
Incompressible Fluid Problems on Embedded Surfaces: Modeling and Variational Formulations

Thomas Jankuhn^{*}, Maxim A. Olshanskii[†] and
Arnold Reusken[‡]

Institut für Geometrie und Praktische Mathematik
Templergraben 55, 52062 Aachen, Germany

^{*} Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany
(jankuhn@igpm.rwth-aachen.de)

[†] Department of Mathematics, University of Houston, Houston, Texas 77204-3008 (mol-shan@math.uh.edu);
Partially supported by NSF through the Division of Mathematical Sciences grant 1522252

[‡] Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany
(reusken@igpm.rwth-aachen.de)

INCOMPRESSIBLE FLUID PROBLEMS ON EMBEDDED SURFACES: MODELING AND VARIATIONAL FORMULATIONS

THOMAS JANKUHN*, MAXIM A. OLSHANSKII†, AND ARNOLD REUSKEN‡

Abstract. Governing equations of motion for a viscous incompressible material surface are derived from the balance laws of continuum mechanics. The surface is treated as a time-dependent smooth orientable manifold of codimension one in an ambient Euclidian space. We use elementary tangential calculus to derive the governing equations in terms of exterior differential operators in Cartesian coordinates. The resulting equations can be seen as the Navier-Stokes equations posed on an evolving manifold. We consider a splitting of the surface Navier-Stokes system into coupled equations for the tangential and normal motions of the material surface. We then restrict ourselves to the case of a geometrically stationary manifold of codimension one embedded in \mathbb{R}^n . For this case, we present new well-posedness results for the simplified surface fluid model consisting of the surface Stokes equations. Finally, we propose and analyze several alternative variational formulations for these surface Stokes problem, including constrained and penalized formulations, which are convenient for Galerkin discretization methods.

1. Introduction. Fluid equations on manifolds appear in the literature on mathematical modelling of emulsions, foams and biological membranes, e.g. [34, 8, 24, 29]; they are also studied as a mathematical problem of its own interest, e.g. [13, 36, 35, 3, 22, 2]. In certain applications, such as the dynamics of liquid membranes [4], one is interested in formulations of fluid equations on evolving (time-dependent) surfaces. Such equations are considered in several places in the literature. The authors of [4] formulate a continuum model of fluid membranes embedded in a bulk fluid, which includes governing equations for a two-dimensional viscous fluid moving on a curved, time-evolving surface. The derivation of a surface strain tensor in that paper uses techniques and notions from differential geometry (k -forms). A similar model was derived from balance laws for mass and momentum and associated constitutive equations in [27]. The derivation and the resulting model uses intrinsic variables on a surface. Equations for surface fluids in the context of two-phase flow are derived or used in [7, 5, 24, 28]. In those papers the surface fluid dynamics is strongly coupled through a no-slip condition with the bulk fluid dynamics. An energetic variational approach was recently used in [19] to derive the dynamical system for the motion of an incompressible viscous fluid on an evolving surface.

Computational methods and numerical analysis of these methods for fluid equations on surfaces is a relatively new field of research. Exploring the line of research starting from the seminal paper [33], it is noted in [4] and [24] that “the equations of motion are formulated intrinsically in a two-dimensional manifold with time-varying metric and make extensive use of the covariant derivative and calculations in local coordinates, which involve the coefficients of the Riemannian connection and its derivatives. The complexity of the equations may explain why they are often written but never solved for arbitrary surfaces.” Recent research addressing the numerical solution of fluid equations on surfaces includes [24, 27, 26, 5, 29, 30, 28].

We discuss the two main contributions of this paper. The first one is related to

*Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany (jankuhn@igpm.rwth-aachen.de).

†Department of Mathematics, University of Houston, Houston, Texas 77204-3008 (molshan@math.uh.edu); Partially supported by NSF through the Division of Mathematical Sciences grant 1522252.

‡Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany (reusken@igpm.rwth-aachen.de).

modeling. Based on fundamental surface continuum mechanical principles treated in [16, 23] we derive fluid equations on an evolving surface from the conservation laws of mass and momentum for a viscous material surface embedded in an ambient continuum medium. We assume that the bulk medium interacts with the fluidic membrane through the area forces. To derive the governing equations, we use *only elementary tangential differential calculus* on a manifold. As a result, the surface PDEs that we derive are formulated in terms of differential operators in the Cartesian coordinates. In particular, we avoid the use of local coordinates. Using exterior differential operators makes the formulation more convenient for numerical purposes and facilitates the application of a level set method or other implicit surface representation techniques (no local coordinates or parametrization involved) to describe the surface evolution. The resulting equations can be seen as the Navier-Stokes equations for a viscous incompressible 2D surface fluid posed on an evolving manifold embedded in \mathbb{R}^3 . The same equations have been derived and studied in the recent paper [19]. In that paper, however, the derivation is based on global energy principles instead of local conservation laws. For gaining some further insight in this rather complex surface Navier-Stokes model, we consider a splitting of the system into coupled equations for the tangential and normal motions of the material surface. We comment on how the surface Navier-Stokes equations that we consider are related to other formulations of surface fluid equations found in the literature (Remark 3.1 and Section 3.2).

The second main contribution of this paper is a derivation of well-posedness results for a strongly simplified case. We restrict ourselves to a geometrically stationary closed smooth manifold of codimension one, embedded in \mathbb{R}^n . For this case, we present new well-posedness results for the surface Stokes equations. Key ingredients in the analysis are a surface Korn's inequality and an inf-sup result for the Stokes bilinear form that couples surface pressure and surface velocity. We propose and analyze several different variational formulations of the surface Stokes problem, including constrained and penalized formulations, which are convenient for Galerkin discretization methods.

The remainder of this paper is organized as follows. Section 2 collects necessary preliminaries and auxiliary results. In section 3 we derive the governing equations for the motion of a viscous material surface, the surface Navier-Stokes system. We also consider a directional splitting of the system and discuss alternative formulations of the surface fluid equations. In section 4 we prove a fundamental surface Korn's inequality and well-posedness of a variational formulation of the surface Stokes problem. In sections 5 and 6 we introduce alternative weak formulations of the surface Stokes problem, which we believe are more convenient for Galerkin discretization methods such as surface finite element methods.

2. Preliminaries. This section recalls some basics of tangential calculus for evolving manifolds of codimension one. Several helpful auxiliary results are also proved in this section. Consider $\Gamma(t) \subset \mathbb{R}^n$, $n \geq 3$, a $(n - 1)$ -dimensional closed, smooth, simply connected evolving manifold. We are mainly interested in $n = 3$, but most of the analysis applies for general n . In the modeling part, section 3, we only consider $n = 3$. The fact that the manifold is embedded in \mathbb{R}^n plays a key role in the derivation and formulation of the PDEs. For example, the surface differential operators will be formulated in terms of differential operators in Euclidean space \mathbb{R}^n , with respect to the standard basis in \mathbb{R}^n .

The outward pointing normal vector on $\Gamma(t)$ is denoted by $\mathbf{n} = \mathbf{n}(x, t)$, and $\mathbf{P} = \mathbf{P}(x, t) = \mathbf{I} - \mathbf{nn}^T$ is the normal projector on the tangential space at $x \in \Gamma(t)$. First we consider $\Gamma = \Gamma(t)$ for some fixed t and introduce spatial differential operators.

For $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the Frechet derivative at $x \in \mathbb{R}^n$ is denoted by $\nabla f(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$, the vector space of linear transformations from \mathbb{R}^n to \mathbb{R}^m . We often skip the argument x in the notation below. The partial derivative is denoted by $\partial_i f = (\nabla f)\mathbf{e}_i \in \mathbb{R}^m$, $i = 1, \dots, n$. Hence $(\nabla f)\mathbf{z} = \sum_{j=1}^n \partial_j f z_j$ for $\mathbf{z} \in \mathbb{R}^n$. Note that for a scalar function f , i.e. $m = 1$, ∇f is a *row* vector; hence $\nabla^T f$ denotes column gradient vector.

The *tangential* derivative (along Γ) is defined as $(\nabla_\Gamma g)\mathbf{P}\mathbf{z} = \sum_{j=1}^n \partial_j g(\mathbf{P}\mathbf{z})_j$ for $\mathbf{z} \in \mathbb{R}^n$. For $m = 1$, i.e. $f : \mathbb{R}^n \rightarrow \mathbb{R}$ the corresponding i -th (covariant) partial derivative is denoted by ∇_i :

$$\nabla_i f = \sum_{j=1}^n \partial_j f(\mathbf{P}\mathbf{e}_i)_j, \quad \text{and} \quad \nabla_\Gamma f := (\nabla_1 f, \dots, \nabla_n f) = (\nabla f)\mathbf{P}. \quad (2.1)$$

We also need such covariant partial derivatives for $m = n$ and $m = n \times n$. For $m = n$ the i -th covariant partial derivative of $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined as

$$\nabla_i \mathbf{v} = \sum_{j=1}^n \mathbf{P} \partial_j \mathbf{v}(\mathbf{P}\mathbf{e}_i)_j, \quad \text{and} \quad \nabla_\Gamma \mathbf{v} := (\nabla_1 \mathbf{v} \dots \nabla_n \mathbf{v}) = \mathbf{P}(\nabla \mathbf{v})\mathbf{P}. \quad (2.2)$$

We shall use the notation $\nabla_\Gamma^T f := (\nabla_\Gamma f)^T$, $\nabla_\Gamma^T \mathbf{v} := (\nabla_\Gamma \mathbf{v})^T$ for the transposed vector and matrix, and similarly for ∇_Γ replaced by ∇ . For $m = n \times n$ the i -th covariant partial derivative of $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is defined as

$$\nabla_i \mathbf{A} = \sum_{j=1}^n \mathbf{P} \partial_j \mathbf{A} \mathbf{P}(\mathbf{P}\mathbf{e}_i)_j, \quad \text{and} \quad \nabla_\Gamma \mathbf{A} := (\nabla_1 \mathbf{A} \dots \nabla_n \mathbf{A}). \quad (2.3)$$

Note that from $\mathbf{n}^T \mathbf{P} = \mathbf{P} \mathbf{n} = 0$ we get $\mathbf{P}(\partial_j \mathbf{P})\mathbf{P} = -\mathbf{P}(\partial_j \mathbf{n} \mathbf{n}^T + \mathbf{n} \partial_j \mathbf{n}^T)\mathbf{P} = 0$, hence $\nabla_i \mathbf{P} = 0$, $i = 1, \dots, n$, i.e., $\nabla_\Gamma \mathbf{P} = 0$. The covariant partial derivatives of f , \mathbf{v} , or \mathbf{A} depend only on the values of these fields on Γ . For scalar functions f, g and vector functions $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have the following product rules:

$$\nabla_\Gamma(fg) = g \nabla_\Gamma f + f \nabla_\Gamma g \quad (2.4)$$

$$\nabla_\Gamma(\mathbf{u} \cdot \mathbf{v}) = \mathbf{v}^T \nabla_\Gamma \mathbf{u} + \mathbf{u}^T \nabla_\Gamma \mathbf{v} \quad (2.5)$$

$$\nabla_\Gamma(f\mathbf{u}) = f \nabla_\Gamma \mathbf{u} + \mathbf{P} \mathbf{u} \nabla_\Gamma f. \quad (2.6)$$

Besides these covariant (partial) derivatives we also need *tangential divergence operators* for $\mathbf{v} : \Gamma \rightarrow \mathbb{R}^n$ and $\mathbf{A} : \Gamma \rightarrow \mathbb{R}^{n \times n}$. These are defined as follows:

$$\operatorname{div}_\Gamma \mathbf{v} := \operatorname{tr}(\nabla_\Gamma \mathbf{v}) = \operatorname{tr}(\mathbf{P}(\nabla \mathbf{v})\mathbf{P}) = \operatorname{tr}(\mathbf{P}(\nabla \mathbf{v})) = \operatorname{tr}((\nabla \mathbf{v})\mathbf{P}), \quad (2.7)$$

$$\operatorname{div}_\Gamma \mathbf{A} := (\operatorname{div}_\Gamma(\mathbf{e}_1^T \mathbf{A}), \dots, \operatorname{div}_\Gamma(\mathbf{e}_n^T \mathbf{A}))^T. \quad (2.8)$$

These tangential differential operators will be used in the modeling of conservation laws in section 3. In particular, the differential operator $\mathbf{P} \operatorname{div}_\Gamma(\nabla_\Gamma \mathbf{v} + \nabla_\Gamma^T \mathbf{v})$, which is the tangential analogon of the $\operatorname{div}(\nabla \mathbf{v} + \nabla^T \mathbf{v})$ operator in Euclidean space, plays a key role. We derive some properties of this differential operator.

