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Abstract

Stochastic quantities of interest are expanded in generalized polynomial chaos expansions using stochastic Galerkin methods. When applied to hyperbolic differential equations, the coefficients of the series do not inherit hyperbolicity in general. Here, we present convex entropies for the systems of coefficients for Burgers' and shallow water equations. This allows to obtain hyperbolicity, wellposedness and energy estimates.

1 Introduction

Wellposedness is an important property that systems of partial differential equations (PDEs) should fulfil. Wellposedness means the solution exists, it is unique and the solution depends continuously on initial conditions [27]. Classical solutions to most hyperbolic conservation laws have this property, which explains, why these equations are widely used to model fluid dynamics [36] and other applications like traffic flow [37]. Most physically motivated systems can be endowed with an entropy that describes the decay of the energy of a hyperbolic system, which in turn guarantees well-posed classical solutions [22, 7, 33]. A famous example is the physical entropy for Euler and shallow water equations, see e.g. [12, 2].

Classical solutions, however, exist in finite time only up to the possible occurence of shocks [49]. Therefore, weak solutions are considered which are not necessarily unique. Existence and uniqueness of bounded weak entropy solutions have been shown in [33] using entropy-entropy flux pairs. All of these entropy-entropy flux pairs must satisfy an entropy inequality. In the *scalar* case a strictly convex flux function and *one* entropy-entropy flux pair is sufficient to characterize the entropy solution uniquely [43, 34]. This result could not been extended to arbitrary systems, when entropies rarely exist or remain unknown [34]. A single entropy-entropy flux pair, however, manages to weed out all but one weak solution, as long as a classical solution exists [12]. Thus, an entropy inherits the wellposedness of classical solutions to a weak formulation.

When initial data are not known exactly, but are given by their probability law or by statistical moments, the deterministic entropy concepts should be extended to the stochastic case. A mathematical framework for random entropy solutions of scalar random hyperbolic equations is developed in [42, 55]. It is shown that existing statistical moments in the initial conditions are inherited to the solution. In this non-intrusive point of view, first *pointwise* entropy solutions are determined, then the expectation is computed.

In contrast, we investigate the case, where the expectation is evaluated first. This intrusive stochastic Galerkin method represents stochastic processes by piecewise orthogonal polynomials, which is known as generalized polynomial chaos (gPC) expansion [58, 5, 18, 61]. Expansions of the stochastic input are substituted into the governing equations and they are projected to obtain deterministic evolution equations for the gPC coefficients. The applications of this procedure has been proven successful for diffusion [62, 16] and kinetic equations [52, 28, 6]. So far, results for general hyperbolic systems are not available [14, 15, 40], since desired properties like hyperbolicity and the existence of entropies are not inherited to the intrusive formulation. A problem is posed by the fact that the deterministic Jacobian of the projected system differs from the random Jacobian of the original system and therefore not even real eigenvalues, which are necessary for hyperbolicity, are guaranteed in general.

It is possible for some systems to first transform the partial differential equations into non-conserved variables and then apply the Galerkin method. See [15, 59] for quasilinear systems, [14, 15] for entropy variables, [45, 17] for Euler equations using the Roe variable transform from [50]. Also, formulations of hyperbolic systems with eigenvectors that are independent of the uncertainty remain hyperbolic [57]. However, for classical fluid-dynamic equations, like the shallow water equations, eigenvectors are stochastic.

Even if the Jacobian has real eigenvalues, which implies unique classical solutions, the system may not be well-posed in the sense that the solution does not depend on initial conditions in a stable way. For this property at least a complete set of eigenvectors must exist. Eigenvalues of most physically motivated hyperbolic systems are separated. Then, a complete set of eigenvectors exists, which implies wellposedness [53]. This argument, however, cannot be used for stochastic Galerkin formulations, when eigenvalues are no more separated.

So far, general well-posed solutions can be established for the gPC systems of scalar conservation laws only [8, 23, 48, 31, 44], since the resulting Jacobian is symmetric and hence diagonalizable with real eigenvalues and a complete set of eigenvectors. In fact, an entropy-entropy flux pair exists for these symmetric systems [21]. This wellposedness result can be extended to hyperbolic systems that do not have necessarily a symmetric Jacobian, but admit an entropy. Then, the system is symmetrizable and hence well-posed [22, 7, 21].

Loss of entropies and hyperbolicity, however, is rather an exception than the rule. It is frequently related to the loss of symmetries [12]. In particular for a stochastic Galerkin formulation of shallow water equations, the loss of hyperbolicity and hence the loss of all entropy-entropy flux pairs is proven in [15, Prop. 2]. Also stochastic Galerkin formulations for isothermal Euler equations are in general not hyperbolic [17, 30]. Fortunately, there is flexibility to truncate and project the series expansion. With the introduction of Roe variables [50, 45] the Galerkin projections preserve symmetries in the flux function. Another open problem [40, Sec. 10.2] is the representation of positive quantities, which may occur with stochastic Galerkin square roots [13, 17]. This issue arises also for splitting and semi-intrusive methods and may lead to the loss of hyperbolicity [10, 9, 51].

In general, entropy solutions exist on finite time domains only. For deterministic Euler and shallow water equations, which have distinct eigenvalues and genuinely nonlinear or linearly degenerate characteristic fields, an entropy solution exists for all $t \in \mathbb{R}_0^+$ as long as the total variation of initial values is sufficently small [2, Th. 7.1]. Although the assumption of genuine nonlinearity can be weakend [38, 39, 2], the eigenvalues of stochastic Galerkin formulations may coincide and the total variation of initial values may be not sufficiently small. Therefore, we expect that weak solutions exist in finite time only and we study the following setting:

A weak formulation on a bounded *time domain* $[0,T)$ *is considered, when an entropy solution exists in the weak sense. If additionally a classical solution exists it should be well-posed also in the weak sense. Furthermore, the solution should coincide in the deterministic case with the physically relevant entropy solution.*

The main contribution of this paper is to introduce the required entropy-entropy flux pair for a stochastic Galerkin formulation of shallow water equations. First, we present two families of entropy-entropy flux pairs for Burgers' equation in Theorem 3.1 that are valid for arbitrary gPC expansions. We use entropies to define the stochastic Galerkin root uniquely. Namely, the positive root can be interpreted as the unique minimum of a strictly convex entropy. Then, we consider a hyperbolic stochastic Galerkin formulation for shallow water equations, which is based on Roe variables from [50, 45]. The main Theorem 4.3 endows this system with an entropy.

Although only truncated series expansions are considered for numerical purposes, a serious problem remains the convergence to the solution. Convergence for Burgers' equation is shown in [15] using an entropy concept. This ansatz, however, is valid for smooth solutions only. Convergence of weak solutions remains still an open question [15]. At least for smooth solutions we answer this question for the presented formulation of shallow water equations by using the similar entropy framework from [33, 12, 15, 19, 20].

The stabilizing effect of entropies is also seen in the *finite* domain of the influence of initial data [12, Th. 4.1.1]. Thus, it is expected that gPC expansions should have *bounded* support. Indeed, bounded initial data are assumed for the pointwise entropies of scalar conservation laws in [42, 55]. For the stochastic Galerkin formulation of Burgers' equation, there are entropies for arbitrary gPC expansions with possibly unbounded support. An intuitive explanation would be a truncation error due to Galerkin projections. For the shallow water equations, however, our analysis is restricted to a class of gPC expansions with bounded support, including the Wiener-Haar expansion [41]. These wavelet expansions are motivated by a robust expansion for solutions that depend on the stochastic input in a non-smooth way and are used for stochastic multiresolution as well as adaptivity in the stochastic space [1, 3, 32, 4, 47, 56].

We illustrate numerically theoretical results for the Wiener-Haar expansion. The results show the hyperbolic character of the system, the smoothness properties of truncated gPC expansions and wellposedness, which follows from the decay of entropies. This confirms the use of wavelet-based gPC expansions in previous works.

2 Cauchy Problem and Weak Solutions

We briefly recall basic results from [12, 36, 35, 2]. A function

$$y \ : \ [0,T)\times\mathbb{R}\to\mathbb{R}^n, \ (t,x)\mapsto y(t,x)$$

is a **weak solution** to the **Cauchy problem**

$$y_t + f(y)_x = 0 \quad \text{with} \quad y(0,x)=\mathcal{I}(x) \quad \text{for} \quad x\in\mathbb{R} \tag{1}$$

if the map $t\mapsto y(t,\cdot)$ is continuous with values in L^1_{loc}, the initial condition is satisfied for every C^1-function φ with compact support contained in the open strip $(0,T)\times\mathbb{R}$ and the solution satisfies

$$\int_0^T\int_{\mathbb{R}}\Big[y(t,x)\varphi_t(t,x) + f\big(y(t,x)\big)\varphi_x(t,x)\Big]\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Given a **strictly convex entropy** η with **entropy flux** μ, a solution is called η**-admissible** if the **entropy inequality**

$$\eta(y)_t + \mu(y)_x \le 0 \tag{2}$$

is satisfied in the distributional sense. For all non-negative testfunctions we have

$$\int_0^T \int_{\mathbb{R}} \Big[\eta\big(y(t,x)\big)\varphi_t(t,x) + \mu\big(y(t,x)\big)\varphi_x(t,x)\Big]\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

