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Feedback boundary control of linear hyperbolic equations with stiff source term

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Abstract

We consider feedback boundary control of hyperbolic systems with stiff source terms. By combining weighted Lyapunov functions, the structure is used to derive novel stabilization results. The result is illustrated with the numerical analysis on the decay rate of the Lyapunov function in terms of the stiff parameter and an application to boundary stabilization of gas dynamics in pipes.

Key words: Stabilization, feedback boundary control, hyperbolic system, stiff source term.

1 Introduction

We are interested in boundary stabilization problems for linear hyperbolic partial differential equations (PDEs) with source terms. Our focus is on stiff source terms and the influence of the stiffness on the design of dissipative feedback laws.

Due to the possible applications in engineering, the control of hyperbolic PDEs has gained interest in the mathematical community. The design of suitable boundary feedback controls has been investigated in great detail in the case of St. Venant equations [10, 21, 9, 2, 11, 3, 23, 19, 6], gas dynamics [15], traffic flows [1] and supply chains [7]. We are interested in problems with linear dynamics and involving source terms similar to e.g. [17, 18]. A general stabilization result is given in [5, Theorem 13.12] or [22] using a smallness assumption on the source terms.

We are concerned with a class of linear hyperbolic PDEs appearing as mathematical models between Boltzmann equation and hyperbolic conservation laws and having a particular relaxation structure [28, 25, 26, 27]. The present structure has been used to prove exponential stability in [17] using Lyapunov functions similar to [5, 4]. As extension to the previous analysis on those models presented in [17] we consider now the case of stiff source terms and their asymptotic expansion. Further, numerical results are presented here. This problem has recently gained interest due to design question of numerical schemes resolving the stiff limit accurately. Among the many publications on this topic we refer to [20, 12, 13, 8] for details and examples. To the best of our knowledge the question of feedback boundary for stiff source terms has not yet been studied.

We consider the spatially one–dimensional linear system with a stiff source term motivated by [17, 27, 10]

$$\begin{cases} u_t + au_x + bq_x = 0, & \text{(1.1a)} \\ q_t + cu_x + dq_x = -\dfrac{e}{\varepsilon}q & \text{(1.1b)} \end{cases}$$

for $x \in [0,1]$, and $t \geq 0$. Here $u \in \mathbb{R}$ and $q \in \mathbb{R}$ are functions of (t,x), and ε is a small positive parameter. In the following we denote by $sgn(a)$ the sign function of a. Introducing the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $U = \begin{pmatrix} u \\ q \end{pmatrix}$ and $Q(U) = \begin{pmatrix} 0 \\ -eq \end{pmatrix}$, the system in the matrix form is

$$U_t + AU_x = \frac{Q(U)}{\varepsilon}. \tag{1.2}$$

Note that contrary to [10] the system is not in characteristic form. As investigated in [27] the following assumptions are fulfilled by many models from mathematical physics; see also [25] for a detailed discussion on the stability under such conditions. As in [17] we assume that

(A1) There exists a symmetric positive definite matrix $A_0 \in \mathbb{R}^{2\times 2}$ such that A_0A is symmetric and $A_0 = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$, where $X_1, X_2 \geq 0$;

(A2) $e > 0$.

Assumption (A1) implies the hyperbolicity of the system. It follows that there exists a transformation matrix $T = (t_{ij}) \in \mathbb{R}^{2\times 2}$ such that

$$T^{-1}AT = \Lambda = \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix},$$

where $\Lambda_\pm$ are diagonal matrices and contain the m positive and $2-m$ negative eigenvalues of A respectively. Note that we assume there is no vanishing eigenvalues. One can show that T^TA_0T is diagonal and positive definite. We consider the boundary stabilization of the system using the Lyapunov function, starting with the definition of a symmetric matrix $\mu(x) = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$:

$$\mu(x) := T^{-T}\exp(-\Lambda x)T^{-1}$$

for $x \in [0,1]$. Then, following [17], the function $L^\varepsilon(t)$ is a Lyapunov function of system (1.1) for $\alpha > 0$:

$$L^\varepsilon(t) := \int_0^1 U^T\big(\alpha A_0 + \mu(x)\big)U\,dx = \alpha\|(u,q)\|^2_{A_0} + \|(u,q)\|^2_\mu. \tag{1.3}$$

Boundary conditions have been proposed in [17, Theorem 3.1] to ensure the exponential decay of $L^\varepsilon(t)$ in time. Here, we are interested in the asymptotic behavior of the Lyapunov function as ε tends to zero. We decompose the solutions (u,q) as

$$u = u^0 + \varepsilon u^1, \qquad q = q^0 + \varepsilon q^1, \tag{1.4}$$

where (u^1, q^1) are the first order perturbation with respect to ε. Plugging the expansion in the original system and using $q^0 = 0$ and $q^1 = -e^{-1}cu^0_x$ we formally obtain up to the zeroth and first order, respectively:

$$\begin{cases} u^0_t + au^0_x = 0, & \text{(1.5a)} \\ q^0 = 0, & \text{(1.5b)} \end{cases}$$

and

$$\begin{cases} u_t^1 + au_x^1 - bce^{-1}u_{xx}^0 = 0, & (1.6a) \\ q_t^1 + cu_x^1 + dq_x^1 = 0. & (1.6b) \end{cases}$$

The problem (1.6) is well-posed if $bce^{-1} > 0$.

