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Red-Green Refinement of Simplicial Meshes in *D* Dimensions

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RED–GREEN REFINEMENT OF SIMPLICIAL MESHES IN D DIMENSIONS

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Abstract. This paper treats the local red–green mesh refinement of consistent, simplicial meshes in d dimensions. A constructive solution to the green closure problem in dimension d is given. Suppose that $\mathcal{T}$ is a mesh and that R is an arbitrary subset of its faces, which is refined with the Coxeter–Freudenthal–Kuhn (red) refinement rule. Green refinements of simplices $S \in \mathcal{T}$ are generated to restore the consistency of the mesh using a particular placing triangulation. No new vertices are created in this process. The green refinements are consistent with the red refinement on R, the unrefined mesh regions, and all other neighboring green refinements.

Key words. mesh refinement, red–green refinement, green closure, simplicial meshes

AMS subject classifications. 65M50, 65N50, 65D18

1. Introduction. This paper treats the local red–green mesh refinement of consistent, simplicial meshes in $\mathbb{R}^d$ with an arbitrary dimension $d \in \mathbb{N}$. A constructive solution to the green closure problem (defined below) in dimension d is given.

Meshes and nested hierarchies of meshes are a cornerstone for finite element methods, finite volume methods, multigrid methods, and other methods in applied mathematics [8, 17]. They also have applications in topology [25]. A mesh is simplicial if all of its elements are simplices (triangles, tetrahedra, pentatopes). It is consistent, if the intersection of mesh elements is again a mesh element. A consistent simplicial mesh is called a triangulation. In applications, $d \leq 3$ prevails, but space-time methods [18, 12] require $d = 4$.

A local mesh refinement $\mathcal{D}$ is produced from a given mesh $\mathcal{C}$ and a (user-specified) subset $R \subseteq \mathcal{C}$ of elements which should be replaced by "smaller" elements. Mesh refinement is a special case of mesh generation [3] which operates under the assumption that only "few" elements of $\mathcal{C}$ require refinement such that the generation of a completely new mesh would be wasteful. Two well-known classes of mesh refinement algorithms are the bisection methods and the red–green methods.

The basic idea of bisection methods is to cut a simplex (repeatedly) into two pieces using a (hyper-) plane. Methods for triangles are treated in [21, 24], for tetrahedra e. g. in [7]. They differ in the criteria for the selection of the cut-planes. Bisection methods are generalized to d-dimensional triangulations in [20, 26].

A red–green refinement algorithm has three components, the red refinement rule, the green refinement rule, and a global component, which coordinates the use of the two refinement rules [4]. The red refinement rule may be applied repeatedly to (the descendants of) a simplex. It is mainly responsible for the stability properties of the full algorithm. The green refinement rule is used to restore the consistency after the red rule has been applied. For triangles, red–green refinement appears in [1]. Tetrahedral meshes are considered, e. g., in [4] and [2, 15].

The red refinement rule uses the Coxeter-Freudenthal-Kuhn method which is available in d dimensions. It is well-known in topology [13, 19], group theory [9], computer graphics [23, 22], applied mathematics [5, 11]. It partitions a d-dimensional simplex S into 2^d smaller simplices. The partition is unique, provided an ordering of the vertices of S is fixed.

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The green refinement rule is typically generated manually or semi-automatically. This gives rise to

PROBLEM 1.1 (Green Closure Problem). *Suppose that S is a simplex in the triangulation $\mathcal{T}$ and that the red refinement has been applied to the subset $R \subseteq \mathcal{T}$. Suppose that F is a face of S and that T is a neighbor of S across F. The green refinement $\mathcal{S}$ of S is a triangulation of S with the following properties.*

1. *If $F \in R$, then $\mathcal{S}$ is consistent with the red refinement of F.*
2. *If F does not have an intersection with any member of R (apart from vertices), then $\mathcal{S}$ does not refine F.*
3. *The refinement of F only depends on "data" on F. (This implies consistency if both S and T are refined with a green refinement rule.)*

The green refinement rules should create as few additional vertices as possible.

For $d = 2$, it is straightforward to solve Problem 1.1 manually and verify the solutions by visual inspection. Already for $d = 3$, the situation is quite complicated. Depending on which of the six edges of a tetrahedron must be subdivided, there are 2^6 cases to consider. In the software packages UG and DROPS [2, 15] the green rule is available for each refinement pattern. No proof of correctness for these rules seems to exist in the literature. Other authors use incomplete sets of green refinement rules [1, 4]. If the situation on a simplex is too complicated, the simplex is refined with the red refinement rule. This can lead to an avalanche effect which spreads out the red refinement. This makes the refined region of the mesh larger than requested by the user. It also generates additional communication in distributed computer programs.

Both bisection methods and red–green methods can produce stable families of refined meshes. In bisection methods, some weak conditions must usually be imposed on the initial mesh to ensure stability. To assess, whether a local refinement algorithm is stable, it is applied repeatedly to a simplex and its descendents. The simplices generated by this process are sorted into equivalence classes with respect to the group of scalings and rigid motions. It is desirable to have few equivalence classes. According to Bey [4], the newest vertex bisection from [21] generally produces $d!\cdot 2^{d-2}$ equivalence classes. Bey also shows that the red refinement generates at most $d!/2$ equivalence classes and that this is the minimal number in a certain class of refinement methods.

Results. The main contribution of this paper is a constructive solution of Problem 1.1, cf. Theorem 3.14. As far as the author knows, this yields the first generalization of red–green refinement to triangulations in d dimensions. The green refinement rule constructed below introduces *no* additional vertices and handles *every* possible refinement pattern. There is *no* avalanche effect. These optimality properties of the resulting red–green refinement method are stated in Theorem 4.6. A further (minor) contribution of this paper is Theorem 5.20, which states that the red refinement of a simplex can be constructed as a placing triangulation.

Overview of the construction. The green refinement rule is based on the placing triangulation from convex geometry [10]. A placing triangulation is constructed step by step from a tuple of points w starting with the empty triangulation $\mathcal{T}$. In step i, the "active" point w^i is connected to all faces of $\mathcal{T}$ which are "visible" from w^i, cf. Section 3.2. An example is shown in Figure 1.1. The placing triangulation has a property that is similar to property (3) in Problem 1.1: The triangulation induced on a boundary face of $\mathcal{T}$ depends only on the sub-tuple of w containing all points in that boundary face.

The ordering of the vertices is important for the result of the placing triangulation. Therefore, one must consider a simplex as a set of points together with an enumeration

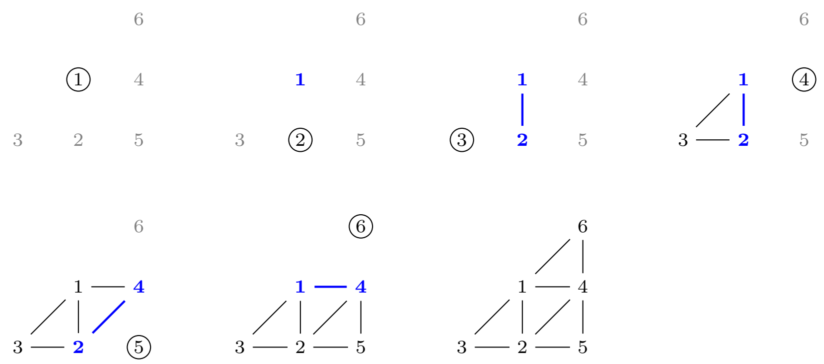

FIG. 1.1. *Construction of the green refinement with all edge-barycenters, $d = 2$. This yields the red refinement. The active node is circled, the visible faces are bold (and blue in the online version).*

or ordering of its vertices. This does not impose additional conditions on the initial meshes for a red–green refinement algorithm: To uniquely define the red refinement of a triangulation for $d \geq 3$, an ordering of the vertices is required, cf. Remark 3.4, Theorem 3.6. Such an enumeration can be constructed on any initial triangulation, cf. Lemma 2.1.

For the green refinement rule, the ordering of the vertices and edge barycenters is constructed such that the placing triangulation of all of these points reproduces the red refinement. This is important for establishing property (1) of Problem 1.1.

Organization of the paper. Section 2 introduces (standard) notation for convex polytopes and triangulations in d dimensions. In Section 3, the red and the green refinement rule are introduced. The focus is on the description of the refinement rules. The main Theorem 3.14 which solves Problem 1.1 is stated, but its proof is delayed to Section 5. In Section 4, some properties of a refinement algorithm with the rules from Section 3 are considered. Theorem 4.6 is proved. Section 5 contains the proof of Theorem 3.14. The three properties of Problem 1.1 are treated in separate sub-sections. In Section 6, a numerical verification of Theorem 3.14 is performed.

2. Polytopes and complexes.

In this section, basic notation of convex geometry is introduced to make the paper broadly accessible. All results on polytopes and complexes stated here are standard; they can be found in classic texts [16, 6] or in the more recent books [23, 10].

Let $V \subset \mathbb{R}^d$ be a finite set of points. The linear hull, the affine hull, and the convex hull of V are denoted by $\operatorname{span} V$, $\operatorname{aff} V$, and $\operatorname{conv} V$, respectively. The empty set is convex and its dimension is -1 by convention.

2.1. Polytopes.

A bounded convex polytope $P \subset \mathbb{R}^d$ is the convex hull of a finite set of points, $P = \operatorname{conv} V$. As this paper only deals with bounded convex polytopes, they are henceforth simply called polytopes. The set V is a vertex description of P. If the dimension of P is k, one says that P is a k-polytope. For any linear functional $\psi\colon \mathbb{R}^d \to \mathbb{R}$, a face of P is given by the set

$$F = \{x \in P \mid \psi(x) = M(P, \psi)\}, \quad M(P, \psi) = \max\{\psi(x) \mid x \in P\}. \tag{2.1}$$

In particular, P is a face of itself. By convention, $\emptyset$ is a face of P. The (finite) set of all faces is $\mathcal{F}(P)$. Set inclusion defines a partial order on $\mathcal{F}(P)$ which turns it into a lattice, the face lattice. All faces are themselves polytopes. The k-dimensional faces are collected in $\mathcal{F}_k(P)$. The 0- and 1-faces are called vertices and edges, the $(k-1)$-faces of a k-polytope are the facets.

A (closed) half-space is a set $\{x \in \mathbb{R}^d \mid \psi(x) \leq c\}$, where ψ is a linear functional, $c \in \mathbb{R}$. A half-space given by a facet F is obtained by taking ψ from (2.1) and $c = M(P, \psi)$. A fundamental theorem of polytope theory is that P is the intersection of all half-spaces given by its facets. This is the facet representation of P.

An elementary way to extend a polytope by adding a new vertex is the pyramid construction. A d-pyramid is defined as $\operatorname{conv}(\{a\} \cup B)$, where the basis B is a $(d-1)$-polytope and the apex a is a point which does not lie in $\operatorname{aff} B$. The pyramid is written as $\operatorname{pyr}(a, B)$. By definition, $\operatorname{pyr}(a, \emptyset) = \{a\}$.

