

# Discretized Feedback Control for Hyperbolic Balance Laws

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#### Abstract

Physical systems such as water and gas networks are usually operated in a state of equilibrium and feedback control is employed to damp small perturbations over time. We consider flow problems on networks, described by hyperbolic balance laws, and analyze the stabilization of steady states. Sufficient conditions for exponential stability in the continuous and discretized setting are presented. Computational experiments illustrate the theoretical findings.

# 1 Introduction

Hyperbolic balance laws can be used to model flow dynamics on networks. Isothermal Euler and shallow water equations form a 2×2 hyperbolic system to model the temporal and spatial evolution of gas and water flow. Boundary control of such systems is subject of current research, see e.g. [3, 5]. In particular, analytical results have been presented in the case of gas flow [15, 2, 14, 18] and water flow [4, 13, 16, 19, 10, 17]. The underlying tool for the study of these problems are Lyapunov functions stabilizing the deviation from steady states in suitable norms, e.g.  $L^2$ ,  $H^2$ . Exponential decay of a continuous Lyapunov function under a so-called dissipative condition has been proven in [8, 11, 6, 7]. Comparisons to other stability concepts are presented in [9]. Stability with respect to the  $H^2$ -norm gives in [7, 3] stability of the nonlinear system.

Most analytical results do not state explicitly decay rates of the Lyapunov function and the influence of the source term is assumed to be small or in intuitive terms, the considered balance laws are viewed as perturbations of conservation laws [8]. In practical applications however there may be a large influence of the source term. Our main result is Corollary 1, where we present explicit decay rates for arbitrary source terms. Recently, explicit decay rates for numerical schemes have been established. In [1] exponential decay on a finite time horizon has been established for conservations laws and in [20] discretizations of linear systems with positive definite and symmetric source term are considered. In Theorem 4.2 we extend those results by presenting a discretized analogue of [9, Theorem 2.3] without source term and [8, Theorem 2] with source term, i.e. we establish a global stabilization result with respect to the  $H^2$ -norm. First, results on Lyapunov stability for the continuous case are presented. Then the discretized case is introduced and the relation to the continuous case is discussed. Numerical experiments, based on isothermal Euler and shallow water equations illustrate the theoretical results.

# 2 Theoretical Results and Basic Notations

We briefly recall results from [3] considering the stability of a steady state of a physical system described by a system of hyperbolic differential equations given by

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t,x) + \frac{\mathrm{d}}{\mathrm{d}x}f\Big(y(t,x)\Big) = -S\Big(y(t,x)\Big) \tag{1}$$

with time and space variables  $(t, x) \in [0, \infty) \times [0, L]$ . We assume a 2×2 system with strictly hyperbolic flux function  $f \in C^1(\mathbb{R}^2; \mathbb{R}^2)$  and source term  $S \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ , but an extension to larger systems is straightforward. To simplify notation we omit time and space variables and abbreviate system (1) by  $y_t + f(y)_x = -S(y)$ , which is for smooth solutions equivalent to solving

$$y_t + D_y f(y) \, y_x = -S(y),$$

where  $D_y f(y)$  denotes the Jacobian of f(y). This system will be analyzed at a steady state.

**Definition 2.1** (Steady State). A steady state (or equilibrium) is a time-invariant possibly space-varying solution  $\bar{y}(x)$  of system (1). It satisfies the system of ordinary differential equations

$$0 = \bar{y}_t = f(\bar{y})_x + S(\bar{y}).$$

Since we are mainly interested in  $H^2$ -stabilization, we assume for now  $\bar{y} \in H^2(0, L)$ . Sobolev's embedding theorem ensures a representant  $\bar{y} \in C^1(0, L)$ .

We first linearize the original system at steady state and than transform the corresponding linear system into Riemann invariants. The opposite approach – first transforming into Riemann invariants and then linearizing – has been discussed in [3]. Both approaches yield stability of the linearized system in Riemann coordinates. Assuming small and smooth perturbations  $\Delta y(t,x) := y(t,x) - \bar{y}(x)$  of the steady state, the flux function and the source term is linearized by  $f(\bar{y}) + A\Delta y$  and  $S(\bar{y}) + \bar{S}\Delta y$ , where A and  $\bar{S}$  denote Jacobian at steady state. We denote by  $A_x$  the matrix, where each entry of A(x) is differentiated with respect to x. Neglecting higher order derivatives, the linearized balance law (1) reads

$$\Delta y_t + A \Delta y_x = -(\bar{S} + A_x) \Delta y. \tag{2}$$

The matrices  $A(x), \bar{S}(x) \in \mathbb{R}^{2 \times 2}$  may depend on the space variable  $x \in [0, L]$ . For a given steady state  $\bar{y}$  we assume an initial perturbation  $\Delta y(0, x) := \Delta y_0(x)$ . Stabilization aims to damp the deviation  $\Delta y$ .

# 2.1 Riemann Invariants

For  $2 \times 2$  systems with hyperbolic flux function the Jacobian A in the quasilinear form (2) is always diagonalizable [12]. Thus there exists a diagonal matrix  $\bar{\Lambda} :=$ 

diag $\{\bar{\lambda}^+, \bar{\lambda}^-\}$ , whose entries are eigenvalues of A and a matrix T, whose columns  $r^{\pm}$  are the corresponding eigenvectors, such that

$$A = T\bar{\Lambda}T^{-1}, \quad \bar{\Lambda} := \operatorname{diag}\{\bar{\lambda}^+, \bar{\lambda}^-\}, \quad T := \left(r^+ \mid r^-\right).$$

We introduce the assumption

$$\bar{\lambda}^{-}(x) < 0 < \bar{\lambda}^{+}(x), \quad \forall \ x \in [0, L],$$
(3)

which holds in the case of isothermal Euler and shallow water equations as long as the flow remains subsonic and subcritical, respectively. Again, the eigenvalues  $\bar{\lambda}^{\pm}(x)$ and eigenvectors  $r^{\pm}(x)$  may depend on x. We introduce the Riemann invariant

$$\zeta := \begin{pmatrix} \zeta^+ \\ \zeta^- \end{pmatrix} := T^{-1} \Delta y \quad \text{and} \quad C := T^{-1} \Big( \bar{S} + A_x \Big) T + \bar{\Lambda} T^{-1} T_x$$

such that for smooth solutions equation (2) is equivalent to  $\zeta_t + \bar{\Lambda}\zeta_x = -C\zeta$ . We prescribe linear feedback boundary conditions by a matrix  $\bar{G} \in \mathbb{R}^{2\times 2}$  such that the system

$$\zeta_t + \Lambda \zeta_x = -C\zeta, \qquad \zeta(0, x) = T^{-1} \Delta y_0(x),$$
  
$$\begin{pmatrix} \zeta^+(t, 0) \\ \zeta^-(t, L) \end{pmatrix} = \bar{G} \begin{pmatrix} \zeta^+(t, L) \\ \zeta^-(t, 0) \end{pmatrix}$$
  
$$(\bar{\mathcal{S}})$$

is well-posed [3]. According to assumption (3) characteristics with  $\bar{\lambda}^+(x) > 0$  and  $\bar{\lambda}^-(x) < 0$  are time-independent and cannot cross. Therefore a solution exists for all  $t \in \mathbb{R}^+_0$  as discussed in [12]. We will show that it is stable with respect to the  $L^2$ -norm in the sense of Definition 2.2, which is taken from [3, Sec. 3], where also the precise definition of a  $L^2$ -solution is found.

**Definition 2.2** (Exponential Stability for the  $L^2$ -Norm [3, Def. 3.1]). The system  $(\bar{S})$  is exponentially stable for the  $L^2$ -norm if there exist  $\mu > 0$  and  $\alpha > 0$  such that for every  $\zeta(0, \cdot) \in L^2([0, L]; \mathbb{R}^2)$  the  $L^2$ -solution of  $(\bar{S})$  satisfies

$$\|\zeta(t,\cdot)\|_{L^{2}([0,L];\mathbb{R}^{2})} \leq \alpha e^{-\mu t} \|\zeta(0,\cdot)\|_{L^{2}([0,L];\mathbb{R}^{2})}, \quad \forall \ t \in \mathbb{R}^{+}_{0}$$

# 2.2 Higher Order Derivatives

A natural extension of exponential stability for the  $H^d$ -norm is given in [3] by replacing the norm  $L^2([0, L]; \mathbb{R}^2)$  by  $H^d([0, L]; \mathbb{R}^2)$ . Additional compatibility conditions must be fulfilled to ensure well-posedness of the initial-boundary value problem (IBVP). We reduce  $H^d$ - to  $L^2$ -stabilization by introducing the Riemann invariant  $\mathcal{R}^{\mathrm{T}} := (\zeta^{(0)}, \ldots, \zeta^{(d)})$ , which contains all derivatives up to order d.

**Theorem 2.3** (IBVP for  $H^d$ ). Given the IBVP  $(\bar{S})$  with possibly space-varying distinct eigenvalues  $\bar{\Lambda}(x) := \text{diag}\{\bar{\lambda}^+(x), \bar{\lambda}^-(x)\}$  with  $\bar{\lambda}^-(x) < 0 < \bar{\lambda}^+(x)$  and source term  $C(x) \in \mathbb{R}^2$ , the Riemann invariant  $\mathcal{R}^{\mathrm{T}} := (\zeta^{(0)}, \ldots, \zeta^{(d)}) \in \mathbb{R}^{2(d+1)}$  fulfills

$$\mathcal{R}_t + \Lambda \mathcal{R}_x = -B\mathcal{R}, \qquad \mathcal{R}^{(j)}(0, x) = \frac{\partial^j}{\partial x^j} \zeta(0, x), \begin{pmatrix} \mathcal{R}^+(t, 0) \\ \mathcal{R}^-(t, L) \end{pmatrix} = G \begin{pmatrix} \mathcal{R}^+(t, L) \\ \mathcal{R}^-(t, 0) \end{pmatrix}$$
(S)

with  $\Lambda := \operatorname{diag}\{\overline{\Lambda}, \ldots, \overline{\Lambda}\} \in \mathbb{R}^{2(d+1)}$  and  $\mathcal{R}^{\pm} := (\mathcal{R}^{(0),\pm}, \ldots, \mathcal{R}^{(d),\pm})^{\mathrm{T}}$ . For  $C, \overline{\Lambda}$  sufficiently smooth the source term is a triangular block matrix of the form

$$B := \begin{pmatrix} B & & & \\ * & B + \bar{\Lambda}_x & & \\ & * & * & \ddots & \\ * & * & * & B + d\bar{\Lambda}_x \end{pmatrix},$$
(4)

where \* denote non-zero entries. Then problem (S) is well-posed for the matrix

$$G = \begin{pmatrix} G_{1,1}^{(0)} & & G_{1,2}^{(0)} & & \\ & \ddots & & \ddots & \\ & & G_{1,1}^{(d)} & & & G_{1,2}^{(d)} \\ & & & G_{2,1}^{(0)} & & & \\ & & \ddots & & & \ddots & \\ & & & & & G_{2,2}^{(d)} & & & & \\ & & & & & & & G_{2,2}^{(d)} \end{pmatrix},$$
(5)

where  $G_{i,j}^{\left(k\right)}$  denotes the i,j-th entry of the matrix

$$G^{(k)} := \begin{pmatrix} \bar{\lambda}^+(0) & \\ & \bar{\lambda}^-(L) \end{pmatrix}^{-k} \bar{G} \begin{pmatrix} \bar{\lambda}^+(L) & \\ & \bar{\lambda}^-(0) \end{pmatrix}^k.$$