For $u^0(t,x)$ solving equation (1.5) we have Lemma (1.1) following directly by [5, Theorem 13.12]. This ensures the exponential stabilization of the limiting equation.

Lemma 1.1. *There exist positive constants* (β_0, β_1) *such that the Lyapunov functions* $\tilde{L}(t)$ *defined as*

$$\tilde{L}(t) = \beta_1 \int_0^1 e^{-sgn(a)\beta_0 x}(u^0)^2\,dx \tag{1.7}$$

decays with exponential rate $|a|\beta_0$ *provided that*

$$BC_5 = -a\beta_1 e^{-sgn(a)\beta_0 x}\big(u^0\big)^2\big|_0^1 \leq 0. \tag{1.8}$$

We denote by $\xi\big|_0^1 = \xi(t,1) - \xi(t,0)$ *for brevity.*

As to the Lyapunov function (1.3), we decompose the Lyapunov function in terms of ε:

$$\begin{aligned} L^\varepsilon(t) = &\int_0^1 (\alpha X_1 + \mu_{11})\big(u^0\big)^2\,dx + \ 2\varepsilon\left[\int_0^1 (\alpha X_1 + \mu_{11})u^0 u^1\,dx - \frac{c}{e}\int_0^1 \mu_{12}u^0 u_x^0\,dx\right] \\ :=\ & L^0(t) + \varepsilon L^1(t). \end{aligned}$$

Theorem 1.2. *The time derivative of* $L^\varepsilon(t)$ *satisfies the inequality:*

$$\frac{d}{dt}(L^0(t) + \varepsilon(L^1(t) + \tilde{L}(t))) \leq -\lambda_1(L^0 + \varepsilon(L^1(t) + \tilde{L}(t))) + BC_1 + \varepsilon(BC_2 + BC_3 + BC_4 + BC_5), \tag{1.9}$$

where

$$\lambda_1 := -\max_{x\in[0,1]} \frac{a\partial_x \mu_{11}}{\alpha X_1 + \mu_{11}} > 0. \tag{1.10}$$

The terms BC_i *for* $i = 1, \ldots, 5$ *are boundary terms defined by equations* (1.11), (1.12), (1.13) *and* (1.8).

Proof. Compute the time derivative of the Lyapunov function:

$$\frac{d}{dt}L^\varepsilon(t) = \frac{d}{dt}L^0(t) + \varepsilon\frac{d}{dt}L^1(t).$$

The zeroth order Lyapunov function $L^0(t)$ gives

$$\frac{d}{dt}L^0(t) = BC_1 + \int_0^1 a\partial_x\mu_{11}\big(u^0\big)^2\,dx,$$

where

$$BC_1 = -a(\alpha X_1 + \mu_{11})\big(u^0\big)^2\big|_0^1. \tag{1.11}$$

By definition we have that $\mu(x)$ is componentwise differentiable and denote the derivative by $\mu_x(x)$. Since A is hyperbolic, we have

$$\mu_x(x)A = T^{-T}DT^{-1}$$

where D is a diagonal matrix with entries $D_{ii} = -\Lambda_{ii}^2 \exp(-\Lambda_{ii}x) < 0$. Therefore, $\mu_x(x)A$ is negative definite. Hence, $(1,0)\mu_x(x)A(1,0)^T = a\partial_x\mu_{11} < 0$.

The first order Lyapunov function $L^1(t)$ yields

$$\begin{aligned}\frac{d}{dt}L^1(t) =& BC_2 + BC_3 - \frac{2bc}{e}\int_0^1 (\alpha X_1 + \mu_{11})\left(u_x^0\right)^2 dx \\ &+ 2a\int_0^1 \partial_x\mu_{11}u^0u^1\, dx - \frac{2c}{e}\int_0^1 (a\partial_x\mu_{12} + b\partial_x\mu_{11})u^0u_x^0\, dx,\end{aligned}$$

where

$$BC_2 = -2a(\alpha X_1 + \mu_{11})u^0u^1\big|_0^1, \quad BC_3 = \frac{2c}{e}\left[a\mu_{12} + b(\alpha X_1 + \mu_{11})\right]u^0u_x^0\big|_0^1. \tag{1.12}$$

Note that $\alpha X_1 + \mu_{11} > 0$. Therefore, $\frac{a\partial_x\mu_{11}}{\alpha X_1+\mu_{11}}$ is well-defined as well as λ_1 in equation (1.10). It follows that

$$\begin{aligned}\frac{d}{dt}L^\varepsilon(t) \le& -\lambda_1 L^\varepsilon(t) + BC_1 + \varepsilon(BC_2 + BC_3) \\ &- \varepsilon\left[\frac{2c}{e}\int_0^1 (a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})u^0u_x^0\, dx + \frac{2bc}{e}\int_0^1 (\alpha X_1 + \mu_{11})\left(u_x^0\right)^2 dx\right].\end{aligned}$$