A d-simplex, $d \geq 0$, is a d-polytope that is the convex hull of exactly $d+1$ points. By convention, the empty set is a simplex. All faces of a simplex are simplices. The face lattice of a d-simplex is isomorphic to the lattice of all subsets of $\{0, \dots, d\}$ ordered by set inclusion. A d-simplex is a pyramid over any of its facets, where the apex is the unique vertex not contained in the facet. For $d \in \{0, \dots, 4\}$, a d-simplex is called a vertex, an edge, a triangle, a tetrahedron, and a pentatope.

2.2. Complexes. A finite set $\mathcal{C}$ of polytopes in $\mathbb{R}^d$ is a complex, if the following two assertions hold,

$$\begin{aligned} &\mathcal{F}(S) \subseteq \mathcal{C} \quad \text{for all } S \in \mathcal{C} \quad &&\text{(face-completeness)},\\ &S \cap T \in \mathcal{C} \quad \text{for all } S, T \in \mathcal{C} \quad &&\text{(intersection property)}. \end{aligned} \tag{2.2}$$

In the literature, the more precise term geometric cell complex is used, but this is not necessary below. The members of $\mathcal{C}$ are called faces (or k-faces to specify the dimension). The set of all k-faces is $\mathcal{F}_k(\mathcal{C})$. In the important case that all members of $\mathcal{C}$ are simplices, $\mathcal{C}$ is called simplicial complex or triangulation.

A sub-complex of $\mathcal{C}$ is a subset, which is itself a complex. Let $A \subseteq \mathcal{C}$. The closure of A, written as $\operatorname{cl} A$, is the smallest sub-complex of $\mathcal{C}$, which contains A as subset. There holds

$$S \in \operatorname{cl} A \quad \Leftrightarrow \quad S \in \mathcal{F}(T) \text{ for some } T \in A. \tag{2.3}$$

The notation $\operatorname{set} \mathcal{C}$ is used for $\cup\{S \mid S \in \mathcal{C}\} \subset \mathbb{R}^d$. The face lattice of a polytope P is a complex. If F is a face of P and $P = \operatorname{set} \mathcal{C}$, the notation $\mathcal{C}|_F$ is used for the restriction of $\mathcal{C}$ to F. That is, $\mathcal{C}|_F = \{T \in \mathcal{C} \mid T \subseteq F\}$.

2.3. Consistent enumerations. Let $\mathcal{C}$ be a complex. An enumeration of a finite set S is a bijective map $e\colon \{0, \dots, |S|-1\} \to S$. It carries the (strict total) ordering of the integers to the elements of S; $e(i)$ is less than $e(j)$, if and only if $i < j$. This is called the ordering given (or induced) by e. Conversely, a strict total ordering of S defines a unique enumeration of S. An enumeration e of $\mathcal{C}$, is a collection of enumerations of the vertices of each member of $\mathcal{C}$,

$$e = (e_S)_{S \in \mathcal{C}}, \quad e_S \text{ is an enumeration of } \mathcal{F}_0(S).$$

Any sub-complex of $\mathcal{D} \subseteq \mathcal{C}$ carries the restricted enumeration $e|_{\mathcal{D}}$ given by all e_S, $S \in \mathcal{D}$. An enumeration of $\mathcal{C}$ is consistent, if, for any $S, T \in \mathcal{C}$, the ordering of the common vertices $\mathcal{F}_0(S \cap T)$ given by e_S and given by e_T is the same. More formally, this means that $e_T^{-1} \circ e_S$ is strictly isotonic on $e_S^{-1}(\mathcal{F}_0(S \cap T))$. A complex with a consistent enumeration is called consistently numbered. A polytope P, particularly a simplex, is called consistently numbered, if its face lattice is.

It was already known to Freudenthal [13] [5, Chp. 3.1.6] that a consistent enumeration of $\mathcal{C}$ can be produced easily from a single enumeration of all vertices of $\mathcal{C}$.

LEMMA 2.1. *If $\mathcal{C}$ is a complex and e is an enumeration of $\mathcal{F}_0(\mathcal{C})$, there is a unique consistent enumeration f of $\mathcal{C}$ such that, for all $S \in \mathcal{C}$, the orders of $\mathcal{F}_0(S)$ given by e and f_S are equal.*

Proof. Let $S \in \mathcal{C}$. As $\mathcal{F}_0(S) \subseteq \mathcal{F}_0(\mathcal{C})$, the ordering given by e restricts to a unique ordering on $\mathcal{F}_0(S)$. There is precisely one isotonic enumeration of $\mathcal{F}_0(S)$ with respect to this ordering, which uniquely defines f_S.

Consistency follows from the fact that f_S and f_T, $S, T \in \mathcal{C}$, both give to $\mathcal{F}_0(S \cap T)$ the same ordering as e. □

Let $x, y \in \mathbb{R}^d$. The lex-order $x \preceq y$ is the total order on $\mathbb{R}^d$ defined by

$$x \preceq y \quad \Leftrightarrow \quad x_i \leq y_i,$$
$$i = \begin{cases} d, & x_d \neq y_d, \\ \min\bigl\{j \in \{1, \dots, d\} \mid x_k = y_k \text{ for all } j < k \leq d\bigr\}, & \text{otherwise.} \end{cases} \tag{2.4}$$

This is a variant of the standard lexicographic order. The tuples are compared with the last position (instead of the first) as the most significant. An enumeration of a (finite) set of points can be produced by enumerating the elements in lex-order; this enumeration is denoted as lex or lex_S if the set under consideration is not clear from the context. The consistent enumeration of a complex $\mathcal{C}$ which is defined by using Lemma 2.1 with $\text{lex}_{\mathcal{F}_0(\mathcal{C})}$ is simply denoted by lex.

Let (S, e) be a consistently numbered d-simplex and $x \in \mathbb{R}^d$. The barycentric coordinates of x with respect to (S, e) are the unique vector $\lambda = (\lambda_0, \dots, \lambda_d)^T \in \mathbb{R}^{d+1}$ with

$$x = \sum_{i=0}^{d} \lambda_i e_S(i), \quad \sum_{i=0}^{d} \lambda_i = 1.$$

2.4. Isotonic affine maps. Let (S, e), (T, f) be consistently numbered k-simplices embedded into $\mathbb{R}^d$ ($k \leq d$). Any bijection $\mathcal{F}_0(S) \to \mathcal{F}_0(T)$ can be extended uniquely to an invertible, affine map $\operatorname{aff} S \to \operatorname{aff} T$ which maps $S \to T$. Let A be the affine map which maps $e_S(i) \mapsto f_T(i)$, $i \in \{0, \dots, k\}$.

LEMMA 2.2. *The map A is the unique invertible, affine map $A\colon \operatorname{aff} S \to \operatorname{aff} T$ with $A(S) = T$, such that $f_{A(F)} = A \circ e_F$ holds for all $F \in \mathcal{F}(S)$.*

Proof. By construction, $f_T = f_{A(S)} = A \circ e_S$. By Lemma 2.1, the consistent enumeration of $\mathcal{F}(T)$ is uniquely determined by this enumeration of $\mathcal{F}_0(T)$. Hence, the enumerations agree on all faces of T, or equivalently, on all faces of S. □

The map A is called the isotonic (or order preserving) affine map $(S, e) \to (T, f)$. Lemma 2.2 can be used as follows: Given (S, e) and an invertible, affine map $A\colon \operatorname{aff} S \to \operatorname{aff} T$, $S \to T$, there is precisely one consistent enumeration f of T which makes A isotonic, namely, $f = Ae = (A \circ e_F)_{F \in \mathcal{F}(S)}$. This is the induced enumeration.

Below, the notation $A(\mathcal{C})$ is used for $\{A(S) \mid S \in \mathcal{C}\}$. This set is a complex if $\mathcal{C}$ is and if A is invertible on set $\mathcal{C}$.

3. Refinement rules for simplices. A d-refinement rule r maps consistently numbered d-simplices (S, e) to pairs $(\mathcal{C}, f) = r(S, e)$, where $\mathcal{C}$ is a triangulation of S (that is, $\operatorname{set} \mathcal{C} = S$) which is consistently numbered by f. Both $\mathcal{C}$ and f may depend on both S and e. The dimension d is usually clear from the context and therefore omitted. One calls S the parent simplex and the members of $\mathcal{C}$ its child simplices.

Suppose that F is a face of S. The restriction $(\mathcal{C}, f)|_F$ is the pair $(\mathcal{C}|_F, f|_F)$, where $f|_F$ is a short-hand notation for the enumeration $f|_{(\mathcal{C}|_F)}$. Suppose that A is an affine map which is invertible on $\operatorname{aff} S$. The image of $(\mathcal{C}, f)$ under A is the pair $(A(\mathcal{C}), Af)$ containing the mapped complex and the mapped enumeration.

3.1. The red refinement rule. Let

$$\hat{v}^0 = 0, \quad \hat{v}^i = \hat{v}^{i-1} + \delta^i, \quad i \in \{1, \ldots, d\}, \tag{3.1}$$

where δ^i is the i-th column of the $d \times d$-identity-matrix. The enumeration of the points $\hat{v}^i$ is given by the lex-order. The scaled reference d-simplex

$$\hat{S} = \operatorname{conv}\{v^0, v^1, \ldots, v^d\} \quad \text{with } v^i = 2\hat{v}^i, i \in \{1, \ldots, d\}, \tag{3.2}$$

is used. The enumeration of its vertices $i \mapsto v^i$, $i \in \{0, \ldots, d\}$, is also given by the lex-order.

A (k, l)-shuffle π is a permutation in $\operatorname{Sym}(k+l)$, the symmetric group on $\{1, \ldots, k+l\}$, with the property that its inverse π^{-1} is isotonic on $\{1, \ldots, k\}$ and on $\{k+1, \ldots, k+l\}$. For $k = 0$ or $l = 0$, the conditions for the empty set appearing in the definition are considered as vacuously true. The set of (k, l)-shuffles is denoted as $\operatorname{Sh}(k, l) \subset \operatorname{Sym}(k + l)$. It is related to the set of binomial coefficients; for example, the number of (k, l)-shuffles is $\binom{k+l}{k}$.

Let $\pi \in \operatorname{Sym}(d)$ operate on $x \in \mathbb{R}^d$ by permuting the components, that is $(\pi x)_i = x_{\pi i}$, $i \in \{1, \ldots, d\}$. Let S_π be the d-simplex

$$S_\pi = \operatorname{conv}\left\{\pi\hat{v}^i \;\middle|\; i \in \{0, \ldots, d\}\right\}. \tag{3.3}$$

An enumeration of the vertices of S_π is given by $e^\pi \colon i \mapsto \pi\hat{v}^i$, $i \in \{0, \ldots, d\}$. By Lemma 2.1, this defines a consistent enumeration of the face lattice of S_π.