We first relate $\mathbf{P} \operatorname{div}_\Gamma(\nabla_\Gamma \mathbf{v})$ to a Laplacian. For this we introduce the space of smooth tangential vector fields $C_T^\infty(\Gamma)^n := \{\mathbf{v} \in C^\infty(\Gamma)^n \mid \mathbf{P}\mathbf{v} = \mathbf{v}\}$, with scalar product $(\mathbf{u}, \mathbf{v})_0 = \int_\Gamma \mathbf{u} \cdot \mathbf{v} ds$, and the space of smooth tangential tensor fields $C_T^\infty(\Gamma)^{n \times n} := \{\mathbf{A} \in C^\infty(\Gamma)^{n \times n} \mid \mathbf{P}\mathbf{A}\mathbf{P} = \mathbf{A}\}$, with scalar product $(\mathbf{A}, \mathbf{B})_0 :=$

$\int_{\Gamma} \text{tr}(\mathbf{A}\mathbf{B}^T) ds$. From the partial integration identity (see, e.g., (14.17) in [15]),

$$\begin{aligned} \int_{\Gamma} \mathbf{v} \cdot (\mathbf{P} \text{div}_{\Gamma} \mathbf{A}) ds &= \int_{\Gamma} \mathbf{v} \cdot \text{div}_{\Gamma} \mathbf{A} ds \\ &= - \int_{\Gamma} \text{tr}(\mathbf{A}^T \nabla_{\Gamma} \mathbf{v}) ds, \quad \mathbf{v} \in C_T^{\infty}(\Gamma)^n, \quad \mathbf{A} \in C_T^{\infty}(\Gamma)^{n \times n}, \end{aligned}$$

it follows that for $\mathcal{L} : C_T^{\infty}(\Gamma)^{n \times n} \rightarrow C_T^{\infty}(\Gamma)^n$ given by $\mathcal{L}(\mathbf{A}) = \mathbf{P} \text{div}_{\Gamma}(\mathbf{A})$ we have

$$(\mathcal{L}(\mathbf{A}), \mathbf{v})_0 = -(\mathbf{A}, \nabla_{\Gamma} \mathbf{v})_0 \quad \text{for all } \mathbf{v} \in C_T^{\infty}(\Gamma)^n, \quad \mathbf{A} \in C_T^{\infty}(\Gamma)^{n \times n}.$$

Hence, $-\mathcal{L}$ is the adjoint of ∇_{Γ} , i.e., $\mathcal{L} = -\nabla_{\Gamma}^*$. Thus we have

$$\mathbf{P} \text{div}_{\Gamma}(\nabla_{\Gamma} \mathbf{v}) = \mathcal{L}(\nabla_{\Gamma} \mathbf{v}) = -\nabla_{\Gamma}^* \nabla_{\Gamma} \mathbf{v} =: \Delta_{\Gamma} \mathbf{v}. \quad (2.9)$$

This *vector Laplacian* Δ_{Γ} is the so-called Bochner Laplacian [31]. It can be extended to a self-adjoint operator on a suitable space of vector fields on Γ .

The mapping $\mathbf{v} \rightarrow \mathbf{P} \text{div}_{\Gamma} \nabla_{\Gamma}^T \mathbf{v}$ requires more calculations. Note that in Euclidean space we have $\text{div}(\nabla^T \mathbf{v})_i = \text{div}(\mathbf{e}_i^T \nabla^T \mathbf{v}) = \text{div}(\partial_i \mathbf{v}) = \partial_i(\text{div} \mathbf{v})$. Hence, for divergence free functions \mathbf{v} we have $\text{div} \nabla^T \mathbf{v} = 0$. For the corresponding surface differential operator we do not have a simple commutation relation, and the analysis becomes more complicated. In [4, 24] this mapping is analyzed with intrinsic tools of differential geometry. It is, however, not clear how the divergence operators used in those papers, which are defined via differential forms, are related to the tangential divergence operator div_{Γ} introduced above, which is defined in Euclidean space \mathbb{R}^n . Lemma 2.1 below shows a representation for $\mathbf{P} \text{div}_{\Gamma} \nabla_{\Gamma}^T \mathbf{v}$. The proof of the lemma is given in the Appendix and it only uses elementary tangential calculus. For a vector field \mathbf{v} on $\Gamma(t)$ we shall use throughout the paper the notion $\mathbf{v}_T = \mathbf{P}\mathbf{v}$ for the tangential part and $v_N = \mathbf{v} \cdot \mathbf{n}$ for the normal coordinate, so that

$$\mathbf{v} = \mathbf{v}_T + v_N \mathbf{n} \quad \text{on } \Gamma(t).$$

LEMMA 2.1. *Let $\mathbf{H} = \nabla_{\Gamma} \mathbf{n} \in \mathbb{R}^n$ be the Weingarten mapping (second fundamental form) on $\Gamma(t)$ and $\kappa := \text{tr}(\mathbf{H})$ the mean curvature. The following holds:*

$$\mathbf{P} \text{div}_{\Gamma} \nabla_{\Gamma}^T \mathbf{v} = \nabla_{\Gamma}^T \text{div}_{\Gamma} \mathbf{v} + (\text{tr}(\mathbf{H})\mathbf{H} - \mathbf{H}^2) \mathbf{v}, \quad \forall \mathbf{v} \in C_T^{\infty}(\Gamma)^n, \quad (2.10)$$

$$\begin{aligned} \mathbf{n} \cdot \text{div}_{\Gamma} \nabla_{\Gamma}^T \mathbf{v} &= \mathbf{n} \cdot \text{div}_{\Gamma}(\nabla_{\Gamma} \mathbf{v}) = -\text{tr}(\mathbf{H} \nabla_{\Gamma} \mathbf{v}) \\ &= -\text{tr}(\mathbf{H} \nabla_{\Gamma} \mathbf{v}_T) - v_N \text{tr}(\mathbf{H}^2), \quad \forall \mathbf{v} \in C^{\infty}(\Gamma)^n, \end{aligned} \quad (2.11)$$

$$\mathbf{P} \text{div}_{\Gamma}(\mathbf{H}) = \nabla_{\Gamma}^T \kappa. \quad (2.12)$$

If $n = 3$, then (2.10) simplifies to

$$\mathbf{P} \text{div}_{\Gamma} \nabla_{\Gamma}^T \mathbf{v} = \nabla_{\Gamma}^T \text{div}_{\Gamma} \mathbf{v} + K \mathbf{v}, \quad \forall \mathbf{v} \in C_T^{\infty}(\Gamma)^3, \quad (2.13)$$

where K is the Gauss curvature, i.e. the product of the two principal curvatures.

We also need an n -dimensional manifold defined by the evolution of Γ ,

$$\mathcal{S} := \bigcup_{t>0} \Gamma(t) \times \{t\};$$

the (space-time) manifold \mathcal{S} is embedded in \mathbb{R}^{n+1} . We assume a flow field $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $V_{\Gamma} = \mathbf{u} \cdot \mathbf{n}$ on \mathcal{S} , where V_{Γ} denotes the *normal velocity* of Γ . For

a smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we consider the material derivative \dot{f} (the derivative along material trajectories in the velocity field \mathbf{u}).

$$\dot{f} = \frac{\partial f}{\partial t} + \sum_{i=1}^n \frac{\partial f}{\partial x_i} u_i = \frac{\partial f}{\partial t} + (\nabla^T f) \cdot \mathbf{u}.$$

The material derivative \dot{f} is a tangential derivative for \mathcal{S} , and hence it depends only on the surface values of f on $\Gamma(t)$. For a vector field \mathbf{v} , we define $\dot{\mathbf{v}}$ componentwise, i.e., $\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \mathbf{u}$. In Lemma 2.2 we derive some useful identities for the material derivative of the normal vector field and normal projector on Γ .

LEMMA 2.2. *The following identities hold on $\Gamma(t)$:*

$$\dot{\mathbf{n}} = \mathbf{H}\mathbf{u}_T - \nabla_\Gamma^T u_N = -\mathbf{P}(\nabla^T \mathbf{u}) \mathbf{n}, \quad (2.14)$$

$$\dot{\mathbf{P}} = \mathbf{P}(\nabla^T \mathbf{u})(\mathbf{I} - \mathbf{P}) + (\mathbf{I} - \mathbf{P})(\nabla \mathbf{u})\mathbf{P}. \quad (2.15)$$

Proof. Let $d(x, t)$ be the signed distance function to $\Gamma(t)$ defined in a neighborhood U_t of $\Gamma(t)$. Define the normal extension of \mathbf{n} and \mathbf{H} to U_t by $\mathbf{n}^T = \nabla d$, $\mathbf{H} = \nabla^2 d$, and consider the closest point projection $p(x, t) = x - d(x, t)\mathbf{n}(x, t)$, $x \in U_t$. We then have

$$\frac{\partial d}{\partial t}(x, t) = -u_N(p(x, t), t), \quad x \in U_t.$$

Using the chain rule we get

$$\nabla[u_N(p(x, t), t)] = \nabla_\Gamma u_N(p(x, t), t)(\mathbf{I} - d(x, t)\mathbf{H}) \quad x \in U_t.$$

Take $x \in \Gamma(t)$, using $d(x, t) = 0$, $p(x, t) = x$ and $\nabla d = \mathbf{n}^T$, we obtain

$$\frac{\partial \mathbf{n}^T}{\partial t} = \frac{\partial}{\partial t} \nabla d = \nabla \frac{\partial d}{\partial t} = -\nabla_\Gamma u_N, \quad \text{on } \Gamma(t). \quad (2.16)$$

Using this and $\mathbf{H}\mathbf{n} = 0$ we get

$$\dot{\mathbf{n}} = \frac{\partial \mathbf{n}}{\partial t} + (\nabla \mathbf{n}) \mathbf{u} = -\nabla_\Gamma^T u_N + \mathbf{H}\mathbf{u}_T,$$

which is the first identity in (2.14). From $\mathbf{u}_T \cdot \mathbf{n} = 0$ we get $\mathbf{n}^T \nabla \mathbf{u}_T = -\mathbf{u}_T^T \nabla \mathbf{n}$ and combined with the symmetry of \mathbf{H} we get

$$\mathbf{H}\mathbf{u}_T = -(\nabla^T \mathbf{u}_T) \mathbf{n}. \quad (2.17)$$

Furthermore, we note that $\nabla(u_N \mathbf{n}) = \mathbf{n} \nabla u_N + u_N \nabla \mathbf{n}$, hence $\mathbf{n}^T \nabla(u_N \mathbf{n}) = \nabla u_N$. Using this, the result in (2.17) and $\mathbf{P}\mathbf{H} = \mathbf{H}$ we get

$$\begin{aligned} -\nabla_\Gamma^T u_N + \mathbf{H}\mathbf{u}_T &= -\mathbf{P}(\nabla^T u_N + (\nabla^T \mathbf{u}_T) \mathbf{n}) \\ &= -\mathbf{P}(\nabla^T(u_N \mathbf{n}) + (\nabla^T \mathbf{u}_T) \mathbf{n}) = -\mathbf{P}(\nabla^T \mathbf{u}) \mathbf{n}, \end{aligned}$$

which is the second identity in (2.14). The result in (2.15) immediately follows from $\dot{\mathbf{P}} = -\dot{\mathbf{n}} \mathbf{n}^T - \mathbf{n} \dot{\mathbf{n}}^T$ and the second identity in (2.14). \square

From (2.14) we see that the vector field $\dot{\mathbf{n}}$ is always tangential to $\Gamma(t)$.

3. Modeling of material surface flows. In this section, we assume $\Gamma(t)$ is a *material* surface (fluidic membrane) embedded in \mathbb{R}^3 as defined in [16, 23], with density distribution $\rho(x, t)$. By $\mathbf{u}(x, t)$, $x \in \Gamma(t)$, we denote the smooth velocity field of the density flow on Γ , i.e. $\mathbf{u}(x, t)$ is the velocity of the material point $x \in \Gamma(t)$. The geometrical evolution of the surface is defined by the normal velocity u_N , for $\mathbf{u}(x, t) = \mathbf{u}_T + u_N \mathbf{n}$. For all $t \in [0, T]$, we assume $\Gamma(t) \subset \mathbb{R}^3$ to be smooth, closed and embedded in an ambient continuum medium, which exerts external (area) forces on the material surface.

Let $\gamma(t) \subset \Gamma(t)$ be a material subdomain. For a smooth $f : \mathcal{S} \rightarrow \mathbb{R}$, we shall make use of the Leibniz rule,

$$\frac{d}{dt} \int_{\gamma(t)} f ds = \int_{\gamma(t)} (\dot{f} + f \operatorname{div}_{\Gamma} \mathbf{u}) ds. \quad (3.1)$$

Inextensibility. We assume that the surface material is inextensible, i.e. $\frac{d}{dt} \int_{\gamma(t)} 1 ds = 0$ must hold. The Leibniz rule yields $\frac{d}{dt} \int_{\gamma(t)} \operatorname{div}_{\Gamma} \mathbf{u} ds = 0$. Since $\gamma(t)$ can be taken arbitrary, we get

$$\operatorname{div}_{\Gamma} \mathbf{u} = 0 \quad \text{on } \Gamma(t). \quad (3.2)$$

Denote by $\kappa := \operatorname{tr}(\mathbf{H}) = \operatorname{div}_{\Gamma} \mathbf{n}$ the (doubled) *mean* curvature. Equation (3.2) can be rewritten as

$$\operatorname{div}_{\Gamma} \mathbf{u}_T = -u_N \kappa \quad \text{on } \Gamma(t). \quad (3.3)$$

Mass conservation. From $\frac{d}{dt} \int_{\gamma(t)} \rho(x, t) ds = 0$, (3.1) and (3.2) we obtain $\dot{\rho} = 0$.

Hence, if $\rho|_{t=0} = \text{const}$, then ρ is constant for all $t > 0$.

Momentum conservation. The conservation of linear momentum for $\gamma(t)$ reads:

$$\frac{d}{dt} \int_{\gamma(t)} \rho \mathbf{u} ds = \int_{\partial\gamma(t)} \mathbf{f}_{\nu} ds + \int_{\gamma(t)} \mathbf{b} ds, \quad (3.4)$$

where \mathbf{f}_{ν} are the contact forces on $\partial\gamma(t)$, $\mathbf{b} = \mathbf{b}(x, t)$ are the area forces on $\gamma(t)$, which include both tangential and normal forces, for example, normal stresses induced by an ambient medium and elastic bending forces.