Since

$$\begin{aligned}&2\int_0^1 (a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})u^0u_x^0\, dx \\ =& \int_0^1 (a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})\left[(u^0)^2\right]_x dx \\ =& (a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})\left(u^0\right)^2\big|_0^1 - \int_0^1 (a\partial_x^2\mu_{12} + b\partial_x^2\mu_{11} + \lambda_1\partial_x\mu_{12})\left(u^0\right)^2 dx,\end{aligned}$$

and $\frac{bc}{e}(\alpha X_1 + \mu_{11}) \ge 0$, we obtain

$$\frac{d}{dt}L^\varepsilon(t) \le -\lambda_1 L^\varepsilon(t) + BC_1 + \varepsilon(BC_2 + BC_3 + BC_4) + \varepsilon\lambda_2\|u^0\|_{L^2}^2,$$

where

$$BC_4 = -\frac{c}{e}(a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})\left(u^0\right)^2\big|_0^1, \tag{1.13}$$

and $\lambda_2 = \max_{x\in[0,1]}\left|\frac{c}{e}\left(a\partial_x^2\mu_{12} + b\partial_x^2\mu_{11} + \lambda_1\partial_x\mu_{12}\right)\right|$. In order to achieve the exponential decay of $L^\varepsilon(t)$, the right-hand side of the inequality (1.9) must be bounded by $-\lambda_1 L^\varepsilon(t)$. Next, we will discuss sufficient conditions. The L^2 norm of u^0 can be bounded by introducing

$$\lambda_3 = \beta_1 \min_{x\in[0,1]} e^{-sgn(a)\beta_0 x}. \tag{1.14}$$

Using Lemma 1.1, we have

$$\tilde{L}(t) \ge \lambda_3\|u^0\|_{L^2}^2.$$

Consider the time derivative of $L^\varepsilon + \varepsilon\tilde{L}$. Combining the result of (1.9) with the assertion of Lemma 1.1,

$$\begin{aligned}
&\frac{d}{dt}(L^\varepsilon(t) + \varepsilon\tilde{L}(t)) \\
\leq &- \lambda_1 L^\varepsilon(t) + BC_1 + \varepsilon(BC_2 + BC_3 + BC_4 + BC_5) + \varepsilon\lambda_2\|u^0\|^2_{L^2} - \varepsilon|a|\beta_0\lambda_3\|u^0\|^2_{L^2} \\
\leq &- \lambda_1(L^\varepsilon(t) + \varepsilon\tilde{L}(t)) + BC_1 + \varepsilon(BC_2 + BC_3 + BC_4 + BC_5) - \varepsilon\left(\gamma|a|\beta_0\lambda_3 - \lambda_2\right)\|u^0\|^2_{L^2}.
\end{aligned}$$

Here, β_0 and β_1 are chosen large enough so that

$$(1-\gamma)|a|\beta_0 \geq \lambda_1 \quad \text{and} \quad \gamma|a|\beta_0\lambda_3 - \lambda_2 \geq 0 \quad \forall\gamma \in (0,1).$$

∎

2 The boundary conditions

In order to obtain exponential decay of the Lyapunov function L^ε with rate λ_1, we need to ensure that the following condition holds:

$$\begin{cases} BC_1 + \varepsilon(BC_2 + BC_3 + BC_4 + BC_5) \leq 0, & \text{(2.1a)} \\ BC_5 \leq 0. & \text{(2.1b)} \end{cases}$$

Sufficient condition for equation (2.1) are formulated below. As in [17] boundary conditions (2.2) are prescribed. Let

$$\begin{pmatrix} u(t,0) \\ q(t,0) \end{pmatrix} = G \begin{pmatrix} u(t,1) \\ q(t,1) \end{pmatrix}, \begin{pmatrix} u_x(t,0) \\ q_x(t,0) \end{pmatrix} = \tilde{G} \begin{pmatrix} u_x(t,1) \\ q_x(t,1) \end{pmatrix}, \tag{2.2}$$

where

$$G = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}, \ \tilde{G} = \begin{pmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{pmatrix}.$$

In the limit $\varepsilon \to 0$, consistent boundary conditions are obtained provided that

$$G_{22} = \tilde{G}_{11}.$$

The boundary conditions for the asymptotic expansion then read

$$\begin{aligned}
u^0(t,0) &= G_{11}u^0(t,1), \quad u^1(t,0) = G_{11}u^1(t,1), \quad q^1(t,0) = G_{22}q^1(t,1), \\
u^0_x(t,0) &= G_{22}u^0_x(t,1), \quad u^1_x(t,0) = G_{22}u^1_x(t,1), \quad q^1_x(t,0) = \tilde{G}_{22}q^1_x(t,1).
\end{aligned}$$

For the matrix A the eigenvalues are ordered such that $\Lambda_1 > \Lambda_2$. Then, a and d always fulfill

$$\Lambda_1 > a, d > \Lambda_2.$$

Depending on the sign of a we obtain different sufficient conditions such that (2.1) holds true provided that non–negative initial data is prescribed.

2.1 Sufficient boundary conditions for $a > 0$

Let $a > 0$. Then, there exists at least one positive eigenvalue Λ_1.