Definition 3.1. *The red refinement of* $(\hat{S}, \operatorname{lex})$ *is the pair* $\operatorname{redref}(\hat{S}, \operatorname{lex}) = \big(\operatorname{redtri}(\hat{S}, \operatorname{lex}), \operatorname{rednum}(\hat{S}, \operatorname{lex})\big)$. *The triangulation* $\operatorname{redtri}(\hat{S}, \operatorname{lex})$ *is the smallest triangulation containing the following* 2^d *simplices of dimension* d*,*

$$\hat{v}^k + S_\pi \quad \textit{for } \pi \in \mathit{Sh}(k, d-k) \quad \textit{for } k \in \{0, \ldots, d\}. \tag{3.4}$$

The enumeration $\operatorname{rednum}(\hat{S}, \operatorname{lex})$ *is given (uniquely) by the mappings* e^π*.*

For a consistently numbered d*-simplex* (S, e)*, let* A *be the isotonic, affine map* $(\hat{S}, \operatorname{lex}) \to (S, e)$*, cf. Lemma 2.2. The red refinement of* (S, e) *is the pair* $\operatorname{redref}(S, e) = A\big(\operatorname{redref}(\hat{S}, \operatorname{lex})\big)$.

Bey [5, La. 3.1.23] proves that $\operatorname{redref}(\hat{S}, \operatorname{lex})$ is a consistently numbered triangulation (cf. also [11]). The following direct characterization of $\operatorname{rednum}(\hat{S}, \operatorname{lex})$ simplifies the definition and analysis of the green refinement rules below. I did not find it in the literature.

Lemma 3.2. *The enumeration* $\operatorname{rednum}(\hat{S}, \operatorname{lex})$ *is given by the lex-order on the vertices of* $\operatorname{redtri}(\hat{S}, \operatorname{lex})$*, that is,* $\operatorname{rednum}(\hat{S}, \operatorname{lex}) = \operatorname{lex}$.

Proof. Owing to [5, La. 3.1.23], one must only show that the order of the vertices of each simplex in (3.4) given by the lex-order. Fix $k \in \{0, \ldots, d\}$, $\pi \in \operatorname{Sh}(k, d-k)$.

Then, using (3.3), one must show that $\hat{v}^k + \pi\hat{v}^i \preceq \hat{v}^k + \pi\hat{v}^j$ is equivalent to $i \leq j$. As the vertices in (3.1) are pairwise different (also when shifted by $\hat{v}^k$) their enumeration in lex-order is unique. Hence, if one shows that $i \leq j$ implies $\hat{v}^k + \pi\hat{v}^i \preceq \hat{v}^k + \pi\hat{v}^j$, the uniqueness of the enumeration in lex-order proves the lemma.

From $i \leq j$ and (3.3) one concludes that every component of the vector $\hat{v}^j - \hat{v}^i$ is nonnegative. Therefore, all components of $\pi(\hat{v}^j - \hat{v}_i) = \pi\hat{v}^j - \pi\hat{v}^i$ are nonnegative, which implies $0 \preceq \pi\hat{v}^j - \pi\hat{v}^i$. Adding $\hat{v}^k$, scaling by $\frac{1}{2}$, and rearranging this inequality yields the required result. ∎

Remark 3.3. *It is well-known that* $\operatorname{redref}(S, e)$ *can be defined without the reference simplex to* $(\hat{S}, \operatorname{lex})$*, cf. [5, Ch. 3.1.4], basically, by expressing* πv^i *in barycentric coordinates (more precisely, the action of* π*). Similar to Lemma 3.2, the enumeration* $\operatorname{rednum}(S, e)$ *can be described without reference to* $(\hat{S}, \operatorname{lex})$ *as given by the lex-order on barycentric coordinates, cf. Lemma 5.8 below.*

Remark 3.4. *A refinement rule is isotonic-affinely invariant, if* $A\big(r(S, e)\big) = r(T, g)$ *holds whenever* A *is the isotonic affine map* $A\colon (S, e) \to (T, g)$*. The red refinement is isotonic-affinely invariant in all dimensions* d*.*

For $d = 2$*, the red refinement is affinely invariant, that is, it commutes with all invertible, affine maps, not only the isotonic one. In this case, the triangulation* $\operatorname{redtri}(\hat{S}, \operatorname{lex})$ *in fact does not depend on* lex*. For* $d = 3$*, this is not true anymore. The triangulation* $\operatorname{redtri}(\hat{S}, e)$ *for some enumeration* e *always contains the four child-tetrahedra at the vertices of* S*. However, the remaining four tetrahedra, which form an octahedron, share one edge (out of a set of three possible ones) which does depend on* e*. As this effect is constrained to the interior of* S*, it is of minor importance for a red–green refinement algorithm for* $d = 3$*.*

For $d \geq 4$*, the choice of* e *influences how* $\operatorname{redtri}(\hat{S}, e)$ *triangulates the boundary of* S*. Hence, a consistent enumeration is required to satisfy the intersection property* (2.2)*.*

An important property of the red refinement rule is that it commutes with restriction to faces,

Lemma 3.5 ([5, La. 3.1.24][11]). *Let* F *be a face of the consistently numbered* d*-simplex* (S, e)*. Then* $\operatorname{redref}(F, e|_{\mathcal{F}(F)}) = \operatorname{redref}(S, e)|_F$*.*

A main theorem, valid in any dimension d, that follows from Lemma 3.5 describes the global red refinement of $\mathcal{C}$ [5, Thm. 3.1.25][11].

Theorem 3.6. *If* $(\mathcal{C}, e)$ *is a consistently numbered simplicial complex, then* $\operatorname{redtri}(\mathcal{C}) = \cup\{\operatorname{redtri}(S, e) \mid S \in \mathcal{C}\}$ *is a consistently numbered simplicial complex. (The numberings* $\operatorname{rednum}(S, e)$ *are consistent with each other.)*

3.2. The placing triangulation. The green refinement rule is defined using a well-known technique from convex geometry to construct triangulations, namely, the placing triangulation, cf. [10, Sec. 4.3], [23]. Let $P \subset \mathbb{R}^d$ be a polytope and $v \in \mathbb{R}^d$ be a point.

Definition 3.7 (Visibility). *A face* F *of* P *is visible from* v*, if there is a linear functional* ψ *on* $\mathbb{R}^d$ *and* $c \in \mathbb{R}$ *such that* $\psi(v) > c$*,* $\psi(x) = c$ *for all* $x \in F$*, and* $\psi(x) \leq c$ *for all* $x \in P$*.*

The definition can also be stated in geometric terms.

Remark 3.8. *If* $v \notin \operatorname{aff} P$*, then every face of* P *(including* P *itself) is visible from* v*. Now suppose that* $v \in \operatorname{aff} P$ *and that* F *is a facet of* P*. There is a unique hyperplane* H *in* $\operatorname{aff} P$ *which contains* F*. The polytope* P *is contained in exactly one of the closed halfspaces of* $\operatorname{aff} P$ *bounded by* H*. The facet* F *is visible from* v*, if and* *only if v is contained in the opposite open halfspace of* $\operatorname{aff} P$. *A face $\tilde{F}$ of P is visible from v if and only if there is a facet F of P with $\tilde{F} \subseteq F$ which is visible from v.*

Definition 3.9 (Placing a vertex). *Let $\mathcal{C}$ be a complex with* $\operatorname{set} \mathcal{C} = P$. *The complex $\mathcal{D}$ with* $\operatorname{set} \mathcal{D} = \operatorname{conv}(P \cup \{v\})$ *that results from placing v in $\mathcal{C}$ is $\mathcal{D} = \mathcal{C} \cup \{\operatorname{pyr}(v, F) \mid F \in \mathcal{C}, F$ is visible from $v\}$.*

If $v \in P$, then $\mathcal{D} = \mathcal{C}$. If $\mathcal{C}$ is a triangulation, so is $\mathcal{D}$.

Definition 3.10 (Placing triangulation). *Given a tuple of n points, $V = (v^i)_{i=1}^n$, the placing triangulation* $\operatorname{pla} V$ *of* $\operatorname{conv}\big\{v^i \bigm| i \in \{1, \dots, n\}\big\}$ *is obtained by beginning with the empty set and placing the vertices of the tuple successively in the order of their subscripts $i = 1, \dots, n$.*

An example is shown in Figure 1.1.

3.3. The green refinement rule. Compared to the red refinement rule, the green refinement rule (on $\hat{S}$) requires a refinement pattern as additional input. This is a set $\hat{R} \subseteq \mathcal{F}(\hat{S})$ of faces of $\hat{S}$, to which the red refinement rule must be applied.

The tuple $\hat{w}$ containing all vertices and edge-barycenters of $\hat{S}$ is defined with the auxiliary tuples

$$\hat{w}_i = (\hat{v}^i + \hat{v}^{d+1-j})_{j=1}^{d+1-i}, \quad i \in \{0, 1, \dots, d\}, \quad \text{as} \quad \hat{w} = \hat{w}_0 \sqcup \hat{w}_1 \sqcup \cdots \sqcup \hat{w}_d. \tag{3.5}$$

Here, $\sqcup$ denotes the concatenation of tuples. The assertion that all vertices and edge-barycenters of $\hat{S}$ occur precisely once in $\hat{w}$ follows from Lemma 5.14 below. A sub-tuple of a given tuple $(v^i)_{i=1}^n$ is a tuple $(v^{i_j})_{j=1}^m$, where i_j is an strictly isotonic map $\{1, \dots, m\} \to \{1, \dots, n\}$.

Definition 3.11. *Let w be the sub-tuple of $\hat{w}$ that contains all vertices of $\hat{S}$ and the barycenters of all edges in* $\operatorname{cl} \hat{R}$ *(cf. (2.3)). The green refinement of* $(\hat{S}, \operatorname{lex})$ *with the refinement pattern $\hat{R}$ is the consistently numbered triangulation* $\operatorname{greenref}(\hat{S}, \operatorname{lex}, \hat{R}) = \big(\operatorname{greentri}(\hat{S}, \operatorname{lex}, \hat{R}), \operatorname{greennum}(\hat{S}, \operatorname{lex}, \hat{R})\big)$. *The triangulation and its consistent numbering are given by*

$$\operatorname{greentri}(\hat{S}, \operatorname{lex}, \hat{R}) = \operatorname{pla}(w), \quad \operatorname{greennum}(\hat{S}, \operatorname{lex}, \hat{R}) = \operatorname{lex}. \tag{3.6}$$

For a consistently numbered simplex (S, e) and the refinement pattern $R_S \subseteq \mathcal{F}(S)$, let A be the isotonic, affine map $(\hat{S}, \operatorname{lex}) \to (S, e)$. *Let $\hat{R} \subseteq \mathcal{F}(\hat{S})$ be the refinement pattern with $A(\hat{R}) = R_S$. The green refinement of (S, e, R_S) is the pair* $\operatorname{greenref}(S, e, R_S) = A\big(\operatorname{greenref}(S, e, \hat{R})\big)$.