*Proof.* We prove statement (4) by induction. Let  $C_x := C_x(x) \in \mathbb{R}^{2 \times 2}$  denote the matrix, where each entry of  $C(x) \in \mathbb{R}^{2 \times 2}$  is differentiated with respect to x. Then for j = 1 the first derivative satisfies

$$0 = \frac{\partial}{\partial x} \Big[ \zeta_t + \bar{\Lambda} \zeta_x + C \zeta \Big] = \mathcal{R}_t^{(1)} + \bar{\Lambda} \mathcal{R}_x^{(1)} + \Big( C + \bar{\Lambda}_x \Big) \mathcal{R}^{(1)} + C_x \mathcal{R}^{(0)}.$$

Let the claim hold for j-1 with  $2 \le j \le d$ . Then we obtain

$$\frac{\partial}{\partial x} \left[ \mathcal{R}_t^{(j-1)} + \bar{\Lambda} \mathcal{R}_x^{(j-1)} + \left( C + (j-1)\bar{\Lambda}_x \right) \mathcal{R}^{(j-1)} + \sum_{i=0}^{j-2} \alpha_i^{(j-1)} \mathcal{R}^{(i)} \right]$$
$$= \mathcal{R}_t^{(j)} + \bar{\Lambda} \mathcal{R}_x^{(j)} + \left( C + j\bar{\Lambda}_x \right) \mathcal{R}^{(j)} + \sum_{i=0}^{j-1} \alpha_i^{(j)} \mathcal{R}^{(i)}$$

for space-variant coefficient matrices  $\alpha_i^{(j)} \in \mathbb{R}^{2 \times 2}$ . The claim follows by induction.

Since each entry  $\mathcal{R}^{(j)} \in \mathbb{R}^2$  consists of one wave  $\mathcal{R}^{(j),+}$  with positive characteristic speed  $\bar{\lambda}^+$  and one with negative speed, they can be reorderd to  $\mathcal{R}^{\pm}$ . Since only the advection part causes a flux over the boundary, balance law (S) simplifies at the boundary to

$$\begin{aligned} \mathcal{R}_{t}^{(j),+}(t,0) &+ \bar{\lambda}^{(j),+}(0) \mathcal{R}_{x}^{(j),+}(t,0) &= 0, \\ \mathcal{R}_{t}^{(j),-}(t,L) &+ \bar{\lambda}^{(j),-}(L) \mathcal{R}_{x}^{(j),-}(t,L) &= 0 \end{aligned}$$

which yields for  $j = 1, \ldots, d$ 

$$\begin{pmatrix} \bar{\lambda}^{+}(0) & \\ & \bar{\lambda}^{-}(L) \end{pmatrix} \begin{pmatrix} \mathcal{R}^{(j),+}(t,0) \\ \mathcal{R}^{(j),-}(t,L) \end{pmatrix} = - \begin{pmatrix} \mathcal{R}^{(j-1),+}(t,0) \\ \mathcal{R}^{(j-1),-}(t,L) \end{pmatrix}_{t}$$
$$= -G^{(j-1)} \begin{pmatrix} \mathcal{R}^{(j-1),+}(t,L) \\ \mathcal{R}^{(j-1),-}(t,0) \end{pmatrix}_{t} = G^{(j-1)} \begin{pmatrix} \bar{\lambda}^{+}(L) & \\ & \bar{\lambda}^{-}(0) \end{pmatrix} \begin{pmatrix} \mathcal{R}^{(j),+}(t,L) \\ \mathcal{R}^{(j),-}(t,0) \end{pmatrix}.$$

# 3 Stabilization in the Continuous Case

In this section we prove exponential stability for a set of Riemann invariants. As basic tool we use a Lyapunov function similar to [15]. But in extension we equip the left boundary for Riemann invariants with positive characteristic speeds by some strictly positive weight  $h_0^{(j)} > 0$  and the right boundary by  $h_L^{(j)} > 0$  for negative speeds, respectively. For now, we assume that they are given, but in following sections we show how they can be determined to ensure exponential stability. We refer to those weights as "ghostcell-weights".

**Definition 3.1** (Lyapunov Function). For a system of Riemann invariants  $\mathcal{R} \in \mathbb{R}^m$  we assume a positive characteristic speed  $\lambda^{(j)}(x) > 0$  for  $j = 1, \ldots, p$  and  $\lambda^{(j)}(x) < 0$  for  $j = p+1, \ldots, m$ , respectively. Let constants  $h_0^{(j)} > 0$  for  $j = 1, \ldots, p$  and  $h_L^{(j)} > 0$  for  $j = p+1, \ldots, m$  be given. We define the diagonal matrix

$$W(x,\mu) := \operatorname{diag}\left\{w^{(1)}(x,\mu), \dots, w^{(p)}(x,\mu), w^{(p+1)}(x,\mu), \dots, w^{(m)}(x,\mu)\right\}$$

with strictly positive entries

$$w^{(j)}(x,\mu) := \frac{h_0^{(j)}}{\lambda^{(j)}(x)} \exp\left(-\mu \int_0^x \frac{1}{\lambda^{(j)}(s)} \,\mathrm{d}s\right), \quad j = 1, \dots, p,$$
$$w^{(j)}(x,\mu) := \frac{-h_L^{(j)}}{\lambda^{(j)}(x)} \exp\left(\mu \int_x^L \frac{1}{\lambda^{(j)}(s)} \,\mathrm{d}s\right), \qquad j = p+1, \dots, m$$

such that for  $x \in [0, L]$  and  $\mu > 0$  fixed, the weighted inner product

$$\left\langle a,b\right\rangle _{W(x)}:=a^{\mathrm{T}}W(x,\mu)b,\quad\forall\;a,b\in\mathbb{R}^{m}$$

and the Lyapunov function

$$\mathcal{L}(t) := \int_{0}^{L} \|\mathcal{R}\|_{W(x)}^{2} \,\mathrm{d}x$$

are well-defined. We define  $h^{(j)}(x,\mu)$  such that  $w^{(j)}(x,\mu) = \frac{h^{(j)}(x,\mu)}{\lambda^{(j)}(x)}$  for  $j = 1, \ldots, p$ and  $w^{(j)}(x,\mu) = -\frac{h^{(j)}(x,\mu)}{\lambda^{(j)}(x)}$  for  $j = p + 1, \ldots, m$ , respectively.

Next we show conditions that guarantee a specific decay rate  $\mu > 0$ . The basic idea of our proof is to ensure non-negative eigenvalues of a matrix  $\mathcal{M}(x)$ , which we will define later. Then we deduce  $\langle \mathcal{R}, \mathcal{M}(x) \mathcal{R} \rangle \geq 0$  for all  $\mathcal{R} \in \mathbb{R}^m$ . Note that

this is only possible if  $\mathcal{M}(x)$  is symmetric. Therefore, we define for non-symmetric weighted source terms the symmetric matrix  $W(x)\mathcal{B}(x)$  by

$$\mathcal{B}(x) := \frac{B(x) + W^{-1}(x)B^{\mathrm{T}}(x)W(x)}{2} \quad \text{such that for all } \mathcal{R} \in \mathbb{R}^{m} \text{ it holds}$$
$$\left\langle \mathcal{R}, \mathcal{B}(x)\mathcal{R} \right\rangle_{W(x)} = \left\langle \mathcal{R}, \frac{W(x)B(x) + B^{\mathrm{T}}(x)W(x)}{2}\mathcal{R} \right\rangle = \left\langle \mathcal{R}, B(x)\mathcal{R} \right\rangle_{W(x)}$$

i.e. the same weighted quadratic form is represented by B and  $\mathcal{B}$ . In the next theorem we state conditions that guarantee a non-increasing Lyapunov function.

**Theorem 3.2** (Lyapunov Stability). Assume  $\mu > 0$  be given such that for all  $x \in [0, L]$  fixed the matrix  $\mathcal{M}(x, \mu) := \mu W(x, \mu) + 2W(x, \mu)\mathcal{B}(x, \mu)$  is positive semidefinite and the matrix  $M(\mu) := G^{\mathrm{T}}H_{\mathbb{G}}G - H_{\mathbb{B}}(\mu)$  is negative semidefinite, where the matrix  $H_{\mathbb{G}} := \mathrm{diag}\{H_0^+, H_L^-\}$  is defined for the given ghostcell-weights as

$$H_0^+ := \operatorname{diag}\left\{h_0^{(1)}, \dots, h_0^{(p)}\right\}, \quad H_L^- := \operatorname{diag}\left\{h_L^{(p+1)}, \dots, h_L^{(m)}\right\}$$

and  $H_{\mathbb{B}}(\mu) := \text{diag} \{ H^+(L,\mu), H^-(0,\mu) \}$  as

$$H^{+}(L,\mu) := \operatorname{diag}\left\{h^{(1)}(L,\mu), \dots, h^{(p)}(L,\mu)\right\},\$$
$$H^{-}(0,\mu) := \operatorname{diag}\left\{h^{(p+1)}(0,\mu), \dots, h^{(m)}(0,\mu)\right\}.$$

Then the derivative of the Lyapunov function is non-increasing, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le 0.$$