Surface diffusion. For the modeling of the contact forces we use results from [16, 23]. In [16], Theorems 5.1 and 5.2, the ‘‘Cauchy-relation’’ $\mathbf{f}_{\nu} = \mathbf{T}\nu$, with a symmetric tangential stress tensor \mathbf{T} is derived (here $\nu = \nu(x, t)$ denotes the normal to $\partial\gamma(t)$ that is tangential to $\Gamma(t)$). We denote this *surface stress tensor* by $\boldsymbol{\sigma}_{\Gamma}$, which has the properties $\boldsymbol{\sigma}_{\Gamma} = \boldsymbol{\sigma}_{\Gamma}^T$ and $\boldsymbol{\sigma}_{\Gamma} = \mathbf{P}\boldsymbol{\sigma}_{\Gamma}\mathbf{P}$. In [16] the following (infinitesimal) *surface rate-of-strain tensor* is derived:

$$E_s(\mathbf{u}) := \frac{1}{2} \mathbf{P}(\nabla \mathbf{u} + \nabla^T \mathbf{u}) \mathbf{P} = \frac{1}{2} (\nabla_{\Gamma} \mathbf{u} + \nabla_{\Gamma}^T \mathbf{u}). \quad (3.5)$$

One needs a constitutive law which relates $\boldsymbol{\sigma}_{\Gamma}$ to this surface strain tensor. We consider a ‘‘Newtonian surface fluid’’, i.e., a constitutive law of the form

$$\boldsymbol{\sigma}_{\Gamma} = -\pi \mathbf{P} + C(\nabla_{\Gamma} \mathbf{u}),$$

with a scalar function π , surface pressure, and a linear mapping C . Assuming *isotropy* and requiring an *independence of the frame of reference* leads to the so-called *Boussinesq–Scriven* surface stress tensor, which can be found at several places in the literature, e.g., [1, 7, 16, 33] :

$$\boldsymbol{\sigma}_\Gamma = -\pi\mathbf{P} + (\lambda - \mu)(\operatorname{div}_\Gamma\mathbf{u})\mathbf{P} + 2\mu E_s(\mathbf{u}),$$

with an interface dilatational viscosity λ and interface shear viscosity $\mu > 0$. We assume λ and μ constant. Due to inextensibility the dilatational term vanishes, and we get

$$\boldsymbol{\sigma}_\Gamma = -\pi\mathbf{P} + 2\mu E_s(\mathbf{u}).$$

Using the relation $\operatorname{div}_\Gamma(\pi\mathbf{P}) = \nabla_\Gamma^T\pi - \pi\kappa\mathbf{n}$, and the Stokes theorem, we obtain the following linear momentum balance for $\gamma(t)$:

$$\frac{d}{dt} \int_{\gamma(t)} \rho\mathbf{u} ds = \int_{\gamma(t)} -\nabla_\Gamma^T\pi + 2\mu \operatorname{div}_\Gamma(E_s(\mathbf{u})) + \mathbf{b} + \pi\kappa\mathbf{n} ds.$$

For the left hand-side of this equation, the Leibniz rule (3.1) gives

$$\frac{d}{dt} \int_{\gamma(t)} \rho\mathbf{u} ds = \int_{\gamma(t)} (\dot{\rho}\mathbf{u} + \rho\dot{\mathbf{u}} + \rho\mathbf{u} \operatorname{div}_\Gamma\mathbf{u}) ds.$$

The inextensibility and mass conservation yield the simplification $\dot{\rho}\mathbf{u} + \rho\dot{\mathbf{u}} + \rho\mathbf{u} \operatorname{div}_\Gamma\mathbf{u} = \rho\dot{\mathbf{u}}$. Hence, we finally obtain the *surface Navier-Stokes equations*:

$$\begin{cases} \rho\dot{\mathbf{u}} = -\nabla_\Gamma^T\pi + 2\mu \operatorname{div}_\Gamma(E_s(\mathbf{u})) + \mathbf{b} + \pi\kappa\mathbf{n}, \\ \operatorname{div}_\Gamma\mathbf{u} = 0. \end{cases} \quad (3.6)$$

Clearly, the area forces \mathbf{b} coming from the adjacent inner and outer media are critical for the dynamics of the the material surface. For the example of an ideal bulk fluid, one may assume normal stresses due to the pressure drop between inner and outer phases, $\mathbf{b} = \mathbf{n}(p^{int} - p^{ext})$, where $p^{int} - p^{ext}$ may depend on the surface configuration, e.g., its interior volume. In an equilibrium with $\mathbf{u} = 0$ this simplifies to the balance of the internal pressure and surface tension forces according to Laplace’s law. Such a balance will be more complex if there is only a shape equilibrium, i.e., $u_N = 0$, but $\mathbf{u}_T \neq 0$, cf. (3.14) below. The area forces \mathbf{b} may also include forces depending on the shape of the surface, such as those due to an elastic bending energy (Willmore energy), cf. for example, [6, 9, 18]. These forces depend on geometric invariants and material parameters. Therefore \mathbf{b} may (implicitly) depend on \mathbf{u} .

The model (3.6) is also derived in [7, 19] and used in [5, 24, 20]. In [5, 20] the interface viscous fluid flow is coupled with outer bulk fluids and for the velocity of the material surface $\mathbf{u} =: \mathbf{u}_\Gamma$ one introduces the condition $\mathbf{u}_\Gamma = (\mathbf{u}^{bulk})|_\Gamma$, which means that both the normal and tangential components of surface and bulk velocities coincide. The condition for the tangential component corresponds to a “no-slip” condition at the interface. The condition $\mathbf{u}_\Gamma = (\mathbf{u}^{bulk})|_\Gamma$, allows to eliminate \mathbf{u}_Γ (using a momentum balance in a small bulk volume element that contains the interface) and to deal with the surface forces (both viscous and \mathbf{b}) through a localized force term in the bulk Navier-Stokes equation. The surface pressure π remains and is used to

satisfy the inextensibility condition $\operatorname{div}_\Gamma \mathbf{u} = 0$. In [24] a simplification of (3.6) for stationary surfaces, cf. (3.13) below, is considered.

In certain cases, for example, when the inertia of the surface material dominates over the viscous forces in the bulk, it may be more appropriate to relax the no-slip condition $\mathbf{u}_\Gamma = (\mathbf{u}^{bulk})|_\Gamma$ and assume the coupling with the ambient medium only through the area forces \mathbf{b} . In such a situation the surface flow can not be “eliminated” and the system (3.6) becomes an important part of the surface–bulk fluid dynamics model. In Section 3.1 below we take a closer look at the normal and tangential dynamics defined by (3.6). As far as we know, in the literature the surface Navier–Stokes equations (3.6), without coupling to bulk fluids, have only been considered for *evolving* surfaces in the recent paper [19]. Results of numerical simulations of such a model for a *stationary* surface, $u_N = 0$, are presented in [24]. This special case $u_N = 0$ will be further addressed in section 3.1.

3.1. Directional splitting of the surface Navier–Stokes equations. The system (3.6) determines $\mathbf{u} = u_N \mathbf{n} + \mathbf{u}_T$ (and thus the evolution of $\Gamma(t)$), and there is a strong coupling between u_N and \mathbf{u}_T . There is, however, a clear distinction between the normal direction and the tangential direction (see, e.g., the difference in the viscous forces in normal and tangential direction in (2.10) and (2.11)). In particular, the geometric evolution of $\Gamma(t)$ is completely determined by u_N (which may depend on \mathbf{u}_T). Therefore, it is of interest to split the equation (3.6) for \mathbf{u} into two coupled equations for u_N and \mathbf{u}_T . We project the momentum equation (3.6) onto the tangential space and normal space, respectively.

First, we compute with the help of identities (2.14)–(2.15)

$$\mathbf{P}\dot{\mathbf{u}} = \dot{\mathbf{u}}_T - \dot{\mathbf{P}}\mathbf{u} = \dot{\mathbf{u}}_T + (\dot{\mathbf{n}} \cdot \mathbf{u}_T)\mathbf{n} + u_N \dot{\mathbf{n}}. \quad (3.7)$$

Note that the last two terms on the right hand-side are orthogonal, since $\mathbf{n} \cdot \dot{\mathbf{n}} = 0$. Applying \mathbf{P} to both sides of (3.7) and using $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}\dot{\mathbf{n}} = \dot{\mathbf{n}}$, we also get

$$\mathbf{P}\dot{\mathbf{u}} = \partial_\Gamma^\bullet \mathbf{u}_T + u_N \dot{\mathbf{n}}, \quad (3.8)$$

where $\partial_\Gamma^\bullet \mathbf{u}_T := \mathbf{P}\dot{\mathbf{u}}_T$ can be interpreted as the covariant material derivative. We also have

$$\mathbf{n} \cdot \dot{\mathbf{u}} = \dot{u}_N - \dot{\mathbf{n}} \cdot \mathbf{u} = \dot{u}_N - \dot{\mathbf{n}} \cdot \mathbf{u}_T.$$

We thus get the following directional splitting of the equations in (3.6):

$$\begin{cases} \rho \dot{\mathbf{u}}_T = -\nabla_\Gamma^T \pi + 2\mu \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}) + \mathbf{b}_T - \rho((\dot{\mathbf{n}} \cdot \mathbf{u}_T)\mathbf{n} + u_N \dot{\mathbf{n}}), \\ \rho \dot{u}_N = 2\mu \mathbf{n} \cdot \operatorname{div}_\Gamma E_s(\mathbf{u}) + \pi \kappa + b_N + \rho \dot{\mathbf{n}} \cdot \mathbf{u}_T, \\ \operatorname{div}_\Gamma \mathbf{u}_T = -u_N \kappa. \end{cases} \quad (3.9)$$

The material derivative of the tangential vector field on the left-hand side of the first equation in (3.9), in general, is not tangential to $\Gamma(t)$. Its normal component is balanced by the term $\rho(\dot{\mathbf{n}} \cdot \mathbf{u}_T)\mathbf{n}$. One can also write this equation only in tangential terms employing the identity (3.8) instead of (3.7). This results in the tangential momentum equation

$$\rho \partial_\Gamma^\bullet \mathbf{u}_T = -\nabla_\Gamma^T \pi + 2\mu \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}) + \mathbf{b}_T - \rho u_N \dot{\mathbf{n}}. \quad (3.10)$$

These equations can be further rewritten using

$$E_s(\mathbf{u}) = E_s(\mathbf{u}_T) + u_N \mathbf{H}. \quad (3.11)$$

From this, the definition of the Bochner Laplacian and the relations in Lemma 2.1 we get

$$\begin{aligned}\mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}) &= \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}_T) + \mathbf{P} \operatorname{div}_\Gamma(u_N \mathbf{H}) \\ &= \frac{1}{2} \mathbf{P} \operatorname{div}_\Gamma(\nabla_\Gamma \mathbf{u}_T) + \frac{1}{2} \mathbf{P} \operatorname{div}_\Gamma(\nabla_\Gamma^T \mathbf{u}_T) + u_N \mathbf{P} \operatorname{div}_\Gamma(\mathbf{H}) + \mathbf{H} \nabla_\Gamma^T u_N \\ &= \frac{1}{2} \Delta_\Gamma \mathbf{u}_T + \frac{1}{2} K \mathbf{u}_T + \frac{1}{2} \nabla_\Gamma^T \operatorname{div}_\Gamma \mathbf{u}_T + u_N \nabla_\Gamma^T \kappa + \mathbf{H} \nabla_\Gamma^T u_N.\end{aligned}$$

We would like to have a representation of $\mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u})$ that does not include derivatives of \mathbf{H} or its invariants. To this end, we note that $\operatorname{div}_\Gamma \mathbf{u}_T = -u_N \kappa$ implies

$$\nabla_\Gamma^T \operatorname{div}_\Gamma \mathbf{u}_T + u_N \nabla_\Gamma^T \kappa = -\nabla_\Gamma^T(u_N \kappa) + u_N \nabla_\Gamma^T \kappa = -\kappa \nabla_\Gamma^T u_N.$$

Combining this we get

$$2\mu \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}) = \mu(\Delta_\Gamma \mathbf{u}_T + K \mathbf{u}_T - \nabla_\Gamma^T(\operatorname{div}_\Gamma \mathbf{u}_T) - 2(\kappa \mathbf{P} - \mathbf{H}) \nabla_\Gamma^T u_N).$$

Note that $\kappa \mathbf{P} - \mathbf{H}$ has the same eigenvalues and eigenvectors as \mathbf{H} , which follows from the relation $\kappa \mathbf{P} - \mathbf{H} = K \mathbf{H}^\dagger$, cf. (8.6). Thus we can rewrite (3.9) as

$$\left\{ \begin{array}{l} \rho \partial_\Gamma^\bullet \mathbf{u}_T = -\nabla_\Gamma^T \pi + \mu(\Delta_\Gamma \mathbf{u}_T + K \mathbf{u}_T - \nabla_\Gamma^T(\operatorname{div}_\Gamma \mathbf{u}_T) - 2(\kappa \mathbf{P} - \mathbf{H}) \nabla_\Gamma^T u_N) \\ \quad + \mathbf{b}_T - \rho u_N \dot{\mathbf{n}} \\ \rho \dot{u}_N = -\mu(\operatorname{tr}(\mathbf{H} \nabla_\Gamma \mathbf{u}_T) + u_N \operatorname{tr}(\mathbf{H}^2)) + \pi \kappa + b_N + \rho \dot{\mathbf{n}} \cdot \mathbf{u}_T \\ \operatorname{div}_\Gamma \mathbf{u}_T = -u_N \kappa. \end{array} \right. \quad (3.12)$$

It is interesting to note that the first equation in (3.12) is of (quasi-)parabolic type, while the equation for the evolution of the normal velocity involves only first order derivatives. Furthermore, $\dot{\mathbf{n}}$ can be expressed in terms of \mathbf{u}_T and u_N ,

$$\dot{\mathbf{n}} = \mathbf{H} \mathbf{u}_T - \nabla_\Gamma^T u_N.$$

Hence the derivatives in the terms $\rho u_N \dot{\mathbf{n}}$ and $\rho \dot{\mathbf{n}} \cdot \mathbf{u}_T$ on the right-hand side of (3.9) and (3.12) are only *tangential* ones (no $\frac{\partial}{\partial t}$ involved). From this we conclude that given $\mathbf{u}(\cdot, t)$ for $t < t_*$ (which determines $\Gamma(t)$, $t < t_*$) the second equation in (3.12) determines the dynamics of $u_N(\cdot, t)$ at $t = t_*$, hence of the surface $\Gamma(t_*)$, and the first equation (3.12) determines the dynamics of $\mathbf{u}_T(\cdot, t)$ at $t = t_*$.