Remark 3.12. *Note that two different orderings of the points w^i appear above. The ordering induced by the tuple $\hat{w}$ is used in the construction of the triangulation* $\operatorname{greentri}(\hat{S}, \operatorname{lex}, \hat{R})$. *However, the enumeration* $\operatorname{greennum}(\hat{S}, \operatorname{lex}, \hat{R})$ *is induced by the lex-order on $\mathbb{R}^d$.*

Remark 3.13. *The construction of a green refinement is straightforward. First, one constructs the placing triangulation of the appropriate sub-tuple of $\hat{w}$. Methods for this are discussed, for example, in [10]. Second, the resulting triangulation is enumerated in lex-order, cf. Remark 5.12.*

The simplices that are generated by a green refinement rule are called green simplices; S is called a green parent simplex.

The refinement pattern R_S may be an arbitrary subset of the face lattice of S. However, due to Lemma 3.5, the red refinement must be applied to all $F \in \operatorname{cl} R_S$ to obtain a triangulation of $\operatorname{cl} R_S$. Complementary to the refinement pattern R_S is the set

$$U_S = \{F \in \mathcal{F}(S) \mid F \cap T \subseteq \mathcal{F}_0(S) \text{ for all } T \in R_S\}. \tag{3.7}$$

It contains the faces of S which do not require a triangulation to be consistent with the red refinement prescribed by R_S. It is easy to see that U_S is always a subcomplex of the face-lattice (contrary to R_S). The *main theorem* of this paper is the following.

THEOREM 3.14. *For all consistently numbered simplices* (S, e)*, all faces* $F \subseteq S$*, and all refinement patterns* $R_S \subseteq \mathcal{F}(S)$*, there holds*

$$\operatorname{greenref}(S, e, R_S)|_F = \operatorname{greenref}\big(F, e|_{\mathcal{F}(F)}, \operatorname{cl} R_S \cap \mathcal{F}(F)\big), \tag{3.8}$$

$$F \in \operatorname{cl} R_S \implies \operatorname{greenref}(S, e, R_S)|_F = \operatorname{redref}(F, e|_{\mathcal{F}(F)}), \tag{3.9}$$

$$F \in U_S \implies \operatorname{greenref}(S, e, R_S)|_F = (\mathcal{F}(F), e|_{\mathcal{F}(F)}). \tag{3.10}$$

The theorem solves the green closure problem 1.1. Assertion (3.8) states that the green refinement of any face is completely determined by the data on this face (the restriction to F of the enumeration and of the refinement pattern). This makes the green refinements consistent with each other wherever two green refinements meet. Assertion (3.9) makes the green refinement consistent with the red refinement wherever this is required by the refinement pattern. Assertion (3.10) makes the green refinement consistent with any unrefined region of the mesh.

The choice $F = S$ is admissible in (3.9). This makes the red refinement of a simplex a placing triangulation, which seems to be a new result, cf. Theorem 5.20 below. Before the proof of Theorem 3.14 is given in Section 5, it is shown in Section 4 how the theorem can be used to derive a global red–green refinement algorithm in any dimension d.

4. Red–green refinement with arbitrary refinement pattern. Let $(\mathcal{C}, e)$ be a consistently numbered triangulation. Suppose that all simplices in the refinement pattern $R \subseteq \mathcal{C}$ must be refined with the red refinement rule. The local refinement $(\mathcal{D}, f)$ is defined from three parts as $\mathcal{D} = \mathcal{R} \cup \mathcal{U} \cup \mathcal{G}$. The first part is generated with red refinement rule as

$$\begin{gathered} \mathcal{R} = \cup \left\{ \operatorname{redtri}(S, e|_{\mathcal{F}(S)}) \;\middle|\; S \in R \right\}, \\ f_F = \operatorname{rednum}(S, e|_{\mathcal{F}(S)})_F \quad \text{for all } S \in R, F \in \mathcal{F}(S). \end{gathered} \tag{4.1}$$

LEMMA 4.1. *The pair* $(\mathcal{R}, f|_{\mathcal{R}})$ *is a consistently numbered complex. The set of all simplices of* $\mathcal{C}$ *which are refined by the pattern* R *is* $\operatorname{cl} R$.

Proof. $\mathcal{R}$ is the union of subcomplexes of $\operatorname{redtri}(\mathcal{C}, e)$ and therefore a simplicial complex by Theorem 3.6. The enumeration in (4.1) is the restriction of $\operatorname{rednum}(\mathcal{C}, e)$ to $\mathcal{R}$ and therefore a consistent enumeration.

Due to Lemma 3.5, any simplex in $\operatorname{cl} R$ is refined by the red refinement rule. Conversely, let $\operatorname{redtri}(S, e) \subseteq \mathcal{R}$ for some $S \in \mathcal{C}$. There is a parent simplex $T \in R$ with $\operatorname{redtri}(S, e) \subseteq \operatorname{redtri}(T, e)$. This gives $\operatorname{set} S \subseteq \operatorname{set} T$. As $\mathcal{C}$ is a complex, one obtains $S \in \mathcal{F}(T) \subseteq \operatorname{cl} R$. □

The second part is the subset of $\mathcal{C}$ that requires no refinement,

$$\mathcal{U} = \{S \in \mathcal{C} \mid S \cap T \subseteq \mathcal{F}_0(\mathcal{C}) \text{ for all } T \in R\}, \quad f_F = e_F \quad \text{for all } f \in \mathcal{U}. \tag{4.2}$$

Note that $\mathcal{U} \cap \operatorname{cl} R$ may contain (many) vertices, but no higher dimensional simplices.

LEMMA 4.2. *The pair* $(\mathcal{U}, f|_{\mathcal{U}})$ *is a consistently numbered complex.*

Proof. Let $S \in \mathcal{U}$, $F \in \mathcal{F}(S)$. As $F \cap T \subseteq S \cap T$ for all T in R, the definition in (4.2) implies that $\mathcal{U}$ is face complete. As subset of $\mathcal{C}$, it satisfies the intersection property. The enumeration f is the restriction of e to the subcomplex $\mathcal{U}$. Therefore, it is consistent. □

The third part comes from the green refinement rule,

$$\begin{aligned} \mathcal{G} &= \cup\left\{\operatorname{greentri}\big(S, e, \operatorname{cl}(R)\cap\mathcal{F}(S)\big) \;\middle|\; S\in\mathcal{C}-(R\cup\mathcal{U})\right\}, \\ f_F &= \operatorname{greennum}\big(S, e, \operatorname{cl}(R)\cap\mathcal{F}(S)\big)_F \quad \text{for all } F\in\mathcal{G}. \end{aligned} \tag{4.3}$$

Remark 4.3. *In addition to the red and green refinement rule, a red–green refinement algorithm has a global component that enforces that a green simplex is not refined further. If this is required by the refinement pattern, the green refinement of the parent simplex is rolled back and replaced with a red refinement. Well-known strategies for this are the worker list based approach of Banks [1] and the level oriented approach of Bastian [4]. Both global components carry over to the d-dimensional situation. Specifically, the algorithm in [4] and its correctness analysis work in d dimensions if one substitutes "d-simplex" for "tetrahedron" and if one removes all references to the case that a green refinement with a specified pattern is not found.*

The remainder of the section is devoted to proving, that $(\mathcal{D}, f)$ is a consistently numbered simplicial complex with $\operatorname{set}\mathcal{D} = \operatorname{set}\mathcal{C}$. A simple partial result is

Lemma 4.4. *The pair $(\mathcal{R}\cup\mathcal{U}, f|_{\mathcal{R}\cup\mathcal{U}})$ is a consistently numbered simplicial complex.*

Proof. Due to the Lemmas 4.1 and 4.2, $(\mathcal{R}, f|_{\mathcal{R}})$ and $(\mathcal{U}, f|_{\mathcal{U}})$ are consistently numbered triangulations. Hence, their union is face-complete. The intersection property must be verified. Owing to (4.2), $\operatorname{set}\mathcal{U}\cap\operatorname{set}R$ is a subset of $\mathcal{F}_0(\mathcal{C})$. By the definition of a refinement rule, there holds $\operatorname{set}\mathcal{R} = \operatorname{set}R$. Thus, there holds $\operatorname{set}\mathcal{R}\cap\operatorname{set}\mathcal{U} = \operatorname{set}R\cap\operatorname{set}\mathcal{U}\subseteq\mathcal{F}_0(\mathcal{C})$. Any vertex v from this set only has the trivial refinement. Hence, the intersection property is satisfied. Finally, the consistency of the enumerations on the intersection must be checked. But as v only has the trivial enumeration, the definitions of f_v in (4.1) and(4.2) agree. □

Theorem 4.5. *The pair $(\mathcal{D}, f)$ is a consistently numbered triangulation of* $\operatorname{set}\mathcal{C}$.

Proof. The set $\mathcal{D}$ is defined as the union of simplicial complexes, cf. the Lemmas 4.1, 4.2, and the definition in (4.3). Thus, $\mathcal{D}$ is face-complete. To prove the intersection property, all parent simplices $S, T\in\mathcal{C}$ are considered. Only the consistency on the boundary of S, T must be checked, as the refinements of S, T are consistently numbered simplicial complexes by the definition of a refinement rule.

By Lemma 4.1, the refinement pattern $\operatorname{cl}R$ contains all simplices of $\mathcal{C}$ which are refined with the red rule. It induces the refinement pattern $R_S = \operatorname{cl}R\cap\mathcal{F}(S)$ on S. From (3.7), one gets the complementary pattern U_S.

Due to Lemma 4.4, one only has to consider the case that S is refined with $\operatorname{greenref}(S, e, R_S)$. Any intersection $F = S\cap T\in\mathcal{C}$ is a face of S. Owing to (3.8), it is refined by $\operatorname{greenref}(F, e, R_F)$.

Consider the three possible cases for T. If $T\in\operatorname{cl}R$, then F is refined by $\operatorname{redref}(F, e|_{\mathcal{F}(F)})$ because of Lemma 3.5. Moreover, $F\in\operatorname{cl}R_S$. Due to property (3.9), $\operatorname{greenref}(S, e, R_S)|_F = \operatorname{redref}(F, e|_{\mathcal{F}(F)})$.

If $T\in\mathcal{U}$, then $\mathcal{F}(F)\subseteq\mathcal{U}$ and $f|_{\mathcal{F}(F)} = e|_{\mathcal{F}(F)}$. Moreover, $F\in U_S$. Due to property (3.10), $\operatorname{greenref}(S, e, R_S)|_F = (\mathcal{F}(F), e|_{\mathcal{F}(F)})$.

If $T\in\mathcal{C}-(\mathcal{U}\cup R)$, then T is refined by $\operatorname{greenref}(T, e, R_T)$. Observing (3.8), F is refined by $\operatorname{greenref}(F, e, R_F)$, which is the same refinement as chosen on S. In summary, $\mathcal{D}$ is a simplicial complex.

The assertion $\operatorname{set}\mathcal{D} = \operatorname{set}\mathcal{C}$ holds because every $S\in\mathcal{C}$ is in one of the sets $\operatorname{cl}R$, $\mathcal{U}$, and $\mathcal{C}-(R\cup\mathcal{U})$. □

Some optimality properties of the red–green refinement $\mathcal{D}$ are collected in the following theorem.