Proposition 2.1. *Assume $a > 0$. Then, the boundary terms $BC_i, i = 1, \ldots, 5$ fulfill the inequalities* (2.1) *provided that*

$$\begin{cases} t_{11}^2 \text{ is sufficiently small enough,} \\ G_{11}^2 = \frac{aX_1+\mu_{11}(1)}{aX_1+\mu_{11}(0)},\ G_{22} = \frac{a\mu_{12}(1)+b\alpha X_1+b\mu_{11}(1)}{a\mu_{12}(0)+b\alpha X_1+b\mu_{11}(0)}, \\ e^{-\beta_0} \geq G_{11}^2, \end{cases} \tag{2.3}$$

where t_{11} is the component of the transformation matrix T.

Proof. Note that

$$\begin{aligned} BC_1 + \varepsilon BC_2 &= -a(\alpha X_1 + \mu_{11})\big[\big(u^0\big)^2 + 2\varepsilon u^0 u^1\big]\big|_0^1, \\ &= -a(\alpha X_1 + \mu_{11})\big[\big(u^0 + \varepsilon u^1\big)^2 - \big(\varepsilon u^1\big)^2\big]\big|_0^1, \\ &= -a(\alpha X_1 + \mu_{11}(1) - (\alpha X_1 + \mu_{11}(0))G_{11}^2)\left(\big(u^0\big)^2 + 2\varepsilon u^0 u^1\right)(t, 1), \end{aligned}$$

and

$$\begin{aligned} BC_3 &= \frac{2c}{e}(a\mu_{12} + b\alpha X_1 + b\mu_{11})u^0 u_x^0\big|_0^1 \\ &= \frac{2c}{e}[a\mu_{12}(1) + b\alpha X_1 + b\mu_{11}(1) - (\mu_{12}(0) + b\alpha X_1 + b\mu_{11}(0))G_{11}G_{22}]u^0(t,1)u_x^0(t,1). \end{aligned}$$

One can verify that the choices of G_{11}^2 and G_{22} lead to

$$BC_1 = 0, \quad BC_2 = 0, \quad BC_3 = 0.$$

Recall the term $BC_5 = -a\beta_1 \exp(-sgn(a)\beta_0 x)(u^0)^2|_0^1$. $BC_5 \leq 0$ yields as sufficient condition

$$e^{-\beta_0} - G_{11}^2 \geq 0.$$

The last boundary condition is BC_4:

$$\begin{aligned} BC_4 = &- \frac{c}{e}(a\partial_x\mu_{12} + b\partial_x\mu_{11} + \lambda_1\mu_{12})\big(u^0\big)^2\Big|_0^1 \\ = &- \frac{c}{e}\big[a\partial_x\mu_{12}(1) + b\partial_x\mu_{11}(1) + \lambda_1\mu_{12}(1) \\ &- G_{11}^2\big(a\partial_x\mu_{12}(0) + b\partial_x\mu_{11}(0) + \lambda_1\mu_{12}(0)\big)\big]\big(u^0(t,1)\big)^2. \end{aligned}$$

The sufficient conditions for $BC_4 \leq 0$ are

$$\begin{aligned} \partial_x\mu_{11}(1) - G_{11}^2\partial_x\mu_{11}(0) &\geq 0, \\ c\big[a\partial_x\mu_{12}(1) + \lambda_1\mu_{12}(1) - G_{11}^2\big(a\partial_x\mu_{12}(0) + \lambda_1\mu_{12}(0)\big)\big] &\geq 0. \end{aligned}$$

Since $\partial_x\mu_{11}(x) < 0$ for $a > 0$, the first inequality above is equivalent to

$$G_{11}^2 \geq \frac{\partial_x\mu_{11}(1)}{\partial_x\mu_{11}(0)}.$$

Because

$$c\big[a\partial_x\mu_{12}(1)+\lambda_1\mu_{12}(1)-G_{11}^2\big(a\partial_x\mu_{12}(0)+\lambda_1\mu_{12}(0)\big)\big]$$
$$=\frac{\Lambda_1-a}{(t_{11}t_{22}-t_{12}t_{21})^2}\left[t_{22}^2(\lambda_1-a\Lambda_1)(e^{-\Lambda_1}-G_{11}^2)-\frac{c}{b}t_{11}^2(\lambda_1-a\Lambda_2)(e^{-\Lambda_2}-G_{11}^2)\right],$$

and $\lambda_1-a\Lambda_1<0$, then $G_{11}^2\geq e^{-\Lambda_1}$ and t_{11}^2 being small enough will ensure the positivity of this term. Because

$$\frac{\mu_{11}(1)}{\mu_{11}(0)}=e^{-\Lambda_1}+\frac{(\Lambda_1-a)^2t_{11}^2(e^{-\Lambda_2}-e^{-\Lambda_1})}{b^2t_{22}^2+(\Lambda_1-a)^2t_{11}^2}\geq e^{-\Lambda_1},$$

and

$$\frac{\mu_{11}(1)}{\mu_{11}(0)}-\frac{\partial_x\mu_{11}(1)}{\partial_x\mu_{11}(0)}=\frac{(\Lambda_1-a)^2b^2t_{11}^2t_{22}^2(\Lambda_1-\Lambda_2)(e^{-\Lambda_2}-e^{-\Lambda_1})}{(b^2t_{22}^2+(\Lambda_1-a)^2t_{11}^2)(b^2t_{22}^2\Lambda_1+(\Lambda_1-a)^2t_{11}^2\Lambda_2)}\geq 0$$

