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On the derivation of the Discontinuous Galerkin method for hyperbolic conservation laws

Dedicated to Professor Chi-Wang Shu on the occasion of his 60th birthday

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Abstract

We re-derive Cockburn and Shu's semi-discrete Discontinuous Galerkin method [Mathematics of Computation, 52:411-435, 1989] by passing to the limit in the classical space-time weak formulation. This leads to a new interpretation of the role of the numerical flux function.

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1 Introduction

In the landmark papers [1], [2], and subsequently in [3] [4] [5] [6] have introduced the Runge-Kutta Discontinuous Galerkin (RKDG) method for systems of hyperbolic conservation laws and convection-dominated problems. Together with the ENO and WENO schemes, the RKDG method is one of the most powerful and widely used high-order accurate methods for compressible flows. It combines the advantages of a shock-capturing, conservative finite volume method with a weak formulation based on piecewise high-order polynomials. This blending is made possible since the polynomials are in general discontinuous at the cell edges, which makes room for numerical fluxes based on approximate Riemann solvers. The method requires smaller time-steps than finite volume schemes, but due to its local stencil it is very well suited for parallelization. More than 25 years after its introduction, the RKDG method thrives with many new developments. We refer the reader to the monograph [7], as well as the recent books [8] [9] for in-depth introductions as a wealth of further references.

In this note, we would like to complement the standard derivation of the semi-discrete weak solution by pointing out the role an infinitesimal version of the integral

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form of the conservation law which is needed at the interior sides of the interfaces. For simplicity, we restrict the discussion to one spatial dimension, so

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0,T), \tag{1.1}$$

where $u = (u_1, \dots u_m)^t$ is the vector of conserved variables, and the Jacobian matrix $f'(u)$ has real eigenvalues and a complete set of eigenvectors. We complete (1.1) with initial values

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}. \tag{1.2}$$

Since classical solutions $u \in C^1(\mathbb{R} \times [0,T))$ break down in finite time, one needs to consider weak solution $u \in BV(\mathbb{R} \times (0,T))$ (see [10], [11]). For any smooth, compactly supported test function $v \in C_0^1(\mathbb{R} \times [0,T))$ with initial values v_0 the weak solution satisfies

$$-\int_0^T \int_{\mathbb{R}} \Big(u v_t + f(u)\, v_x \Big) dx dt - \int_{\mathbb{R}} u_0 v_0 \, dx = 0. \tag{1.3}$$

The RKDG method is based on a weak formulation which uses *piecewise* smooth test functions and a semi-discrete limit. In the following, we will derive this directly from (1.3).

The outline of the paper is as follows: in Section 2 we review the original derivation of the DG scheme. In Section 3 we localize the test function v to have support in a single space-time cell. In Section 4, we pass to the semi-discrete limit in the localized weak form. From this, we recover the original semi-discrete DG scheme by restricting the localized, semi-discrete weak form to the piecewise polynomial ansatz space. Section 5 concludes with a discussion of the new derivation.

2 Original derivation of the semi-discrete DG scheme

Cockburn and Shu start their derivation of the RKDG method from the following semi-discrete weak formulation [2, (2.7)]: Let $x_j := j\Delta x \in \mathbb{R}$ be the cell centers and $I_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ the cells of the computational grid. Let V_h^k be the space of piecewise polynomials of degree at most k. Then the DG solution is a family of functions $u^h : [0,T) \to V_h^k$ such that for all $t \in (0,T)$ and all test functions $v^h \in V_h^k$,

$$\int_{I_j} \Big(\frac{d}{dt} u^h(x,t) \Big) v^h(x) \, dx + \int_{I_j} \frac{\partial}{\partial x} f(u^h(x,t))\, v^h(x) \Big) \, dx = 0. \tag{2.4}$$