REMARK 3.1. The model (3.9), or equivalently the one in (3.12), differs from the fluid model on evolving surfaces derived in [4]. In the latter a tangential momentum equation (eq. (3) in [4]) is introduced, which is similar to, but different from, the first equation in (3.9). The model in [4] is based on a ‘‘conservation of linear momentum *tangentially* to the surface’’, which is not precisely specified. Our model is derived based on a conservation of total momentum (i.e. for \mathbf{u} , not for \mathbf{u}_T) as in (3.4). The ‘‘tangential’’ equation (1.2) in the paper [19] is the same as the one obtained by applying the projection \mathbf{P} to the first equation in (3.6). We just showed that this projected equation equals (3.10) and the first equations in (3.9) and (3.12).

We next discuss two special cases.

Firstly, assume that the system evolves to an equilibrium with $\Gamma(t)$ stationary, i.e., $u_N = 0$. Then the equations in (3.9) reduce to the following surface incompressible Navier-Stokes equations for the tangential velocity \mathbf{u}_T on a *stationary* surface Γ :

$$\left\{ \begin{array}{l} \rho \left(\frac{\partial \mathbf{u}_T}{\partial t} + (\mathbf{u}_T \cdot \nabla_\Gamma) \mathbf{u}_T \right) = -\nabla_\Gamma^T \pi + 2\mu \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}_T) + \mathbf{b}_T \\ \operatorname{div}_\Gamma \mathbf{u}_T = 0. \end{array} \right. \quad (3.13)$$

For the derivation of the first equation in (3.13) we used (3.10), $u_N = 0$, and

$$\begin{aligned}\partial_\Gamma^\bullet \mathbf{u}_T &= \mathbf{P} \left(\frac{\partial \mathbf{u}_T}{\partial t} + (\nabla \mathbf{u}_T) \mathbf{u} \right) = \mathbf{P} \left(\frac{\partial \mathbf{u}_T}{\partial t} + (\nabla \mathbf{u}_T) \mathbf{u}_T \right) \\ &= \frac{\partial \mathbf{u}_T}{\partial t} + (\nabla_\Gamma \mathbf{u}_T) \mathbf{u}_T =: \frac{\partial \mathbf{u}_T}{\partial t} + (\mathbf{u}_T \cdot \nabla_\Gamma) \mathbf{u}_T.\end{aligned}$$

The second equation in (3.9), or (3.12), reduces to

$$b_N = \mu \operatorname{tr}(\mathbf{H} \nabla_\Gamma \mathbf{u}_T) - \pi \kappa - \rho \mathbf{u}_T \cdot \mathbf{H} \mathbf{u}_T, \quad (3.14)$$

which describes the reaction force b_N of the surface flow \mathbf{u}_T . If there is no surface flow, i.e., $\mathbf{u}_T = 0$, this reaction force is the usual surface tension $\pi \kappa$, with a surface tension coefficient π .

In the second case, $\Gamma(0)$ is taken equal to the plane $z = 0$ in \mathbb{R}^3 . This is not a closed surface, but the derivation above also applies to connected surfaces without boundary, which may be unbounded. We consider $b_N = 0$, $u_N(0) = 1$. Only easily checks that independent of \mathbf{u}_T the second equation in (3.9) is satisfied for $u_N(\cdot, t) = 1$, $\dot{\mathbf{n}} = 0$, $\mathbf{H} = 0$ for all $t \geq 0$. Hence, the evolving surface is given by the plane $\Gamma(t) = \{(x, y, z) = (0, 0, t)\}$. The first and the third equations in (3.9) reduce to the standard planar Navier-Stokes equations for \mathbf{u}_T .

3.2. Other formulations of the surface Navier–Stokes equations. Different formulations of the surface Navier–Stokes equations are found in the literature. Some of them are formally obtained by substituting Cartesian differential operators by their geometric counterparts [36, 10] rather than from first mechanical principles. This leads to surface formulations which are not necessarily equivalent. The diagram below and identities (3.15) illustrate some “correspondences” between Cartesian and surface operators, where for the surface velocities we assume $u_N = 0$, i.e., $\mathbf{u} = \mathbf{u}_T$,

$$\begin{array}{ccccc} \mathbb{R}^{n-1} : & -\operatorname{div}(\nabla \mathbf{u} + \nabla^T \mathbf{u}) & \stackrel{\operatorname{div} \mathbf{u}=0}{=} & -\Delta \mathbf{u} & = & (\operatorname{rot}^T \operatorname{rot} - \nabla \operatorname{div}) \mathbf{u} \\ & \wr & & \wr & & \wr \\ \text{Manifold} : & \underbrace{-\mathbf{P} \operatorname{div}_\Gamma(2E_s(\mathbf{u}))}_{\substack{\text{surface} \\ \text{diffusion}}} & \stackrel{\operatorname{div}_\Gamma \mathbf{u}=0}{\neq} & \underbrace{-\Delta_\Gamma \mathbf{u}}_{\substack{\text{Bochner} \\ \text{Laplacian}}} & \neq & \underbrace{-\Delta_\Gamma^H \mathbf{u}}_{\substack{\text{Hodge} \\ \text{Laplacian}}} \end{array}$$

Moreover, for a surface in \mathbb{R}^3 we have, cf. (2.9), (2.13) and the Weitzenböck identity [31], the following equalities for \mathbf{u} such that $\operatorname{div}_\Gamma \mathbf{u} = 0$:

$$-\mathbf{P} \operatorname{div}_\Gamma(2E_s(\mathbf{u})) = -\Delta_\Gamma \mathbf{u} - K \mathbf{u} = -\Delta_\Gamma^H \mathbf{u} - 2K \mathbf{u}. \quad (3.15)$$

Formulations of the surface momentum equations employing the identity

$$-\mathbf{P} \operatorname{div}_\Gamma(2E_s(\mathbf{u})) = -\Delta_\Gamma^H \mathbf{u} - 2K \mathbf{u},$$

with the Hodge–de Rham Laplacian $-\Delta_\Gamma^H$ can be convenient for rewriting the problem in surface stream-function – vorticity variables, see, e.g., [24]. However, such a formulation is less convenient for the analysis of well-posedness, since the Gauss curvature K in general does not have a fixed sign. Moreover, in a numerical approximation of (3.13) one would have to approximate the Gauss curvature K based on a “discrete” (e.g., piecewise planar) surface approximation, which is known to be a delicate numerical issue.

In the remainder we restrict our discussion to the formulation with the surface rate-of-strain tensor $\mathbf{P} \operatorname{div}_\Gamma(E_s(\mathbf{u}))$. Using (3.15) we see that the Navier-Stokes system (3.13), which is the special case of (3.6) for a stationary surface, coincides with the Navier-Stokes equations (on a stationary surface) considered in [35] (see [35] section 6). We note that the authors of [19] also considered equations (3.6) simplified to the case of a stationary surface. They, however, claim to obtain a system different from the one in [35].

If the evolution of the surface is known *a priori*, then u_N is given, and the first and the third equations in (3.9) or (3.12) define a closed system for \mathbf{u}_T and π . The continuum mechanics corresponding to such a closed system is less clear to us, since the fundamental momentum balance (3.4) used to derive the equations does not assume any *a priori* constraint on u_N . Nevertheless, the resulting system may be of its own interest from a mathematical or numerical point of view.

3.3. Surface Stokes problem. The mathematical analysis of well-posedness of a problem as in (3.9) (or (3.6)) is a largely open question. In this paper, we study the well-posedness of a relatively simple special case, namely a *Stokes* problem on a *stationary* surface. We assume that $u_N = 0$ (stationary surface) and assume that the viscous surface forces dominate and thus it is reasonable to skip the nonlinear $\mathbf{u}_T \cdot \nabla_\Gamma \mathbf{u}_T$ term in the material derivative. Furthermore, we first restrict to the equilibrium flow problem, i.e., $\frac{\partial \mathbf{u}_T}{\partial t} = 0$. We thus obtain the *stationary surface Stokes* problem

$$\begin{aligned} -2\mu \mathbf{P} \operatorname{div}_\Gamma(E_s(\mathbf{u}_T)) + \nabla_\Gamma^T \pi &= \mathbf{b}_T, \\ \operatorname{div}_\Gamma \mathbf{u}_T &= 0. \end{aligned} \quad (3.16)$$

One readily observes that all constant pressure fields and tangentially rigid surface fluid motions are in the kernel of the differential operator on the left-hand side of the equation. Integration by parts, immediately implies the necessary consistency condition for the right-hand side of (3.16),

$$\int_\Gamma \mathbf{b}_T \mathbf{v}_T \, ds = 0 \quad \text{for all } \mathbf{v}_T \text{ s.t. } E_s(\mathbf{v}_T) = 0. \quad (3.17)$$

In the following sections we analyze different weak formulations of this Stokes problem.

The subspace of all tangential vector fields \mathbf{v}_T on Γ satisfying $E_s(\mathbf{v}_T) = 0$ plays an important role in the analysis of the surface Stokes problem. In the literature, such fields are known as *Killing vector fields*, see, e.g., [32]. For a smooth two-dimensional Riemannian manifold, Killing vector fields form a Lie Algebra, which dimension is at most 3. For a compact smooth surface Γ embedded in \mathbb{R}^3 the dimension of the algebra is 3 iff Γ is isometric to a 2D sphere.

4. A well-posed variational surface Stokes equation. Assume that Γ is a closed sufficiently smooth manifold. We introduce the space $V := H^1(\Gamma)^n$, with norm

$$\|\mathbf{u}\|_1^2 := \int_\Gamma \|\mathbf{u}(s)\|_2^2 + \|\nabla \mathbf{u}^e(s)\|_2^2 \, ds, \quad (4.1)$$

where $\|\cdot\|_2$ denotes the vector and matrix 2-norm. Here \mathbf{u}^e denotes the constant extension along normals of $\mathbf{u} : \Gamma \rightarrow \mathbb{R}^n$. We have $\nabla \mathbf{u}^e = \nabla(\mathbf{u} \circ p) = \nabla \mathbf{u}^e \mathbf{P}$, where p is the closest point projection onto Γ , hence only tangential derivatives are included in this H^1 -norm. We define the spaces

$$V_T := \{ \mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{n} = 0 \}, \quad E := \{ \mathbf{u} \in V_T \mid E_s(\mathbf{u}) = 0 \}. \quad (4.2)$$

Note that E is a closed subspace of V_T and $\dim(E) \leq 3$. We use an orthogonal decomposition $V_T = V_T^0 \oplus E$ with the Hilbert space $V_T^0 = E^\perp$ (hence $V_T^0 \sim V_T/E$). We also need the factor space $L_0^2(\Gamma) := \{p \in L^2(\Gamma) \mid \int_\Gamma p \, dx = 0\} \sim L^2(\Gamma)/\mathbb{R}$. We introduce the bilinear forms

$$a(\mathbf{u}, \mathbf{v}) := 2\mu \int_\Gamma E_s(\mathbf{u}) : E_s(\mathbf{v}) \, ds = 2\mu \int_\Gamma \text{tr}(E_s(\mathbf{u})E_s(\mathbf{v})) \, ds, \quad \mathbf{u}, \mathbf{v} \in V, \quad (4.3)$$

$$b(\mathbf{u}, p) := - \int_\Gamma p \, \text{div}_\Gamma \mathbf{u} \, ds, \quad \mathbf{u} \in V, p \in L^2(\Gamma). \quad (4.4)$$

We take $f \in V'$, such that $f(\mathbf{v}_T) = 0$ for all $\mathbf{v}_T \in E$, and consider the following variational Stokes problem: determine $(\mathbf{u}_T, p) \in V_T^0 \times L_0^2(\Gamma)$ such that

$$\begin{aligned} a(\mathbf{u}_T, \mathbf{v}_T) + b(\mathbf{v}_T, p) &= f(\mathbf{v}_T) \quad \text{for all } \mathbf{v}_T \in V_T, \\ b(\mathbf{u}_T, q) &= 0 \quad \text{for all } q \in L^2(\Gamma). \end{aligned} \quad (4.5)$$

This weak formulation is consistent to the strong one in (3.16) for $f(\mathbf{v}_T) = (\mathbf{b}_T, \mathbf{v}_T)_0$. Note that $E_s(\mathbf{v}_T) = 0$ implies $\text{tr}(\nabla_\Gamma \mathbf{v}_T) = 0$ and thus $\text{div}_\Gamma \mathbf{v}_T = 0$, hence, $b(\mathbf{v}_T, p) = 0$ for all $\mathbf{v}_T \in E$. From this it follows that the first equation in (4.5) is always satisfied for all $\mathbf{v}_T \in E$, hence it is not relevant whether we use V_T or V_T^0 as space of test functions. For the analysis of well-posedness a surface Korn's inequality is a crucial ingredient. Although there are results in the literature on Korn's type equalities on surfaces, e.g. [11, 21], these are related to surface models of thin shells, such as Koiter's model, which contain derivatives in the direction of the normal displacement. In the literature we did not find a result of the type given in (4.6) below, and therefore we include a proof.