For this homogeneous system the entropy and the entropy flux are smooth functions defined on a open ball $\mathbb{H} \subset \mathbb{R}^n$ [12] and satisfy the **compatibility condition**

$$\mathrm{D}_y\mu(y) = \mathrm{D}_y\eta(y)\mathrm{D}_y f(y). \tag{3}$$

Similar to [40, 13, 46, 60, 54, 24] we extend the Cauchy problem (1) to have initial conditions depending on a possibly multidimensional random parameter ξ, which we call similar to [40] "germ". We consider the weak formulation

$$\int_0^T \int_{\mathbb{R}} \mathbb{E}\Big[\Big(y(t,x;\xi)\varphi_t(t,x) + f\big(y(t,x;\xi)\big)\varphi_x(t,x)\Big)\phi_k(\xi)\Big]\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $k = 0,\dots,K$, where the orthogonal polynomials ϕ_k form a basis of

$$\mathbb{L}^2(\Omega,\mathbb{P}) \coloneqq \Big\{Z \;\Big|\; Z:\Omega\to\mathbb{R} \;\text{ measurable},\;\; \|Z\| < \infty\Big\} \quad \text{with}$$
$$\langle Z_1, Z_2\rangle \coloneqq \int Z_1 Z_2\,\mathrm{d}\mathbb{P}.$$

We introduce a **generalized polynomial chaos** (gPC) as a set of orthogonal subspaces $\hat{\mathcal{S}}_k \subset \mathbb{L}^2(\Omega,\mathbb{P})$ with

$$\mathcal{S}_K \coloneqq \bigoplus_{k=0}^{K} \hat{\mathcal{S}}_k \;\to\; \mathbb{L}^2(\Omega,\mathbb{P}) \;\text{ for }\; K\to\infty.$$

We refer to an orthogonal basis of $\mathcal{S}_K$ as a gPC basis $\{\phi(\xi)\}_{k=0}^K$ with germ $\xi \sim \mathbb{P}$. We assume $y(t,x;\cdot) \in \mathbb{L}^2(\Omega,\mathbb{P})$ and approximate for any fixed (t,x) the solution by

$$\mathcal{G}_K[y](t,x;\xi) \coloneqq \sum_{k=0}^{K} \hat{y}_k(t,x)\phi_k(\xi), \quad \hat{y}_k(t,x) \coloneqq \frac{\big\langle y(t,x;\cdot),\phi_k(\cdot)\big\rangle}{\|\phi_k\|^2},$$

where $\mathcal{G}_K$ denotes the projection operator of the stochastic process $y(t,x;\xi)$ onto the gPC basis of degree $K \in \mathbb{N}_0$. Due to the Cameron-Martin Theorem [5] the expansion converges in the sense $\big\|\mathcal{G}_K[y](t,x;\cdot) - y(t,x;\cdot)\big\| \to 0$ for $K\to\infty$. We will assume normed basis polynomials with $\|\phi_k\| = 1$. Then, the **Galerkin product** is defined as

$$\hat{\mathcal{G}}_k[y,z](t,x;\xi) \coloneqq \sum_{k=0}^{K} (\hat{y} * \hat{z})_k(t,x)\phi_k(\xi) \quad \text{with}$$
$$(\hat{y} * \hat{z})_k(t,x) \coloneqq \sum_{i,j=0}^{K} \hat{y}_i(t,x)\hat{z}_j(t,x)\big\langle \phi_i\phi_j,\phi_k\big\rangle.$$

The **third and fourth moment** are approximated by

$$\begin{aligned}&\hat{\mathcal{G}}_k^{(\ell)}[y](t,x;\xi) \coloneqq \sum_{k=0}^{K} \hat{y}_k^{\ell *}(t,x)\phi_k(\xi) \quad \text{with}\\ &\hat{y}^{3*} \coloneqq \big((\hat{y} * \hat{y}) * \hat{y}\big) \;\text{ and }\; \hat{y}^{4*} \coloneqq \big((\hat{y} * \hat{y}) * (\hat{y} * \hat{y})\big).\end{aligned} \tag{4}$$

Similar to [45, 46], we express these terms with the symmetric matrix

$$\mathcal{P}(\hat{y}) \coloneqq \sum_{\ell=0}^{K} \hat{y}_\ell \mathcal{M}_\ell \quad \text{with} \quad \mathcal{M}_\ell \coloneqq \Big(\langle \phi_\ell, \phi_i \phi_j \rangle\Big)_{i,j=0,\dots,K}$$

such that $\hat{y} * \hat{z}$. The Galerkin product is symmetric, but not associative [13, 54], i.e. $(\hat{y} * \hat{z}) * \hat{q} \neq \hat{y} * (\hat{z} * \hat{q})$. An intuitive explanation would be the truncation errors that arise from disregarding the components of the product (yz) which are orthogonal to $\mathcal{S}_K$. Therefore, the definitions (4) are rather arbitrary and we refer the interested reader to [40, 13], where other approximations of the moments are discussed. In particular, the choice (4) allows an extension of desired properties, e.g. hyperbolicity, to the stochastic case.

3 Intrusive Formulation of Burgers' Equation

We consider the stochastic Galerkin formulation of the **Burgers' equation**. The stochastic Galerkin method applied to the flux function $f(\alpha) \coloneqq \frac{\alpha^2}{2}$ leads to

$$\hat{\alpha}_t + \hat{f}(\hat{\alpha})_x = 0 \quad \text{for} \quad \hat{f}(\hat{\alpha}) \coloneqq \frac{\hat{\alpha} * \hat{\alpha}}{2}. \tag{5}$$

The Jacobian of the projected flux function is $\mathrm{D}_{\hat{\alpha}} \hat{f}(\hat{\alpha}) = \mathcal{P}(\hat{\alpha})$. Note that there is no restriction on the gPC expansion and also bases with unbounded support are admitted.

3.1 Entropie and Entropy Flux Pairs

We state two families of entropy-entropy flux pairs for the stochastic Galerkin formulation of Burgers' equation (5).

Theorem 3.1 (Burgers' Equation)**.** *Define the entropy-entropy flux pairs*

$$\Big(\eta_1(\hat{\alpha}), \mu_1(\hat{\alpha})\Big) = \left(\frac{\|\hat{\alpha}\|_2^2}{2}, \frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) \hat{\alpha}}{3}\right),$$
$$\Big(\eta_2(\hat{\alpha}), \mu_2(\hat{\alpha})\Big) = \left(\frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) \hat{\alpha}}{3}, \frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}^2(\hat{\alpha}) \hat{\alpha}}{4}\right).$$

The pair (η_1, μ_1) is a strictly convex entropy-entropy flux pair of system (5) for all $\hat{\alpha} \in \mathbb{R}^{K+1}$. The pair (η_2, μ_2) is a strictly convex entropy-entropy flux pair on the convex set

$$\mathbb{H}^+ \coloneqq \left\{\hat{\alpha} \in \mathbb{R}^{K+1} \;\middle|\; \mathcal{P}(\hat{\alpha}) \text{ is strictly positive definite}\right\}.$$

Furthermore, shifted entropy-entropy flux pairs are given by

$$\Big(\eta_i(\hat{\alpha}; \hat{h}), \mu_i(\hat{\alpha}; \hat{h})\Big) = \Big(\eta_i(\hat{\alpha}), \mu_i(\hat{\alpha})\Big) + \left(\hat{h}^{\mathrm{T}} \hat{\alpha}, \hat{h}^{\mathrm{T}} \frac{\hat{\alpha} * \hat{\alpha}}{2}\right) \quad \text{for all} \quad \hat{h} \in \mathbb{R}^{K+1}.$$

Proof. The set $\mathbb{H}^+$ is convex, since for arbitrary $\hat{\alpha}, \hat{\beta} \in \mathbb{H}^+$ the matrix

$$\mathcal{P}\Big(\lambda \hat{\alpha} + (1-\lambda)\hat{\beta}\Big) = \sum_{k=0}^{K} \Big(\lambda \hat{\alpha} + (1-\lambda)\hat{\beta}\Big)_k \mathcal{M}_k = \lambda \mathcal{P}(\hat{\alpha}) + (1-\lambda)\mathcal{P}(\hat{\beta})$$

is strictly positive definite for all $\lambda \in [0,1]$ as a sum of strictly positive definite matrices. The gradients and the Hessian of the auxiliary functions

$$\varphi_0(\hat{\alpha}) \coloneqq \frac{1}{2}\hat{\alpha}^{\mathrm{T}}\hat{\alpha}, \qquad \varphi_1(\hat{\alpha}) \coloneqq \frac{1}{3}\hat{\alpha}^{\mathrm{T}}\mathcal{P}(\hat{\alpha})\hat{\alpha}, \qquad \varphi_2(\hat{\alpha}) \coloneqq \frac{1}{4}\hat{\alpha}^{\mathrm{T}}\mathcal{P}^2(\hat{\alpha})\hat{\alpha}$$

are $\nabla_{\hat{\alpha}}\varphi_0(\hat{\alpha}) = \hat{\alpha}$, $\nabla^2_{\hat{\alpha}}\varphi_0(\hat{\alpha}) = \mathbb{1}$, $\nabla_{\hat{\alpha}}\varphi_1(\hat{\alpha}) = \hat{\alpha}^{2*}$, $\nabla^2_{\hat{\alpha}}\varphi_1(\hat{\alpha}) = 2\mathcal{P}(\hat{\alpha})$ and $\nabla_{\hat{\alpha}}\varphi_2(\hat{\alpha}) = \mathcal{P}^2(\hat{\alpha})\hat{\alpha}$. This follows from the symmetry of the Galerkin product and