*Proof.* The analysis for the advection part  $\mathcal{R}_t + \Lambda \mathcal{R}_x = 0$  can be reduced to a single Riemann invariant  $\mathcal{L}^{(j)} := \int \mathcal{R}^{(j)}(t,x)^2 w^{(j)}(x,\mu) \, dx$ . For positive characteristic speeds  $j = 1, \ldots, p$  we get by using integration by parts

$$\begin{split} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{(j)}(t) &= -\int \frac{\mathrm{d}}{\mathrm{d}x} \Big[ \mathcal{R}^{(j)}(t,x)^2 \Big] h^{(j)}(x,\mu) \,\mathrm{d}x \\ &= - \Big[ \mathcal{R}^{(j)}(t,\cdot)^2 h^{(j)}(\cdot,\mu) \Big]_0^L + \int \mathcal{R}^{(j)}(t,x)^2 \frac{\mathrm{d}}{\mathrm{d}x} \Big[ h^{(j)}(x,\mu) \Big] \,\mathrm{d}x \\ &= - \Big[ \mathcal{R}^{(j)}(t,L)^2 h^{(j)}(L,\mu) - \mathcal{R}^{(j)}(t,0)^2 h_0^{(j)} \Big] - \mu \mathcal{L}^{(j)}(t). \end{split}$$

It follows analogously for  $j = p + 1, \ldots, m$ 

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{(j)}(t) = \left[\mathcal{R}^{(j)}(t,L)^2 h_L^{(j)} - \mathcal{R}^{(j)}(t,0)^2 h^{(j)}(0,\mu)\right] - \mu \mathcal{L}^{(j)}(t)$$

Summing up yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le - \sum_{j=1}^{p} \left[ \mathcal{R}^{(j)}(t,L)^2 h^{(j)}(L,\mu) - \mathcal{R}^{(j)}(t,0)^2 h_0^{(j)} \right]$$
(6)

$$+\sum_{j=p+1}^{m} \left[ \mathcal{R}^{(j)}(t,L)^2 h_L^{(j)} \qquad -\mathcal{R}^{(j)}(t,0)^2 h^{(j)}(0,\mu) \right]$$
(7)

$$-\mu \mathcal{L}(t). \tag{8}$$

The boundary terms (6) and (7) read

$$\begin{pmatrix} \mathcal{R}^{+}(t,0) \\ \mathcal{R}^{-}(t,L) \end{pmatrix}^{\mathrm{T}} H_{\mathbb{G}} \begin{pmatrix} \mathcal{R}^{+}(t,0) \\ \mathcal{R}^{-}(t,L) \end{pmatrix} - \begin{pmatrix} \mathcal{R}^{+}(t,L) \\ \mathcal{R}^{-}(t,0) \end{pmatrix}^{\mathrm{T}} H_{\mathbb{B}}(\mu) \begin{pmatrix} \mathcal{R}^{+}(t,L) \\ \mathcal{R}^{-}(t,0) \end{pmatrix}^{\mathrm{T}} \\ = \begin{pmatrix} \mathcal{R}^{+}(t,L) \\ \mathcal{R}^{-}(t,0) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G^{\mathrm{T}} H_{\mathbb{G}} G & - & H_{\mathbb{B}}(\mu) \end{bmatrix} \begin{pmatrix} \mathcal{R}^{+}(t,L) \\ \mathcal{R}^{-}(t,0) \end{pmatrix}^{\mathrm{T}}$$

and are non-positive if the matrix  $M(\mu)$  is negative semidefinite. The Lyapunov function for the reaction part  $\mathcal{R}_t = -B\mathcal{R}$  reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) = \int \mathcal{R}_t^{\mathrm{T}} W(x,\mu)\mathcal{R} + \mathcal{R}^{\mathrm{T}} W(x,\mu)\mathcal{R}_t \,\mathrm{d}x$$

$$= -\int \mathcal{R}^{\mathrm{T}} \Big( B^{\mathrm{T}}(x)W(x,\mu) + W(x,\mu)B(x) \Big)\mathcal{R} \,\mathrm{d}x$$

$$= -2\int \Big\langle \mathcal{R}, W(x,\mu)\mathcal{B}(x)\mathcal{R} \Big\rangle \,\mathrm{d}x.$$
(9)

Summing up (8) and (9) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\mu \int \left\langle \mathcal{R}, W(x,\mu)\mathcal{R} \right\rangle - 2 \int \left\langle \mathcal{R}, W(x,\mu)\mathcal{B}(x)\mathcal{R} \right\rangle \mathrm{d}x$$
$$= -\int \left\langle \mathcal{R}, \mathcal{M}(x,\mu)\mathcal{R} \right\rangle \mathrm{d}x \leq 0.$$

Theorem 3.2 only guarantees a non-increasing Lyapunov function. Exponential decay and an explicit decay rate are established in the following corollary.

**Corollary 1** (Exponential Stability). Assume  $\mu > 0$  be given such that the matrix  $M(\mu)$ , defined in Theorem 3.2, is negative semidefinite and define the matrix

$$\tilde{\mathcal{B}}(x,\mu) := \frac{1}{2} \Big[ W^{1/2}(x,\mu) B(x) W^{-1/2}(x,\mu) + W^{-1/2}(x,\mu) B^{\mathrm{T}}(x) W^{1/2}(x,\mu) \Big]$$

with eigenvalues  $d^{(j)}(x,\mu)$ . Then the Lyapunov function decays exponentially, i.e.  $\mathcal{L}(t) \leq e^{-\tilde{\mu}t}\mathcal{L}(0)$  with rate  $\tilde{\mu} \leq \mu + 2d_{\min}(\mu)$ , where  $d_{\min}(\mu) := \min_{x,j} \left\{ d^{(j)}(x,\mu) \right\}$ denotes the smallest eigenvalue of  $\tilde{\mathcal{B}}(x,\mu)$ . The smallest eigenvalue is estimated as

$$d_{\min}(\mu) \ge -\text{cond}_{\sqrt{W}}(\mu) \max_{x \in [0,L]} \left\{ \left\| B(x) \right\|_2 \right\}, \quad \text{cond}_{\sqrt{W}}(\mu) := \max_{\substack{x \in [0,L], \\ i,j=1,\dots,m}} \sqrt{\frac{w^{(i)}(x,\mu)}{w^{(j)}(x,\mu)}}$$

*Proof.* The symmetric matrix  $\tilde{\mathcal{B}}(x,\mu) = V^{\mathrm{T}}(x,\mu)D(x,\mu)V(x,\mu)$  is diagonalizable by an orthogonal matrix  $V(x,\mu)$ . For  $\tilde{\mu} \leq \mu + 2d_{\min}(\mu)$  the matrix

$$\mu - \tilde{\mu} + 2\tilde{\mathcal{B}}(x,\mu) = V^{\mathrm{T}}(x,\mu) \Big[ \mu - \tilde{\mu} + 2D(x,\mu) \Big] V(x,\mu)$$

is also diagonalizable with non-negative eigenvalues and consequently is positive semidefinite as well as the symmetric matrix

$$(\mu - \tilde{\mu})W(x,\mu) + 2W(x,\mu)\mathcal{B}(x,\mu).$$
(10)

The claim follows by Theorem 3.2 since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\tilde{\mu}\mathcal{L}(t) - \int \left\langle \mathcal{R}, \left( (\mu - \tilde{\mu})W(x,\mu) + 2W(x,\mu)\mathcal{B}(x,\mu) \right) \mathcal{R} \right\rangle \mathrm{d}x \leq -\tilde{\mu}\mathcal{L}(t).$$

The spectral radius of a matrix A satisfies for any p-norm  $\rho\{A\} \leq ||A||_p$  and the 2-norm  $||A||_2 := \sqrt{\rho\{AA^T\}}$  satisfies  $||\cdot||_2 = ||(\cdot)^T||_2$  and specifically for a diagonal matrix with positive entries  $||W(x,\mu)||_2 = \max_{j=1,\dots,m} \{w^{(j)}(x,\mu)\}$ . Thus we get

$$\begin{aligned} d_{\min}(\mu) &\geq -\rho \big\{ \tilde{\mathcal{B}}(x,\mu) \big\} \geq - \big\| \tilde{\mathcal{B}}(x,\mu) \big\|_2 \geq - \big\| W^{1/2}(x,\mu) \big\|_2 \big\| W^{-1/2}(x,\mu) \big\|_2 \big\| B(x) \big\|_2 \\ &\geq - \mathrm{cond}_{\sqrt{W}}(\mu) \max_{x \in [0,L]} \Big\{ \big\| B(x) \big\|_2 \Big\}. \end{aligned}$$

The guaranteed decay rate  $\tilde{\mu}$  not only depends on the definiteness of the source term but also on weights. The estimate of  $d_{\min}(\mu)$  guarantees the existence of  $\mu > 0$ that yields an exponential decay with rate  $\tilde{\mu} > 0$  as long as the condition number  $\operatorname{cond}_{\sqrt{W}}(\mu)$  remains bounded. The analysis covers the pathological case  $\tilde{\mu} \leq 0$ . In this case there is no decaying Lyapunov function guaranteed. If  $W(x)\mathcal{B}(x)$  is positive semidefinite or in the case of conservation laws we get the decay rate  $\tilde{\mu} := \mu$ .

An estimate that is independent of  $\mu$  is possible if weights are of the form  $W(x,\mu) := w(x,\mu)\mathbb{1}$  with  $w(x,\mu) \in \mathbb{R}^+$  and identity matrix  $\mathbb{1} \in \mathbb{R}^{m \times m}$ . Then the guaranteed decay rate is estimated by

$$\tilde{\mu} \le \mu + 2d_{\min}, \quad d_{\min} := \min_{\substack{x \in [0,L], \ j=1,\dots,m}} \left\{ d^{(j)}(x) \right\},$$

where  $d^{(j)}(x)$  denote the eigenvalues of  $\frac{1}{2}[B(x) + B^{\mathrm{T}}(x)]$ . This is seen when diagonalizing by an orthogonal matrix V(x) such that  $\frac{1}{2}[B(x)+B^{\mathrm{T}}(x)] = V^{\mathrm{T}}(x)D(x)V(x)$ . Then the symmetric matrix (10) is diagonalizable by

$$\begin{split} & w(x,\mu) \Big[ (\mu - \tilde{\mu}) \mathbb{1} + B(x) + B^{\mathrm{T}}(x) \Big] \\ &= \Big( V(x) \sqrt{w(x,\mu)} \Big)^{\mathrm{T}} \Big[ (\mu - \tilde{\mu}) \mathbb{1} + 2D(x) \Big] \Big( V(x) \sqrt{w(x,\mu)} \Big), \end{split}$$

where eigenvalues  $\mu - \tilde{\mu} + 2d^{(j)}(x)$  are non-negative for all  $j = 1, \ldots, m$  due to the assumption  $\tilde{\mu} \leq \mu + 2d_{\min}$ . This case is considered in [20] in a discretized setting. Note that a not necessarily symmetric source term with positive eigenvalues does not imply positive definiteness of the symmetric matrix  $B^{\mathrm{T}}(x) + B(x)$ .