LEMMA 4.1. *Assume Γ is C^2 smooth. There exists $c_K > 0$ such that*

$$\|E_s(\mathbf{u})\|_{L^2(\Gamma)} \geq c_K \|\mathbf{u}\|_1 \quad \text{for all } \mathbf{u} \in V_T^0. \quad (4.6)$$

Proof. Let $\mathbf{u} = \mathbf{u}_T \in V_T^0$ be given. Throughout this proof, the extension \mathbf{u}^e is also denoted by \mathbf{u} . Since $\nabla \mathbf{u}^e = \nabla \mathbf{u}$ includes only tangential derivatives we introduce the notation

$$\nabla_P \mathbf{u} := (\nabla \mathbf{u})\mathbf{P} = \nabla \mathbf{u}^e$$

for the tangential derivative. Furthermore, the symmetric part of the tangential derivative tensor is denoted by $\mathbf{e}_s(\mathbf{u}) := \frac{1}{2}(\nabla_P \mathbf{u} + \nabla_P^T \mathbf{u})$. Below we derive the following inequality:

$$\|\mathbf{u}\|_{L^2(\Gamma)} + \|\mathbf{e}_s(\mathbf{u})\|_{L^2(\Gamma)} \geq c \|\mathbf{u}\|_1 \quad \text{for all } \mathbf{u} \in V_T. \quad (4.7)$$

Recall (2.17), $\mathbf{H}\mathbf{u} = -(\nabla^T \mathbf{u})\mathbf{n}$. Using this and $\mathbf{P} = \mathbf{I} - \mathbf{n}\mathbf{n}^T$ we get $\nabla_\Gamma^T \mathbf{u} = \mathbf{P}\nabla^T \mathbf{u}\mathbf{P} = \mathbf{P}\nabla^T \mathbf{u} - \mathbf{P}(\nabla^T \mathbf{u})\mathbf{n}\mathbf{n}^T = \nabla_P^T \mathbf{u} + \mathbf{H}\mathbf{u}\mathbf{n}^T$, and thus we get the identity

$$E_s(\mathbf{u}) = \mathbf{e}_s(\mathbf{u}) + \frac{1}{2}(\mathbf{H}\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T\mathbf{H}).$$

Since the surface is C^2 -smooth this equality implies $\|\mathbf{e}_s(\mathbf{u})\|_{L^2(\Gamma)} \leq \|E_s(\mathbf{u})\|_{L^2(\Gamma)} + c\|\mathbf{u}\|_{L^2(\Gamma)}$, and combining this with (4.7) yields

$$\|\mathbf{u}\|_{L^2(\Gamma)} + \|E_s(\mathbf{u})\|_{L^2(\Gamma)} \geq c\|\mathbf{u}\|_1 \quad \text{for all } \mathbf{u} \in V_T, \quad (4.8)$$

with some $c > 0$. We now apply the Petree-Tartar Lemma, e.g. Lemma A.38 in [14] to $E_s \in \mathcal{L}(V_T^0, L^2(\Gamma)^{3 \times 3})$, which is injective, and the compact embedding $\text{id} : V_T^0 \rightarrow L^2(\Gamma)^3$. Application of this lemma yields the desired result.

It remains to proof the inequality (4.7). We use a local parametrization of Γ and a standard Korn's inequality in Euclidean space.

Let $\omega \subset \mathbb{R}^{n-1}$ be a bounded open connected domain and $\Phi : \omega \rightarrow \Gamma$ a local parametrization of Γ ; $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}\}$ denotes the Cartesian basis in \mathbb{R}^{n-1} . Partial derivatives of $\Phi(\xi) = \Phi(\xi_1, \dots, \xi_{n-1})$ are denoted by $\mathbf{a}_\alpha(\xi) := \frac{\partial \Phi(\xi)}{\partial \xi_\alpha} \in \mathbb{R}^n$, $\alpha = 1, \dots, n-1$. Below we often skip the argument $\xi \in \omega$. Greek indices always range from 1 to $n-1$, and roman indices from 1 to n . We furthermore define $\mathbf{a}_n := \mathbf{n}$. The dual basis (or contravariant basis) is given by \mathbf{a}^β such that $\mathbf{P}\mathbf{a}^\beta = \mathbf{a}^\beta$ and $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = 0$ for $\alpha \neq \beta$ and $\mathbf{a}^\beta \cdot \mathbf{a}_\beta = 1$. Furthermore $\mathbf{a}^n := \mathbf{a}_n$. Note that $\mathbf{P}\mathbf{a}_\alpha = \mathbf{a}_\alpha$, $\mathbf{P}\mathbf{a}^\alpha = \mathbf{a}^\alpha$, $\mathbf{P}\mathbf{a}_n = \mathbf{P}\mathbf{a}^n = 0$. A given vector function $\mathbf{u} : \Gamma \rightarrow \mathbb{R}^n$ is pulled back to ω as follows:

$$\vec{\mathbf{u}} = (\vec{u}_1, \dots, \vec{u}_{n-1}) : \omega \rightarrow \mathbb{R}^{n-1}, \quad \vec{u}_\alpha := (\mathbf{u} \circ \Phi) \cdot \mathbf{a}_\alpha.$$

Note that $\mathbf{u} \circ \Phi = \vec{u}_\alpha \mathbf{a}^\alpha$ (Einstein summation convention). We also use the standard notation $\vec{u}_{\alpha,\beta} := \frac{\partial \vec{u}_\alpha}{\partial \xi_\beta}$. Note that $(\mathbf{a}^\lambda \cdot \mathbf{a}_\alpha)_{,\beta} = 0$ and thus $\mathbf{a}^\lambda \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{a}_{\lambda,\beta} \cdot \mathbf{a}_\alpha$ holds. Using this we get

$$\begin{aligned} \vec{u}_{\alpha,\beta} &= \mathbf{a}_\alpha \cdot \nabla(\mathbf{u} \circ \Phi) \boldsymbol{\xi}_\beta + (\mathbf{u} \circ \Phi) \cdot \mathbf{a}_{\alpha,\beta} = \mathbf{a}_\alpha \cdot (\nabla \mathbf{u} \circ \Phi) \mathbf{a}_\beta + (\vec{u}_\lambda \mathbf{a}^\lambda) \cdot \mathbf{a}_{\alpha,\beta} \\ &= \mathbf{a}_\alpha \cdot (\nabla_P \mathbf{u} \circ \Phi) \mathbf{a}_\beta + \vec{u}_\lambda (\mathbf{a}^\lambda \cdot \mathbf{a}_{\alpha,\beta}) = \mathbf{a}_\alpha \cdot (\nabla_P \mathbf{u} \circ \Phi) \mathbf{a}_\beta - \vec{u}_\lambda \mathbf{a}_{\lambda,\beta} \cdot \mathbf{a}_\alpha. \end{aligned}$$

Now note that for $\xi \in \omega$ and $x := \Phi(\xi)$ we have

$$\begin{aligned} \nabla_P(\mathbf{a}^\lambda \circ \Phi^{-1}(x)) \mathbf{a}_\beta(\xi) &= \nabla(\mathbf{a}^\lambda \circ \Phi^{-1}(x)) \mathbf{a}_\beta(\xi) = \nabla \mathbf{a}^\lambda(\xi) \nabla \Phi^{-1}(x) \mathbf{a}_\beta(\xi) \\ &= \nabla \mathbf{a}^\lambda(\xi) [\nabla \Phi(\xi)]^{-1} \mathbf{a}_\beta(\xi) = \nabla \mathbf{a}^\lambda(\xi) \boldsymbol{\xi}_\beta = \frac{\partial \mathbf{a}^\lambda(\xi)}{\partial \xi_\beta} = \mathbf{a}_{\lambda,\beta}(\xi). \end{aligned}$$

Using this in the relation above we obtain

$$\vec{u}_{\alpha,\beta}(\xi) = \mathbf{a}_\alpha(\xi) \cdot (\nabla_P \mathbf{u}(x) - \vec{u}_\lambda(\xi) \nabla_P(\mathbf{a}^\lambda \circ \Phi^{-1})(x)) \mathbf{a}_\beta(\xi), \quad \xi \in \omega, \quad x = \Phi(\xi). \quad (4.9)$$

The symmetric part of the Jacobian in \mathbb{R}^{n-1} is denoted by $E(\vec{\mathbf{u}})_{\alpha\beta} = \frac{1}{2}(\vec{u}_{\alpha,\beta} + \vec{u}_{\beta,\alpha})$. Thus we get (we skip the arguments again):

$$E(\vec{\mathbf{u}})_{\alpha\beta} = \mathbf{a}_\alpha \cdot (\mathbf{e}_s(\mathbf{u}) - \vec{u}_\lambda \mathbf{e}_s(\mathbf{a}^\lambda \circ \Phi^{-1})) \mathbf{a}_\beta. \quad (4.10)$$

From this we get, using the C^2 smoothness of the manifold:

$$\|E(\vec{\mathbf{u}})(\xi)\|_2 \leq c(\|\mathbf{e}_s(\mathbf{u})(x)\|_2 + \|\vec{\mathbf{u}}(\xi)\|_2) \leq c(\|\mathbf{e}_s(\mathbf{u})(x)\|_2 + \|\mathbf{u}(x)\|_2), \quad (4.11)$$

for $\xi \in \omega$, $x = \Phi(\xi)$. Now we derive a bound for $\|\nabla_P \mathbf{u}(x)\|_2$ in terms of $\|\nabla \vec{\mathbf{u}}(\xi)\|_2$. Let e_i be the standard basis in \mathbb{R}^n . Note that $e_i = (e_i \cdot \mathbf{a}^l) \mathbf{a}_l$. Using this, $(\nabla_P \mathbf{u}) \mathbf{n} = 0$ and (4.9) we get (we skip the arguments ξ and x):

$$\begin{aligned} e_j \cdot \nabla_P \mathbf{u} e_i &= (e_i \cdot \mathbf{a}^l)(e_j \cdot \mathbf{a}^m) \mathbf{a}_m \nabla_P \mathbf{u} \mathbf{a}_l \\ &= (e_i \cdot \mathbf{a}^\beta)(e_j \cdot \mathbf{a}^\alpha) \mathbf{a}_\alpha \cdot \nabla_P \mathbf{u} \mathbf{a}_\beta + (e_i \cdot \mathbf{a}^\beta)(e_j \cdot \mathbf{n}) \mathbf{n} \cdot \nabla_P \mathbf{u} \mathbf{a}_\beta \\ &= (e_i \cdot \mathbf{a}^\beta)(e_j \cdot \mathbf{a}^\alpha) (\vec{u}_{\alpha,\beta} + \vec{u}_\lambda \mathbf{a}_\alpha \cdot \nabla_P(\mathbf{a}^\lambda \circ \Phi^{-1}) \mathbf{a}_\beta) \\ &\quad + (e_i \cdot \mathbf{a}^\beta)(e_j \cdot \mathbf{n}) \mathbf{n} \cdot \nabla_P \mathbf{u} \mathbf{a}_\beta. \end{aligned}$$

Note that

$$\mathbf{n} \cdot \nabla_P \mathbf{u} \mathbf{a}_\beta = \mathbf{n} \cdot (\nabla \mathbf{u}) \mathbf{P} \mathbf{a}_\beta = \mathbf{n} \cdot (\nabla \mathbf{u}) \mathbf{a}_\beta = (\nabla^T) \mathbf{u} \mathbf{n} \cdot \mathbf{a}_\beta = -\mathbf{H} \mathbf{u} \cdot \mathbf{a}_\beta = -\mathbf{u} \cdot \mathbf{H} \mathbf{a}_\beta.$$

Using this in the relation above and using the smoothness of Γ then yields

$$\|\nabla_P \mathbf{u}(x)\|_2 \leq c(\|\nabla \bar{\mathbf{u}}(\xi)\|_2 + \|\bar{\mathbf{u}}(\xi)\|_2 + \|\mathbf{u}(x)\|_2) \leq c(\|\nabla \bar{\mathbf{u}}(\xi)\|_2 + \|\bar{\mathbf{u}}(\xi)\|_2), \quad (4.12)$$

for $\xi \in \omega$, $x = \Phi(\xi)$. For $\omega \subset \mathbb{R}^{n-1}$ we have the Korn inequality

$$\int_\omega (\|E(\bar{\mathbf{u}})\|_2^2 + \|\bar{\mathbf{u}}\|_2^2) d\xi \geq c_K \int_\omega \|\nabla \bar{\mathbf{u}}\|_2^2 d\xi, \quad (4.13)$$

with $c_K = c_K(\omega) > 0$. Since Γ is compact, there is a finite number of maps $\Phi_i : \omega_i \rightarrow \Phi_i(\omega_i) \subset \Gamma$, $i = 1, \dots, N$, which form a parametrization of Γ . Using the results in (4.12), (4.13) and (4.11) we then get

$$\begin{aligned} \|\mathbf{u}\|_1^2 &= \int_\Gamma \|\nabla_P \mathbf{u}(x)\|_2^2 + \|\mathbf{u}(x)\|_2^2 dx \leq N \max_{1 \leq i \leq N} \int_{\Phi_i(\omega_i)} \|\nabla_P \mathbf{u}(x)\|_2^2 + \|\mathbf{u}(x)\|_2^2 dx \\ &\leq c \int_{\omega_i} (\|\nabla \bar{\mathbf{u}}(\xi)\|_2^2 + \|\bar{\mathbf{u}}(\xi)\|_2^2) |\det(\nabla \Phi_i)| d\xi \\ &\leq c \int_{\omega_i} \|E(\bar{\mathbf{u}})(\xi)\|_2^2 + \|\bar{\mathbf{u}}(\xi)\|_2^2 |\det(\nabla \Phi_i)| d\xi \\ &\leq c \int_{\Phi_i(\omega_i)} \|\mathbf{e}_s(\mathbf{u})(x)\|_2^2 + \|\mathbf{u}(x)\|_2^2 dx \leq c \int_\Gamma \|\mathbf{e}_s(\mathbf{u})(x)\|_2^2 + \|\mathbf{u}(x)\|_2^2 dx, \end{aligned}$$

from which the inequality in (4.7) easily follows. \square

Korn's inequality implies ellipticity of the bilinear form $a(\cdot, \cdot)$ on V_T^0 . In the next lemma we treat the second main ingredient needed for well-posedness of the Stokes saddle point problem, namely an inf-sup property of $b(\cdot, \cdot)$.