$$\begin{aligned}
3\nabla_{\hat{\alpha}}\varphi_1(\hat{\alpha}) &= \nabla_{\tilde{\alpha}}\left[\sum_{k=0}^{K}\tilde{\alpha}_k\hat{\alpha}^{\mathrm{T}}\mathcal{M}_k\hat{\alpha}\right]\Bigg|_{\tilde{\alpha}=\hat{\alpha}} &&+ \sum_{k=0}^{K}\hat{\alpha}_k\nabla_{\hat{\alpha}}\left[\hat{\alpha}^{\mathrm{T}}\mathcal{M}_k\hat{\alpha}\right] \\
&= \left(\hat{\alpha}^{\mathrm{T}}\mathcal{M}_k\hat{\alpha}\right)_{k=0,\dots,K} &&+ 2\sum_{k=0}^{K}\hat{\alpha}_k\mathcal{M}_k\hat{\alpha} \\
&= \left(\hat{\alpha}_k^{2*}\right)_{k=0,\dots,K} &&+ 2\mathcal{P}(\hat{\alpha})\hat{\alpha} \\
&= 3\hat{\alpha} * \hat{\alpha}, \\
\nabla^2_{\hat{\alpha}}\varphi_1(\hat{\alpha}) &= \mathrm{D}_{\hat{\alpha}}\left[\nabla_{\hat{\alpha}}\varphi_1(\hat{\alpha})\right] = 2\mathcal{P}(\hat{\alpha}), \\
4\nabla_{\hat{\alpha}}\varphi_2(\hat{\alpha}) &= 2\nabla_{\hat{\alpha}}\left[\mathcal{P}(\hat{\alpha})\hat{\alpha}\right]\mathcal{P}(\hat{\alpha})\hat{\alpha} = 4\hat{\alpha}^{3*}.
\end{aligned}$$

The Hessian $\nabla^2_{\hat{\alpha}}\varphi_0(\hat{\alpha}) = \mathbb{1}$ is strictly positive definite with all eigenvalues being one independently of the solution $\hat{\alpha}$. Therefore, the choice $\eta_0(\hat{\alpha}) \coloneqq \varphi_0(\hat{\alpha})$ is an entropy for all $\hat{\alpha} \in \mathbb{R}^{K+1}$. The corresponding entropy flux $\mu_1(\hat{\alpha}) = \varphi_1(\hat{\alpha})$ satisfies

$$\mathrm{D}_{\hat{\alpha}}\mu_1(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}}\varphi_1(\hat{\alpha}) = \hat{\alpha}^{\mathrm{T}}\mathcal{P}(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}}\eta_1(\hat{\alpha})\mathrm{D}_{\hat{\alpha}}f(\hat{\alpha}). \tag{6}$$

The Hessian $\nabla^2_{\hat{\alpha}}\varphi_1(\hat{\alpha}) = \mathcal{P}(\hat{\alpha})$ is strictly positive definite for all $\hat{\alpha} \in \mathbb{H}^+$ and hence, $\eta_2(\hat{\alpha})$ is an entropy on the restricted set $\mathbb{H}^+$. The corresponding entropy flux $\mu_2(\hat{\alpha}) = \varphi_2(\hat{\alpha})$ satisfies

$$\mathrm{D}_{\hat{\alpha}}\mu_2(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}}\varphi_2(\hat{\alpha}) = \hat{\alpha}^{\mathrm{T}}\mathcal{P}^2(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}}\eta_2(\hat{\alpha})\mathrm{D}_{\hat{\alpha}}\hat{f}(\hat{\alpha}). \tag{7}$$

Since the Hessian matrices of the shifts $\hat{h}^{\mathrm{T}}\hat{\alpha}$ are zero, there is no influence on the convexity of entropies. We conclude with equations (6) and (7)

$$\mathrm{D}_{\hat{\alpha}}\mu_i(\hat{\alpha};\hat{h}) = \mathrm{D}_{\hat{\alpha}}\eta_i(\hat{\alpha})\mathrm{D}_{\hat{\alpha}}\hat{f}(\hat{\alpha}) + \hat{h}^{\mathrm{T}}\mathcal{P}(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}}\eta_i(\hat{\alpha};\hat{h})\mathrm{D}_{\hat{\alpha}}\hat{f}(\hat{\alpha}).$$

□

In fact, the choice $\eta_1(\hat{\alpha}) = \|\hat{\alpha}\|_2^2$ gives an entropy for general systems with symmetric Jacobian [21, Ex. 3.2] and it is already used in [15].

3.2 Stochastic Galerkin Square Root

The Galerkin square root of gPC modes $\hat{h} \in \mathbb{R}^{K+1}$ is introduced e.g. in [40] as the solution of the nonlinear system $\hat{\alpha} * \hat{\alpha} = \hat{h}$. It is already remarked in [13] that the representation of positive physical quantities is difficult. To illustrate the point, we consider an expansion with Hermite polynomials for $K = 1$. The solutions read

$$\begin{aligned}
\hat{\alpha}^+ &\coloneqq \frac{1}{2}\begin{pmatrix}\sqrt{\hat{h}_0+\hat{h}_1}+\sqrt{\hat{h}_0-\hat{h}_1}\\ \sqrt{\hat{h}_0+\hat{h}_1}-\sqrt{\hat{h}_0-\hat{h}_1}\end{pmatrix}, &\qquad \hat{\alpha}^- &\coloneqq -\frac{1}{2}\begin{pmatrix}\sqrt{\hat{h}_0+\hat{h}_1}+\sqrt{\hat{h}_0-\hat{h}_1}\\ \sqrt{\hat{h}_0+\hat{h}_1}-\sqrt{\hat{h}_0-\hat{h}_1}\end{pmatrix}, \\
\tilde{\alpha}^{(1)} &\coloneqq \frac{1}{2}\begin{pmatrix}\sqrt{\hat{h}_0+\hat{h}_1}-\sqrt{\hat{h}_0-\hat{h}_1}\\ \sqrt{\hat{h}_0+\hat{h}_1}+\sqrt{\hat{h}_0-\hat{h}_1}\end{pmatrix}, &\qquad \tilde{\alpha}^{(2)} &\coloneqq -\frac{1}{2}\begin{pmatrix}\sqrt{\hat{h}_0+\hat{h}_1}-\sqrt{\hat{h}_0-\hat{h}_1}\\ \sqrt{\hat{h}_0+\hat{h}_1}+\sqrt{\hat{h}_0-\hat{h}_1}\end{pmatrix}.
\end{aligned}$$

We observe that the solution of the nonlinear system $\hat{\alpha} * \hat{\alpha} = \hat{h}$ may neither be unique nor real, which is similar to the deterministic case. Therefore, a more precise characterization is necessary, which is based on the following observations:

(i) Let a deterministic state with $\hat{h}_0 > 0$, $\hat{h}_1 = 0$ be given. The solutions $\tilde{\alpha}^{(1)}$ and $\tilde{\alpha}^{(2)}$ yield stochastic expansions, which are not meaningful. The solution $\hat{\alpha}^+$ gives the positive and $\hat{\alpha}^-$ the negative root.

(ii) The matrix $\mathcal{P}(\hat{\alpha}^+)$ is positive definite and $\mathcal{P}(\hat{\alpha}^-)$ is negative definite. Both solutions are related by $\mathcal{P}(\hat{\alpha}^-) = -\mathcal{P}(\hat{\alpha}^+)$. On the other hand, the solutions $\tilde{\alpha}^{(1)}$, $\tilde{\alpha}^{(2)}$ yield indefinite matrices.

(iii) If the variance $\hat{h}_1^2$ is sufficiently large, there is no real valued solution.

Corollary 1 generalizes these observations for arbitrary expansions by identifying the positive square root as the unique minimum of the entropy $\eta_2(\cdot; \hat{h})$ on the set $\mathbb{H}^+$. Figure 3.2 shows the sets $\mathbb{H}^\pm$ in terms of $\hat{\alpha}$, where $\mathbb{H}^-$ denotes states of a negative definite matrix $\mathcal{P}(\hat{\alpha})$. For given $\hat{h}$ the entropy $\eta_2(\cdot; \hat{h})$ and contours are plotted in the third dimension. The local extrema, which are projected on the $(\hat{\alpha}_0, \hat{\alpha}_1)$-plain, are the square roots. Contours illustrate that extrema are unique on the sets $\mathbb{H}^\pm$ only. Note that we do not claim that a solution $\hat{\alpha} \in \mathbb{H}^+$ with $\hat{\alpha} * \hat{\alpha} = \hat{h}$ exists. Corollary 1 is only a *uniqueness* result and we refer the interested reader to [17, Sec. 4], where existence has been discussed.