THEOREM 4.6. *The triangulation $\mathcal{R}$ is the minimal complex containing the red refinement of R. Complementary, $\mathcal{U}$ is the maximal subcomplex of $\mathcal{C}$ which is compatible with $\mathcal{R}$. That is, for any complex $\tilde{\mathcal{D}}$ with $\mathcal{R} \subseteq \tilde{\mathcal{D}}$, there holds $\mathcal{C} \cap \tilde{\mathcal{D}} \subseteq \mathcal{U}$. The new vertices in $\mathcal{D}$ with respect to $\mathcal{C}$ are precisely the barycenters of the edges in* $\operatorname{cl} R$.

Proof. Assume $\tilde{\mathcal{R}}$ contains $\operatorname{redtri}(S, e|_{\mathcal{F}(S)})$ for all $S \in R$. Then, $\mathcal{R} \subseteq \tilde{\mathcal{R}}$ because of (4.1). Hence, $\mathcal{R}$ is minimal. This proves the first assertion of the theorem.

By definition, $\mathcal{U} \subseteq \mathcal{C}$. Let $S \in \mathcal{C} - \mathcal{U}$ be a simplex. By (4.2), there is $T \in R$ such that $F = S \cap T \not\subseteq \mathcal{F}_0(\mathcal{C})$. The dimension of F is at least 1. Due to Lemma 3.5, $\mathcal{R}$ contains the barycenters of the edges of F. Hence, S cannot be a member of a complex $\tilde{\mathcal{D}}$ which contains $\mathcal{R}$. The second assertion of the theorem follows by taking the contrapositive.

There holds $\mathcal{F}_0(\mathcal{R}) = \mathcal{F}_0\big(\operatorname{cl}(R)\big) \cup B$, where B is the set of all barycenters of $\mathcal{F}_1\big(\operatorname{cl}(R)\big)$. This follows from (3.1) and (3.4), cf. Lemma 5.14 below. The vertices of $\mathcal{U}$ satisfy $\mathcal{F}_0(\mathcal{U}) \subseteq \mathcal{F}_0(\mathcal{C})$ due to the definition in (4.2).

Let $S \in \mathcal{C} - (\mathcal{R} \cup \mathcal{U})$ and $\mathcal{S} = \operatorname{greentri}(S, e, \operatorname{cl}(R) \cap \mathcal{F}(S))$. From (3.5) and Definition 3.10, it follows that $\mathcal{F}_0(\mathcal{S}) = \mathcal{F}_0(S) \cup B_S$, where B_S contains the barycenters of the edges in $\operatorname{cl}(R) \cap \mathcal{F}(S)$. From (4.3), one gets $\mathcal{F}_0(\mathcal{G}) \subseteq \mathcal{F}_0(\mathcal{C}) \cup B$. The third assertion of the theorem follows from $\mathcal{D} = \mathcal{R} \cup \mathcal{U} \cup \mathcal{G}$. □

5. The proof of Theorem 3.14. Each of the properties (3.8)–(3.10) is treated in a separate subsection below. In Section 5.1, key properties of the placing triangulation are stated.

The proofs of property (3.8) (Section 5.2) and property (3.9) (Section 5.3) are both rather technical. For property (3.8), one must deal with many objects: a simplex, one of its faces, the corresponding reference simplices, the affine mappings between these simplices, the tuples defining the respective green refinements, etc. On the other hand, the placing triangulation commutes with invertible, affine maps and with the restriction to faces (cf. Section 5.1), which simplifies the analysis.

In the proof of property (3.9), a partition of the reference simplex into many parts is used. The simplex is partitioned into cubes, and the cubes are partitioned further by halfspaces. The visible parts of the tentative triangulations in the computation of the placing triangulation are traced through the partition, which makes the proof technical. Furthermore, the partitioning is shown to be consistent with the red refinement. (That is, the interiors of the d-dimensional simplices are not cut.) This enables an inductive proof, in which the red refinement and the green refinement with the full refinement pattern are compared.

5.1. Properties of the placing triangulation. The following uniqueness property of the placing triangulation is required in Section 5.3.

LEMMA 5.1 ([10, La. 4.3.2]). *The complex $\mathcal{D}$ obtained by placing a point v in a complex $\mathcal{C}$ is the unique complex with* $\operatorname{set}\mathcal{D} = \operatorname{conv}(\operatorname{set}\mathcal{C} \cup \{v\})$ *that contains $\mathcal{C}$ as subcomplex.*

Let $V = (v^i)_{i=1}^n$ be a tuple of points and $P = \operatorname{conv}\big\{v^i \;\big|\; i \in \{1, \dots, n\}\big\}$.

LEMMA 5.2. *The placing triangulation commutes with invertible affine mappings, that is* $\operatorname{pla}\big((Av^i)_{i=1}^n\big) = A\big(\operatorname{pla} V\big)$*, where A is an affine mapping which is invertible on* $\operatorname{aff} P$.

Proof. The proof is by induction. If the tuple contains only one point, there holds $\operatorname{pla}\big((Av^i)_{i=1}^1\big) = A\big(\operatorname{pla}(v^i)_{i=1}^1\big)$. Let $\mathcal{C} = \operatorname{pla}(v^i)_{i=1}^{m-1}$ for some $m \leq n$, $A(\mathcal{C})$ be the mapped complex. In view of Definition 3.9, one must show that for all $F \in \mathcal{C}$

$$F \text{ is visible from } v^m \quad \Leftrightarrow \quad A(F) \text{ is visible from } Av^m. \tag{5.1}$$

As A is invertible on aff P, it is sufficient to prove only the implication from left to right. If F is visible from v^m, one gets from Definition 3.7 the functional ψ on $\mathbb{R}^d$ and $c \in \mathbb{R}$ with $\psi(v^m) > c$, $\psi(x) = c$ for all $x \in F$, and $\psi(x) \leq c$ for all $x \in S$. Clearly, $\phi = \psi \circ A^{-1}$ is a linear functional. One gets $\phi(Av^m) > c$, $\phi(y) = c$ for all $y \in A(F)$, and $\phi(y) \leq c$ for all $y \in A(S)$. Thus, $A(F)$ is visible from Av^m. Using (5.1), one obtains $\mathrm{pla}\big((Av^i)_{i=1}^m\big) = A\big(\mathrm{pla}(v^i)_{i=1}^m\big)$ from the induction hypothesis. ∎

Given a tuple of weights $\alpha = (\alpha_i)_{i=1}^n \subset \mathbb{R}$, the regular complex of P is obtained as follows [10, Sec. 4.3] [23].

Definition 5.3 (Regular complex). *Let $Q \subset \mathbb{R}^{d+1}$ be the convex hull of the n points in $\{(v^1, \alpha_1), \dots, (v^n, \alpha_n)\} \subset \mathbb{R}^{d+1}$. A lower facet of Q is a facet as in* (2.1) *with $\psi(x) = a^T x$, $a \in \mathbb{R}^{d+1}$, where $a_{d+1} < 0$. The regular complex* $\mathrm{rc}(V, \alpha)$ *of P is the smallest complex which contains the projections of all lower facets of Q onto $\mathbb{R}^d \times \{0\}$.*

Let F be a face of P and $V_F = (v^{i_j})_{j=1}^m$ be the sub-tuple of V containing all $v^i \in F$; let $\alpha_F = (\alpha_{i_j})_{j=1}^m$.

Lemma 5.4 ([10, La. 2.3.15]). *The regular triangulation commutes with the restriction to faces. That is,* $\mathrm{rc}(V_F, \alpha_F) = \big(\mathrm{rc}(V, \alpha)\big)|_F$.

Lemma 5.5 ([10, La. 4.3.4]). *All placing triangulations are regular complexes. Moreover, there is a constant $c \geq 1$ (depending on V) such that* $\mathrm{pla}(V) = \mathrm{rc}(V, \alpha)$, *if the weights satisfy*

$$\alpha_1 > 0, \quad \alpha_{i+1} > c\alpha_i, \quad i \in \{1, \dots, n-1\}. \tag{5.2}$$

Lemma 5.6. *The placing triangulation commutes with the restriction to faces. That is,* $\mathrm{pla}(V_F) = \big(\mathrm{pla}(V)\big)|_F$.

Proof. Due to Lemma 5.5, there is a constant c_S and weights $(\alpha_i^S)_{i=1}^n$ satisfying (5.2) such that $\mathrm{pla}(V) = \mathrm{rc}(V, \alpha^S)$. Using Lemma 5.5 again, there is a constant c_F and weights $(\alpha_i^F)_{i=1}^m$ satisfying (5.2) such that $\mathrm{pla}(V_F) = \mathrm{rc}(V_F, \alpha^F)$. Let $c = \max\{c_S, c_F\}$ and $\alpha_1 > 0$, $\alpha_{i+1} > c\alpha_i$, $1 \leq i \leq n-1$. The tuple α satisfies (5.2) with the constant c_S, and the tuple $(\alpha_{i_j})_{j=1}^m$ satisfies (5.2) with the constant c_F. Hence,

$$\mathrm{pla}(V)|_F = \mathrm{rc}(V, \alpha)|_F = \mathrm{rc}\big(V_F, (\alpha_{i_j})_j\big) = \mathrm{pla}(V_F). \qquad ∎$$

5.2. Consistency within the green refinement. In this section, property (3.8) is proved. One must show that both $\mathrm{greennum}(S, e, R_S)|_F$ and $\mathrm{greentri}(S, e, R_S)|_F$ only depend on the data on F, which are the enumeration e_F and the restriction of (the closure of) the refinement pattern to F. There are two key lemmas: Lemma 5.9 relates the lex-order on the reference k-simplex and the lex-order on a k-face of the reference d-simplex. This is important for enumeration of the green refinement. Lemma 5.10 relates the tuple $\hat{w}$ from (3.5) on the k-dimensional and the d-dimensional reference simplex. Together with the invariance properties of the placing triangulation, this makes the green refinement of F independent of its embedding into S.

Let (S, e) be an ordered d-simplex and $F \subseteq S$ be a k-face. Let $R_S \subseteq \mathcal{F}(S)$ be a refinement pattern on S. The following is well-known.

Lemma 5.7. *Barycentric coordinates are invariant under isotonic invertible affine maps. That is, if $A\colon (S, e) \to (T, f)$ is such a map, and if λ are the barycentric* *coordinates of $x \in \operatorname{aff} S$ with respect to (S, e), then λ are the barycentric coordinates of $Ax \in \operatorname{aff} T$ with respect to (T, f).*

LEMMA 5.8. *Let $x, y \in \mathbb{R}^d$. Let $\lambda, \mu \in \mathbb{R}^{d+1}$ be their barycentric coordinates with respect to $(\hat{S}, \mathrm{lex})$. Then, $x \preceq y$ is equivalent to $\lambda \preceq \mu$.*

Proof. The case $x = y$ or $\lambda = \mu$ is trivial. Assume $x \neq y$, $\lambda \neq \mu$. The definition of barycentric coordinates and (3.1), (3.2) imply

$$x_i = 2 \sum_{j=i}^{d} \lambda_j, \quad i \in \{1, \dots, d\}.$$

Elementary manipulations reveal

$$\lambda_0 = \frac{1}{2}(1 - x_1), \quad \lambda_i = \frac{1}{2}(x_i - x_{i+1}) \quad \text{for } i \in \{1, \dots, d-1\}, \quad \lambda_d = \frac{1}{2} x_d.$$

These relations also hold with x, λ replaced by y, μ, respectively.