with t_{11}^2 being small enough, we only need to show that $G_{11}^2\geq\frac{\mu_{11}(1)}{\mu_{11}(0)}$. Thanks to $\mu_{11}(0)>\mu_{11}(1)>0$, the calculation below

$$G_{11}^2-\frac{\mu_{11}(1)}{\mu_{11}(0)}=\frac{\alpha X_1(\mu_{11}(0)-\mu_{11}(1))}{\mu_{11}(0)(\alpha X_1+\mu_{11}(0))}>0$$

completes the proof. ∎

2.2 Sufficient boundary conditions for $a<0$

Let $a<0$. Then, there exists at least one negative eigenvalue Λ_2. We omit the proof and state the conclusion only due to the close similarity.

Proposition 2.2. *Assume $a<0$. Then, the boundary terms $BC_i, i=1,\dots,5$ fulfill the inequalities* (2.1) *provided that*

$$\begin{cases} t_{22}^2 \text{ and } X_1 \text{ are sufficiently small,}\\ G_{11}^2=\frac{aX_1+\mu_{11}(1)}{aX_1+\mu_{11}(0)},\ G_{22}=\frac{a\mu_{12}(1)+b\alpha X_1+b\mu_{11}(1)}{a\mu_{12}(0)+b\alpha X_1+b\mu_{11}(0)},\\ e^{\beta_0}\leq G_{11}^2. \end{cases}\tag{2.4}$$

Some remarks are in order.

Remark 2.1. *The conditions proposed are independent of the value ε and solutions, and therefore we expect uniform exponential decay in the numerical results.*

Moreover, the proof of Proposition 2.1 shows the sufficiency of the boundary conditions. The decay property of the Lyapunov function are observed numerically even in the case when (2.3) *or* (2.4) *are not fulfilled exactly. This shows that the estimates in the proof for the decay of the Lyapunov function are rather pessimistic.*

3 Numerical results

For the numerical simulations, we examine the behavior of the system as the positive perturbation parameter ε approaches to zero and the mesh for the spatial discretization is refined respectively. We employ the IMEX-SSP2(3,2,2) stiffly accurate scheme proposed in [24] for the system (1.1) and an Upwind scheme for zeroth-order system (1.5). An IMEX method is used in order to avoid stiff integrations due to the smallness of ε.

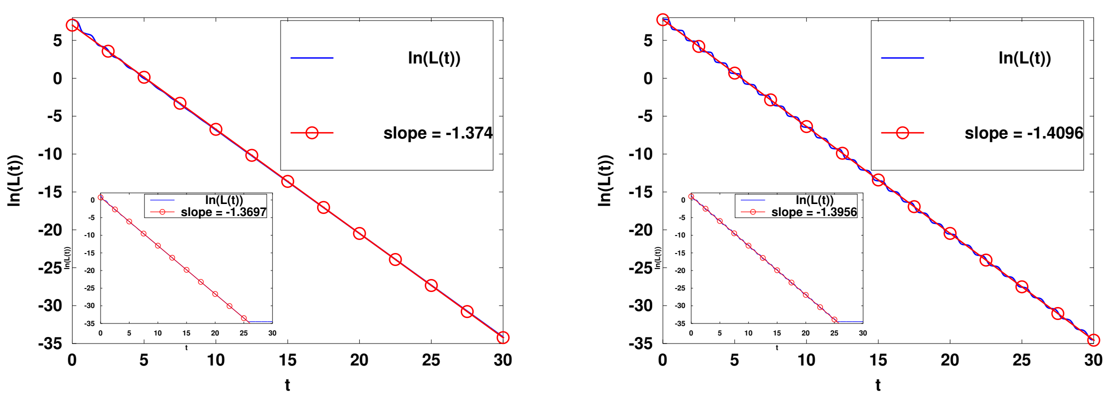
3.1 Exponential stability and asymptotic expansion

Let the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 0.75 \end{pmatrix}$ and $e = 1$. It follows that the eigenvalues are $\Lambda_1 = 1.88, \Lambda_2 = -0.133$. Let $X_1 = |c| = 1, X_2 = |b| = 1$, $t_{11} = 1, t_{22} = -1$, $\alpha = 1$, $\beta_0 = 0.1$ and $\beta_1 = 1$. The boundary conditions are $G = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}$, $\tilde{G} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

We compute the numerical solutions (u, q) and the Lyapunov functions at different time steps. By varying the values of ε, we expect that the Lyapunov functions will approach to the one of the zeroth-order system (1.5). The initial condition is taken as the Gaussian type function with a trigonometric perturbation:

$$u_0(x) = 100 \exp\left[-\frac{(x-0.5)^2}{0.002}\right]\left[1 + \varepsilon \sin(8\pi x + \pi)\right], \qquad q_0(x) = \varepsilon \sin(8\pi x + \pi). \tag{3.1}$$

Let N denote the number of meshes for the spatial discretization and Δt the time step. Figure 1 shows the Lyapunov function $L^\varepsilon(t)$ in the log scale using the blue solid line with a truncation at the machine precision 10^{-15}. The red line with circles is the fitting to a straight line using linear regression. The numerical decay rate given by this example is around 1.4, significantly larger than the analytical prediction $\lambda_1 = 0.04$. The inserts at the bottom left corner are the results generated with a cubic polynomial initial data. One observes that different initial data show the similar decay rate.