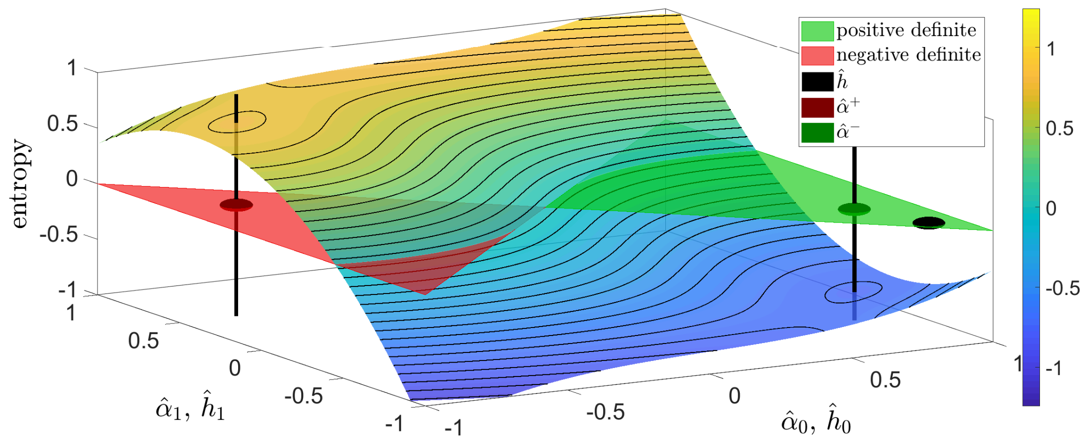

Fig. 1: Contour plot of the entropy $\eta_2(\hat{\alpha}; -\hat{h})$. Solutions $\hat{\alpha}^+ = (0.8, -0.5)^{\mathrm{T}}$ and $\hat{\alpha}^- = (-0.8, 0.5)^{\mathrm{T}}$ for $\hat{h} = (0.89, -0.8)^{\mathrm{T}}$.

Corollary 1 (Stochastic Galerkin Square Root)**.** *Let a state $\hat{h} \in \mathbb{R}^+ \times \mathbb{R}^K$ be given such that there exists $\hat{\alpha} \in \mathbb{H}^+$ satisfying $\hat{\alpha} * \hat{\alpha} = \hat{h}$. Then, the minimum*

$$\hat{\alpha}^+ = \underset{\hat{\alpha} \in \mathbb{H}^+}{\operatorname{argmin}} \left\{ \eta_2(\hat{\alpha}; -\hat{h}) \right\} \quad \textit{of the entropy} \quad \eta_2(\hat{\alpha}; -\hat{h}) = \frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) \hat{\alpha}}{3} - \hat{h}^{\mathrm{T}} \hat{\alpha}$$

is unique and a solution of stochastic Galerkin square roots, i.e. $\hat{\alpha}^+ \in \left\{ \hat{\alpha} \in \mathbb{R}^{K+1} \,\middle|\, \hat{\alpha} * \hat{\alpha} = \hat{h} \right\}$.

Proof. Due to Theorem 3.1 the entropy $\eta_2(\hat{\alpha}; -\hat{h})$ is a strictly convex function on $\mathbb{H}^+$. Therefore, it has a unique minimum that is attained for

$$0 = \nabla_{\hat{\alpha}} \eta_2(\hat{\alpha}; -\hat{h}) = \hat{\alpha}^+ * \hat{\alpha}^+ - \hat{h} \quad \Leftrightarrow \quad \hat{\alpha}^+ * \hat{\alpha}^+ = \hat{h}.$$

□

There is a numerical implication of Corollary 1. The initial guess of numerical solvers should be in the set $\mathbb{H}^+$. Flat level sets close to extrema suggest that gradient free methods should be used to find roots numerically. Namely, the minimum of $\eta_2(\hat{\alpha}; \hat{h})$ may be determined without the roots of the gradient $\nabla_{\hat{\alpha}} \eta_2(\hat{\alpha}; \hat{h})$.

4 Shallow Water Equations

We consider the **shallow water equations**

$$\frac{\partial}{\partial t}\begin{pmatrix} h(t,x) \\ q(t,x) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q(t,x) \\ \frac{q^2(t,x)}{h(t,x)} + \frac{1}{2} g h^2(t,x) \end{pmatrix} = 0 \tag{$\mathcal{C}$}$$

with the conserved quantities height, momentum $y := (h, q)^{\mathrm{T}}$ and the gravitational constant $g > 0$. For smooth solutions an equivalent formulation is

$$y_t + \mathrm{D}_y f(y)\, y_x = 0, \quad \mathrm{D}_y f(y) = \begin{pmatrix} 0 & 1 \\ gh - u^2(y) & 2u(y) \end{pmatrix}, \quad u(y) := \frac{q}{h}.$$

The eigenvalues of the Jacobian read $\lambda^{\pm}(y) = u(y) \pm \sqrt{gh}$ with the velocity $u(y)$ as auxiliary variable. We use the idea of **Roe variables** [50, 35, 45].

Definition 4.1 (Roe Variables)**.** With velocity $u(y) := q/h$ as auxiliary variable the Roe variables are defined as $\omega := (\alpha, \beta) := \left(\sqrt{h}, \sqrt{h} u(y)\right)$. The gPC modes are denoted as $\hat{\omega} := (\hat{\alpha}, \hat{\beta})$. The mapping between Roe and conserved variables is

$$\begin{aligned}
\mathcal{Y} &: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}, && \hat{\omega} \mapsto \begin{pmatrix} \alpha^2 \\ \alpha\beta \end{pmatrix} = y \quad \text{for} \quad K = 0, \\
\hat{\mathcal{Y}} &: \mathbb{H}^+ \times \mathbb{R}^{K+1} \to \left(\mathbb{R}^+ \times \mathbb{R}^K\right) \times \mathbb{R}^{K+1}, && \hat{\omega} \mapsto \begin{pmatrix} \hat{\alpha} * \hat{\alpha} \\ \hat{\alpha} * \hat{\beta} \end{pmatrix} = \hat{y} \quad \text{for} \quad K \in \mathbb{N}.
\end{aligned}$$

Note that the expected water height $\hat{h}_0 = (\hat{\alpha} * \hat{\alpha})_0 = \|\hat{\alpha}\|_2^2 > 0$ is positive and that mappings are bijective due to Corollary 1. We parameterize the shallow water equations by the germ ξ to obtain the equivalent formulations:

$$\frac{\partial}{\partial t}\begin{pmatrix} h(t,x;\xi) \\ q(t,x;\xi) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q(t,x;\xi) \\ \frac{q^2(t,x;\xi)}{h(t,x;\xi)} + \frac{1}{2} g h^2(t,x;\xi) \end{pmatrix} = 0 \quad \mathbb{P}\text{-a.s.} \tag{$\mathcal{C}(\xi)$}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha^2(t,x;\xi) \\ (\alpha\beta)(t,x;\xi) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} (\alpha\beta)(t,x;\xi) \\ \beta^2(t,x;\xi) + \frac{1}{2} g \alpha^4(t,x;\xi) \end{pmatrix} = 0 \quad \mathbb{P}\text{-a.s.} \tag{$\mathcal{R}(\xi)$}$$

We substitute the truncated gPC expansions into the systems ($\mathcal{C}(\xi)$) and ($\mathcal{R}(\xi)$) to obtain

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathcal{G}_K[h](t,x;\xi) \\ \mathcal{G}_K[q](t,x;\xi) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \mathcal{G}_K[q](t,x;\xi) \\ \frac{\mathcal{G}_K^2[q](t,x;\xi)}{\mathcal{G}_K[h](t,x;\xi)} + \frac{1}{2} g \mathcal{G}_K^2[h](t,x;\xi) \end{pmatrix} = 0, \tag{$\mathcal{C}_K(\xi)$}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{\mathcal{G}}_K[\alpha,\alpha](t,x;\xi) \\ \hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi) \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi) \\ \hat{\mathcal{G}}_K[\beta,\beta](t,x;\xi) + \frac{1}{2} g \hat{\mathcal{G}}_K^{(4)}[\alpha](t,x;\xi) \end{pmatrix} = 0. \tag{$\mathcal{R}_K(\xi)$}$$

The truncated systems ($\mathcal{C}_K(\xi)$) and ($\mathcal{R}_K(\xi)$) are no more equivalent and should be solved for the gPC modes in $\mathbb{L}^2(\Omega, \mathbb{P})$-sense. The solution, however, does in general not exist due to truncation errors. Indeed, it is shown in [15] that equation ($\mathcal{C}_K(\xi)$) leads to a non-hyperbolic system due to the term q^2/h. Similarily for isothermal Euler equations, a stochastic Galerkin method that is only based on conserved variables does not preserve hyperbolicity [17, 30]. This issue can be circumvented by introducing Roe variables, which preserve the symmetry of the term β^2. The gPC modes of the formulation ($\mathcal{R}_K(\xi)$) are described by the system

$$\begin{pmatrix} \hat{\alpha} * \hat{\alpha} \\ \hat{\alpha} * \hat{\beta} \end{pmatrix}_t + \begin{pmatrix} \hat{\alpha} * \hat{\beta} \\ \hat{\beta} * \hat{\beta} + \frac{1}{2} g\,(\hat{\alpha} * \hat{\alpha}) * (\hat{\alpha} * \hat{\alpha}) \end{pmatrix}_x = 0, \tag{$\hat{\mathcal{R}}_K$}$$

which we will endow with an entropy. We reformulate it in terms of the conserved variables $\hat{y} = \hat{\mathcal{Y}}(\hat{\omega})$ to obtain the conservative formulation $\hat{y}_t + \hat{f}(\hat{y}) = 0$ with flux function

$$\hat{f}(\hat{y}) := \hat{f}_1(\hat{y}) + \hat{f}_2(\hat{y}) \quad \text{for} \quad \hat{f}_1(\hat{y}) := \begin{pmatrix} \hat{q} \\ \frac{1}{2} g\, \hat{h} * \hat{h} \end{pmatrix} \quad \text{and} \quad \hat{f}_2(\hat{y}) := \tilde{f}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big) := \begin{pmatrix} \mathbb{O} \\ \hat{\beta} * \hat{\beta} \end{pmatrix}. \tag{8}$$

Note that the first part $f_1(\hat{y})$ of the flux in equation (8) is expressed in terms of conserved variables alone, which motivates the choice of the 4-th moment (4). The proof of the following Lemma is moved to the appendix.