# 4 Stabilization in the Discretized Case

Using a space discretization  $\Delta x$ , the space interval [0, L] is divided into N cells such that  $\Delta xN = L$  with centers  $x_i := (i + \frac{1}{2})\Delta x$  for  $i = 0, \ldots, N - 1$  and edges  $x_{i+\frac{1}{2}} := i\Delta x$  for  $i = 0, \ldots, N$ . Outside the domain ghostcells with centers  $x_0$  and  $x_{N+1}$  are added. The discrete time steps are denoted by  $t_k := k\Delta t$  for  $k \in \mathbb{N}_0$  and  $\Delta t > 0$  such that the CFL-condition

$$\text{CFL} := \max_{\substack{x \in [0,L], \\ j=1,...,m}} \left| \lambda^{(j)}(x) \right| \; \frac{\Delta t}{\Delta x} \le 1$$

holds, where  $\lambda^{(j)}$  are eigenvalues of  $(\mathcal{S})$ . Cell averages at  $t_k$  are approximated by

$$\mathcal{R}_{i}^{k} := \begin{pmatrix} \mathcal{R}_{i}^{k,+} \\ \mathcal{R}_{i}^{k,-} \end{pmatrix} \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \mathcal{R}^{+}(t_{k},x) \\ \mathcal{R}^{-}(t_{k},x) \end{pmatrix} \, \mathrm{d}x \in \mathbb{R}^{m}.$$

The advection part can be approximated by a left and right sided upwind-scheme and the reaction part by the explicit Euler method, i.e.

$$\mathcal{R}_{i}^{k+1} = \mathcal{R}_{i}^{k} - \frac{\Delta t}{\Delta x} \Lambda_{i} \begin{pmatrix} \mathcal{R}_{i}^{k,+} - \mathcal{R}_{i-1}^{k,+} \\ \mathcal{R}_{i+1}^{k,-} - \mathcal{R}_{i}^{k,-} \end{pmatrix} - \Delta t B_{i} \mathcal{R}_{i}^{k}$$
(11)

where  $\Lambda_i := \operatorname{diag}\{\lambda_i^{(1)}, \ldots, \lambda_i^{(m)}\}$  contains positive characteristic speeds  $\lambda_i^{(j)} := \lambda^{(j)}(\xi^+)$  with  $\xi^+ \in [x_{i-1}, x_i]$  and negative speeds  $\lambda_i^{(j)} := \lambda^{(j)}(\xi^-)$  with  $\xi^- \in [x_i, x_{i+1}]$ , respectively. The discretized source term is  $B_i := B(x_i) \in \mathbb{R}^{m \times m}$  and  $\mathcal{B}_i := \mathcal{B}(x_i)$ . Boundary conditions determine ghostcell values by

$$\begin{pmatrix} \mathcal{R}_0^{k,+} \\ \mathcal{R}_{N+1}^{k,-} \end{pmatrix} = G \begin{pmatrix} \mathcal{R}_N^{k,+} \\ \mathcal{R}_1^{k,-} \end{pmatrix}.$$

# 4.1 Analysis of the Discretized Lyapunov Function

To improve the readability of the text we omit in the discretized case the dependency on the decay rate. The continuous Lyapunov function for a single Riemann invariant with positive constant characteristic speed  $\lambda^+$  is of the form

$$\mathcal{L}^+(t) = c \int \mathcal{R}^+(t,x)^2 e^{-\frac{\mu}{\lambda^+}x} \,\mathrm{d}x$$

and a straightforward discretized analogue in  $t_k$  would be

$$\mathcal{L}^{k,+} = c \sum_{i=1}^{N} \left( \mathcal{R}_{i}^{k,+} \right)^{2} e^{-\frac{\mu}{\lambda^{+}} x_{i}} \Delta x,$$

which is proposed in [1, 20]. There Lyapunov stability was only proven for a finite time interval, as a blow up in derivatives occur in contrast to the continuous case. To circumvent this problem we approximate the continuous derivative

$$\frac{\partial}{\partial s}h^{(j)}(s) = -\mu \frac{h^{(j)}}{\lambda^+}(s) \quad \text{by} \quad \frac{h_i^{(j)} - h_{i-1}^{(j)}}{\Delta x} = -\mu \frac{h_{i-1}^{(j)}}{\lambda_i^{(j)}} \quad \text{for} \quad j = 1, \dots, p$$
(12)

$$\frac{\partial}{\partial s}h^{(j)}(s) = -\mu \frac{h^{(j)}}{\lambda^{-}}(s) \quad \text{by} \quad \frac{h^{(j)}_{i+1} - h^{(j)}_{i}}{\Delta x} = -\mu \frac{h^{(j)}_{i+1}}{\lambda^{(j)}_{i}} \quad \text{for} \quad j = p+1,\dots,m.$$
(13)

The one-sided difference quotients (12) and (13) explain the choices (14) and (15) in the next definition.

**Definition 4.1** (Discretized Lyapunov Function). For a system of Riemann invariants  $\mathcal{R}_i^k \in \mathbb{R}^m$  with i = 1, ..., N we assume a positive characteristic speed

 $\lambda_i^{(j)} > 0$  for  $j = 1, \ldots, p$  and  $\lambda_i^{(j)} < 0$  for  $j = p + 1, \ldots, m$ , respectively. Let constants  $h_0^{(j)} > 0$  for  $j = 1, \ldots, p$  and  $h_{N+1}^{(j)} > 0$  for  $j = p + 1, \ldots, m$  be given. We define the diagonal matrix

$$W_i := \operatorname{diag} \left\{ w_i^{(1)}, \dots, w_i^{(p)}, \ w_i^{(p+1)}, \dots, w_i^{(m)} \right\}$$

with strictly positive entries  $w_i^{(j)} = \frac{h_{i-1}^{(j)}}{\lambda_i^{(j)}}$  for  $j = 1, \ldots, p$  and  $w_i^{(j)} = -\frac{h_{i+1}^{(j)}}{\lambda_i^{(j)}}$  for  $j = p+1, \ldots, m$ , respectively, where  $h_i^{(j)}$  is recursively defined by (12) and (13), i.e.

$$h_i^{(j)} := h_0^{(j)} \prod_{\ell=1}^i \left( 1 - \Delta x \frac{\mu}{\lambda_\ell^{(j)}} \right), \quad j = 1, \dots, p,$$
(14)

$$h_i^{(j)} := h_{N+1}^{(j)} \prod_{\ell=N}^{i} \left( 1 - \Delta x \frac{\mu}{|\lambda_\ell^{(j)}|} \right), \quad j = p+1, \dots, m.$$
(15)

We introduce for all i = 1, ..., N the weighted inner product

$$\langle a, b \rangle_{W_i} := a^{\mathrm{T}} W_i b, \quad \forall \ a, b \in \mathbb{R}^m$$
 (16)

and the Lyapunov function

$$\mathcal{L}^k := \sum_{i=1}^N \left\| \mathcal{R}_i^k \right\|_{W_i}^2 \Delta x.$$
(17)

From now on we assume a space restriction

$$\Delta x \in (0, \lambda_{\min}/\mu) \quad \text{with} \quad \lambda_{\min} := \min_{\substack{x \in [0, L], \\ j=1, \dots, m}} \left| \lambda^{(j)}(x) \right| \tag{18}$$

which ensures strictly positive weights  $W_i$  such that the inner product (16) and consequently the Lyapunov function (17) are well-defined. We show a discretized analogue of Theorem 3.2.

**Theorem 4.2** (Lyapunov Stability in the Discretized Case). Assume that for all k, i there exists  $\theta < \infty$  such that

$$\left\| \begin{pmatrix} \mathcal{R}_{i}^{k,+} - \mathcal{R}_{i-1}^{k,+} \\ \mathcal{R}_{i+1}^{k,-} - \mathcal{R}_{i}^{k,-} \end{pmatrix} \right\|_{W_{i}} \left\| \mathcal{R}_{i}^{k} \right\|_{W_{i}} \le \frac{\theta}{2} \left\| \mathcal{R}_{i}^{k} \right\|_{W_{i}}^{2}.$$
(19)

Let the matrix  $\mathcal{M}_i := \mu W_i + 2W_i \mathcal{B}_i - \Delta t B_i^{\mathrm{T}} W_i B_i - \theta \frac{\Delta t}{\Delta x} \| W_i^{-1/2} B_i^{\mathrm{T}} W_i \Lambda_i W_i^{-1/2} \| \mathbb{1}$ be positive semidefinite for i = 1, ..., N and the matrix  $M := G^{\mathrm{T}} H_{\mathbb{G}} G - H_{\mathbb{B}}$  be negative semidefinite, where ghostcell-weights  $H_{\mathbb{G}}$  for  $\mathcal{R}_0^{k,+}$  and  $\mathcal{R}_{N+1}^{k,-}$  are defined as in Theorem 3.2, and the matrix  $H_{\mathbb{B}} := \mathrm{diag} \{ H_1^+, H_N^- \}$  by the discretizations

$$H_N^+ := \operatorname{diag}\left\{h_N^{(1)}, \dots, h_N^{(p)}\right\}$$
 and  $H_1^- := \operatorname{diag}\left\{h_1^{(p+1)}, \dots, h_1^{(m)}\right\}$ 

Then the discrete derivative of Lyapunov function (17) is non-increasing, i.e.

$$\frac{\mathcal{L}^{k+1} - \mathcal{L}^k}{\Delta t} \le 0$$

*Proof.* The numerical scheme (11) reads in splitting form

$$\mathcal{R}_i^{k+1} = \tilde{\mathcal{R}}_i^k - \Delta t B_i \mathcal{R}_i^k, \quad \tilde{\mathcal{R}}_i^k := \mathcal{R}_i^k - \frac{\Delta t}{\Delta x} \Lambda_i \begin{pmatrix} \mathcal{R}_i^{k,+} - \mathcal{R}_{i-1}^{k,+} \\ \mathcal{R}_{i+1}^{k,-} - \mathcal{R}_i^{k,-} \end{pmatrix},$$

where  $\tilde{\mathcal{R}}_i^k$  is the discretization of the upwind-scheme. Again the analysis for the advection part is reduced to a single discretized Riemann invariant

$$\mathcal{L}_{\text{upwind}}^{k,(j)} \coloneqq \sum_{i=1}^{N} (\tilde{\mathcal{R}}_i^{k,(j)})^2 w_i^{(j)} \Delta x, \quad j = 1, \dots, m.$$