Let $v_l^{(j)}(x)$ be the Legendre polynomials over I_j, and define the degrees of freedom of $u^h(\cdot, t)$ by

$$u_j^{(l)}(t) := \frac{1}{\Delta x^{l+1}} \int_{I_j} u^h(x,t) v_l^{(j)}(x) \, dx.$$

Inserting this into (2.4), integrating the spatial flux by parts, and suppressing time t gives the method of lines [2, (2.8)]

$$\begin{aligned} \frac{d}{dt} u_j^{(l)} &+ \frac{1}{\Delta x^{l+1}} \Big(v_l^{(j)}(x) f(u^h(x)) \Big) \Big|_{x = x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \\ &- \frac{1}{\Delta x^{l+1}} \int_{I_j} f(u^h(x)) \frac{d}{dx} v_l^{(j)}(x) \, dx \; = \; 0. \end{aligned} \tag{2.5}$$

Note that the boundary terms resulting from the integration by parts, $v_l^{(j)}(x_{j\pm\frac{1}{2}})$ and $f(u^h(x_{j\pm\frac{1}{2}}))$, are the traces at the interior side with respect to cell I_j. But Cockburn and Shu introduce the innocent-looking notation

$$f_{j\pm\frac{1}{2}} := f(u^h(x_{j\pm\frac{1}{2}})), \tag{2.6}$$

and then note that this is not well-defined, since u^h is discontinuous at the interface. Then they comment: "*It is this freedom which gives us a chance to adopt the successful finite difference non-oscillatory methodology.*" Subsequently, they replace the interface flux by an approximate Riemann-solver

$$f_{j+\frac{1}{2}} := h_{j+\frac{1}{2}} := h(u^h(x_{j+\frac{1}{2}}-0),\, u^h(x_{j+\frac{1}{2}}+0)) \tag{2.7}$$

and arrive at the final semi-discrete scheme [2, (2.10)],

$$\begin{aligned} \frac{d}{dt}u_j^{(l)} + \frac{1}{\Delta x^{l+1}} \Big(v_l^{(j)}(x)h(x)\Big)\Big|_{x=x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} & \\ - \frac{1}{\Delta x^{l+1}} \int_{I_j} f(u^h(x))\frac{d}{dx}v_l^{(j)}(x)\,dx \;&=\; 0. \end{aligned} \tag{2.8}$$

For the piecewise constant approximation ($k=0$), this is the semi-discrete first order finite volume scheme. The choice of numerical flux (2.7) is commonly described as stabilization by adding upwind viscosity to the scheme.

In the following, we present an alternative derivation of (2.8) which follows directly from the classical form of the weak solution (1.3), instead of using (2.4) – (2.7).

3 Localizing the weak formulation

In this section we localize the weak formulation (1.3). Let $v \in C_0^1(\mathbb{R}\times[0,T))$ be any test function, and let

$$I_j^n := I_j \times (t^n, t^{n+1}) = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (t^n, t^{n+1}) \tag{3.9}$$

be a space-time cell. Given $\varepsilon \ll \Delta t$, define the interior of the cell by

$$I_{j,\varepsilon}^n := \{(x,t)\in I_j^n \mid \operatorname{dist}\big((x,t),\partial I_j^n\big) > \varepsilon\}. \tag{3.10}$$

Let $v_\varepsilon := v \star \kappa_\varepsilon$ be a localization of the test function such that

$$v_\varepsilon(x,t) = \begin{cases} v(x,t), & \text{if } (x,t)\in I_{j,\varepsilon}^n \\ 0, & \text{if } (x,t)\notin I_j^n. \end{cases} \tag{3.11}$$

Suppose now that $u \in BV(\mathbb{R}\times(0,T))$ is a weak solution of (1.1). By (1.3),

$$\begin{aligned} 0 &= -\iint_{\mathbb{R}\times(0,T))} \Big(u\,\partial_t v_\varepsilon + f(u)\,\partial_x v_\varepsilon\Big)dxdt \\ &= -\iint_{I_{j,\varepsilon}^n} \Big(u\,\partial_t v + f(u)\,\partial_x v\Big)dxdt \\ &\quad -\iint_{I_j^n\setminus I_{j,\varepsilon}^n} \Big(u\,\partial_t v_\varepsilon + f(u)\,\partial_x v_\varepsilon\Big)dxdt \end{aligned} \tag{3.12}$$