(a) $\varepsilon = 0.01$.　(b) $\varepsilon = 0.0001$.

Figure 1: The Lyapunov function $L^\varepsilon(t)$ in the log scale. The final time is $t = 30$. $N = 1000$ and $\Delta t = 10^{-4}$. The blue solid line shows the function $\ln(L^\varepsilon(t))$ The red line with circles is the fitting to a straight line using linear regression. (a) and (b): The initial data is given by (3.1). The inserts at the bottom left corner are the results generated with a cubic polynomial initial data.

In Figure 2 we analyse the dependence of the decay rate on ε. Due to the asymptotic expansion of L^ε in terms of L^0, we expect that the decay rate tends towards the decay rate of L^0 for $\varepsilon \to 0$. The rate of this convergence will be analysed in the next section. Here, we only show the different decay rates for different values of ε. As expected for small values for ε there is no difference in the rates between L^ε and L^0.

In Figure 3 we show the mesh independence of the derived results. We compute the discrete Lyapunov function in logarithmic scale for different spatial meshes and two choices for ε. Since

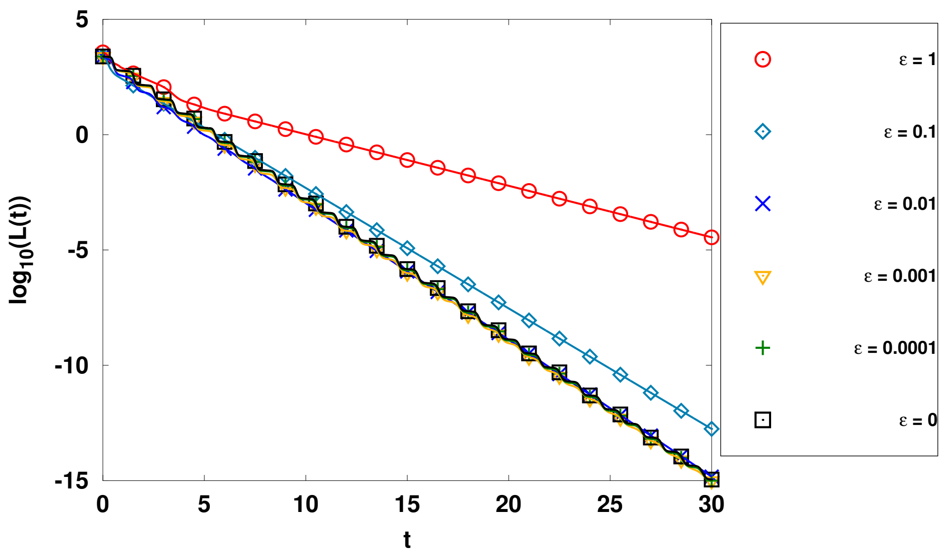
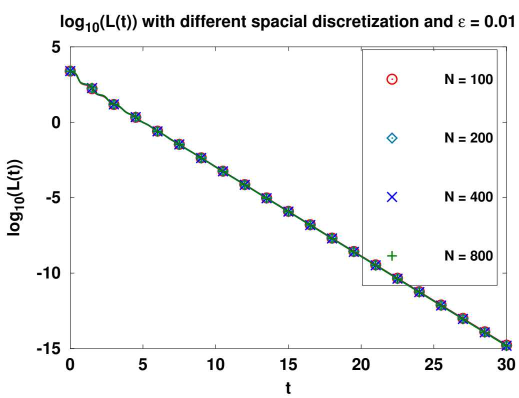
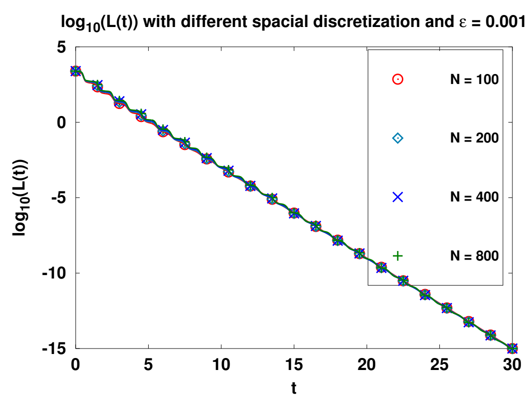


Figure 2: The Lyapunov function $L^\varepsilon(t)$ in the log scale with various perturbation parameter ε. The final time is $t = 30$. The time step $\Delta t = 10^{-4}$ and $N = 1000$.

the analytical result is independent of numerical mesh we do not expect different decay properties for different meshes, as confirmed in Figure 3.



(a) $\varepsilon = 0.01$.



(b) $\varepsilon = 0.001$.