Due to  $[(1 - \gamma)\alpha + \gamma\beta]^2 \leq (1 - \gamma)\alpha^2 + \gamma\beta^2$  for  $\gamma \in [0, 1]$  with the definition  $\mathbf{D}_i^{(j)} := \frac{\Delta t}{\Delta x} \lambda_i^{(j)} \in [0, 1]$ , we obtain for  $j = 1, \dots, p$ 

$$(\tilde{\mathcal{R}}_{i}^{k,(j)})^{2} \leq (1 - \mathcal{D}_{i}^{(j)})(\mathcal{R}_{i}^{k,(j)})^{2} + \mathcal{D}_{i}^{(j)}(\mathcal{R}_{i-1}^{k,(j)})^{2}.$$

Then the time derivative of the discretized Lyapunov function satisfies

$$\begin{split} \frac{\mathcal{L}_{\text{upwind}}^{k+1,(j)} - \mathcal{L}^{k,(j)}}{\Delta t} &= \sum_{i=1}^{N} \frac{1}{\Delta t} \Big[ (\tilde{\mathcal{R}}_{i}^{k,(j)})^{2} - (\mathcal{R}_{i}^{k,(j)})^{2} \Big] \quad \frac{h_{i-1}^{(j)}}{\lambda_{i}^{(j)}} \Delta x \\ &\leq -\sum_{i=1}^{N} \frac{D_{i}^{(j)}}{\Delta t} \Big[ (\mathcal{R}_{i}^{k,(j)})^{2} - (\mathcal{R}_{i-1}^{k,(j)})^{2} \Big] \quad \frac{h_{i-1}^{(j)}}{\lambda_{i}^{(j)}} \Delta x \\ &= -\sum_{i=1}^{N} \Big[ (\mathcal{R}_{i}^{k,(j)})^{2} - (\mathcal{R}_{i-1}^{k,(j)})^{2} \Big] \qquad h_{i-1}^{(j)} \\ &= -\Big[ (\mathcal{R}_{N}^{k,(j)})^{2} h_{N}^{+} - (\mathcal{R}_{0}^{k,(j)})^{2} h_{0}^{(j)} \Big] + \sum_{i=1}^{N} (\mathcal{R}_{i}^{k,(j)})^{2} \frac{h_{i-1}^{(j)} - h_{i-1}^{(j)}}{\Delta x} \Delta x \\ &= -\Big[ (\mathcal{R}_{N}^{k,(j)})^{2} h_{N}^{(j)} - (\mathcal{R}_{0}^{k,(j)})^{2} h_{0}^{(j)} \Big] - \mu \mathcal{L}^{k,(j)}. \end{split}$$

With similar calculations for  $j = p + 1, \ldots, m$  we obtain altogether

$$\frac{\mathcal{L}_{\text{upwind}}^{k+1} - \mathcal{L}^{k}}{\Delta t} \leq - \sum_{j=1}^{p} \left[ \left( \mathcal{R}_{N}^{k,(j)} \right)^{2} h_{N}^{(j)} - \left( \mathcal{R}_{0}^{k,(j)} \right)^{2} h_{0}^{(j)} \right]$$
(20)

$$+\sum_{j=p+1}^{m} \left[ \left( \mathcal{R}_{N+1}^{k,(j)} \right)^2 h_{N+1}^{(j)} - \left( \mathcal{R}_1^{k,(j)} \right)^2 h_1^{(j)} \right]$$
(21)

$$-\mu \mathcal{L}^k. \tag{22}$$

Analogously to the continuous case boundary terms (20) and (21) are non-positive if M is negative semidefinite. Assumption (19) yields

$$\left| \left\langle \mathcal{R}_{i}^{k} - \tilde{\mathcal{R}}_{i}^{k}, B_{i} \mathcal{R}_{i}^{k} \right\rangle_{W_{i}} \right| = \frac{\Delta t}{\Delta x} \left| \left\langle B_{i}^{\mathrm{T}} W_{i} \Lambda_{i} \begin{pmatrix} \mathcal{R}_{i}^{k,+} - \mathcal{R}_{i-1}^{k,+} \\ \mathcal{R}_{i+1}^{k,-} - \mathcal{R}_{i}^{k,-} \end{pmatrix}, \mathcal{R}_{i}^{k} \right\rangle \right|$$
  
$$\leq \frac{\theta}{2} \frac{\Delta t}{\Delta x} \left\| W_{i}^{-1/2} B_{i}^{\mathrm{T}} W_{i} \Lambda_{i} W_{i}^{-1/2} \right\| \left\| \mathcal{R}_{i}^{k} \right\|_{W_{i}}^{2}.$$
(23)

Using  $\mathcal{R}_i^{k+1} = \tilde{\mathcal{R}}_i^k - \Delta t B_i \mathcal{R}_i^k$  we obtain for

$$\mathcal{L}^{k+1} = \sum_{i=1}^{N} \left[ \left\| \tilde{\mathcal{R}}_{i}^{k} \right\|_{W_{i}}^{2} - 2\Delta t \left\langle \tilde{\mathcal{R}}_{i}^{k}, B_{i} \mathcal{R}_{i}^{k} \right\rangle_{W_{i}} + \Delta t^{2} \left\| B_{i} \mathcal{R}_{i}^{k} \right\|_{W_{i}}^{2} \right] \Delta x$$

estimates (22), (23) and due to the positive semidefiniteness of  $\mathcal{M}_i$ 

$$\frac{\mathcal{L}^{k+1} - \mathcal{L}^{k}}{\Delta t} = \frac{\mathcal{L}^{k+1}_{\text{upwind}} - \mathcal{L}^{k}}{\Delta t} - \sum_{i=1}^{N} \left[ \left\langle \mathcal{R}^{k}_{i}, (2\mathcal{B}_{i} - \Delta t B^{\text{T}}_{i} B_{i}) \mathcal{R}^{k}_{i} \right\rangle_{W_{i}} + 2 \left\langle \mathcal{R}^{k}_{i} - \tilde{\mathcal{R}}^{k}_{i}, B_{i} \mathcal{R}^{k}_{i} \right\rangle_{W_{i}} \right] \Delta x \leq - \sum_{i=1}^{N} \left\langle \mathcal{R}^{k}_{i}, \mathcal{M}_{i} \mathcal{R}^{k}_{i} \right\rangle \Delta x \leq 0.$$

We establish an exponential decaying Lyapunov function similar to Corollary 1.

**Corollary 2** (Exponential Stability in the Discretized Case). We define the matrix  $\tilde{\mathcal{M}}_i(\tilde{\mu}) := (\mu - \tilde{\mu})W_i + 2W_i\mathcal{B}_i - \Delta t B_i^{\mathrm{T}}W_iB_i - \theta \frac{\Delta t}{\Delta x} \|W_i^{-1/2}B_i^{\mathrm{T}}W_i\Lambda_iW_i^{-1/2}\|\mathbb{1}$ . Under the assumptions of Theorem 4.2 the Lyapunov function decays exponentially in the following sense:

$$\mathcal{L}^{k} \leq e^{-\tilde{\mu}k\Delta t} \mathcal{L}^{0} \quad \text{for } \tilde{\mu} \leq \inf\left\{\bar{\mu} > 0 \ \Big| \ \tilde{\mathcal{M}}_{i}(\bar{\mu}) \text{ is positive semidefinite for all } i\right\}$$

*Proof.* For  $\mathcal{M}_i(\tilde{\mu})$  positive semidefinite the discretized derivative is estimated by

$$\frac{\mathcal{L}^{k+1} - \mathcal{L}^k}{\Delta t} \le -\tilde{\mu}\mathcal{L}^k - \sum_{i=1}^N \left\langle \mathcal{R}_i^k, \tilde{\mathcal{M}}_i(\tilde{\mu})\mathcal{R}_i^k \right\rangle \Delta x \le -\tilde{\mu}\mathcal{L}^k.$$

The claim follows from

$$\mathcal{L}^{k} \leq (1 - \Delta t \tilde{\mu}) \mathcal{L}^{k-1} \leq (1 - \Delta t \tilde{\mu})^{k} \mathcal{L}^{0} \leq e^{-\tilde{\mu}k\Delta t} \mathcal{L}^{0}.$$

We end this section by discussing the assumptions of Theorem 4.2. Assumption (19) is a restriction on the spatial discretization. For the given CFL-condition and given  $\theta > 0$  it is satisfied for  $\Delta x > 0$  small enough, since for sufficiently smooth solutions the left-hand sides scales as  $\mathcal{O}(\Delta x)$ . Since the ratio  $\frac{\Delta t}{\Delta x}$  is fixed the influence of the term  $\theta \frac{\Delta t}{\Delta x} ||W_i^{-1/2} B_i^{\mathrm{T}} W_i \Lambda_i W_i^{-1/2}||$  is of order  $\mathcal{O}(\theta)$ . Therefore, in the formal limit we do not have an additional restriction compared with Theorem 3.2. Note that we have to account for the steady state  $\mathcal{R}_i^k = 0$ , therefore assumption (19) does not simplify to

$$\left\| \begin{pmatrix} \mathcal{R}_i^{+,k} - \mathcal{R}_{i-1}^{+,k} \\ \mathcal{R}_{i+1}^{-,k} - \mathcal{R}_i^{-,k} \end{pmatrix} \right\|_{W_i} \le \frac{\theta}{2} \| \mathcal{R}_i^k \|_{W_i}.$$

The matrix  $\mathcal{M}_i$  consists of the sum of two positive semidefinite matrices and two negative semidefinite matrices. The terms  $\mu W_i + 2W_i \mathcal{B}_i$  are in accordance to Theorem 3.2 and the term  $\theta \frac{\Delta t}{\Delta x} \| W_i^{-1/2} B_i^{\mathrm{T}} W_i \Lambda_i W_i^{-1/2} \|$  has been discussed. But the symmetric and negative semidefinite term  $-\Delta t B_i^{\mathrm{T}} W_i B_i$  makes the positive definiteness assumption of  $\mathcal{M}_i$  stronger, since the additional term introduces a restriction on the time step  $\Delta t$ . In the case of a positive semidefinite source term  $B_i$ this restriction coincides with the stability region of the explicit Euler method, i.e.  $\Delta t \max_{i=1,...,N} |\rho(B_i)| \in [0, 2].$ 

To show this we assume that  $B_i$  is positive semidefinite for all i = 1, ..., N. Then it holds for all eigenvectors  $v_i \in \mathbb{R}^m$  with eigenvalues  $\delta_i \in \mathbb{R}^+_0$  such that  $B_i v_i = \delta_i v_i$ :

$$\langle v_i, W_i \mathcal{B}_i v_i \rangle = \langle v_i, \delta_i v_i \rangle_{W_i} = \delta_i ||v_i||_{W_i}^2$$
  
 
$$\langle v_i, B_i^{\mathrm{T}} W_i B_i v_i \rangle = \langle B_i v_i, B_i v_i \rangle_{W_i} = \delta_i^2 ||v_i||_{W_i}^2$$