LEMMA 4.2. *The following inf-sup estimate holds:*

$$\inf_{p \in L_0^2(\Gamma)} \sup_{\mathbf{v}_T \in V_T^0} \frac{b(\mathbf{v}_T, p)}{\|\mathbf{v}_T\|_1} \geq c > 0. \quad (4.14)$$

Proof. Take $p \in L_0^2(\Gamma)$. Let $\phi \in H^1(\Gamma) \cap L_0^2(\Gamma)$ be the solution of

$$\Delta_\Gamma \phi = p \quad \text{on } \Gamma.$$

For ϕ we have the regularity estimate $\|\phi\|_{H^2(\Gamma)} \leq c\|p\|_{L^2}$. Take $\mathbf{v}_T := -\nabla_\Gamma^T \phi \in V_T$, and the orthogonal decomposition $\mathbf{v}_T = \mathbf{v}_T^0 + \tilde{\mathbf{v}}$, with $\mathbf{v}_T^0 \in V_T^0$, $\tilde{\mathbf{v}} \in E$. We have $\|\mathbf{v}_T^0\|_1 \leq \|\mathbf{v}_T\|_1 \leq c\|\phi\|_{H^2(\Gamma)} \leq c\|p\|_{L^2}$. Furthermore, $E_s(\tilde{\mathbf{v}}) = 0$ implies $\text{div}_\Gamma \tilde{\mathbf{v}} = 0$ and thus $b(\mathbf{v}_T^0, p) = b(\mathbf{v}_T, p)$. Using this we get

$$\frac{b(\mathbf{v}_T^0, p)}{\|\mathbf{v}_T^0\|_1} = \frac{b(\mathbf{v}_T, p)}{\|\mathbf{v}_T\|_1} = \frac{\int_\Gamma \Delta_\Gamma \phi p ds}{\|\mathbf{v}_T^0\|_1} = \frac{\|p\|_{L^2}^2}{\|\mathbf{v}_T^0\|_1} \geq c\|p\|_{L^2}, \quad (4.15)$$

which completes the proof. \square

THEOREM 4.3. *The weak formulation (4.5) is well-posed.*

Proof. Note that $\|E_s(\mathbf{u})\|_{L^2} \leq \|\nabla \mathbf{u}^e\|_{L^2}$ and $\|\text{div}_\Gamma \mathbf{u}\|_{L^2} \leq n\|\nabla_\Gamma \mathbf{u}\|_{L^2} = n\|\nabla \mathbf{u}^e\|_{L^2}$ hold. From this it follows that the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous on $V_T \times V_T$ and $V_T \times L_0^2(\Gamma)$, respectively. Ellipticity of $a(\cdot, \cdot)$ follows from Lemma 4.1 and the inf-sup property of $b(\cdot, \cdot)$ is derived in Lemma 4.2. \square

5. A well-posed variational Stokes problem with Lagrange multiplier.

In the formulation (4.5) the velocity \mathbf{u}_T is tangential to the surface. For Galerkin discretization methods, such as a finite element method, this may be less convenient, cf. Remark 6.2. In this section we consider a variational formulation in a space, which does not contain the restriction $\mathbf{n} \cdot \mathbf{u} = 0$. The latter constraint is treated using a Lagrange multiplier.

We recall the notation $\mathbf{u} = \mathbf{u}_T + u_N \mathbf{n}$ for $\mathbf{u} \in V$ and we define the following Hilbert space:

$$V_* := \{ \mathbf{u} \in L^2(\Gamma)^n : \mathbf{u}_T \in V_T, u_N \in L^2(\Gamma) \}, \quad \text{with } \|\mathbf{u}\|_{V_*}^2 := \|\mathbf{u}_T\|_1^2 + \|u_N\|_{L^2(\Gamma)}^2.$$

Note that $V_* \sim V_T \oplus L^2(\Gamma)$ and $E \subset V_T \subset V_*$ is a closed subspace of V_* . Thus the space $V_*^0 := E^\perp \subset V_* \sim V_T^0 \oplus L^2(\Gamma)$ is a Hilbert space. We introduce the bilinear form

$$\tilde{b}(\mathbf{u}, \{p, \lambda\}) = - \int_{\Gamma} \operatorname{div}_{\Gamma} \mathbf{u}_T p \, ds + \int_{\Gamma} \lambda u_N \, ds = b(\mathbf{u}_T, p) + (\lambda, u_N)_{L^2(\Gamma)}.$$

on $V_* \times (L_0^2(\Gamma) \times L^2(\Gamma))$. Based on the identity (3.11) we introduce (with an abuse of notation, cf. (4.3)) the bilinear form

$$a(\mathbf{u}, \mathbf{v}) := 2\mu \int_{\Gamma} \operatorname{tr}((E_s(\mathbf{u}_T) + u_N \mathbf{H})(E_s(\mathbf{v}_T) + v_N \mathbf{H})) \, ds, \quad \mathbf{u}, \mathbf{v} \in V_*. \quad (5.1)$$

In this bilinear form we need $H^1(\Gamma)$ smoothness of the tangential component \mathbf{u}_T and only $L^2(\Gamma)$ smoothness of the normal component u_N . If the latter component has also $H^1(\Gamma)$ smoothness, then from (3.11) we get

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_{\Gamma} \operatorname{tr}(E_s(\mathbf{u})E_s(\mathbf{v})) \, ds, \quad \text{for } \mathbf{u}, \mathbf{v} \in V. \quad (5.2)$$

The bilinear form $a(\cdot, \cdot)$ is continuous:

$$a(\mathbf{u}, \mathbf{v}) \leq c \|\mathbf{u}\|_{V_*} \|\mathbf{v}\|_{V_*} \quad \forall \mathbf{u}, \mathbf{v} \in V_*.$$

For $f \in V_*'$ such that $f(\mathbf{v}_T) = 0$ for all $\mathbf{v}_T \in E$, we consider the modified Stokes weak formulation: Determine $(\mathbf{u}, \{p, \lambda\}) \in V_*^0 \times (L_0^2(\Gamma) \times L^2(\Gamma))$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + \tilde{b}(\mathbf{v}, \{p, \lambda\}) &= f(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V_*^0, \\ \tilde{b}(\mathbf{u}, \{q, \nu\}) &= 0 \quad \text{for all } \{q, \nu\} \in L_0^2(\Gamma) \times L^2(\Gamma). \end{aligned} \quad (5.3)$$

One easily checks that this weak formulation is consistent to the strong one in (3.16). Below in Remark 5.1 we explain that the test space V_*^0 in the first equation in (5.3) can be replaced by V_* .

THEOREM 5.1. *The problem (5.3) is well-posed. Furthermore, its unique solution satisfies $\mathbf{u} \cdot \mathbf{n} = 0$.*

Proof. The bilinear forms $a(\cdot, \cdot)$ and $\tilde{b}(\cdot, \{\cdot, \cdot\})$ are continuous on $V_* \times V_*$ and $V_* \times (L_0^2(\Gamma) \times L^2(\Gamma))$, respectively. It is not clear whether $a(\cdot, \cdot)$ is elliptic on V_*^0 . For well-posedness, however, it is sufficient to have ellipticity of this bilinear form on the kernel of $\tilde{b}(\cdot, \{\cdot, \cdot\})$:

$$\mathcal{K} := \{ \mathbf{u} \in V_*^0 \mid \tilde{b}(\mathbf{u}, \{p, \lambda\}) = 0 \text{ for all } \{p, \lambda\} \in L_0^2(\Gamma) \times L^2(\Gamma) \}.$$

Note that

$$\mathcal{K} \subset \mathcal{K}_0 := \{ \mathbf{u} \in V_*^0 \mid \tilde{b}(\mathbf{u}, \{0, \lambda\}) = 0 \text{ for all } \lambda \in L^2(\Gamma) \} = \{ \mathbf{u} \in V_*^0 \mid u_N = 0 \}.$$

Using Lemma 4.1 it follows that

$$a(\mathbf{u}, \mathbf{u}) = a(\mathbf{u}_T, \mathbf{u}_T) \geq 2\mu c_K^2 \|\mathbf{u}_T\|_1^2 = 2\mu c_K^2 \|\mathbf{u}\|_{V_*}^2 \text{ for all } \mathbf{u} \in \mathcal{K}_0, \quad (5.4)$$

and thus we have ellipticity of $a(\cdot, \cdot)$ on the kernel of $\tilde{b}(\cdot, \{\cdot, \cdot\})$. It remains to check the inf-sup condition for $\tilde{b}(\cdot, \{\cdot, \cdot\})$. Take $\{p, \lambda\} \in L_0^2(\Gamma) \times L^2(\Gamma)$. Take $\mathbf{v}_T^0 \in V_T^0$ such that

$$\frac{b(\mathbf{v}_T^0, p)}{\|\mathbf{v}_T^0\|_1} \geq c\|p\|_{L^2}$$

holds, with $c > 0$, cf. Lemma 4.2. Take $\mathbf{v} := \mathbf{v}_T^0 + \lambda \mathbf{n} \in V_*^0$, hence $\|\mathbf{v}\|_{V_*}^2 = \|\mathbf{v}_T^0\|_1^2 + \|\lambda\|_{L^2(\Gamma)}^2$. We get:

$$\begin{aligned} \tilde{b}(\mathbf{v}, \{p, \lambda\}) &= b(\mathbf{v}_T^0, p) + \|\lambda\|_{L^2}^2 \geq c\|p\|_{L^2} \|\mathbf{v}_T^0\|_1 + \|\lambda\|_{L^2}^2 \\ &\geq c(\|p\|_{L^2}^2 + \|\lambda\|_{L^2}^2)^{\frac{1}{2}} \|\mathbf{v}\|_{V_*}. \end{aligned}$$

Hence, the required inf-sup property holds, from which the well-posedness result follows. If in the second equation in (5.3) we take $q = 0$ and $\nu \in L^2(\Gamma)$ arbitrary, it follows that for the solution \mathbf{u} we have $u_N = 0$, i.e., $\mathbf{u} \cdot \mathbf{n} = 0$ holds. \square

REMARK 5.1. If in the first equation in (5.3) we take $v_N = 0$, $\mathbf{v}_T \in E$, it follows from $E_s(\mathbf{v}_T) = 0$, $\tilde{b}(\mathbf{v}, \{p, \lambda\}) = b(\mathbf{v}_T, p) = 0$, $f(\mathbf{v}) = f(\mathbf{v}_T) = 0$ that the first equation in (5.3) is satisfied for all $\mathbf{v}_T \in E$, hence the test space V_*^0 can be replaced by V_* (which is convenient in a Galerkin method).

For the unique solution \mathbf{u} we have $u_N = 0$, and taking $v_N = 0$, $\nu = 0$ it follows that if $f(\mathbf{v}) = f(\mathbf{v}_T)$ then (\mathbf{u}_T, p) coincides with the unique solution of (4.5). In this sense, the problem (5.3) for $(\mathbf{u}, \{p, \lambda\}) \in V_*^0 \times (L_0^2(\Gamma) \times L^2(\Gamma))$ is a *consistent generalization* of the problem (4.5) for $(\mathbf{u}_T, p) \in V_T^0 \times L_0^2(\Gamma)$.