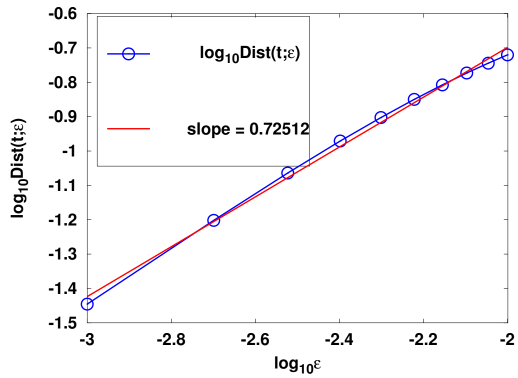
Figure 3: The Lyapunov function $L^\varepsilon(t)$ in the log scale with different spatial discretization. The final time is $t = 30$. The time step $\Delta t = 10^{-3}$ for $N \leq 400$ and $\Delta t = 10^{-4}$ for $N = 800$. The overlapping of the profiles shows that the Lyapunov function is independent of the mesh discretization.

3.2 Numerical analysis of the convergence rate of the Lyapunov functions as $\varepsilon \to 0$.

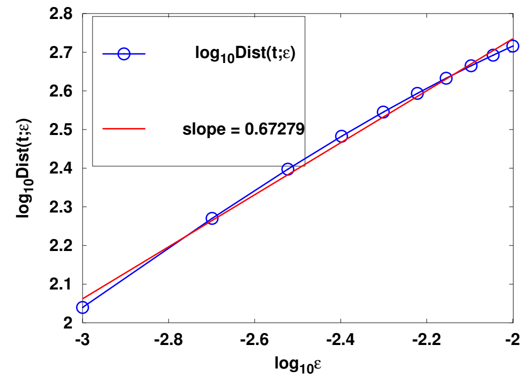
We study the convergence rate of the Lyapunov functions $L^\varepsilon(t)$ as $\varepsilon \to 0$. We measure the distance between $L^\varepsilon(t)$ and $L^0(t)$ using the L^1 norm over the time interval $[0, t]$. More precisely, we compute

$$\mathrm{Dist}(t; \varepsilon) = \int_0^t |L^\varepsilon(s) - L^0(s)|\, ds.$$

A linear decay of $\mathrm{Dist}(t; \varepsilon)$ with respect to ε is predicted by the decomposition of the Lyapunov functions. Therefore, we plot this distance in the log scale.



(a) The polynomial-type initial data.



(b) The Gaussian initial data.

Figure 4: The final time is $t = 5$. The time step $\Delta t = 10^{-4}$ and $N = 1000$. The blue curve with circles is the profile of $\log_{10} \mathrm{Dist}(t; \varepsilon)$ as a function of $\log_{10} \varepsilon$. The circles correspond to the ten numerical tests with $\varepsilon = 0.001, 0.002, 0.003, \ldots, 0.008, 0.009, 0.01$ respectively. The red solid line is the parameter fitting result using linear regression. Note that the slope is around 0.7, smaller than 1, the theoretical prediction. This results from the errors of the numerical schemes since the spatial discretization with meshsize $\Delta x = 0.001$ is relatively coarse compared to the values of ε.

3.3 The application to gas flow governed by the isentropic Euler equations

We consider the gas flow in pipes described by the isentropic Euler equations (written in (ρ, u) variables):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (3.2a) \\ \partial_t u + \dfrac{p'(\rho)}{\rho}\partial_x \rho + u \partial_x u = -\dfrac{c_f}{\varepsilon} u^2, & (3.2b) \end{cases}$$

where $\rho(t, x)$ and $u(t, x)$ are the density and the velocity of the gas flow respectively. $p(\rho)$ is the pressure function and we consider the case where $p(\rho) = \rho^2$. Note that $\frac{p'(\rho)}{\rho} = 2$. c_f is the friction coefficient.

The steady states of the problem (3.2) are denoted by $(\rho_s(x), u_s(x))$ and satisfy the following system:

$$\begin{cases} \dfrac{d}{dx}(\rho_s u_s) = 0, & (3.3a) \\ \dfrac{p'(\rho_s)}{\rho_s}\dfrac{d}{dx}\rho_s + u_s\dfrac{d}{dx}u_s = -\dfrac{c_f}{\varepsilon}u_s^2. & (3.3b) \end{cases}$$

Note that $\rho_s u_s \equiv c$ for some positive constant c. The equation (3.3b) gives

$$\frac{d}{dx}\rho_s = \frac{c_f c^2 \rho_s}{\varepsilon\big(c^2 - \rho_s^2 p'(\rho_s)\big)}. \tag{3.4}$$

Given some boundary condition $\rho_s(0)$, the ordinary differential equation (3.4) is solved using a second order Runge-Kutta scheme. We refer to e.g. [14] for more details on the stationary states.