The stability region of the explicit Euler method yields for the guaranteed decay rate  $\tilde{\mu} := \mu - \theta \frac{\Delta t}{\Delta x} \| W_i^{-1/2} B_i^{\mathrm{T}} W_i \Lambda_i W_i^{-1/2} \| W_i^{-1}$  and for all eigenvectors

$$\left\langle v_i, \tilde{\mathcal{M}}_i(\tilde{\mu})v_i \right\rangle = 2\left\langle v_i, W_i \mathcal{B}_i v_i \right\rangle - \Delta t \left\langle v_i, B_i^{\mathrm{T}} W_i B_i v_i \right\rangle = \left(2 - \Delta t \delta_i\right) \delta_i \|v_i\|_{W_i}^2 \ge 0.$$

To sum up we observe the guaranteed decay rate in the discretized case is slightly smaller than in the continuous case, because Corollary 2 yields an exponential decay for  $\tilde{\mu} < \mu$ , where discretization in space and time makes the guaranteed rate smaller. Furthermore, we note that eigenvalues of  $B_i$  do not coincide with those of  $\mathcal{B}_i$ , only in the sense  $\langle v_i, \mathcal{B}_i v_i \rangle_{W_i} = \langle v_i, B_i v_i \rangle_{W_i} = \langle v_i, \delta_i v_i \rangle_{W_i}$ . As in the continuous case our analysis covers the pathological case  $\tilde{\mu} \leq 0$ .

# 5 Suitable Boundary Conditions for Exponential Decay

Due to [8, 9] exponential stability is ensured if the matrix G, which describes feedback boundary conditions, satisfies a *dissipative* condition. The following theorem is proven based on the norm

$$\rho_1(G) := \inf_{\mathcal{D} \in \mathbb{D}} \Big\{ \big\| \mathcal{D} G \mathcal{D}^{-1} \big\| \Big\},\,$$

where  $\mathbb{D}$  denotes the set of real  $m \times m$  diagonal matrices with strictly positive entries.

**Theorem 5.1** (Dissipative Condition [8, Th. 2]). If  $\rho_1(G) < 1$ , there exists  $\varepsilon > 0$  such that, if  $||G|| < \varepsilon$ , then the IBVP (S) is exponentially stable in the sense of Definition 2.2.

We state a similar criterion for source terms with possibly large operator norm. To obtain the corresponding result in the discretized case we assume the space restriction (18), i.e.  $\Delta x \in (0, \lambda_{\min}/\mu)$  for  $\lambda_{\min} := \min_{\substack{x \in [0,L], \ j=1,...,m}} |\lambda^{(j)}(x)|$ , and we define

$$\rho_{\mu,\Delta x}(G,\mathcal{D}) := \left(1 - \Delta x \frac{\mu}{\lambda_{\min}}\right)^{-N/2} \left\| \mathcal{D}G\mathcal{D}^{-1} \right\|_2$$
$$\rho_{\mu}(G,\mathcal{D}) := e^{\mu \frac{L}{2\lambda_{\min}}} \left\| \mathcal{D}G\mathcal{D}^{-1} \right\|_2.$$

**Theorem 5.2** (Discretized Dissipative Boundary Condition). Let a decay rate  $\mu > 0$ , discretization  $\Delta x \in (0, \lambda_{\min}/\mu)$  be given and assume  $\rho_{\mu,\Delta x}(G, \mathcal{D}) \leq 1$  for

some  $\mathcal{D} \in \mathbb{D}$ . Then the matrix  $M(\mu) := G^{\mathrm{T}} H_{\mathbb{G}} G - H_{\mathbb{B}}(\mu)$  is negative semidefinite for  $H_{\mathbb{G}} := \mathcal{D}^2$  and  $H_{\mathbb{B}}(\mu) := H_{\mathbb{G}} \Pi^2(\mu)$ , where  $\Pi^2(\mu) := \prod_{i=1}^N \operatorname{diag}\{\Lambda_i^+(\mu), \Lambda_i^-(\mu)\}$ is defined by

$$\Lambda^{+}(\mu) := \operatorname{diag}\left\{1 - \Delta x \frac{\mu}{\lambda_{i}^{(1)}}, \dots, 1 - \Delta x \frac{\mu}{\lambda_{i}^{(p)}}\right\},$$
$$\Lambda^{-}(\mu) := \operatorname{diag}\left\{1 - \Delta x \frac{\mu}{|\lambda_{i}^{(p+1)}|}, \dots, 1 - \Delta x \frac{\mu}{|\lambda_{i}^{(m)}|}\right\}.$$

*Proof.* According to Definition 4.1 the weights at ghostcells  $H_{\mathbb{G}}$  are related to  $H_{\mathbb{B}}(\mu)$  by  $H_{\mathbb{B}}(\mu) = H_{\mathbb{G}}\Pi^2(\mu)$ . Defining  $\mathcal{D}_{\mathbb{B}}(\mu) := \mathcal{D}\Pi(\mu)$  the boundary matrix reads

$$M(\mu) := G^{\mathrm{T}} H_{\mathbb{G}} G - H_{\mathbb{B}}(\mu) = G^{\mathrm{T}} \mathcal{D}^{2} G - \mathcal{D}_{\mathbb{B}}^{2}(\mu).$$

The inverse  $\Pi^{-1}(\mu)$  exists for  $\Delta x < \lambda_{\min}/\mu$ . Since the  $\|\cdot\|_2$ -norm of a matrix B satisfies  $\|B\|_2^2 = \rho\{B^T B\}$ , the inverse is estimated by

$$\begin{split} \left\|\Pi^{-1}(\mu)\right\|_{2}^{2} &\leq \prod_{i=1}^{N} \left\|\operatorname{diag}\left\{\Lambda_{i}^{+}(\mu), \Lambda_{i}^{-}(\mu)\right\}^{-1/2}\right\|_{2}^{2} = \prod_{i=1}^{N} \rho\left\{\operatorname{diag}\left\{\Lambda_{i}^{+}(\mu), \Lambda_{i}^{-}(\mu)\right\}^{-1}\right\} \\ &\leq \left(1 - \Delta x \frac{\mu}{\lambda_{\min}}\right)^{-N}. \end{split}$$

Then the assumption  $\rho_{\mu,\Delta x}(G, \mathcal{D}) \leq 1$  implies that the matrix  $M(\mu)$  is negative semidefinite:

In practical applications decay rate and boundary conditions are prescribed and weights of a Lyapunov function can be determined by the minimization problem

$$\mathcal{D} := \underset{\tilde{\mathcal{D}} \in \mathbb{D}}{\operatorname{argmin}} \Big\{ \big\| \tilde{\mathcal{D}} G \tilde{\mathcal{D}}^{-1} \big\|_2 \Big\}.$$

The set  $\mathbb{D}$  may be restricted for technical reasons. Furthermore, the minimum must neither be unique nor actually exist as long as  $\rho_{\mu,\Delta x}(G, \mathcal{D}) \leq 1$  holds. Of course, the minimization problem must be solvable, which in the discretized case is a slightly stronger assumption. In the limit  $\Delta x \to 0$  the assumption tends to the continuous case. To see this, we assume that  $\mathcal{D}$  is the infimum, i.e.  $\rho_1(G) = \|\mathcal{D}G\mathcal{D}^{-1}\|$ . Then the norm  $\rho_{\mu,\Delta x}(\cdot, \mathcal{D})$  is an approximation of  $\rho_{\mu}(\cdot, \mathcal{D}) = \exp(\mu \frac{L}{2\lambda_{\min}})\rho_1(\cdot) > \rho_1(\cdot)$ . More precisely, there is convergence from above, i.e.  $\rho_{\mu,\Delta x}(\cdot, \mathcal{D}) \searrow \rho_{\mu}(\cdot, \mathcal{D}) \searrow \rho_1(\cdot)$ for  $\Delta x, \mu \to 0$ . To sum up the decay of the Lyapunov function is split into two parts. Firstly, there is a decay in the interior of the interval (0, L) which is described in Theorem 3.2 and Corollary 1 and that is higher for a larger decay rate  $\mu$ . Secondly, dissipative boundary conditions cause a decay which is smaller for a larger decay rate  $\mu$ . This means that both parts are coupled. But we balance the decay in the interior by appropriate boundary conditions.

#### 6 Numerical Results

#### 6.1 Isothermal Euler Equations

As a first example we consider isothermal Euler equations, defined by

$$\begin{pmatrix} \rho \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{\rho} + a^2 \rho \end{pmatrix}_x = - \begin{pmatrix} 0 \\ \frac{f}{D} \frac{q|q|}{2\rho} \end{pmatrix},$$

where  $\rho(t, x)$  is the density of the gas, q(t, x) the mass flux in the pipe, f a friction factor, D the diameter of the pipe and a is the speed of sound. In a steady state there is no variation of the momentum  $\bar{q} = m$  and we will assume w.l.g.  $m \ge 0$ . The steady state is determined by the initial value problem

$$\left(\frac{m^2}{\bar{\rho}} + a^2 \bar{\rho}\right)_x + \frac{f}{D} \frac{m^2}{2\bar{\rho}} = 0$$
  
$$\Leftrightarrow \qquad a^2 \bar{\rho}^2(x) - 2m^2 \ln\left(\bar{\rho}(x)\right) = a^2 \bar{\rho}_0^2 - 2m^2 \ln\left(\bar{\rho}_0\right) - \frac{f}{D} m^2 x$$

with  $\bar{\rho}_0 := \bar{\rho}(0) > 0$ . Linearization at steady state gives

$$A = \begin{pmatrix} 0 & 1\\ a^2 - (\frac{\bar{q}}{\bar{\rho}})^2 & 2\frac{\bar{q}}{\bar{\rho}} \end{pmatrix}, \quad \bar{S} = \frac{f}{D} \frac{\bar{q}}{\bar{\rho}} \begin{pmatrix} 0 & 0\\ -\frac{1}{2}\frac{\bar{q}}{\bar{\rho}} & 1 \end{pmatrix}.$$

Assuming a subsonic flow with  $|\bar{q}/\bar{\rho}| < a$  the Jacobian A is diagonalizable with characteristic speeds  $\lambda^{\pm} = \bar{q}/\bar{\rho} \pm a$  satisfying  $\lambda^{-} < 0 < \lambda^{+}$ .