Lemma 4.2. *Assume there is an eigenvalue decomposition $\mathcal{P}(\hat{\alpha}) = V D_{\mathcal{P}}(\hat{\alpha}) V^{\mathrm{T}}$ with constant, orthonormal eigenvectors. Define the variables $\hat{u}(\hat{\omega}) := \mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}$, $\widehat{u^2}(\hat{\omega}) := \mathcal{P}_1^2(\hat{\omega})\hat{\beta}$ and the matrices $\mathcal{P}_1(\hat{\omega}) := \mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha})$, $\mathcal{P}_2(\hat{\omega}) := \mathcal{P}(\hat{\beta})\mathcal{P}^{-2}(\hat{\alpha})$. Then, it holds*

$$\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}(\hat{h}) = (\hat{\alpha} * \hat{\beta})^{\mathrm{T}}, \tag{9}$$

$$\mathrm{D}_{\hat{\omega}}\Big[\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}\Big] = \Big(-\hat{\beta}^{\mathrm{T}}\mathcal{P}_1^2(\hat{\omega}), 3\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\Big), \tag{10}$$

$$\mathrm{D}_{\hat{\omega}}\Big[(\hat{\alpha} * \hat{\alpha})^{\mathrm{T}}(\hat{\alpha} * \hat{\beta})\Big] = \Big(3\hat{\alpha}^{\mathrm{T}}\mathcal{P}(\hat{\alpha})\mathcal{P}(\hat{\beta}), \hat{\alpha}^{\mathrm{T}}\mathcal{P}^2(\hat{\alpha})\Big), \tag{11}$$

$$\mathrm{D}_{\hat{\alpha}}\Big[\hat{u}(\hat{\omega})\Big] = \Big(-\mathcal{P}_2(\hat{\omega}), \mathcal{P}^{-1}(\hat{\alpha})\Big), \tag{12}$$

$$\mathrm{D}_{\hat{\omega}}\Big[\mathcal{P}_2(\hat{\omega})\hat{\beta}\Big] = \Big(-2\mathcal{P}_1(\hat{\omega})\mathcal{P}_2(\hat{\omega}), 2\mathcal{P}_2(\hat{\omega})\Big). \tag{13}$$

Finally, we state an entropy-entropy flux pair for shallow water equations in our main theorem.

Theorem 4.3 (Shallow Water Equations)**.** *Assume there is an eigenvalue decomposition with constant eigenvectors, i.e. $\mathcal{P}(\hat{\alpha}) = V D_{\mathcal{P}}(\hat{\alpha}) V^{\mathrm{T}}$. Let states in the open, admissible set*

$$\mathbb{H} := \Big\{ \hat{y} := (\hat{h}, \hat{q})^{\mathrm{T}} \in \big(\mathbb{R}^+ \times \mathbb{R}^K\big) \times \mathbb{R}^{K+1} \;\Big|\; \hat{\alpha} \in \mathbb{H}^+ \text{ for } (\hat{\alpha}, \hat{\beta})^{\mathrm{T}} = \hat{\mathcal{Y}}^{-1}(\hat{y}) \Big\}$$

be given. Then, the Jacobian of the flux function (8) is

$$\mathrm{D}_{\hat{y}}\hat{f}(\hat{y}) = \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) - \mathcal{P}_1^2(\hat{\omega}) & 2\mathcal{P}_1(\hat{\omega}) \end{pmatrix}$$

for $\hat{\omega} = (\hat{\alpha}, \hat{\beta})$. Its eigenvalues are real and read

$$\sigma\big\{\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\big\} = D_{\mathcal{P}}(\hat{\beta})D_{\mathcal{P}}^{-1}(\hat{\alpha}) \pm \sqrt{g}\, D_{\mathcal{P}}(\hat{h})D_{\mathcal{P}}^{-1}(\hat{\alpha}).$$

An entropy-entropy flux pair is $(\eta, \mu)(\hat{y}) := \big(\eta_1 + \eta_2, \mu_1 + \mu_2\big)(\hat{y})$ with

$$\eta_1(\hat{y}) := \frac{g}{2}\,\|\hat{h}\|_2^2 \quad \text{and} \quad \eta_2(\hat{y}) := \tilde{\eta}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big) := \frac{1}{2}\|\hat{\beta}\|_2^2,$$

$$\mu_1(\hat{y}) := g\,\hat{h}^{\mathrm{T}}\hat{q} \quad \text{and} \quad \mu_2(\hat{y}) := \tilde{\mu}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big) := \frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}.$$

Proof. The Jacobian of the flux function and the Jacobian of the entropy read

$$\begin{aligned}
\mathrm{D}_{\hat{y}}\hat{f}(\hat{y}) &= \mathrm{D}_{\hat{y}}\hat{f}_1(\hat{y}) + \mathrm{D}_{\hat{\omega}}\tilde{f}_2(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \\
&= \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) & \mathbb{O} \end{pmatrix} + \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 2\mathcal{P}(\hat{\beta}) \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathcal{P}^{-1}(\hat{\alpha}) & \mathbb{O} \\ -\frac{1}{2}\mathcal{P}_2(\hat{\omega}) & \mathcal{P}^{-1}(\hat{\alpha}) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) - \mathcal{P}_1^2(\hat{\omega}) & 2\mathcal{P}_1(\hat{\omega}) \end{pmatrix}, \\
\mathrm{D}_{\hat{y}}\eta(\hat{y}) &= \mathrm{D}_{\hat{y}}\eta_1(\hat{y}) + \mathrm{D}_{\hat{\omega}}\tilde{\eta}_2(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) = (g\hat{h}^{\mathrm{T}}, \mathbb{O}) + (\mathbb{O}, \hat{\beta}^{\mathrm{T}})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \\
&= \Big(g\hat{h}^{\mathrm{T}} - \frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega}), \hat{\beta}^{\mathrm{T}}\mathcal{P}^{-1}(\hat{\alpha})\Big).
\end{aligned}$$

The compatibility condition (3) is equivalent to

$$\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y}) = \mathrm{D}_{\hat{y}}\mu(\hat{y}) = \mathrm{D}_{\hat{\omega}}\big[\tilde{\mu}(\hat{\omega})\big][\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \quad \text{for} \quad \tilde{\mu}(\omega) := \mu\Big(\hat{\mathcal{Y}}(\hat{\omega})\Big).$$

This holds due to Lemma 4.2, which yields

$$\begin{aligned}
&\Big(\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\Big)\hat{\mathcal{Y}}(\hat{\omega}) \\
&\quad = \Big(g\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}(\hat{h}) - \hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}_1^2(\hat{\omega}),\ g\hat{h}^{\mathrm{T}} + \frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega})\Big)\hat{\mathcal{Y}}(\hat{\omega}) \\
&\quad = \Big(g(\hat{\alpha} * \hat{\beta})^{\mathrm{T}} - \hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}_1^2(\hat{\omega}),\ g\hat{h}^{\mathrm{T}} + \frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega})\Big)\hat{\mathcal{Y}}(\hat{\omega}) \\
&\quad = \Big(3g(\hat{h} * \hat{\beta})^{\mathrm{T}} - \frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1^2(\hat{\omega}),\quad g(\hat{h} * \hat{\alpha})^{\mathrm{T}} + \frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\Big) \\
&\quad = \mathrm{D}_{\hat{\omega}}\big[\tilde{\mu}(\hat{\omega})\big].
\end{aligned}$$

We define the auxiliary variables $\widetilde{\nabla\eta}(\hat{\omega}) := \nabla_{\hat{y}}\eta\big(\hat{\mathcal{Y}}(\hat{\omega})\big)$ and $D_1(\hat{\omega}) := D_{\mathcal{P}}(\hat{\beta})D_{\mathcal{P}}^{-1}(\hat{\alpha})$. Using Lemma 4.2 we obtain the eigenvalue decomposition of the Hessian

$$\begin{aligned}
\nabla_{\hat{y}}^2\eta\big(\hat{\mathcal{Y}}(\hat{y})\big) &= \mathrm{D}_{\hat{\omega}}\widetilde{\nabla\eta}(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) = \begin{pmatrix} g\mathbb{1} + \mathcal{P}_2^2(\hat{\omega}) & -\mathcal{P}_1(\hat{\omega})\mathcal{P}^{-2}(\hat{\alpha}) \\ -\mathcal{P}_1(\hat{\omega})\mathcal{P}^{-2}(\hat{\alpha}) & \mathcal{P}^{-2}(\hat{\alpha}) \end{pmatrix} \\
&= T(\hat{\alpha})D_\eta(\hat{\omega})T^{-1}(\hat{\alpha}) \quad \text{with} \\
T(\hat{\alpha}) &:= \begin{pmatrix} \mathcal{P}^{-1}(\hat{\alpha})V & \\ & \mathcal{P}^{-1}(\hat{\alpha})V \end{pmatrix}, \quad D_\eta(\hat{\omega}) := \begin{pmatrix} gD_{\mathcal{P}}^2(\hat{\alpha}) + D_1^2(\hat{\omega}) & -D_1(\hat{\omega}) \\ -D_1(\hat{\omega}) & \mathbb{1} \end{pmatrix}.
\end{aligned}$$

Due to the block diagonal structure of the similar and symmetric matrix $D_\eta(\hat{\omega})$ we calculate the real eigenvalues componentwise as

$$\begin{aligned}
\sigma\big\{D_\eta(\hat{\omega})\big\} &= \frac{1}{2}\Big(gD_{\mathcal{P}}^2(\hat{\alpha}) + D_1^2(\hat{\omega}) + 1\Big) \\
&\quad \pm \frac{1}{2}\bigg[\Big(gD_{\mathcal{P}}^2(\hat{\alpha}) + D_1^2(\hat{\omega}) + 1\Big)^2 - 4gD_{\mathcal{P}}^2(\hat{\alpha})\bigg]^{1/2}.
\end{aligned}$$

They are strictly positive if and only if $\big|D_{\mathcal{P}}(\hat{\alpha})\big| \neq 0$.