6. Well-posed augmented variational formulations. Another way to relax the tangential constraint in the test and trial spaces is to augment the weak formulation (4.5) with a normal term such that the augmented bilinear form defines an inner product in V_* . The augmentation can be done for the bilinear form $a(\cdot, \cdot)$ used in (4.5) as well as for the one used in (5.3). Given an augmentation parameter $\tau \geq 0$, we define

$$\begin{aligned} a_\tau(\mathbf{u}, \mathbf{v}) &:= 2\mu \int_\Gamma E_s(\mathbf{u}_T) : E_s(\mathbf{v}_T) ds + \tau \int_\Gamma u_N v_N ds \\ &= a(\mathbf{u}_T, \mathbf{v}_T) + \tau(u_N, v_N)_{L^2(\Gamma)}, \\ \hat{a}_\tau(\mathbf{u}, \mathbf{v}) &:= 2\mu \int_\Gamma E_s(\mathbf{u}) : E_s(\mathbf{v}) ds + \tau \int_\Gamma u_N v_N ds \\ &= a(\mathbf{u}, \mathbf{v}) + \tau(u_N, v_N)_{L^2(\Gamma)}, \end{aligned} \quad (6.1)$$

for $\mathbf{u}, \mathbf{v} \in V_*$. We consider, for $\tau > 0$, the following two problems: determine $(\mathbf{u}, p) \in V_*^0 \times L_0^2(\Gamma)$ such that

$$(a) \begin{cases} a_\tau(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}_T, p) = f(\mathbf{v}_T), \\ b(\mathbf{u}_T, q) = 0, \end{cases} \quad \text{or} \quad (b) \begin{cases} \hat{a}_\tau(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}_T, p) = f(\mathbf{v}_T), \\ b(\mathbf{u}_T, q) = 0, \end{cases} \quad (6.2)$$

for all $\mathbf{v} \in V_*$, $q \in L^2(\Gamma)$. Well-posedness of these formulations is given in the following theorem.

THEOREM 6.1. *The problem (6.2)(a) is well-posed. The problem (6.2)(b) is well-posed for sufficiently large $\tau > 0$. In (6.2)(b) we take $\tau > 0$ sufficiently large such that this problem is well-posed. The unique solution \mathbf{u} of (6.2)(a) satisfies $\mathbf{u} \cdot \mathbf{n} = 0$ and \mathbf{u}_T coincides with the unique solution of (4.5). For the tangential part $\hat{\mathbf{u}}_T$ of $\hat{\mathbf{u}}$, the unique solution of (6.2)(b), the following estimate holds*

$$\|\hat{\mathbf{u}}_T - \mathbf{u}_T\|_1 \leq C \tau^{-\frac{1}{2}} \|f\|_{V'}, \quad (6.3)$$

where C depends only on Γ .

Proof. Note that due to Korn's inequality in Lemma 4.1 we have

$$a_\tau(\mathbf{u}, \mathbf{u}) \geq 2\mu c_K^2 \|\mathbf{u}_T\|_1^2 + \tau \|u_N\|_{L^2}^2 \geq \min\{2\mu c_K^2, \tau\} \|\mathbf{u}\|_{V_*}^2.$$

Hence for any $\tau > 0$, $a_\tau(\mathbf{u}, \mathbf{v})$ defines a scalar product on V_* . We already discussed in section 5 that the bilinear form $\hat{a}_\tau(\mathbf{u}, \mathbf{v})$ is well-defined on V_* due to the identity (3.11). If τ is sufficiently large, for example, $\tau > \|\mathbf{H}\|_{L^\infty(\Gamma)}$, then $\hat{a}_\tau(\mathbf{u}, \mathbf{v})$ also defines a scalar product on V_* . The inf-sup property for $b(\cdot, \cdot)$ on $V_*^0 \times L_0^2(\Gamma)$ immediately follows from the one on $V_T^0 \times L_0^2(\Gamma)$, i.e., (4.14). Hence we obtain the well-posedness of both problems. It is easy to check that $\mathbf{u} = \mathbf{u}_T$, with \mathbf{u}_T the solution of (4.5), solves the augmented problem in (6.2)(a). Denote by $\hat{\mathbf{u}}, \hat{p}$ the solution of (6.2)(b). By testing the weak formulation with $\mathbf{v} = \hat{\mathbf{u}}$, $q = p$, and applying Korn's inequality we obtain the estimate for the normal part of $\hat{\mathbf{u}}$,

$$\|\hat{u}_N\|_{L^2(\Gamma)} \leq C \tau^{-\frac{1}{2}} \|f\|_{V'}.$$

For arbitrary $\mathbf{v}_T \in V_T$ we have thanks to (3.11), (4.5) and (6.2),

$$\begin{aligned} a_\tau(\hat{\mathbf{u}}_T - \mathbf{u}_T, \mathbf{v}_T) &= -2\mu \int_\Gamma \hat{u}_N \mathbf{H} : E_s(\mathbf{v}_T) ds + b(\mathbf{v}_T, p - \hat{p}) \\ &\leq C \|\hat{u}_N\|_{L^2(\Gamma)} \|\mathbf{v}_T\|_1 + b(\mathbf{v}_T, p - \hat{p}) \leq C \tau^{-\frac{1}{2}} \|f\|_{V'} \|\mathbf{v}_T\|_1 + b(\mathbf{v}_T, p - \hat{p}). \end{aligned}$$

Taking $\mathbf{v}_T = \hat{\mathbf{u}}_T - \mathbf{u}_T$ the pressure term vanishes and using Korn's inequality for the left-hand side leads to (6.3). \square

The well-posedness statements in the theorem above still hold if $f(\mathbf{v}_T)$ is replaced by $f(\mathbf{v})$, with $f \in V_*'$. We close this section with a few remarks.

REMARK 6.1. We briefly address properties of the different variational formulations (4.5), (5.3) and (6.2) that we consider relevant for discretization by Galerkin methods such as fitted or unfitted finite element methods for PDEs posed on surfaces [12, 25]. In such finite element methods one usually approximates a smooth surface Γ by a triangulated Lipschitz surface Γ_h . The normal vector field \mathbf{n}_h to such a surface is no longer continuous. Enforcing strongly the tangential condition $\mathbf{u} \cdot \mathbf{n}_h = 0$ for the numerical solution can be inconvenient if standard $H^1(\Gamma)^3$ -conforming finite elements are used. Formulations (5.3) and (6.2) allow to enforce the tangential condition weakly and occur to us more suitable for numerical purposes. In (5.3) one needs a suitable finite element space for the Lagrange multiplier λ . This is avoided in (6.2), but that formulation requires a suitable value for the penalty parameter τ . Note that the formulations in (5.3) and (6.2)(a) are consistent with (4.5), in particular the solution $\mathbf{u} \in V_*^0$ has the property $\mathbf{u} \cdot \mathbf{n} = 0$. The problem in (6.2)(b) is not consistent. However,

compared to (6.2)(a) the formulation in (6.2)(b) has the attractive property that one has to approximate $\nabla_\Gamma \mathbf{u} = \mathbf{P} \nabla \mathbf{u} \mathbf{P}$ instead of $\nabla_\Gamma \mathbf{u}_T = \mathbf{P} \nabla \mathbf{u}_T \mathbf{P} = \mathbf{P} \nabla (\mathbf{P} \mathbf{u}) \mathbf{P}$. Hence, in (6.2)(b) differentiation of \mathbf{P} is avoided. A finite element discretization for a vector surface Laplace problem (instead of Stokes) based on an augmented formulation very similar to the one in (6.2)(b) has been studied in the recent paper [17]. Finally note that $b(\mathbf{u}, p) = - \int_\Gamma p \operatorname{div}_\Gamma \mathbf{u}_T ds = - \int_\Gamma p \operatorname{div}_\Gamma (\mathbf{P} \mathbf{u}) ds$, used in both (5.3) and (6.2), requires a differentiation of \mathbf{P} . If in the finite element method we have $p = p_h \in H^1(\Gamma)$ we can use $b(\mathbf{u}, p) = \int_\Gamma \mathbf{u}_T \nabla_\Gamma p ds$ and thus avoid this differentiation. be considered.

REMARK 6.2. The formal extension of the weak formulations in (4.5), (5.3) and (6.2) to the Navier-Stokes equations (3.13) on stationary surfaces is straightforward, but not studied in this paper.

7. Conclusions and outlook. Based on surface mass and momentum conservation laws we derived the surface Navier-Stokes equations (3.6), which can be found in several other papers in the literature. All differential operators used are defined in terms of first (partial) derivatives in the outer Euclidean space \mathbb{R}^3 . Relations to formulations presented in the setting of differential geometry (e.g., Bochner and Hodge-de Rham Laplacians) are briefly addressed. Well-posedness results of several variational formulations of a Stokes problem on a stationary surface are presented. For this a surface Korn's inequality and an inf-sup property for the Stokes bilinear form $b(\cdot, \cdot)$ are derived.

In a forthcoming paper we will present results of numerical experiments with finite element methods applied to the different variational formulations of the Stokes problem. Furthermore, we plan to develop error analyses for these finite element discretization methods. Clearly, there are many other related topics that can be addressed in future research. For example, an extension of the well-posedness analysis presented in this paper to the case of a Stokes problem on an evolving surface, the extension from Stokes to an Oseen or Navier-Stokes equation on a stationary (or even evolving) surface, or an analysis of a coupled surface-bulk flow problem. Related to the latter we note that first results on well-posedness of such a coupled problem have recently been presented in [20]. Furthermore, a further study and validation of such surface Navier-Stokes equations (coupled with bulk fluids) based on numerical simulations is an open research field.

REFERENCES

- [1] R. ARIS, *Vectors, tensors and the basic equations of fluid mechanics*, Courier Corporation, 2012.
- [2] M. ARNAUDON AND A. B. CRUZEIRO, *Lagrangian Navier–Stokes diffusions on manifolds: variational principle and stability*, Bulletin des Sciences Mathématiques, 136 (2012), pp. 857–881.
- [3] V. I. ARNOL'D, *Mathematical methods of classical mechanics*, vol. 60, Springer Science & Business Media, 2013.
- [4] M. ARROYO AND A. DESIMONE, *Relaxation dynamics of fluid membranes*, Physical Review E, 79 (2009), p. 031915.
- [5] J. W. BARRETT, H. GARCKE, AND R. NÜRNBERG, *A stable numerical method for the dynamics of fluidic membranes*, Numerische Mathematik, 134 (2016), pp. 783–822.
- [6] A. BONITO, R. NOCETTO, AND M. PAULETTI, *Parametric FEM for geometric membranes*, J. of Comp. Phys., 229 (2010), pp. 3171–3188.
- [7] D. BOTHE AND J. PRÜSS, *On the two-phase Navier–Stokes equations with Boussinesq–Scriven surface fluid*, Journal of Mathematical Fluid Mechanics, 12 (2010), pp. 133–150.
- [8] H. BRENNER, *Interfacial transport processes and rheology*, Elsevier, 2013.

- [9] P. B. CANHAM, *The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell*, Journal of theoretical biology, 26 (1970), pp. 61N777–76IN881.
- [10] C. CAO, M. A. RAMMAHA, AND E. S. TITI, *The Navier–Stokes equations on the rotating 2-d sphere: Gevrey regularity and asymptotic degrees of freedom*, Zeitschrift für angewandte Mathematik und Physik ZAMP, 50 (1999), pp. 341–360.
- [11] P. CIARLET, *Mathematical Elasticity, Volume III: Theory of shells*, North-Holland, 2000.
- [12] G. DZIUK AND C. M. ELLIOTT, *Finite element methods for surface PDEs*, Acta Numerica, 22 (2013), pp. 289–396.
- [13] D. G. EBIN AND J. MARSDEN, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Annals of Mathematics, (1970), pp. 102–163.
- [14] A. ERN AND J.-L. GUERMOND, *Theory and practice of finite elements*, Springer, New York, 2004.
- [15] S. GROSS AND A. REUSKEN, *Numerical Methods for Two-phase Incompressible Flows*, Springer, Berlin, 2011.
- [16] M. E. GURTIN AND A. I. MURDOCH, *A continuum theory of elastic material surfaces*, Archive for Rational Mechanics and Analysis, 57 (1975), pp. 291–323.
- [17] P. HANSBO, M. G. LARSON, AND K. LARSSON, *Analysis of finite element methods for vector Laplacians on surfaces*, arXiv preprint arXiv:1610.06747, (2016).
- [18] W. HELFRICH, *Elastic properties of lipid bilayers: theory and possible experiments*, Zeitschrift für Naturforschung C, 28 (1973), pp. 693–703.
- [19] H. Koba, C. LIU, AND Y. GIGA, *Energetic variational approaches for incompressible fluid systems on an evolving surface*, Quarterly of Applied Mathematics, (2016).
- [20] D. LENGELER, *On a Stokes-type system arising in fluid vesicle dynamics*, arXiv preprint arXiv:1506.08991, (2015).
- [21] S. MARDARE, *Inequality of Korn’s type on compact surfaces without boundary*, Chin. Ann. Math., 24B (2003), pp. 191–204.
- [22] M. MITREA AND M. TAYLOR, *Navier-Stokes equations on Lipschitz domains in Riemannian manifolds*, Mathematische Annalen, 321 (2001), pp. 955–987.
- [23] A. MURDOCH AND H. COHEN, *Symmetry considerations for material surfaces*, Archive for Rational Mechanics and Analysis, 72 (1979), pp. 61–98.
- [24] I. NITSCHKE, A. VOIGT, AND J. WENSCH, *A finite element approach to incompressible two-phase flow on manifolds*, Journal of Fluid Mechanics, 708 (2012), pp. 418–438.
- [25] M. A. OLSHANSKII, A. REUSKEN, AND J. GRANDE, *A finite element method for elliptic equations on surfaces*, SIAM J. Numer. Anal., 47 (2009), pp. 3339–3358.
- [26] M. RAHIMI, A. DESIMONE, AND M. ARROYO, *Curved fluid membranes behave laterally as effective viscoelastic media*, Soft Matter, 9 (2013), pp. 11033–11045.
- [27] P. RANGAMANI, A. AGRAWAL, K. K. MANDADAPU, G. OSTER, AND D. J. STEIGMANN, *Interaction between surface shape and intra-surface viscous flow on lipid membranes*, Biomechanics and modeling in mechanobiology, (2013), pp. 1–13.
- [28] A. REUSKEN AND Y. ZHANG, *Numerical simulation of incompressible two-phase flows with a Boussinesq-Scriven surface stress tensor*, Numerical Methods in Fluids, 73 (2013), pp. 1042–1058.
- [29] S. REUTHER AND A. VOIGT, *The interplay of curvature and vortices in flow on curved surfaces*, Multiscale Modeling & Simulation, 13 (2015), pp. 632–643.
- [30] D. S. RODRIGUES, R. F. AUSAS, F. MUT, AND G. C. BUSCAGLIA, *A semi-implicit finite element method for viscous lipid membranes*, Journal of Computational Physics, 298 (2015), pp. 565–584.
- [31] S. ROSENBERG, *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*, no. 31, Cambridge University Press, 1997.
- [32] T. SAKAI, *Riemannian geometry*, vol. 149, American Mathematical Soc., 1996.
- [33] L. SCRIVEN, *Dynamics of a fluid interface equation of motion for Newtonian surface fluids*, Chemical Engineering Science, 12 (1960), pp. 98–108.
- [34] J. C. SLATTERY, L. SAGIS, AND E.-S. OH, *Interfacial transport phenomena*, Springer Science & Business Media, 2007.
- [35] M. E. TAYLOR, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Communications in Partial Differential Equations, 17 (1992), pp. 1407–1456.
- [36] R. TEMAM, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1988.