According to [15] stationary states exist as smooth solutions only on a finite space interval, until a critical length is reached. There, a blow-up in the derivatives occurs. The density at steady state is decreasing, i.e.  $\bar{\rho}_x < 0$  and consequently eigenvalues are increasing as  $\lambda_x^{\pm} = -m\bar{\rho}^{-2}\bar{\rho}_x > 0$ . Characteristics do not cross and a stable solution exists on unbounded time domains. This is related to [18].

#### Numerical Results

We begin the analysis with periodic boundary conditions stated in Riemann coordinates, i.e. a simple loop such that there are no dissipative boundary conditions, and the system is only stabilized by the source term. In the context of system  $(\mathcal{S})$  the boundary matrix G simplifies to the identity. Furthermore, we consider the pathological case  $\mu := 0$ . For all simulations we use a unit time and space interval with CFL = 1 and the parameters a = 1,  $\frac{f}{D} = 1$ . Steady state is determined by  $(\bar{\rho}_0, m) := (3, 0.2)$  and initial perturbations in Riemann coordinates are  $\zeta^{\pm}(0, x) := \cos(2\pi x)$ . The continuous and discretized Lyapunov functions are normalized such that  $\mathcal{L}(0) = 1$ . The upwind-scheme let us expect a linear convergence, which is indeed observed in Figure 1 and Table 1, where the logarithm (causing negative values) of the error

$$\hat{L}_x^1 := \sum_{i=1}^N \sum_{j=1}^m \left| \mathcal{R}_i^{(j), \text{ref}} - \mathcal{R}_i^{(j)} \right| \Delta x$$

in t = 1 is shown together with the empirical order of convergence (EOC). The reference solution  $\mathcal{R}^{\text{ref}}$  is calculated on a refined grid with  $\Delta x = 2^{-12}$  and is summed up on the corresponding coarser grid, which causes no additional error for finite volume methods.



Fig. 1:  $\log_2$  of  $\hat{L}_x^1$ -error for Riemann invariants

			$\hat{L}_x^1$ -erro	or	EOC					
$\Delta x$	d = 0	d = 1	d = 2	d = 3	d = 4	d = 0	d = 1	d = 2	d = 3	d = 4
$2^{-4}$	0.52	3.61	24.11	146.27	953.65					
$2^{-5}$	0.27	1.87	12.46	75.48	492.38	0.95	0.95	0.95	0.95	0.95
$2^{-6}$	0.13	0.94	6.27	37.97	247.28	0.99	0.99	0.99	0.99	0.99
$2^{-7}$	0.07	0.46	3.11	18.80	121.91	1.01	1.01	1.01	1.01	1.02
$2^{-8}$	0.03	0.23	1.51	9.12	58.63	1.04	1.04	1.04	1.04	1.06
$2^{-9}$	0.02	0.11	0.70	4.26	26.98	1.10	1.10	1.10	1.10	1.12

Tab. 1:  $\hat{L}_x^1$ -error and EOC for Riemann invariants

Next we focus on Lyapunov functions with respect to the  $L^2, H^1, \ldots, H^4$  norm, illustrated in Figure 2. Due to friction effects we observe a decay in the Lyapunov function. However, close to t = 0.25, 0.75 the Lyapunov functions stay almost constant, which hints that estimates are sharp.

Table 2 shows discretization errors and corresponding EOCs. For the Lyapunov function there is no theoretical linear convergence anymore. Therefore we show



Fig. 2: Lyapunov functions for a simple loop

additionally an approximation of the  $L^2$ - and  $L^{\infty}$ -error

$$L_t^p := \left(\int_0^1 \left| \mathcal{L}^{\mathrm{ref}}(t) - \mathcal{L}^{(\Delta x)}(t) \right|^p \mathrm{d}t \right)^{1/p}, \quad L_t^\infty := \max_{t \in [0,1]} \left| \mathcal{L}^{\mathrm{ref}}(t) - \mathcal{L}^{(\Delta x)}(t) \right|,$$

where the reference value  $\mathcal{L}^{\text{ref}}$  is again calculated on a refined grid with  $\Delta x = 2^{-12}$ and the coarser approximation is linearly interpolated on the refined grid. Integrals are approximated by the trapezoidal rule.

We observe the rate of convergence of the upwind-scheme is inherited to corresponding Lyapunov functions. We refer the interested reader to [20], where this is only observed for CFL = 0.75.

#### **Compressor Stations**

Since friction makes the pressure decrease along pipes, compressors are used to amplify pressure, as illustrated in Figure 3 for a pipeline with two compressors. Like in [15, 14] compressors are modelled by assuming conservation of mass and by applying a compressor power  $u(t) \ge 0$ , i.e.

$$q^{\rm in}(t,L) = q^{\rm out}(t,0) \quad \text{and} \quad u(t) = q^{\rm out}(t,0) \left[ \left( \frac{\rho^{\rm out}(t,0)}{\rho^{\rm in}(t,L)} \right)^{\kappa} - 1 \right],$$
(24)

where the superindices "in" and "out" stand in the case  $u \ge 0$  for the left-sided and right-sided pipe of the corresponding compressor. The specific heat ratio  $\kappa$ depends on the considered gas, we use  $\kappa = 0.5$ . Additional boundary conditions are needed to close the system. So we assume a given steady state in the left and right end. Compressor powers  $u_1$ ,  $u_2$  are determined by the steady state and affect boundary conditions. An interesting question is how to set power to obtain dissipative boundary conditions. An analytical answer is given in [15] for a finite time horizon. In extension to [15], Figure 3 shows a stability domain for compressor power

$$\left\{ (u_1, u_2) \in \mathbb{R}_0^+ \mid \rho_{\mu, \Delta x}(G, \mathcal{D}) \le 1 \text{ for some } \mathcal{D} \right\}$$

			$L_t^1$ -error	:		EOC				
$\Delta x$	d = 0	d = 1	d = 2	d = 3	d = 4	d = 0	d = 1	d = 2	d = 3	d = 4
$2^{-4}$	6.90	7.08	7.08	7.08	7.00					
$2^{-5}$	3.61	3.70	3.70	3.70	3.63	0.93	0.93	0.93	0.93	0.95
$2^{-6}$	1.84	1.88	1.88	1.88	1.81	0.98	0.98	0.98	0.98	1.00
$2^{-7}$	0.91	0.94	0.94	0.94	0.87	1.01	1.01	1.01	1.01	1.06
$2^{-8}$	0.45	0.46	0.46	0.46	0.39	1.04	1.04	1.04	1.04	1.17
$2^{-9}$	0.21	0.21	0.21	0.21	0.14	1.10	1.10	1.10	1.10	1.42
			EOC							
$\Delta x$	d = 0	d = 1	d = 2	d = 3	d = 4	d = 0	d = 1	d = 2	d = 3	d = 4
$2^{-4}$	7.90	8.04	8.04	8.04	7.96					
$2^{-5}$	4.15	4.23	4.23	4.23	4.15	0.93	0.93	0.93	0.93	0.94
$2^{-6}$	2.12	2.16	2.16	2.16	2.08	0.97	0.97	0.97	0.97	1.00
$2^{-7}$	1.06	1.08	1.08	1.07	1.00	1.00	1.00	1.00	1.00	1.06
$2^{-8}$	0.51	0.52	0.52	0.52	0.44	1.04	1.04	1.04	1.04	1.16
$2^{-9}$	0.24	0.25	0.25	0.25	0.17	1.09	1.09	1.09	1.09	1.42
			$L_t^{\infty}$ -erro	r	EOC					
$\Delta x$	d = 0	d = 1	d = 2	d = 3	d = 4	d = 0	d = 1	d = 2	d = 3	d = 4
$2^{-4}$	13.90	13.92	13.92	13.91	13.78					
$2^{-5}$	7.41	7.43	7.43	7.42	7.29	0.91	0.91	0.91	0.91	0.92
$2^{-6}$	3.79	3.79	3.79	3.79	3.66	0.97	0.97	0.97	0.97	0.99
$2^{-7}$	1.87	1.87	1.87	1.87	1.74	1.02	1.02	1.02	1.02	1.07
$2^{-8}$	0.89	0.90	0.90	0.90	0.77	1.06	1.06	1.06	1.06	1.19
$2^{-9}$	0.42	0.42	0.42	0.42	0.29	1.09	1.09	1.09	1.09	1.40

Tab. 2:  $L_t^1$ -,  $L_t^2$ -,  $L_t^\infty$ - error and EOC for Lyapunov functions, units in 0.01

that guarantees a dissipativity condition for  $\mu := 1$  and thus yields stability for unbounded time intervals. We have used the matlab function fmincon to approximate the minimum  $\min_{\mathcal{D} \in \mathbb{D}} \{\rho_{\mu,\Delta x}(G, \mathcal{D})\}$  on the z-axis.



Fig. 3: Pipeline with two compressors and stability domain for  $L^2$ -norm

For this simulation we have obtained the linear boundary conditions

$$\begin{pmatrix} \Delta\rho^{(2)}(t,0)\\ \Delta\rho^{(3)}(t,0)\\ \Deltaq^{(1)}(t,L)\\ \Deltaq^{(2)}(t,L) \end{pmatrix} = \begin{pmatrix} \bar{c}_1^{(1)} & \bar{c}_2^{(1)} \\ & \bar{c}_1^{(2)} & \bar{c}_2^{(2)} \\ & & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \Delta\rho^{(1)}(t,L)\\ \Delta\rho^{(2)}(t,L)\\ \Deltaq^{(2)}(t,0)\\ \Deltaq^{(3)}(t,0) \end{pmatrix}$$

by linearizing the nonlinear energy equation (24)

$$\rho^{\mathrm{out}}(t,0) = \rho^{\mathrm{in}}(t,L) \left(\frac{u}{q^{\mathrm{out}}(t,0)} + 1\right)^{1/\kappa} \approx \bar{\rho}^{\mathrm{out}}(0) + (\bar{c}_1,\bar{c}_2) \left(\frac{\Delta\rho^{\mathrm{in}}(t,L)}{\Delta q^{\mathrm{out}}(t,0)}\right)$$

for some coefficients  $\bar{c}_i$ . The superindices (j) denote the *j*-th pipe and the *j*-th compressor, respectively. Recall that at the left and right end additional boundary conditions are needed to close the system, in our simulations  $\Delta \rho^{(1)}(t,0) = 0$  and  $\Delta q^{(3)}(t,L) = 0$ . The transformation into Riemann invariants is given by the following theorem.