□

Note that the entropy-entropy flux pair reduces to the physical entropy in the deterministic case, see e.g. [12] for

$$\eta_0(y) := \frac{1}{2}\frac{q^2}{h} + \frac{g}{2}h^2, \quad \mu_0(y) := \frac{1}{2}\frac{q^3}{h^2} + gqh. \tag{14}$$

Remark 1. Isothermal Euler equations describe the density of gas ρ and read

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho(t,x)\\ q(t,x)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}q(t,x)\\ \frac{q^2(t,x)}{\rho(t,x)}+a^2\rho(t,x)\end{pmatrix}=0$$

with the speed of sound $a>0$. An intrusive formulation is

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat\rho(t,x)\\ \hat q(t,x)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\hat q(t,x)\\ \hat\beta(t,x)*\hat\beta(t,x)+a^2\hat\rho(t,x)\end{pmatrix}=0. \tag{15}$$

It has been shown in [17] for *arbitrary* gPC bases that the eigenvalues of system (15) are real and there is a full set of eigenvectors provided that $\hat\rho\in\mathbb{H}^+$ holds. We cannot show symmetric hyperbolicity for arbitrary bases and we cannot state an entropy. At least for bases with eigenvalue decompositions of the form $\mathcal{P}(\hat\alpha)=VD_{\mathcal{P}}(\hat\alpha)V^{\mathrm{T}}$, however, the system remains symmetrizable: We define the matrix

$$H(\hat y)\coloneqq\tilde H\left(\hat{\mathcal{Y}}^{-1}[\hat y]\right)\coloneqq\begin{pmatrix}\mathcal{P}^2(\hat\beta)\mathcal{P}^{-4}(\hat\alpha)+a^2\mathcal{P}^{-2}(\hat\alpha) & -\mathcal{P}(\hat\beta)\mathcal{P}^{-3}(\hat\alpha)\\ -\mathcal{P}(\hat\beta)\mathcal{P}^{-3}(\hat\alpha) & \mathcal{P}^{-2}(\hat\alpha)\end{pmatrix},$$

which reduces in the deterministic case for the entropy $\eta(y)={}^{q^2}\!/_{2\rho}+a^2\rho\ln(\rho)$ to the Hessian $\nabla_y^2\eta(y)=H(y)$. Provided that $\hat\alpha\in\mathbb{H}^+$ holds, the matrix $H(\hat y)$ is strictly positive definite and the product $H(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)$ is symmetric.

The assumption of constant eigenvectors is borrowed from [45]. It holds for the Wiener-Haar basis [45, Appendix B]. Then, corresponding eigenvectors are related to the Haar matrix [26]. In [45] this property is also shown for piecewise linear multi-wavelets with sufficiently small number of gPC truncation K.

5 Energy Estimates

We summarize our findings and state the notion of hyperbolicity in more detail. Similar to [25, 21] we call a system

weakly hyperbolic	if eigenvalues of the Jacobian are real,
strongly hyperbolic	if eigenvalues are real and there exists a complete set of eigenvectors,
strictly hyperbolic	if eigenvalues of the Jacobian are real and distinct,
symmetric hyperbolic	if a symmetric, strictly positive definite matrix $H(\hat y)$ exists so that the product $H(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)$ is symmetric.

Note that weakly hyperbolic systems are not necessarily stable [25]. All systems in this paper are at least strongly hyperbolic and hence stable. As illustrated in Figure 2, symmetric and strictly hyperbolic systems form important classes, which will be elaborated below. Deterministic Burgers', Euler and shallow water equations are both symmetric and strictly hyperbolic. In general, stochastic Galerkin formulations fail to have distinct eigenvalues. As an example one may consider the state $\hat\alpha\coloneqq(\hat\alpha_0,0,\dots,0)$, where the Jacobian of Burgers' equation reads $\mathcal{P}(\hat\alpha)=\hat\alpha_0\mathbb{1}$. The presented formulations, however, remain symmetric hyperbolic, since they are endowed with entropy-entropy flux pairs [7, 22]. This allows energy estimates, which do not hold for general strongly hyperbolic systems [25].

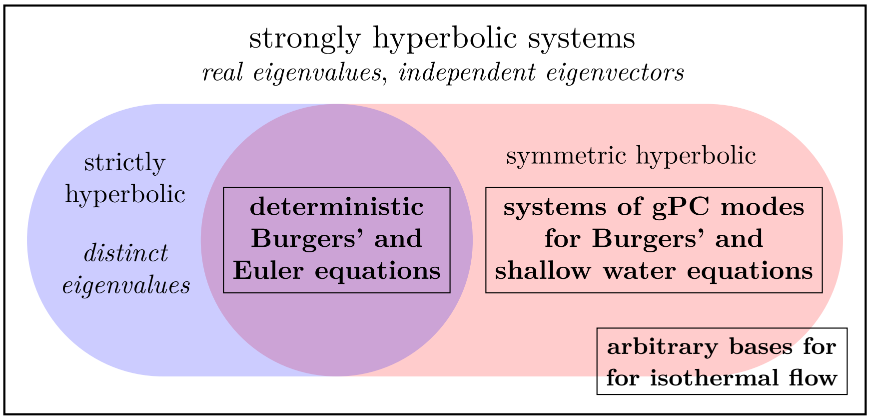

Fig. 2: **Summary of considered hyperbolic systems**

5.1 Wellposedness of Cauchy Problems

We state important results on stable hyperbolic systems endowed with entropies [12, Th. 5.3.1].

Corollary 2 (Wellposedness of Classical Solutions)**.** *Let $\hat{y}^*(t,x) \in \mathbb{H}_c$ denote a Lipschitz continuous classical solution of the Cauchy problem (1) on a finite time domain $[0,T)$ with initial data $\hat{\mathcal{I}}^*(x)$ which takes values in a convex, compact subset $\mathbb{H}_c \subset \mathbb{H}$. Let $\hat{y}(t,x) \in \mathbb{H}_c$ be any η-admissible weak solution with initial values $\hat{\mathcal{I}}(x)$. Then, we obtain:*

(i) The classical solution exists up to some point in time $T > 0$.

(ii) The classical solution $\hat{y}^*$ with initial values $\hat{\mathcal{I}}^*$ is the unique η-admissible weak solution.

(iii) For any $r > 0$, $t \in [0,T)$, there are positive constants $a, b, s \geq 0$ such that

$$\int\limits_{\|x\|<r} \big\|\hat{y}^*(t,x) - \hat{y}(t,x)\big\| \,\mathrm{d}x \leq a e^{bt} \int\limits_{\|x\|<r+st} \big\|\hat{\mathcal{I}}^*(x) - \hat{\mathcal{I}}(x)\big\| \,\mathrm{d}x.$$

The constant b depends on the Lipschitz continuity of the classical solution $\hat{y}^*$.