8. Appendix. We give an elementary proof of the results given in Lemma 2.1. For this it is very convenient to introduce a tensor notation and the Einstein summation convention for the differential operators ∇_i (covariant partial derivative) and div_Γ (surface divergence). For a scalar function f we have, cf. (2.1):

$$\nabla_i f = \partial_k f P_{ki} = P_{ik} \partial_k f.$$

(scalar entries of the matrix \mathbf{P} are denoted P_{ij}). For the vector function $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have, cf. (2.2):

$$\nabla_i u_j := (\nabla_i \mathbf{u})_j = (\nabla_\Gamma \mathbf{u})_{ji} = P_{jl} \partial_k u_l P_{ki} = P_{ik} \partial_k u_l P_{lj},$$

and for matrix valued functions we get, cf. (2.3):

$$\nabla_i A_{sl} := (\nabla_i \mathbf{A})_{sl} = P_{sm} \partial_k A_{mn} P_{nl} P_{ki} = P_{ik} \partial_k A_{mn} P_{ms} P_{nl}.$$

For the divergence operators we have the representations:

$$\begin{aligned} \text{div}_\Gamma \mathbf{u} &= (\nabla_\Gamma \mathbf{u})_{ii} = P_{ik} \partial_k u_l P_{li} = P_{lk} \partial_k u_l \\ (\text{div}_\Gamma \mathbf{A})_i &= \text{div}_\Gamma (e_i^T \mathbf{A}) = P_{lk} \partial_k A_{il}. \end{aligned}$$

Below, functions $\mathbf{u} \in C^2(\Gamma)^n$ are always extended to a neighborhood of Γ by taking constant values along the normal \mathbf{n} .

LEMMA 8.1. *The following identities hold:*

$$(\mathbf{P} \text{div}_\Gamma \nabla_\Gamma^T \mathbf{u})_i = \nabla_k (\nabla_\Gamma \mathbf{u})_{ki} =: \nabla_k \nabla_i u_k \quad (8.1)$$

$$\nabla_i (\text{div}_\Gamma \mathbf{u}) = \nabla_i (\nabla_\Gamma \mathbf{u})_{kk} =: \nabla_i \nabla_k u_k. \quad (8.2)$$

Proof. We use the representations introduced above and thus get

$$(\mathbf{P} \text{div}_\Gamma \nabla_\Gamma^T \mathbf{u})_i = P_{is} \text{div}_\Gamma (\nabla_\Gamma^T \mathbf{u})_s = P_{is} P_{lk} \partial_k (\nabla_\Gamma \mathbf{u})_{ls}. \quad (8.3)$$

Furthermore,

$$\nabla_k (\nabla_\Gamma \mathbf{u})_{ki} = P_{kr} \partial_r (\nabla_\Gamma \mathbf{u})_{ls} P_{si} P_{lk} = P_{is} P_{lr} \partial_r (\nabla_\Gamma \mathbf{u})_{ls},$$

and comparing this with (8.3) proves the result in (8.1). Note that using $P_{lk} n_k = P_{ms} n_m = 0$ (where n_j denotes the j -th component of the normal vector \mathbf{n}) we get

$$(\nabla_\Gamma \mathbf{u})_{km} \partial_r P_{mk} = -P_{ms} \partial_s u_l P_{lk} ((\partial_r n_m) n_k + n_m (\partial_r n_k)) = 0.$$

Using this we get

$$\begin{aligned} \nabla_i (\nabla_\Gamma \mathbf{u})_{kk} &= P_{ir} \partial_r (\nabla_\Gamma \mathbf{u})_{nm} P_{mk} P_{nk} = P_{ir} \partial_r (\nabla_\Gamma \mathbf{u})_{nm} P_{mn} = P_{ir} \partial_r ((\nabla_\Gamma \mathbf{u})_{nm} P_{mn}) \\ &= P_{ir} \partial_r (P_{mk} \partial_k u_l P_{ln} P_{mn}) = P_{ir} \partial_r (P_{mk} \partial_k u_l P_{lm}) \\ &= P_{ir} \partial_r (P_{lk} \partial_k u_l) = P_{ir} \partial_r (\text{div}_\Gamma \mathbf{u}) = \nabla_i (\text{div}_\Gamma \mathbf{u}), \end{aligned} \quad (8.4)$$

and thus the identity (8.2) holds. \square

We now derive a result for the commutator $\nabla_k \nabla_i u_k - \nabla_i \nabla_k u_k$.

LEMMA 8.2. *Let $\mathbf{H} = \nabla \mathbf{n}$ be the Weingarten mapping. Then for $\mathbf{u} \in C^2(\Gamma)^n$ with $\mathbf{P}\mathbf{u} = \mathbf{u}$ the identity*

$$\nabla_k \nabla_i u_k - \nabla_i \nabla_k u_k = ((\text{tr}(\mathbf{H})\mathbf{H} - \mathbf{H}^2)\mathbf{u})_i, \quad i = 1, \dots, n,$$

holds.

Proof. By definition we have

$$\nabla_k \nabla_i u_k = P_{kr} \partial_r (\nabla_\Gamma \mathbf{u})_{nm} P_{mi} P_{nk} = \partial_r (P_{ms} \partial_s u_l P_{ln}) P_{mi} P_{nr}.$$

We use the product rule, $H_{rl} = \partial_r n_l$, $\partial_r P_{ms} = -\partial_r (n_m n_s) = -H_{rm} n_s - H_{rs} n_m$, $P_{mi} n_m = 0$, and thus obtain

$$\begin{aligned} \nabla_k \nabla_i u_k &= (\partial_r P_{ms} \partial_s u_l P_{ln} + P_{ms} \partial_r \partial_s u_l P_{ln} + P_{ms} \partial_s u_l \partial_r P_{ln}) P_{mi} P_{nr} \\ &= -H_{rm} n_s \partial_s u_l P_{lr} P_{mi} + P_{is} P_{lr} \partial_s \partial_r u_l - H_{rn} n_l \partial_s u_l P_{is} P_{nr}. \end{aligned}$$

We also have, cf. (8.4),

$$\begin{aligned} \nabla_i \nabla_k u_k &= P_{ir} \partial_r (P_{lk} \partial_k u_l) = P_{ir} \partial_r P_{lk} \partial_k u_l + P_{ir} P_{lk} \partial_r \partial_k u_l \\ &= -P_{ir} (H_{rl} n_k + H_{rk} n_l) \partial_k u_l + P_{ir} P_{lk} \partial_r \partial_k u_l. \end{aligned}$$

Hence, for the difference we get

$$\begin{aligned} \nabla_k \nabla_i u_k - \nabla_i \nabla_k u_k &= H_{rl} n_k \partial_k u_l P_{ir} - H_{rm} n_s \partial_s u_l P_{lr} P_{mi} + H_{rk} n_l \partial_k u_l P_{ir} - H_{rn} n_l \partial_s u_l P_{is} P_{nr}. \end{aligned}$$

Using $\mathbf{P}\mathbf{u} = \mathbf{u}$ we get

$$\begin{aligned} H_{rm} n_s \partial_s u_l P_{lr} P_{mi} &= H_{rm} n_s \partial_s (P_{lr} u_l) P_{mi} - H_{rm} n_s u_l \partial_s P_{lr} P_{mi} \\ &= H_{mr} n_s \partial_s u_r P_{im} - H_{rm} n_s \partial_s P_{lr} P_{mi} u_l. \end{aligned}$$

Furthermore, using $\mathbf{H}\mathbf{n} = 0$, we get

$$H_{rm} n_s \partial_s P_{lr} P_{mi} u_l = -H_{rm} n_s (H_{sl} n_r + H_{sr} n_l) P_{mi} u_l = 0.$$

Combining these results we get

$$\nabla_k \nabla_i u_k - \nabla_i \nabla_k u_k = H_{rk} n_l \partial_k u_l P_{ir} - H_{rn} n_l \partial_s u_l P_{is} P_{nr}.$$

Using $\mathbf{n}^T \mathbf{u} = 0$ (in a neighborhood of Γ) we get $\partial_k (n_l u_l) = 0$ and in combination with $\mathbf{H}\mathbf{P} = \mathbf{P}\mathbf{H} = \mathbf{H}$ we get

$$\begin{aligned} H_{rk} n_l \partial_k u_l P_{ir} &= -H_{rk} \partial_k n_l P_{ir} u_l = -H_{rk} H_{kl} P_{ir} u_l \\ &= -H_{ik} H_{kl} u_l = -(H^2)_{il} u_l = -(\mathbf{H}^2 \mathbf{u})_i. \end{aligned}$$

Finally note that

$$\begin{aligned} H_{rn} n_l \partial_s u_l P_{is} P_{nr} &= -H_{rn} \partial_s n_l P_{is} P_{nr} u_l = -H_{rn} H_{sl} P_{is} P_{nr} u_l \\ &= -H_{rr} H_{il} u_l = -\text{tr}(\mathbf{H})(\mathbf{H}\mathbf{u})_i. \end{aligned}$$

Combining these results completes the proof. \square

By combining the results of Lemma 8.1 and Lemma 8.2 we have proved the result (2.10). Let \mathbf{A} be an $n \times n$ matrix with $\mathbf{P}\mathbf{A} = \mathbf{A}\mathbf{P} = \mathbf{A}$, hence $\mathbf{A}\mathbf{n} = \mathbf{A}^T \mathbf{n} = 0$. We then have

$$\begin{aligned} \mathbf{n} \cdot \text{div}_\Gamma \mathbf{A} &= n_i (\text{div}_\Gamma \mathbf{A})_i = n_i P_{lk} \partial_k A_{il} = P_{lk} \partial_k (n_i A_{il}) - P_{lk} \partial_k n_i A_{il} \\ &= -P_{lk} H_{ki} A_{il} = -H_{li} A_{il} = -(\mathbf{H}\mathbf{A})_{ll} = -\text{tr}(\mathbf{H}\mathbf{A}), \end{aligned}$$

and combining this with $\text{tr}(\mathbf{H}\mathbf{A}) = \text{tr}(\mathbf{A}^T\mathbf{H}) = \text{tr}(\mathbf{H}\mathbf{A}^T)$ one obtains the result in (2.11). The result in (2.12) follows from (we use $\mathbf{H}\mathbf{n} = 0$, $\mathbf{n}^T\mathbf{H} = 0$):

$$\begin{aligned} (\mathbf{P} \operatorname{div}_\Gamma(\mathbf{H}))_i &= P_{ij}P_{lk}\partial_k H_{jl} = P_{ij}P_{lk}\partial_k\partial_j n_l = P_{ij}P_{lk}\partial_j\partial_k n_l = P_{ij}P_{lk}\partial_j H_{kl} \\ &= P_{ij}\partial_j(P_{lk}H_{kl}) - P_{ij}(\partial_j P_{lk})H_{kl} \\ &= P_{ij}\partial_j H_{ll} + P_{ij}((\partial_j n_l)n_k + (\partial_j n_k)n_l)H_{kl} = P_{ij}\partial_j \kappa = (\nabla_\Gamma \kappa)_i. \end{aligned}$$

LEMMA 8.3. *For $n = 3$ the identity*

$$\text{tr}(\mathbf{H})\mathbf{H} - \mathbf{H}^2 = K\mathbf{P},$$

with K the Gauss curvature, holds.

Proof. We apply the Cayley-Hamilton theorem to the linear mapping $\mathbf{P}\mathbf{H}\mathbf{P} = \mathbf{H} : \text{range}(\mathbf{P}) \rightarrow \text{range}(\mathbf{P})$. Note that $\dim(\text{range}(\mathbf{P})) = 2$. This yields

$$\mathbf{H}^2 - \text{tr}(\mathbf{H})\mathbf{H} + \det(\mathbf{H})\mathbf{P} = 0, \quad (8.5)$$

and using $\det(\mathbf{H}) = K$ we obtain the desired result. \square

The result in (2.13) follows from (2.10) and Lemma 8.3. As a corollary of (8.5) we obtain for $n = 3$ the identity

$$\kappa\mathbf{P} - \mathbf{H} = K\mathbf{H}^\dagger, \quad (8.6)$$

where \mathbf{H}^\dagger is the generalized inverse of \mathbf{H} . Note that $K\mathbf{H}^\dagger$ has the same eigenvalues and eigenvectors as \mathbf{H} , but the eigenpairs are not the same.