As $\varepsilon \to 0$, the first integral clearly converges to

$$-\iint_{\mathbb{R}\times(0,T))} \Big(u\,\partial_t v + f(u)\,\partial_x v\Big) dxdt.$$

We claim that the second integral converges to

$$\int_{I_j} \hat{u}_j^n(x,t)v(x,t)|_{t=t^n}^{t^{n+1}}\,dx + \int_{t^n}^{t^{n+1}} \int_{\partial I_j^n} f(\hat{u}_j^n(x,t))\,v(x,t)\cdot n(x)\,dxdt, \tag{3.13}$$

where $n(x)$ is the outside unit normal of I_j and

$$\hat{u}_j^n : \partial I_j^n \to \mathbb{R}^m \tag{3.14}$$

is the interior trace of u on the boundary of the space-time cell I_j^n. For instance,

$$\begin{aligned}
&\int_{t^n}^{t^n+\varepsilon} \int_{I_j^n} u\,\partial_t v_\varepsilon\,dxdt \\
&= \int_{I_j^n} \hat{u}_j^n(x,t^n) \int_{t^n}^{t^n+\varepsilon} \partial_t v_\varepsilon\,dtdx + \int_{I_j^n} \int_{t^n}^{t^n+\varepsilon} (u(x,t) - \hat{u}_j^n(x,t^n))\partial_t v_\varepsilon\,dtdx \\
&=: B_\varepsilon + C_\varepsilon.
\end{aligned} \tag{3.15}$$

Since $v_\varepsilon(x,t^n) \equiv 0$,

$$\lim_{\varepsilon\to 0} B_\varepsilon = \lim_{\varepsilon\to 0} \int_{I_j^n} \hat{u}_j^n(x)(v_\varepsilon(x,t^n+\varepsilon) - v_\varepsilon(x,t^n))dx = \int_{I_j^n} \hat{u}_j^n(x)v(x,t^n)dx, \tag{3.16}$$

so B_ε converges to the corresponding term in (3.15). The term C_ε vanishes in the limit, since u is of bounded variation:

$$\lim_{\varepsilon\to 0} |C_\varepsilon| \le \int_{I_j^n} \Big(\max_{t^n\le t\le t^n+\varepsilon} |u(x,t) - \hat{u}_j^n(x,t^n)| \int_{t^n}^{t^n+\varepsilon} |\partial_t v_\varepsilon(x,t)|\,dt\Big) dx = 0. \tag{3.17}$$

Hence we have proven the following theorem:

Theorem 3.1 (localized weak solution)**.** *Suppose that $u \in BV(\mathbb{R}\times(0,T))$ is a weak solution of (1.1) and that $v \in C_0^1(\mathbb{R}\times[0,T))$ is a test function. Then for any space-time cell $K := (a,b)\times(t^n,\, t^{n+1})$ and the corresponding interior trace $\hat{u}_K$ on the boundary of K, we have*

$$\begin{aligned}
0 = &-\int_{t^n}^{t^{n+1}} \int_a^b \Big(u\,\partial_t v + f(u)\,\partial_x v\Big) dxdt \\
&+ \int_a^b \hat{u}_K(x,t)v(x,t)|_{t=t^n}^{t^{n+1}}\,dx \\
&+ \int_{t^n}^{t^{n+1}} f(\hat{u}_K(x,t))\,v(x,t)|_{x=a}^b\,dt.
\end{aligned} \tag{3.18}$$

As a side remark, we note that that the theorem may be used to motivate a space-time DG method by choosing (a,b) to be a computational cell, and space-time polynomials for u and v.