Next, we decompose solutions (ρ, u) as

$$\rho = \rho_s + \sigma\tilde{\rho}, \qquad u = u_s + \sigma\tilde{u},$$

and linearize (3.2) around the steady states:

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} u_s & \rho_s \\ 2 & u_s \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} \tilde{u} & \tilde{\rho} \\ 0 & \tilde{u} \end{pmatrix} \partial_x \begin{pmatrix} \rho_s \\ u_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2c_f u_s}{\varepsilon}\tilde{u} \end{pmatrix}. \tag{3.5}$$

Remark 3.1. *The ratio of the friction coefficient c_f and the relaxation parameter ε is small enough so that the steady states (ρ_s, u_s) remain almost constant in a finite domain, i.e. $\partial_x \rho_s = \partial_x u_s = 0$. Therefore, ρ_s and u_s are taken as constants and we drop the third term on the left-hand side of equation* (3.5)*.*

With ρ_s and u_s being constant, we apply the boundary stabilization to the system below,

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} u_s & \rho_s \\ 2 & u_s \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2c_f u_s}{\varepsilon}\tilde{u} \end{pmatrix}. \tag{3.6}$$

Similarly, we decompose the solutions $(\tilde{\rho}, \tilde{u})$ as

$$\tilde{\rho} = \rho^0 + \varepsilon\rho^1, \qquad \tilde{u} = u^0 + \varepsilon u^1.$$

The choices of parameters are given by

$$c_f = 0.0001, \quad \varepsilon = 0.01, \quad c = 0.6, \quad (\rho_s, u_s) = (0.5, 1.2), \tag{3.7}$$

which is supersonic. The prediction of the decay rate is $\lambda_1 = 0.0947$. The initial data is given by a cubic polynomial perturbed by a trigonometric function. Figure 5 shows that the Lyapunov function of the system (3.6) with $\varepsilon = 0.01$ is decreasing in an exponential rate 0.44358 that is larger than the theoretical prediction.

Note that $\partial_x^2 \mu_{11} > 0$ holds true unconditionally. Hence $\partial_x \mu_{11}$ becomes positive for x sufficiently large. This has also been observed in a theoretical study for stabilization in [16] and is related to the shape of the stationary states in the case of isentropic Euler equations.

Numerically, we may consider a case where the prerequisite $a\partial_x \mu_{11} < 0$ is violated. Consider a subsonic case with $(\rho_s, u_s) = (1.2, 0.5)$ that violates this assumption on the domain $\{x : x \in [0, 1]\}$. Then, the Lyapunov function of the system (3.6) with nonnegative ε is increasing with time, as shown in Figure 6.

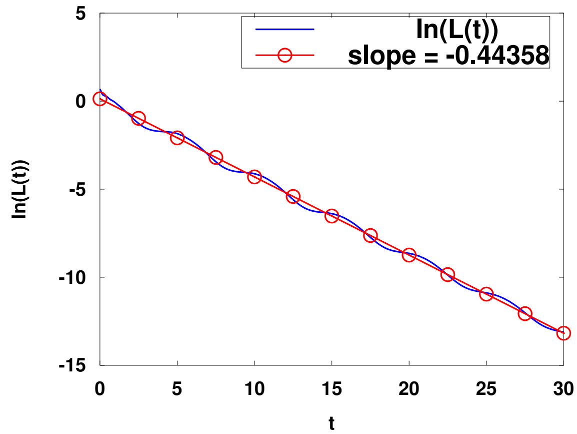


Figure 5: The supersonic case. The blue solid line shows the function $\ln(L^{\varepsilon}(t))$ with $\varepsilon = 0.01$. The red line with circles is the fitting to a straight line using linear regression. The parameters are given in equation (3.7). The final time is $t = 30$. The time step $\Delta t = 10^{-4}$ and $N = 1000$. The decay rate is given by the absolute value of the slope 0.44358.

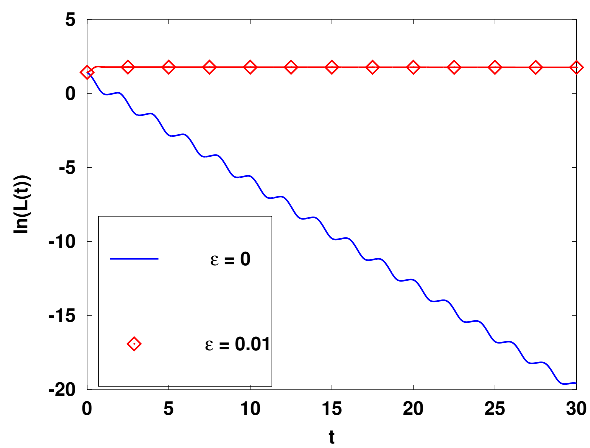


Figure 6: The subsonic case. The blue solid line is the function $\ln(L^0(t))$ and the red line with diamonds is the function $\ln(L^{\varepsilon}(t))$ with $\varepsilon = 0.01$. The parameters are given in equation (3.7) with $(\rho_s, u_s) = (1.2, 0.5)$. The final time is $t = 30$. The time step $\Delta t = 10^{-4}$ and $N = 1000$. The Lyapunov functions with $\varepsilon = 0$ and 0.01 do not decay in time.

4 Conclusion

We presented an expansion of a Lyapunov function in terms of the stiff parameter in the source term. The additional conditions on the boundary terms for exponential stability are stated and numerical results are presented. The obtained conditions are independent of ε, but dependent on the solution. In forthcoming discussions we plan to extend this result to linear systems of higher dimension as well as to other boundary conditions.

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