Assume $x \preceq y$. If there holds $i = d$ in (2.4), then one gets $\lambda_d \leq \mu_d$, which implies $\lambda \preceq \mu$. Otherwise $i \in \{1, \dots, d-1\}$ and $x_k = y_k$ for all $k \in \{i+1, \dots, d\}$. Thus, $\lambda_k = \mu_k$ for all $k \in \{i+1, \dots, d\}$. From $x_i \leq y_i$, one gets

$$\lambda_i = \frac{1}{2}(x_i - x_{i+1}) \leq \frac{1}{2}(y_i - y_{i+1}) = \mu_i,$$

and the inequality is strict if $x_i < y_i$. Therefore, $\lambda \preceq \mu$.

Assume $\lambda \preceq \mu$ (with the caveat that the components of the barycentric coordinates are counted from 0 to d, not from 1 to $d+1$). If $i = d$ in (2.4), then one gets $x_d \leq y_d$, which implies $x \preceq y$. Otherwise, $i \in \{1, \dots, d-1\}$. The case $i = 0$ does not occur because of the assumption $\lambda \neq \mu$. One finds $x_k = y_k$ for all $k \in \{i+1, \dots, d\}$. Hence,

$$x_i = \lambda_i + \sum_{j=i+1}^{d} \lambda_j \leq \mu_i + \sum_{j=i+1}^{d} \mu_j = y_i,$$

and the inequality is strict if $\lambda_i < \mu_i$. Therefore, $x \preceq y$. □

Let $\hat{F}$ be a k-face of $\hat{S}$, let $\hat{S}^k$ be the k-dimensional reference simplex, and let $\hat{A}$ be the isotonic, affine map $(\hat{S}^k, \mathrm{lex}) \to (\hat{S}, \mathrm{lex})$, $\hat{S}^k \to \hat{F}$.

LEMMA 5.9. *For all $x, y \in \mathbb{R}^k$, there holds $x \preceq y \Leftrightarrow \hat{A}x \preceq \hat{A}y$, where the first lex-order is on $\mathbb{R}^k$ and the second is on $\mathbb{R}^d$.*

Proof. Let λ, μ be the barycentric coordinates of x, y with respect to $(\hat{S}^k, \mathrm{lex})$. Due to Lemma 5.7, λ, μ are the barycentric coordinates of $\hat{A}x, \hat{A}y$ with respect to $(\hat{F}, \mathrm{lex})$. Let $\bar{\lambda}, \bar{\mu}$ be the barycentric coordinates of $\hat{A}x, \hat{A}y$ with respect to $(\hat{S}, \mathrm{lex})$. As F is a face of $\hat{S}$, λ is a sub-tuple of $\bar{\lambda}$, and μ is a sub-tuple of $\bar{\mu}$. The mapping of the indices $f \colon \{0, \dots, k\} \to \{0, \dots, d\}$ depends only on k, d, and $\hat{F}$. It is isotonic. The components of $\bar{\lambda}, \bar{\mu}$ which are not in the sub-tuple defined by f are all equal to 0. This implies $\lambda \preceq \mu \Leftrightarrow \bar{\lambda} \preceq \bar{\mu}$. Using Lemma 5.8, this is equivalent to $\hat{A}x \preceq \hat{A}y$ for the lex-order on $\hat{S}$. □

Recall the tuple $\hat{w}$ defined in (3.5). Let $\tilde{w}$ be the same tuple on the k-dimensional reference simplex $\hat{S}^k$ (instead of $\hat{S} = \hat{S}^d$).

LEMMA 5.10. *The tuple $(\hat{A}\tilde{w}^j)_j$ is a sub-tuple of $\hat{w}$.*

Proof. Let $y \in \mathbb{R}^d$ be an element of the tuple $\hat{w}$ in (3.5), and let $\bar{\lambda}$ be its barycentric coordinates with respect to $(\hat{S}, \mathrm{lex})$. The point y is of the form $\hat{v}^i + \hat{v}^{d+1-j}$ with $i \in \{0, \dots, d\}$, $j \in \{1, \dots, d+1-i\}$. If $j = d+1-i$, then y is a vertex of $\hat{S}$.

Otherwise, y is the barycenter of the edge with the vertices $2\hat{v}^i$ and $2\hat{v}^{d+1-j}$. In both cases, the barycentric coordinates of y with respect to $(\hat{S}, \mathrm{lex})$ are given by

$$\bar{\lambda} = \frac{1}{2}(\delta^i + \delta^{d+1-j}) \in \mathbb{R}^{d+1}. \tag{5.3}$$

For the auxiliary tuples $\hat{w}_i$ in (3.5), one has the following characterization:

$$y \in \hat{w}_i \quad \Leftrightarrow$$
$$\Big(\bar{\lambda}_i \in \big\{\tfrac{1}{2}, 1\big\}, \quad \bar{\lambda}_j = 0 \quad \text{for all } 0 \le j < i, \quad \bar{\lambda}_j \in \{0, \tfrac{1}{2}\} \quad \text{for all } i < j \le d\Big). \tag{5.4}$$

The implication "$\Rightarrow$" follows from (5.3) and $j \le d+1-i$. To prove the implication "$\Leftarrow$", two cases are considered. If $\bar{\lambda}_i = 1$, then all other barycentric coordinates are zero. Thus, y is the vertex in $\hat{w}_i$ given by $j = d+1-i$. If $\bar{\lambda}_i = \frac{1}{2}$, then there is precisely one index k, $i < k \le d$, with $\bar{\lambda}_k = \frac{1}{2}$. Writing $k = d+1-l$ with $1 \le l < d+1-i$, one gets $x \in \hat{w}_i$.

Let x be an element of $\tilde{w}$ with the barycentric coordinates λ with respect to $(\hat{S}^k, \mathrm{lex})$. One obtains analogues of (5.3) and (5.4) for x and λ by replacing d with k and using indices $l \in \{0, \dots, k\}$, $m \in \{1, \dots, k+1-l\}$ instead of i, j. Let $y = \hat{A}x$. Due to Lemma 5.7, λ is a sub-tuple of $\bar{\lambda}$. The mapping of the indices f, cf. the proof of Lemma 5.9, depends only on k, d, and $\hat{F}$. Hence, by (5.4), y is in $\hat{w}_i$ with $i = f(l)$. Therefore, the auxiliary tuple $\tilde{w}_l$ is mapped by $\hat{A}$ to an enumeration of a subset of the range of $\hat{w}_i$.

After these preliminaries, consider the points $x = \tilde{w}^\alpha$ and $\xi = \tilde{w}^\beta$ with indices $\alpha < \beta$. If x, ξ are in different auxiliary tuples $\tilde{w}_l$, $\tilde{w}_m$, respectively, then $l < m$. As the mapping f of the indices is strictly isotonic, $\hat{A}x$ is in $\hat{w}_i$, and $\hat{A}\xi$ is in $\hat{w}_j$ for some indices $i < j$. That is, the mapped points occur in the same order as x and ξ. Otherwise, x and ξ are in the same auxiliary tuple $\tilde{w}_l$. Due to the preceding paragraph, $\hat{A}x$ and $\hat{A}\xi$ are in the same tuple $\hat{w}_i$. Observing (3.5), (5.3), and (5.4), one finds that the tuples $\tilde{w}_l$ and $\hat{w}_i$ both enumerate their points from large to small with respect to the lex-order. Hence, $\xi \preceq x$. Using Lemma 5.9, one obtains $\hat{A}\xi \preceq \hat{A}x$. Therefore, $\hat{A}x$ is enumerated before $\hat{A}\xi$ in $\hat{w}_i$. □

Lemma 5.11. *Property* (3.8) *is satisfied.*

Proof. Suppose that

$$A_F \colon \mathbb{R}^k \to \operatorname{aff} F, \quad (\hat{S}^k, \mathrm{lex}) \to (F, e|_{\mathcal{F}(F)}),$$
$$A_S \colon \mathbb{R}^d \to \operatorname{aff} S, \qquad (\hat{S}, \mathrm{lex}) \to (S, e).$$

With $\hat{A}$, $\hat{F}$ as before, there hold $A_S(\hat{F}) = F$ and, due to Lemma 2.2, $A_F = A_S \circ \hat{A}$. To verify property (3.8), one must consider the assertion on the triangulations and the assertion on the enumerations.

Let $\hat{R} \subseteq \mathcal{F}(\hat{S})$ be the refinement pattern on $\hat{S}$ with $A_S(\hat{R}) = R_S$. Then, $\mathrm{greentri}(\hat{S}, \mathrm{lex}, \hat{R})$ is the placing triangulation $\mathrm{pla}(w)$ with the sub-tuple w of $\hat{w}$ given in Definition 3.11. By Lemma 5.6, there holds $\mathrm{greentri}(\hat{S}, \mathrm{lex}, \hat{R})|_{\hat{F}} = \mathrm{pla}\big((w^{i_j})_j\big)$, where $(w^{i_j})_j$ is the sub-tuple of w containing all points in $\hat{F}$. Using Lemma 5.10, this can be written as $(w^{i_j})_j = (\hat{A}\tilde{w}^{i_j})_j$. The tuple $(\tilde{w}^{i_j})_j \subset \mathbb{R}^k$ contains the vertices and edge-barycenters (of $\hat{S}^k$) for the refinement pattern $\hat{R}_F \subseteq \mathcal{F}(\hat{S}^k)$ with $\hat{A}(\hat{R}_F) = \mathrm{cl}\,\hat{R} \cap \mathcal{F}(\hat{F})$. The appearance of $\mathrm{cl}\,\hat{R}$ instead of $\hat{R}$ in the right-hand side is due to the definition of w in Definition 3.11. Owing to Lemma 5.2,

$$\begin{aligned}\operatorname{greentri}(\hat S, \operatorname{lex}, \hat R)|_{\hat F} &= \operatorname{pla}\big((w^{i_j})_j\big) \\ &= \hat A\big(\operatorname{pla}((\tilde w^{i_j})_j)\big) = \operatorname{greentri}(\hat F, \operatorname{lex}, \operatorname{cl}\hat R \cap \mathcal{F}(\hat F)).\end{aligned}$$

Mapping the preceding chain of equations with A_S, one obtains $\operatorname{greentri}(S, e, R_S)|_F = \operatorname{greentri}(F, e|_{\mathcal{F}(F)}, \operatorname{cl} R_S \cap \mathcal{F}(F))$.