4 Semi-discrete limit

For simplicity of notation, we set $t^n = 0$, $t^{n+1} = \Delta t$, and $x_L := x_{j-\frac{1}{2}}$ and $x_R := x_{j+\frac{1}{2}}$. We consider piecewise smooth weak solutions $u(x,t)$ such that $u(x,0)$ is smooth in (x_L, x_R). In order to pass to the semi-discrete limit in the localized weak formulation (3.18), we choose a piecewise smooth test functions $v(x)$ which is independent of time. Let $\rho(A)$ be the spectral radius of an $m \times m$-matrix A, and let

$$s := \max_{\mathbb{R}\times(0,\Delta t)} \rho(f'(U)) \tag{4.19}$$

be the maximal propagation speed of the weak solution. We suppose that $s\Delta t \ll \Delta x$. To highlight the role of the interior fluxes $f(\hat{u}_K)$, which correspond to the boundary fluxes in (2.5), we introduce the interior points

$$y_L := x_L + s\Delta t, \quad y_R := x_R - s\Delta t. \tag{4.20}$$

Note that $u(x,t)$ is smooth at $y_{L/R}$. We apply (3.18) with $(a,b) = (y_L, y_R)$. Dividing by Δt we obtain

$$\begin{aligned} 0 &= -\frac{1}{\Delta t}\int_0^{\Delta t}\int_{y_L}^{y_R} f(u(x,t))\,\partial_x v(x)\,dxdt + \int_{y_L}^{y_R} \frac{\hat{u}(x,\Delta t) - \hat{u}(x,0)}{\Delta t} v(x)\,dx \\ &\quad + \frac{1}{\Delta t}\int_0^{\Delta t} \big(f(u(y_R,t))\,v(y_R) - f(u(y_L,t))\,v(y_L)\big)\,dt \qquad (4.21) \\ &= -\frac{1}{\Delta t}\int_0^{\Delta t}\int_{y_L}^{y_R} f(u(x,t))\,\partial_x v(x)\,dxdt + \int_{y_L}^{y_R} \frac{\hat{u}(x,\Delta t) - \hat{u}(x,0)}{\Delta t} v(x)\,dx \\ &\quad + \frac{v(x_R)}{\Delta t}\int_0^{\Delta t} f(u(y_R,t)\,dt - \frac{v(x_L)}{\Delta t}\int_0^{\Delta t} f(u(y_L,t))\,dt + \mathcal{O}(\Delta t). \qquad (4.22) \end{aligned}$$

Now we consider the infinitesimal cells $K_L := (x_L, y_L) \times (0, \Delta t)$ respectively $K_R := (y_R, x_R) \times (0, \Delta t)$ which are of size $O(\Delta t^2)$. Since u is a weak solution, it satisfies the integral relations

$$0 = \int_{x_L}^{y_L} u(x,t)|_{t=0}^{\Delta t}\,dx + \int_0^{\Delta t} f(u(x,t))\,|_{x_L}^{y_L}\,dt, \tag{4.23}$$

$$0 = \int_{y_R}^{x_R} u(x,t)|_{t=0}^{\Delta t}\,dx + \int_0^{\Delta t} f(u(x,t))\,|_{y_R}^{x_R}\,dt. \tag{4.24}$$