**Theorem 6.1** (Transform from Physical into Riemann Coordinates). Define the set of densities  $\rho := (\rho^{(1)}, \ldots, \rho^{(n)})$  and mass fluxes  $q := (q^{(1)}, \ldots, q^{(n)})$ , where n denotes the number of coupled pipes, and assume boundary conditions of the form

$$\begin{pmatrix} \Delta\rho(t,0)\\ \Delta q(t,L) \end{pmatrix} = K \begin{pmatrix} \Delta\rho(t,L)\\ \Delta q(t,0) \end{pmatrix}, \qquad K := \begin{pmatrix} k_{1,1} & k_{1,2}\\ k_{2,1} & k_{2,2} \end{pmatrix}$$
(25)

with  $k_{i,j} \in \mathbb{R}^{n \times n}$  and an existing inverse  $k_{2,2}^{-1}$ . Then the matrix  $\overline{G} \in \mathbb{R}^{2n \times 2n}$  for  $(\overline{S})$  is given by the  $n \times n$  block matrices

$$g_{1,1} := -q_{1,1} + g_{1,2}g_{2,2}^{-1}g_{2,1}, \qquad g_{1,2} := -q_{1,2}g_{2,2}, g_{2,1} := g_{2,2}q_{2,1}, \qquad g_{2,2} := -q_{2,2}^{-1},$$

where  $1 \in \mathbb{R}^{n \times n}$  denotes the identity matrix and  $q_{i,j} \in \mathbb{R}^{n \times n}$  the blocks of

$$Q := T^{-1}(0)K_1^{-1}K_2T(L) \quad with$$
  
$$K_1 := \begin{pmatrix} -\mathbb{1} & k_{1,2} \\ & k_{2,2} \end{pmatrix}, \quad K_2 := \begin{pmatrix} k_{1,1} \\ & k_{2,1} & -\mathbb{1} \end{pmatrix}.$$

*Proof.* With the definitions  $y := (\rho, q)^{\mathrm{T}}$  and  $\mathbb{O} := \mathrm{diag}\{0, \ldots, 0\} \in \mathbb{R}^{n \times n}$  boundary conditions (25) read

$$\begin{pmatrix} \mathbb{O} & \\ & \mathbb{O} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \\ & \mathbb{O} \end{pmatrix} \Delta y(t,0) + \begin{pmatrix} \mathbb{O} & \\ & \mathbb{1} \end{pmatrix} \Delta y(t,L) \\ -K \begin{bmatrix} \begin{pmatrix} \mathbb{O} & \\ & \mathbb{1} \end{pmatrix} \Delta y(t,0) + \begin{pmatrix} \mathbb{1} & \\ & \mathbb{O} \end{pmatrix} \Delta y(t,L) \end{bmatrix},$$

which is by assumption equivalent to

$$\begin{aligned} \Delta y(t,0) &= -K_1^{-1}K_2 \qquad \Delta y(t,L) \\ \Leftrightarrow & T(0)\zeta(t,0) = -K_1^{-1}K_2T(L) \qquad \zeta(t,L) \\ \Leftrightarrow & \zeta(t,0) \qquad = -\underbrace{T^{-1}(0)K_1^{-1}K_2T(L)}_{=Q}\zeta(t,L). \end{aligned}$$

We define the matrices

$$A := \begin{pmatrix} -\mathbb{1} & g_{1,2} \\ & g_{2,2} \end{pmatrix}, \quad B := \begin{pmatrix} g_{1,1} \\ g_{2,1} & -\mathbb{1} \end{pmatrix},$$

where the inverse  $A^{-1}$  exists, since  $q_{2,2}^{-1}$  exists by assumption. Similar calculations for Riemann coordinates show

$$\zeta(t,0) = -A^{-1}B\zeta(t,L).$$

The ansatz  $A^{-1}B \stackrel{!}{=} Q$  yields the claim.

# 6.2 Shallow Water Equations

As second example we consider the shallow water equations with conserved quantities height h and momentum hu. Since we do not consider shocks, we use the equivalent formulation with respect to velocity u.

$$\binom{h}{u}_{t} + \binom{hu}{\frac{1}{2}u^{2} + gh}_{x} = -g \binom{0}{C\frac{u^{2}}{h} - S_{B}},$$

where g denotes the gravitational constant, C describes friction and  $S_B := -B_x$  the slope of the possibly space-varying bottom topography B(x). To obtain a subcritical flow we assume Froud's number  $Fr := |u|/\sqrt{gh} < 1$  such that the Jacobian of the flux function

$$A = \begin{pmatrix} \bar{u} & \bar{h} \\ g & \bar{u} \end{pmatrix}, \quad \bar{S} = gC \begin{pmatrix} 0 & 0 \\ -\left(\frac{\bar{u}}{\bar{h}}\right)^2 & 2\frac{\bar{u}}{\bar{h}} \end{pmatrix}$$

is diagonalizable with distinct eigenvalues  $\lambda^{\pm} = \bar{u} \pm \sqrt{g\bar{h}}$ . The momentum in a steady state is again constant, i.e.  $\bar{h}\bar{u} = m$ . Exponential stability in  $L^2$ -norm for a cascade of n pools is investigated analytically in [10, 3], where the simple model  $S_B := C\bar{u}^2/\bar{h}$  is assumed. Then each constant pair  $(\bar{h}, \bar{u})$  yields a steady state. Our

analysis may be viewed as an extension to a non-monotone bottom topography. More generally, a steady state is characterized by the ODE

$$0 = \left(\frac{1}{2}\bar{u}^2 + g\bar{h}\right)_x + g\left(C\frac{\bar{u}^2}{\bar{h}} - S_B\right) = \bar{h}_x\left(-\frac{m^2}{\bar{h}^3} + g\right) + g\left(C\frac{m^2}{\bar{h}^3} - S_B\right).$$

Note that in the last equation it holds

$$-\frac{m^2}{\bar{h}^3} + g = -\frac{\bar{u}^2}{\bar{h}} + g > 0 \ \, \text{for} \ \, \mathrm{Fr} < 1,$$

so the ODE does not degenerate. We assume the bottom topography

$$S_B := s_1 + C s_2$$
 with  $s_1 := \left(1 - \frac{\bar{u}^2}{g\bar{h}}\right) \bar{h}_x, \quad s_2 := \frac{\bar{u}^2}{\bar{h}},$ 

which gives immediately a steady state. For the simple model  $S_B := C\bar{u}^2/\bar{h}$  eigenvalues are constant and space-varying for a non-monotone bottom topography. A subcritical flow ensures in both cases distinct eigenvalues  $\bar{\lambda}^- < 0 < \bar{\lambda}^+$ , which implies again an existing solution on unbounded time intervals [12].

#### **Numerical Results**



Fig. 4: Lyapunov functions for a simple loop for reference solution with steady state  $\bar{h}(x) := 3$  and  $\bar{h}(x) := 3 + 10^{-3} \cos(2\pi x)$ 

In most of the literature stability analysis is restricted to boundary conditions, see e.g. [3]. In extension we highlight the influence of source terms by comparing a constant height  $\bar{h}(x) := 3$  with perturbations  $\bar{h}(x) := 3 + 10^{-p} \cos(2\pi x)$  for  $p = 1, \ldots, 4$  and m = 0.2,  $\Delta x = 2^{-12}$ , g = 1, C = 1. We assume  $\mu = 0.25$  with dissipative boundary conditions  $G := \text{diag}\{0.9, \ldots, 0.9\}$  such that assumptions of Theorem 5.2 are fulfilled.

Figure 4 shows on the left side decaying Lyapunov functions for a steady state with constant height and momentum and on the right for a perturbed height with non-constant bottom slope. In contrast Figure 5 shows under the same setting an

Estimated rate									
	constant	p = 4	p = 3	p=2	p = 1				
d = 0	0.2499	0.2498	0.2485	0.2254	-0.1563				
d = 1	0.2499	0.2495	0.2420	0.0374	-2.2991				
d = 2	0.2499	0.2458	0.1288	-1.3167	-15.8406				
Guaranteed rate									
	constant	p = 4	p = 3	p=2	p = 1				
d = 0	0.2499	0.2498	0.2487	0.2279	-0.1154				
d = 1	0.2499	0.2495	0.2428	0.0591	-2.0424				
d=2	0.2499	0.2463	0.1412	-1.1566	-14.2202				
Observed rate									
	constant	p = 4	p = 3	p = 2	p = 1				
d = 0	0.2572	0.2572	0.2572	0.2537	0.1343				
d = 1	0.3461	0.3459	0.3416	0.2126	-1.3244				
d = 2	0.2632	0.2615	0.2089	-0.6659	-10.7687				

Tab. 3: Estimated decay rate  $\mu^{e}$  (top), guaranteed rate  $\mu^{g}$  (middle) and observed rate  $\mu^{o}$  (bottom) for  $\mu := 0.25$  and reference solution with constant and perturbed steady states



Fig. 5: Non-decaying Lyapunov functions for a simple loop for reference solution with steady state  $\bar{h}(x) := 3 + 0.1 \cos(2\pi x)$ 

unstable steady state. More specifically, Table 3 shows the in Corollary 1 guaranteed decay rate with estimated smallest eigenvalue and the observed rate, namely:

- guaranteed rate:  $\mu^{g} := \mu + 2d_{\min}(\mu)$
- estimated rate:  $\mu^{e} := \mu - \operatorname{cond}_{\sqrt{W}}(\mu) \max_{x \in [0,L]} \left\{ \|B(x)\|_{2} \right\}$ • observed rate:  $\mu^{o} := \min_{t \in [0,T]} \left\{ -\frac{1}{t} \ln \left( \frac{\mathcal{L}^{\operatorname{ref}}(t)}{\mathcal{L}^{\operatorname{ref}}(0)} \right) \right\}$

We see that stability is only guaranteed if the perturbations are small enough, as for  $\mu^{g} < 0$  exponential stability is not guaranteed by Corollary 1. The case p = 1, where stability is not guaranteed and which is gray shaded in Table 3, is shown in Figure 5. Furthermore, the estimated decay rate seems as a good lower bound and the observed rate is indeed larger than the guaranteed rate, i.e.  $\mu^{g} \leq \mu^{g} \leq \mu^{o}$ . This is because Corollary 1 takes only the smallest eigenvalue into account whereas positive eigenvalues make Lyapunov functions decrease further.

# 7 Summary

Exponential Lyapunov stability with explicit decay rate has been proven by a theoretical and numerical analysis. We have shown how weights can be set to ensure exponential stability. Results are generalized to stability with respect to higher Sobolev norms. Influence of source terms has been investigated and stability on unbounded time domains has been established.

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