Note that general classical solutions of strongly hyperbolic systems are not well-posed in the weak sense, which explains the classes in Figure 2. Another reason is a stronger stability result for **initial boundary value problems** (IBVP) on a bounded space interval $\Omega_b \subset \mathbb{R}$

$$\hat{y}_t + \hat{f}(\hat{y})_x = 0 \text{ on } (0,T) \times \Omega_b, \quad \hat{y}(0,x) = \hat{\mathcal{I}}(x) \text{ on } \overline{\Omega_b}, \quad \hat{y}(t,x) = \hat{\mathcal{B}}(t,x) \text{ on } [0,T] \times \partial\Omega_b$$

with energy estimates of the form

$$\begin{aligned} &\int_{\Omega_b} \big\|\hat{y}(t,x)\big\|^2 \,\mathrm{d}x + \int_0^t \int_{\partial\Omega_b} \big\|\hat{y}(s,x)\big\|^2 \,\mathrm{d}x \,\mathrm{d}s \\ &\leq c(t) \bigg(\int_{\overline{\Omega_b}} \big\|\hat{\mathcal{I}}(x)\big\|^2 \,\mathrm{d}x + \int_0^t \int_{\partial\Omega_b} \big\|\hat{\mathcal{B}}(s,x)\big\|^2 \,\mathrm{d}x \,\mathrm{d}s \bigg). \end{aligned}$$

Then, the solution is bounded by initial data plus the growth due to boundary data [25]. Estimates are derived in [23] for the advection equation and in [44] for Burgers' equation under the assumption of *compatible* boundary conditions. We elaborate this point for subcritical water flows: Gravitational forces dominate the relatively slow velocity and eigenvalues satisfy $\lambda^-(y) < 0 < \lambda^+(y)$, i.e. $\sqrt{gh} > |q/\rho|$. As stochastic analogue, we assume

$$\sqrt{g}\,\mathcal{G}_K[h] > \big|\mathcal{G}_K[\beta]\big| \quad \mathbb{P}\text{-a.s.} \tag{16}$$

We obtain with [59, Th. 2.1] the eigenvalue estimate

$$\sqrt{g}\,\mathcal{G}_K[h] > \big|\mathcal{G}_K[\beta]\big|\ \mathbb{P}\text{-a.s} \ \Leftrightarrow\ \mathcal{G}_K[\sqrt{g}\,h \pm \beta] > 0\ \mathbb{P}\text{-a.s} \ \Rightarrow\ \sqrt{g} D_{\mathcal{P}}(\hat{h}) \pm D_{\mathcal{P}}(\hat{\beta}) > 0.$$

Thus, the inequality $\big|D_{\mathcal{P}}(\hat{\beta}) D_{\mathcal{P}}^{-1}(\hat{\alpha})\big| < \sqrt{g} D_{\mathcal{P}}(\hat{h}) D_{\mathcal{P}}^{-1}(\hat{\alpha})$ is satisfied and due the eigendecomposition in Theorem 4.3 at each boundary $K+1$ conditions must be imposed.

5.2 Error Estimates for Truncated Polynomial Chaos Expansions

We define the residuum $\mathcal{R}(\hat{y}) \coloneqq \hat{y}_t + \hat{f}(\hat{y})_x$ and we assume that the infinite gPC expansion with modes $\hat{y}^*_k$, $k \in \mathbb{N}_0$ solves the underlying system, i.e. $\mathcal{R}(\hat{y}^*) = 0$. For the orthogonal projection $\hat{y} \in \ell_2$ we have $\big\|\mathcal{R}(\hat{y})\big\| \to 0$ for $K \to 0$. However, it is not clear if the solution itself converges against the exact solution. Similarly to [12, 15, 20] we discuss this question for smooth solutions and introduce the relative entropy and the relative entropy flux as

$$\begin{aligned}
\eta(\hat{y}^*|\hat{y}) &\coloneqq \eta(\hat{y}^*) - \eta(\hat{y}) - \mathrm{D}_{\hat{y}}\eta(\hat{y})\big(\hat{y}^* - \hat{y}\big),\\
\mu(\hat{y}^*|\hat{y}) &\coloneqq \mu(\hat{y}^*) - \mu(\hat{y}) - \mathrm{D}_{\hat{y}}\eta(\hat{y})\big(f(\hat{y}^*) - f(\hat{y})\big).
\end{aligned}$$

Then, for general systems that are endowed with an entropy the following Lemma is proven in the appendix. It is related to Kruzhkov's entropy framework [33] and is similar to [12, Th. 5.2.1], [19, Lemma 2.7], [20, Th. 3.8].

Lemma 5.1. *Assume the approximation $\hat{y}$ is Lipschitz continuous in space $x \in \mathbb{R}$. Then, the following inequality holds:*

$$\begin{aligned}
\int_{\mathbb{R}} \eta\Big(\hat{y}^*(t,x)\Big|\hat{y}(t,x)\Big)\,\mathrm{d}x \le{}& \int_{\mathbb{R}} \eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x\\
&- \int_0^t \int_{\mathbb{R}} \mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big)\\
&+ \hat{y}_x^{\mathrm{T}}(s,x)\nabla^2_{\hat{y}}\eta\big(\hat{y}(s,x)\big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big)\\
&- \Big(\hat{f}\big(\hat{y}^*(s,x)\big) - \hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

The inner product $\langle \hat{y}^*, \hat{y}\rangle_{\nabla^2} \coloneqq \big\langle \hat{y}^*, \nabla^2_{\hat{y}}\eta(\hat{y})\hat{y}\big\rangle$ is well-defined for strictly convex entropies. Second-order Taylor approximations of the scalar entropy and the vector valued flux function yield the expressions

$$\begin{aligned}
\eta(\hat{y}^*|\hat{y}) &= \frac{1}{2}\|\hat{y}^* - \hat{y}\|^2_{\nabla^2} + \mathcal{O}\big(\|\hat{y}^* - \hat{y}\|_2^2\big),\\
\Big\|\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^* - \hat{y}) - \big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\big)\Big\| &\le \frac{c_{\hat{f}}(\hat{y})}{2}\|\hat{y}^* - \hat{y}\|^2_{\nabla^2},
\end{aligned}$$

where $c_{\hat f}(\hat y)$ depends only on the flux function and on the approximated states. With this second order approximation, Lemma 5.1, Cauchy-Schwarz and Young's inequality for products we obtain

$$\begin{aligned}
&\int \|\hat y^* - \hat y\|_{\nabla^2}^2 \,\mathrm{d}x - \int \|\hat{\mathcal{I}}^* - \hat{\mathcal{I}}\|_{\nabla^2}^2 \,\mathrm{d}x \\
&\le 2\int_0^T \int_{\mathbb{R}} \left|\left\langle \hat y^* - \hat y, \mathcal{R}(\hat y)\right\rangle_{\nabla^2}\right| + \left|\left\langle \mathrm{D}_{\hat y}\hat f(\hat y)(\hat y^* - \hat y) - \big(\hat f(\hat y^*) - \hat f(\hat y)\big), \hat y_x\right\rangle_{\nabla^2}\right| \mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^T \int_{\mathbb{R}} \|\hat y^* - \hat y\|_{\nabla^2}^2 + \big\|\mathcal{R}(\hat y)\big\|_{\nabla^2}^2 + \|\hat y_x\|_{\nabla^2} c_{\hat f}(\hat y)\|\hat y^* - \hat y\|_{\nabla^2}^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^T c(s;\hat y) \int \|\hat y^* - \hat y\|_{\nabla^2}^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}} \big\|\mathcal{R}(\hat y)\big\|_{\nabla^2}^2 \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

with the constant $c(s;\hat y) \coloneqq \max_{x\in\mathbb{R}} \Big\{1 + \big\|\hat y_x(s,x)\big\|_{\nabla^2} c_{\hat f}\big(\hat y(s,x)\big)\Big\}$.

Gronwall's inequality yields the **a posteriori estimate**

$$\begin{aligned}
\int_{\mathbb{R}} \Big\|\hat y^*(t,x) - \hat y(t,x)\Big\|_{\nabla^2}^2 \mathrm{d}x \le \Bigg[\int_{\mathbb{R}} \Big\|\hat{\mathcal{I}}^*(x) - \hat{\mathcal{I}}(x)\Big\|_{\nabla^2}^2 \mathrm{d}x \\
+ \int_{\mathbb{R}} \Big\|\mathcal{R}\big(\hat y(s,x)\big)\Big\|_{\nabla^2}^2 \mathrm{d}x\,\mathrm{d}s\Bigg] \exp\Big(\int_0^t c(s;\hat y)\,\mathrm{d}x\Big).
\end{aligned} \tag{17}$$

Once an estimate of the form (17), without second order approximation, is derived, the convergence of the gPC expansion in the PDE is inherited to the solution – at least for smooth solutions. We will derive an estimate for shallow water equations. Similar estimates for Burgers' equation are given in [15, 20].

Theorem 5.2 (Convergence of Shallow Water Equations)**.** *Define the auxiliary functions*

$$\begin{aligned}
\hat V\big(\hat\omega(t,x)\big) &\coloneqq \mathcal{P}^{-1}\big(\hat\alpha(t,x)\big)\frac{\partial}{\partial x}\hat\beta(t,x) - \mathcal{P}^{-2}\big(\hat\alpha(t,x)\big)\mathcal{P}\big(\hat\beta(t,x)\big)\frac{\partial}{\partial x}\hat\alpha(t,x), \\
\mathcal{P}_0\big(\hat\alpha^*(t,x), \hat\alpha(t,x)\big) &\coloneqq \mathcal{P}^{-1}\big(\hat\alpha(t,x)\big)\mathcal{P}\big(\hat\alpha^*(t,x)\big)
\end{aligned}$$

and assume the approximation $\hat y$ is Lipschitz continuous. For states $\hat y^*, \hat y \in \mathbb{H}_c \subset \mathbb{H}$ there is a constant $c \in [1,\infty)$ such that the spectral radius $\sigma_{\max}\{\mathcal{P}_0(\hat\alpha^*, \hat\alpha)\} < \sqrt{c}$ is bounded and there is the estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \eta\Big(\hat y^*(t,x)\Big|\hat y(t,x)\Big)\,\mathrm{d}x \le \Big[\int_{\mathbb{R}} \eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x \\
&+ c\int_0^t \int_{\mathbb{R}} \Big\|\mathcal{R}\big(\hat y(s,x)\big)\Big\|_2^2 \mathrm{d}x\,\mathrm{d}s\Big] \exp\Bigg(\int_0^t \frac{1}{2} + \max_{x\in\mathbb{R}}\bigg\{\Big\|\mathcal{P}\Big(\hat V\big(\hat\omega(s,x)\big)\Big)\Big\|_2\bigg\}\,\mathrm{d}s\Bigg).
\end{aligned}$$

