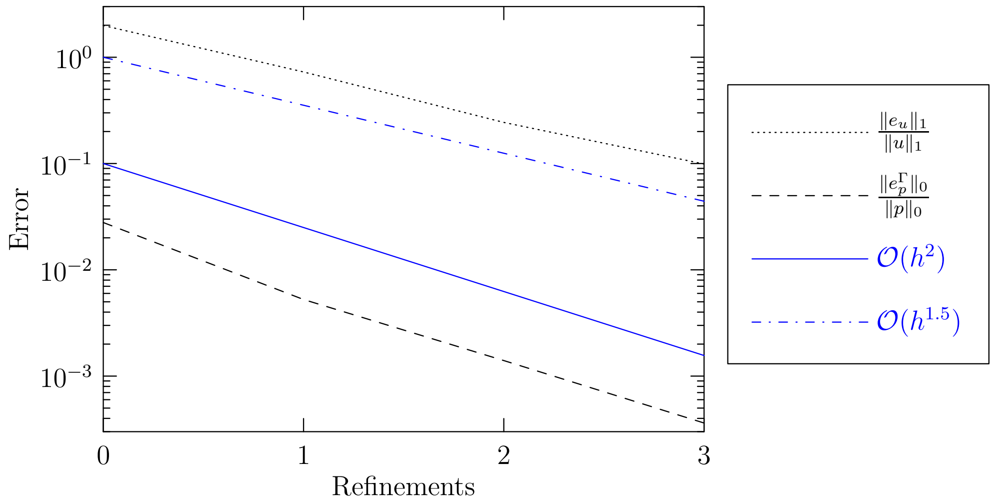



FIG. 7.5. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ and a realistic parameter setting corresponding to an air bubble in water.*

$L = 10^{-3}\,m$ and a characteristic velocity $U = 10^{-2}\,m/s$. Figure 7.4 shows a plot of the velocity and pressure along the $x$-axis on the finest grid (refinement level 3). Note the large scaling of the pressure and velocity solution, yielding $\|p\|_0 = 2.15 \cdot 10^3$ and $\|u\|_1 = 2.97 \cdot 10^3$, whereas in the previous examples both norms are of order 1. Due to the kink of the velocity at the interface (see $u_2$ in Figure 7.4), which is not aligned with the triangulation, for the standard velocity space without enrichment one expects a poor convergence order of 0.5. Figure 7.5 shows the convergence behavior for different grid refinement levels. We observe a convergence order of 1.5, showing that the surface tension discretization error dominates the error induced by the velocity kink. A reduced order of 0.5 is expected on fine enough grids, which, however, could not be tested in this experiment due to memory limitations. The results in this experiment are in accordance with our experience that for the simulation of realistic two-phase flows usually the pressure jump enrichment and the discretization of the surface tension force are essential, wheras the velocity kink enrichment (often) seems to be of minor importance. A similar experience is reported in [16].

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