The enumeration $\operatorname{greennum}(S, e, R_S)$ is defined as the mapping by A_S of lex on $\operatorname{greentri}(\hat S, \operatorname{lex}, \hat R)$. As $A_F = A_S \circ \hat A$, the proof of (3.8) can be completed by showing the following: For any $x, y \in \mathbb{R}^k$, the statement $x \preceq y$ is equivalent to $\hat A x \preceq \hat A y$. This is precisely the statement of Lemma 5.9. □

Remark 5.12. *The green refinement rules are defined on the reference simplex and then mapped to specific simplices. This is convenient in an implementation to precompute results or take advantage of the fact that only integer arithmetic is required to compute the placing triangulation on $\hat S$. However, the proof of Lemma 5.11 shows that the green refinement of a pattern R_S can be computed as the placing triangulation on S. Moreover, the enumeration of the refinement is given by the lex-order of the barycentric coordinates with respect to (S, e).*

Remark 5.13. *Property* (3.8) *is used to simplify the proof of* (3.9) *and* (3.10). *If the two latter properties hold for $F = S$, then they hold for all $F \in \mathcal{F}(F)$. This is because* (3.8) *implies that the embedding of F into S does not alter the green refinement of F.*

5.3. Consistency between the red and the green refinement. The basic idea of the proof of property (3.9) is to carefully trace the visible parts and the current point of $\hat w$ through the placing triangulation of $\hat S$. For this, $\hat S$ is partitioned into cubes C_i; each cube is partitioned with halfspaces into polytopes $\tilde H_{i,j}$. The visible parts from specific points of $\hat w$ are characterized (Lemmas 5.15 and 5.17). Furthermore, the $\tilde H_{i,j}$ are compatible with the red refinement, cf. Lemma 5.16. These facts are combined in an inductive proof which compares the placing triangulation with the red refinement.

The facet representation of $\hat S$ and S_π, $\pi \in \operatorname{Sym}(d)$, in (3.2) and (3.3) is given by

$$\hat S = \left\{x \in \mathbb{R}^d \;\middle|\; 2 \ge x_1 \ge x_2 \ge \cdots \ge x_d \ge 0\right\}, \tag{5.5}$$

$$S_\pi = \left\{x \in \mathbb{R}^d \;\middle|\; 1 \ge x_{\pi 1} \ge x_{\pi 2} \ge \cdots \ge x_{\pi d} \ge 0\right\}, \tag{5.6}$$

respectively. The vertices and edge-barycenters of the simplex $\hat S$ have integer coordinates, cf. (3.1), (3.2). Such points are called integral points.

The cubes C_i and the hyperplanes H_i are given by

$$\begin{aligned} C_i &= \hat v^i + [0,1]^d, \quad i \in \{0, \ldots, d\}, \\ H_i &= \left\{x \in \mathbb{R}^d \;\middle|\; x_i = 1\right\}, \quad i \in \{1, \ldots, d\}, \end{aligned}$$

cf. Figure 5.1. The common facet $C_{i-1} \cap C_i$ of C_{i-1} and C_i is contained in H_i for $i \in \{1, \ldots, d\}$. Furthermore, H_i separates any C_j from any C_k for $0 \le j < i \le k \le d$. For $i \in \{0, \ldots d\}$, let

$$U_i = \cup\left\{C_j \mid 0 \le j < i\right\}, \quad \hat W_i = \left\{x \in \hat S \cap C_i - U_i \;\middle|\; x \text{ is an integral point.}\right\} \tag{5.7}$$

The following lemma classifies the integral points of $\hat S$.

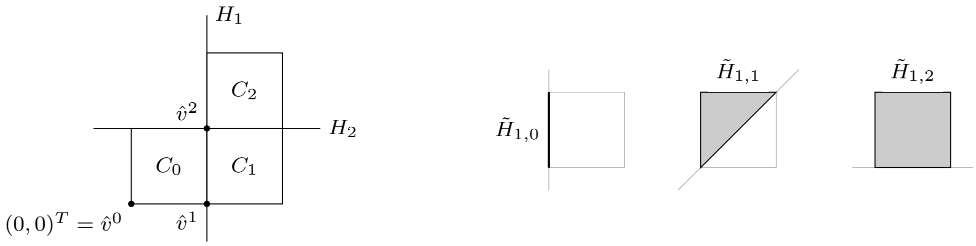

FIG. 5.1. *Left: The cubes C_i and hyper-planes H_i for $d = 2$. Right: The polytopes $\tilde{H}_{1,j} \subseteq C_1$, $j \in \{0, 1, 2\}$, for $d = 2$.*

LEMMA 5.14. *The set $\hat{W}_i$ contains precisely the $d + 1 - i$ points $\hat{v}^i + \hat{v}^{i+j}$, $j \in \{0, \dots, d - i\}$, $i \in \{0, \dots, d\}$.*

Proof. The set U_0 is empty. Thus $\hat{W}_0$ contains precisely the $d + 1$ vertices of S_{id}. Suppose $i \in \{1, \dots, d\}$. The integral points in C_i are of the form $x = \hat{v}^i + (u_j)_{j=1}^d$, where each u_j is in $\{0, 1\}$. All integral points in the intersection of C_i and U_i are in H_i. Hence, the satisfy $x_i = 1$. Thus, the remaining integral points x have $x_i = 2$. As $x \in \hat{S}$, (5.5) implies $x_j = 2$, $1 \le j \le i$. For $j > i$, there holds $\hat{v}^i_j = 0$, which implies $x_j \le 1$. The monotonicity of x_j as function of j, cf. (5.5), implies that there can be at most one transition from 1 to 0 in $(x_j)_{j=i+1}^d$. All integral points satisfying this condition are given by x^j, $j \in \{0, \dots, d - i\}$, with components

$$x^j_k = \begin{cases} 2, & 1 \le k \le i, \\ 1, & i + 1 \le k \le i + j, \\ 0, & i + j + 1 \le k \le d. \end{cases} \qquad \square \tag{5.8}$$

The auxiliary tuple $\hat{w}_i$ in (3.5) enumerates $\hat{W}_i$. Thus, Lemma 5.14 proves that $\hat{w}$ enumerates the vertices and edge-barycenters of $\hat{S}$. For the remainder of Section 5.3, an arbitrary $i \in \{1, \dots, d\}$ is fixed. The family of $d + 2 - i$ halfspaces

$$\begin{gathered} H_{i,0} = \hat{v}^i + \left\{x \in \mathbb{R}^d \;\middle|\; x_i \le 0\right\}, \quad H_{i,d+1-i} = \hat{v}^i + \left\{x \in \mathbb{R}^d \;\middle|\; x_i \le 1\right\}, \\ H_{i,j} = \hat{v}^i + \left\{x \in \mathbb{R}^d \;\middle|\; x_i \le x_{d+1-j}\right\}, \quad 1 \le j \le d - i. \end{gathered} \tag{5.9}$$

is needed. An equivalent representation of these halfspaces is

$$\begin{gathered} H_{i,0} = \left\{x \in \mathbb{R}^d \;\middle|\; x_i \le 1\right\}, \quad H_{i,d+1-i} = \left\{x \in \mathbb{R}^d \;\middle|\; x_i \le 2\right\}, \\ H_{i,j} = \left\{x \in \mathbb{R}^d \;\middle|\; x_i - 1 \le x_{d+1-j}\right\}, \quad 1 \le j \le d - i. \end{gathered} \tag{5.10}$$

For any $H_{i,j}$, let $\tilde{H}_{i,j} = H_{i,j} \cap \hat{S} \cap C_i$, cf. Figure 5.1. Let $\mathcal{C}\tilde{H}_{i,j} = \overline{\tilde{H}_{i,d+1-i} - \tilde{H}_{i,j}}$ be the closure of the complement of $\tilde{H}_{i,j}$ in $\tilde{H}_{i,d+1-i}$.

LEMMA 5.15. *The sets $\tilde{H}_{i,j}$ are an ascending chain with respect to set inclusion; that is, $\tilde{H}_{i,j} \subseteq \tilde{H}_{i,k}$ for all $0 \le j \le k \le d+1-i$. Furthermore, $\tilde{H}_{i,0} = \{x_i = 1\} \cap \hat{S} \cap C_i$ and $\tilde{H}_{i,d+1-i} = \hat{S} \cap C_i$.*

Proof. First consider $k = d + 1 - i$. From the definitions of C_i and $\hat{S}$ and (5.10), one concludes immediately $\tilde{H}_{i,d+1-i} = \hat{S} \cap C_i$. This also yields $\tilde{H}_{i,j} \subseteq \tilde{H}_{i,d+1-i}$ for all $0 \le j \le d + 1 - i$.

Now consider $j = 0$. From the definition of C_i and (5.10), one gets $\tilde{H}_{i,0} = \{x_i = 1\} \cap \hat{S} \cap C_i$. Hence, for any $x \in \tilde{H}_{i,0}$, there holds $x_i - 1 = 0$. Using $x \in C_i$, one gets $x_{d+1-k} \ge 0$ for $1 \le k \le d - i$. One concludes $x \in \tilde{H}_{i,k}$ for all $0 \le k \le d - i$.

It remains to treat the cases $1 \le j \le k \le d-i$. Let $x \in \tilde{H}_{i,j}$. Thus, $x_i - 1 \le x_{d+1-j}$. As $x \in \hat{S}$ and $k \ge j$, one gets from (5.5) that there holds $x_{d+1-k} \ge x_{d+1-j}$. One concludes $x \in \tilde{H}_{i,k}$. ◻

LEMMA 5.16. *For any $j \in \{1, \dots, d+1-i\}$, and any simplex $S = \hat{v}^i + S_\pi$, $\pi \in Sh(i, d-i)$, there holds: If $\pi^{-1} i \ge \pi^{-1}(d+1-j)$, then $S \subseteq \tilde{H}_{i,j}$, otherwise $S \subseteq \mathcal{C}\tilde{H}_{i,j}$.*

Proof. First, consider the case $i \in \{1, \dots, d-1\}$, $1 \le j \le d-i$. One has $i < d+1-j$. Hence, for any shuffle $\pi \in \mathrm{Sh}(i, d-i)$, there holds either $\pi^{-1} i > \pi^{-1}(d+1-j)$ or $\pi^{-1} i < \pi^{-1}(d+1-j)$. Consider the case $\pi^{-1} i > \pi^{-1}(d+1-j)$. Observing that $\pi^{-1} i$ is the position at which i occurs in the tuple $(\pi 1, \pi 2, \dots, \pi d)$ and using (5.5), one obtains $S_\pi \subseteq \{x \in \mathbb{R}^d \mid x_i \le x_{d+1-j}\}$. By (5.9), there holds $S = \hat{v}^i + S_\pi \subseteq H_{i,j}$. From Definition 3.1, one infers $S \subseteq \hat{S} \cap C_i$.

Now, consider $\pi^{-1} i < \pi^{-1}(d+1-j)$. One gets $S \subseteq \hat{v}^i + \{x \in \mathbb{R}^d \mid x_i \ge x_{d+1-j}\}$. This is the closure of $\mathbb{R}^d - H_{i,j}$. As $S \subseteq \hat{S} \cap C_i$, one obtains $S \subseteq \mathcal{C}\tilde{H}_{i,j}$.