Using (4.23) and (4.24) in (4.21) yields

$$
\begin{aligned}
0 = &-\frac{1}{\Delta t}\int_0^{\Delta t}\int_{y_L}^{y_R} f(u(x,t))\,\partial_x v(x)\,dxdt + \int_{y_L}^{y_R}\frac{\hat{u}(x,\Delta t)-\hat{u}(x,0)}{\Delta t}v(x)\,dx \\
&+\frac{v(x_R)}{\Delta t}\left\{\int_0^{\Delta t} f(u(x_R,t)\,dt + \int_{y_R}^{x_R} u(x,t)|_{t=0}^{\Delta t}\,dx\right\} \\
&-\frac{v(x_L)}{\Delta t}\left\{\int_0^{\Delta t} f(u(x_L,t))\,dt - \int_{x_L}^{y_L} u(x,t)|_{t=0}^{\Delta t}\,dx\right\} + \mathcal{O}(\Delta t) \\
= &-\frac{1}{\Delta t}\int_0^{\Delta t}\int_{y_L}^{y_R} f(u(x,t))\,\partial_x v(x)\,dxdt + \int_{x_L}^{x_R}\frac{\hat{u}(x,\Delta t)-\hat{u}(x,0)}{\Delta t}v(x)\,dx \\
&+\frac{v(x_R)}{\Delta t}\int_0^{\Delta t} f(u(x_R,t)\,dt - \frac{v(x_L)}{\Delta t}\int_0^{\Delta t} f(u(x_L,t))\,dt + \mathcal{O}(\Delta t),
\end{aligned}
\tag{4.25}
$$

where

$$u(x_L,t) = u(x_{j-\frac{1}{2}},t) = w_{\text{Riem}}(0; u(x_{j-\frac{1}{2}}-0,0), u(x_{j-\frac{1}{2}}-+0,0) \tag{4.26}$$

$$u(x_R,t) = u(x_{j+\frac{1}{2}},t) = w_{\text{Riem}}(0; u(x_{j+\frac{1}{2}}-0,0), u(x_{j+\frac{1}{2}}-+0,0) \tag{4.27}$$

are the solutions of the Riemann problem at the interfaces. Now we pass to the limit in Δt and obtain the semi-discrete weak formulation:

Theorem 4.1. *Any piecewise smooth space-time weak solution according to* (1.3) *of a system of hyperbolic conservation laws satisfies the semi-discrete limit*

$$
\begin{aligned}
0 = &\frac{d}{dt}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,0)v(x)\,dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u(x,0))\,\partial_x v(x)\,dxdt \\
&+ v(x)f(u(x,0))|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}
\end{aligned}
\tag{4.28}
$$

for all test functions $v \in C^1([x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}])$, where $u(x_{j\pm\frac{1}{2}},0)$ are the exact solutions of the Riemann problem (4.26), (4.27).

Remark 4.2. *(i) We recover Cockburn and Shu's semi-discrete DG scheme* (2.8) *by choosing piecewise polynomials $u(\cdot,0)$ and v in* (4.28).

(ii) The flux function $f(u(x_{j\pm\frac{1}{2}},0))$ in (4.28) *is Godunov's flux. It is usually replaced by an approximate Riemann solver, see [2], [9].*

(iii) The semi-discrete scheme (2.8) *has also been derived using a smooth solution together with piecewise smooth test functions, and finally replacing the fluxes across the interfaces by an approximate Riemann solver, see for instance [9, Chapter 8.2.2].*

(iv) We would like to recall that the space-time weak formulation of conservation laws, when applied to piecewise smooth solutions, implies the Rankine-Hugoniot condition. This in turn implies that the space-time flux is continuous across the discontinuities. For stationary shocks, the spatial flux is constant, which gives uniqueness in (2.6)*. For non-stationary shocks, a piece of information is missing. In the present derivation, we recover this information by the integral relations* (4.23) *and* (4.24)*. All derivations coincide once a conservative numerical has been introduced.*

5 Conclusion

We have re-derived Cockburn and Shu's semi-discrete discontinuous Galerkin scheme for systems of hyperbolic conservation laws by localizing the space-time weak formulation and passing to the semi-discrete limit. A crucial ingredient is an infinitesimal version of the integral form of the conservation law, which we use to the interior of the cell interfaces. This highlights that the conservative numerical flux function (i.e. the approximate Riemann solver) is not primarily a convenient stabilization. Rather, it is a necessary ingredient for the consistency of the scheme with the weak solution.

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