The remaining cases are $i \in \{1, \dots, d\}$, $j = d+1-i$, which correspond to $\pi^{-1} i = \pi^{-1}(d+1-j)$. Note that for these choices of i, j one has $\tilde{H}_{i,j} = \hat{S} \cap C_i$. Hence, $\hat{v}^i + S_\pi \subseteq \tilde{H}_{i,j}$ for all $\pi \in \mathrm{Sh}(i, d-i)$ by (3.4). ◻

LEMMA 5.17. *The point $\hat{v}^i + \hat{v}^{d-j}$ is the only integral point in $\tilde{H}_{i,j+1} - \tilde{H}_{i,j}$, $0 \le j \le d-i$.*

Proof. Due to Lemma 5.14, one only has to consider the points in $\hat{W}_i$. Hence, one proves that $\hat{v}^i + \hat{v}^{d-j}$ is the only point in $\hat{W}_i \cap (H_{i,j+1} - H_{i,j})$, $0 \le j \le d-i$. Let $x = \hat{v}^{d-j}$. From $0 \le j \le d-i$, one gets $i \le d-j$, and with (3.1) this implies $x_i = 1$. Let $1 \le k \le d+1-i$. Using (5.5), (5.9), and $\hat{W}_i \subset H_{i,d+1-i}$, one concludes that $\hat{v}^i + x \in H_{i,k}$ holds if and only if

$$(x_{d+1-k} = 1, \quad 1 \le k \le d-i) \quad \text{or} \quad k = d+1-i.$$

This is equivalent to $d+1-k \le d-j$ for $1 \le k \le d-i$ or $k = d+1-i$. Rearranging these conditions yields that $\hat{v}^i + x \in H_{i,k}$ if and only if $j+1 \le k \le d+1-i$. This proves the assertion. ◻

Let $\mathcal{T}_{i,0}$ be the ($(d-1)$-dimensional) subtriangulation of the regular refinement $\mathrm{redtri}(\hat{S}, \mathrm{lex})$ with $\mathrm{set}\,\mathcal{T}_{i,0} = \hat{S} \cap \{x_i = 1\}$. Furthermore, let

$$\mathcal{T}_{i,j} = \mathrm{cl}\left\{S \in \mathrm{redtri}(\hat{S}, \mathrm{lex}) \;\middle|\; S \subseteq \tilde{H}_{i,j}\right\}, \quad 1 \le j \le d+1-i. \tag{5.11}$$

Note that $\mathcal{T}_{i,d+1-i}$ is the restriction of $\mathrm{redtri}(\hat{S}, \mathrm{lex})$ to C_i.

LEMMA 5.18. *For $0 \le j \le d+1-i$ there holds* $\mathrm{set}\,\mathcal{T}_{i,j} = \tilde{H}_{i,j}$.

Proof. The two cases $j \in \{0, d+1-i\}$ follow immediately from (3.4), Definition 3.1, and the definition of $\tilde{H}_{i,j}$. Let $1 \le j \le d-i$. Due to (5.11), there holds $\mathrm{set}\,\mathcal{T}_{i,j} \subseteq \tilde{H}_{i,j}$.

Let $x \in \tilde{H}^\circ_{i,j}$ be an arbitrary point from the interior of $\tilde{H}_{i,j}$. Using $\mathrm{set}\,\mathcal{T}_{i,d+1-i} = \hat{S} \cap C_i$, one finds a d-simplex $S \in \mathcal{T}_{i,d+1-i}$ that contains x. As S meets $\tilde{H}^\circ_{i,j}$, Lemma 5.16 implies $S \subseteq \tilde{H}_{i,j}$. Therefore, $\tilde{H}^\circ_{i,j} \subseteq \mathrm{set}\,\mathcal{T}_{i,j}$; taking the closure completes the proof. ◻

LEMMA 5.19. *Placing the vertices in $\hat{w}_i$ from* (3.5) *successively in $\mathcal{T}_{i,0}$, yields the triangulation $\mathcal{T}_{i,d+1-i}$.*

Proof. Let $\mathcal{S}_0 = \mathcal{T}_{i,0}$, and let $\mathcal{S}_j$ be the triangulation obtained from placing $\hat{v}^i + \hat{v}^{d+1-j}$ in $\mathcal{S}_{j-1}$, $1 \le j \le d+1-i$. One shows $\mathcal{T}_{i,j} = \mathcal{S}_j$ by induction on $0 \le j \le d+1-i$. The case $j = 0$ holds by definition.

Assume that $\mathcal{T}_{i,j} = \mathcal{S}_j$ for some $1 < j \le d - i$. One shows that this implies $\mathcal{T}_{i,j+1} = \mathcal{S}_{j+1}$. Let $v = \hat{v}^i + \hat{v}^{d+1-(j+1)}$ be the vertex to be placed in $\mathcal{S}_j$ to obtain $\mathcal{S}_{j+1}$. From (5.7), one gets $v \in \partial\hat{S} \cap \partial C_i$. As $\partial\tilde{H}_{i,j} \subseteq \partial\hat{S} \cup \partial C_i \cup \partial H_{i,j}$, the facet $F = (\partial H_{i,j}) \cap \hat{S} \cap C_i$ is the only facet of $\tilde{H}_{i,j}$ which is visible from v.

By the induction hypothesis, $\mathcal{T}_{i,j}$ and $\mathcal{S}_j$ induce the same $(d-1)$-dimensional triangulation $\mathcal{T}_F$ on F, $\mathcal{T}_F = \{S \in \mathcal{T}_{i,j} \mid S \subseteq F\}$. Let $P = \overline{\tilde{H}_{i,j+1} - \tilde{H}_{i,j}}$. By Lemma 5.18, there is a subtriangulation $\tilde{\mathcal{T}}$ of $\mathcal{T}_{i,j+1}$ with set $\tilde{\mathcal{T}} = P$. There holds $\mathcal{T}_F \subset \tilde{\mathcal{T}}$. All d-simplices of $\tilde{\mathcal{T}}$ include at least one vertex which is not in F. From Lemma 5.17 one concludes that this vertex is v. Hence, $\tilde{\mathcal{T}} = \{\mathrm{pyr}(v, S) \mid S \in \mathcal{T}_F\}$, and P is the pyramid $\mathrm{pyr}(v, F)$. Using $\tilde{H}_{i,j+1} = \tilde{H}_{i,j} \cup P$, one gets $\mathcal{T}_{i,j+1} = \mathcal{T}_{i,j} \cup \{\mathrm{pyr}(v, S) \mid S \in \mathcal{T}_{i,j}$ is visible from $v.\}$. Thus, $\mathcal{T}_{i,j+1}$ is obtained by placing v in $\mathcal{T}_{i,j}$. Invoking Lemma 5.1 and the induction hypothesis, one gets $\mathcal{T}_{i,j+1} = \mathcal{S}_{j+1}$. ◻

Theorem 5.20. *There holds* $\mathrm{pla}(\hat{w}) = \mathrm{redtri}(\hat{S}, \mathrm{lex})$.

Proof. Placing the $d+1$ points of $\hat{w}_0$ yields the face lattice of $\hat{v}^0 + S_{\mathrm{id}} = S_{\mathrm{id}}$. The restriction of this lattice to H_1 is $\mathcal{T}_{1,0}$. Lemma 5.19 is applied in an induction argument; in each step $i \in \{1, \dots, d\}$, the restriction of $\mathrm{redtri}(\hat{S}, \mathrm{lex})$ to C_i is added to the tentative triangulation. This proves the assertion provided that the only facet of $H_{i,0} \cap \hat{S}$, $i \in \{1, \dots, d\}$, which is visible from the points in $\hat{w}_i$ is the one which is triangulated by $\mathcal{T}_{i,0}$. There holds

$$\partial(H_{i,0} \cap \hat{S}) = (H_i \cap \hat{S}) \cup (H_{i,0} \cap \partial\hat{S})$$

Let $x \in \hat{W}_i$. Using $x \in \hat{S}$ and Definition 3.7, one finds that no part of $H_{i,0} \cap \partial\hat{S}$ is visible from x. As $x_i = 2$, the facet $H_i \cap \hat{S}$ is visible from x. Using Lemma 5.19 inductively, one obtains $\mathrm{pla}(\hat{w}) = \mathrm{redtri}(\hat{S}, \mathrm{lex})$. ◻

Corollary 5.21. *Property* (3.9) *is satisfied.*

Proof. Taking into account Remark 5.13, one must only consider $F = S$ in (3.9). Let (S, e) be a consistently numbered d-simplex. As $S \in \mathrm{cl}\, R_S$ all vertices and edge-barycenters of S are used, that is $w = \hat{w}$ in (3.6). From Theorem 5.20, one obtains $\mathrm{greentri}(\hat{S}, \mathrm{lex}, \hat{R}) = \mathrm{redtri}(\hat{S}, \mathrm{lex})$. Both the green refinement and the red refinement of S are defined by mapping the refinements of $\hat{S}$ to S. Thus, $\mathrm{greentri}(S, e, R) = \mathrm{redtri}(S, e)$.

The enumeration $\mathrm{greennum}(S, e, R_S)$ is the mapping of the lex-enumeration of $\hat{w}$ on $\hat{S}$. Due to Lemma 3.2, this agrees with the enumeration $\mathrm{rednum}(S, e)$. ◻

5.4. Consistency between the green refinement and unrefined simplices.

Lemma 5.22. *Property* (3.10) *is satisfied.*

Proof. Thanks to Remark 5.13, one only has to consider the case $F = S \in U_S$ in the proof of (3.10). The condition $S \in U_S$ implies that no edge-barycenters appear in the refinement pattern of S. Hence, the sub-tuple w of $\hat{w}$ in (3.5) consists of the vertices of $\hat{S}$, and the placing triangulation in (3.6) is $\mathcal{F}(\hat{S})$. Finally, the enumeration lex on $\hat{S}$ is mapped to the enumeration e given on S due to Definition 3.11. ◻

6. Numerical experiment. The validity of the properties (3.8) and (3.9) is confirmed with a computer program in several dimensions d. To compute the green refinement of $\hat{S}$ for a given refinement pattern, first the tuple of points w in (3.6) is generated. The placing triangulation is generated with the software tool box `polymake` [14] for convex geometry.

Using this tool, the following test is performed: Suppose a refinement pattern $\hat{R}$ of $\hat{S}$, a facet $\hat{F}$ of $\hat{S}$, and the restriction $\hat{R}_F$ of $\hat{R}$ to $\hat{F}$ is given. The green refinement greentri$(\hat{F}, \text{lex}, \hat{R}_F)$ is computed in two ways. First, directly from Definition 3.11 as a $(d-1)$-dimensional green refinement of $\hat{F}$. Second, as the restriction greentri$(\hat{S}, \text{lex}, \hat{R})|_{\hat{F}}$. These triangulations of $\hat{F}$ are compared for equality. The test is performed for all refinement patterns and all facets of $\hat{S}$ for $d \in \{1, 2, 3, 4, 5\}$. Property (3.8) is satisfied in all cases.

To verify Theorem 5.20 and property (3.9), the red refinement of $\hat{S}$ is generated from the definition (3.4). This is compared to pla$(\hat{w})$ for $d \in \{1, 2, \ldots, 20\}$. The triangulations are equal in all cases.

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