Proof. Due to the equality

$$\begin{aligned}
\mathrm{D}_{\hat y}\eta(\hat y)(\hat y^* - \hat y) &= \Big[g\hat h - \frac{1}{2}\mathcal{P}_2(\hat\omega)\hat\beta, \mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big]^{\mathrm{T}}(\hat y^* - \hat y) \\
&= g\hat h^{\mathrm{T}}(\hat h^* - \hat h) - \frac{1}{2}\|\hat\beta\|_2^2 - \frac{1}{2}\big\|\mathcal{P}_0(\hat\omega^*, \hat\omega)\hat\beta\big\|_2^2 + \hat\beta^{\mathrm{T}}\mathcal{P}_0(\hat\omega^*, \hat\omega)\hat\beta^*
\end{aligned}$$

we obtain the relative entropy

$$\begin{aligned}\eta(\hat{y}^*|\hat{y}) &= \Big[\frac{g}{2}\|\hat{h}^*\|_2^2 + \frac{1}{2}\|\hat{\beta}^*\|_2^2\Big] \\ &\quad - \Big[\frac{g}{2}\|\hat{h}\|_2^2 + \frac{1}{2}\|\hat{\beta}\|_2^2\Big] - \Big[g\hat{h} - \frac{1}{2}\mathcal{P}_2(\hat{\omega})\hat{\beta}, \mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}\Big](\hat{y}^* - \hat{y}) \\ &= \frac{g}{2}\|\hat{h}^* - \hat{h}\|_2^2 + \frac{1}{2}\big\|\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big\|_2^2.\end{aligned}$$

By definition we have $\hat{V}(\hat{\omega}) = (\mathbb{O}, \mathbb{1})\, \nabla_{\hat{y}}^2 \eta(\hat{y})\hat{y}_x$ and we calculate

$$\begin{aligned}&\left|\Big(\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^* - \hat{y}) - \big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\big)\Big)^{\mathrm{T}} \nabla_{\hat{y}}^2\eta(\hat{y})\hat{y}_x\right| \\ &= \left|\begin{pmatrix}\mathbb{O} \\ \frac{g}{2}\big(\hat{h}^* - \hat{h}\big)^{2*} + \big(\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big)^{2*}\end{pmatrix}^{\mathrm{T}} \nabla_{\hat{y}}^2\eta(\hat{y})\hat{y}_x\right| \\ &= \left|\Big(\frac{g}{2}\big(\hat{h}^* - \hat{h}\big)^{2*} + \big(\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big)^{2*}\Big)^{\mathrm{T}} \hat{V}(\hat{\omega})\right| \\ &\leq \left|\frac{g}{2}\big(\hat{h}^* - \hat{h}\big)^{\mathrm{T}}\mathcal{P}\big(\hat{V}(\hat{\omega})\big)\big(\hat{h}^* - \hat{h}\big)\right| \\ &\quad + \left|\big(\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big)^{\mathrm{T}}\mathcal{P}\big(\hat{V}(\hat{\omega})\big)\big(\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big)\right| \\ &\leq 2\left\|\mathcal{P}\big(\hat{V}(\hat{\omega})\big)\right\|_2 \eta(\hat{y}^*|\hat{y}).\end{aligned} \tag{18}$$

For states $\hat{y}^*, \hat{y} \in \mathbb{H}_c$ and $\hat{\alpha}^*, \hat{\alpha} \in \mathbb{H}^+$ the constant $c := \max\big\{1, c_1^2\big\} \in [1, \infty)$ with

$$c_1 := \max_{\substack{t\in[0,T],\\ x\in\mathbb{R}}} \Big\{\sigma_{\min}^{-1}\big\{\mathcal{P}\big(\hat{\alpha}(t,x)\big)\big\}\sigma_{\max}\big\{\mathcal{P}\big(\hat{\alpha}^*(t,x)\big)\big\}\Big\} \geq \sigma_{\max}\Big\{\mathcal{P}_0\big(\hat{\alpha}^*(t,x), \hat{\alpha}(t,x)\big)\Big\}.$$

is bounded. Then, we obtain the estimate

$$\begin{aligned}\big\|\hat{y}^* - \hat{y}\big\|_{\nabla^2}^2 &= \begin{pmatrix}\hat{h}^* - \hat{h} \\ \hat{q}^* - \hat{q}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} g(\hat{h}^* - \hat{h}) - \mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\mathcal{P}(\hat{\beta})\big[\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big] \\ \mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}(\hat{\alpha}^*)\big[\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big]\end{pmatrix} \\ &= g\|\hat{h}^* - \hat{h}\|_2^2 + \big[\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big]^{\mathrm{T}}\mathcal{P}_0^2(\hat{\omega}^*, \hat{\omega})\big[\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big] \\ &\leq g\|\hat{h}^* - \hat{h}\|_2^2 + c_1^2\big\|\hat{\beta}^* - \mathcal{P}_0(\hat{\omega}^*, \hat{\omega})\hat{\beta}\big\|_2^2 \\ &\leq 2\max\{1, c_1^2\}\eta(\hat{y}^*|\hat{y}).\end{aligned} \tag{19}$$

Estimate (19), Cauchy-Schwarz and Young's inequality for products imply

$$\begin{aligned}\Big|\big\langle\mathcal{R}(\hat{y}), \hat{y}^* - \hat{y}\big\rangle_{\nabla^2}\Big| &\leq \big\|\mathcal{R}(\hat{y})\big\|_{\nabla^2}\big\|\hat{y}^* - \hat{y}\big\|_{\nabla^2} \leq \big\|\mathcal{R}(\hat{y})\big\|_{\nabla^2}\sqrt{2}\max\{1, c_1\}\eta(\hat{y}^*|\hat{y})^{1/2} \\ &\leq \frac{\max\{1, c_1^2\}}{2}\big\|\mathcal{R}(\hat{y})\big\|_{\nabla^2}^2 + \eta(\hat{y}^*|\hat{y}).\end{aligned} \tag{20}$$

Lemma 5.1 with the estimates (18) and (20) yield

$$
\begin{aligned}
&\int_{\mathbb{R}} \eta(\hat{y}^*|\hat{y})\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}} \eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \frac{c}{2}\big\|\mathcal{R}(\hat{y})\big\|_{\nabla^2}^2 + \eta(\hat{y}^*|\hat{y}) + 2\left\|\mathcal{P}\big(\hat{V}(\hat{\omega})\big)\right\|_2 \eta(\hat{y}^*|\hat{y})\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_{\mathbb{R}} \eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}} \frac{c}{2}\big\|\mathcal{R}(\hat{y})\big\|_{\nabla^2}^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \left(1 + 2\max_{x\in\mathbb{R}}\left\{\left\|\mathcal{P}\big(\hat{V}(\hat{\omega})\big)\right\|_2\right\}\right)\int_{\mathbb{R}} \eta(\hat{y}^*|\hat{y})\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

The claim follows from Gronwall's inequality.

□

The presented estimate shows that the convergence of the residuum is inherited to the solution as long as the solution remains smooth and η-admissible. In general, this holds local in time only, since we do not guarantee that the solution is admissible on unbounded time domains. This is an expected result, since normally also the solutions of deterministic hyperbolic systems cease to exist [2].

6 Numerical Illustration of the Theoretical Results

First, we show the solutions and the entropies for truncated Wiener-Haar expansions. In particular, we highlight their smoothness properties and state statistics of interest. Then, we illustrate the decay of entropies, which mimics the stability of the system.

To this end, an interval $[0, x_{\text{end}}]$ is divided into N cells by a space discretization $\Delta x > 0$ with $\Delta x N = x_{\text{end}}$. The centers are $x_j \coloneqq \big(j + \frac{1}{2}\big)\Delta x$ and the edges are $x_{j+1/2} \coloneqq j\Delta x$ for $j = 0, \dots, N$. The evolution of cell averages are desribed by the ordinary differential equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{y}}_j(t) = -\frac{1}{\Delta x}\left[\hat{f}\Big(\hat{y}(t, x_{j+1/2})\Big) - \hat{f}\Big(\hat{y}(t, x_{j-1/2})\Big)\right], \quad \bar{\mathbf{y}}_j(t) \coloneqq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \hat{y}(t,x)\,\mathrm{d}x.
$$

To obtain a semi-discretization in space for an approximation $\hat{\mathbf{y}}_j \approx \bar{\mathbf{y}}_j$, we use the local Lax-Friedrichs flux

$$
\hat{\mathbf{f}}(\hat{\mathbf{y}}_\ell, \hat{\mathbf{y}}_r) \coloneqq \frac{1}{2}\Big(\hat{f}(\hat{\mathbf{y}}_\ell) + \hat{f}(\hat{\mathbf{y}}_r)\Big) - \frac{1}{2}\max_{j=\ell,r}\left\{\sigma\Big\{\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\big|_{\hat{y}=\hat{\mathbf{y}}_j}\Big\}\right\}(\hat{\mathbf{y}}_r - \hat{\mathbf{y}}_\ell),
$$

where the spectrum $\sigma\big\{\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\big\}$ is given in Theorem 4.3. Furthermore, the central, weighted, essentially non oscillatory (CWENO) reconstruction from [11] is applied. We denote the reconstruction at the left side of a cell interface by $\hat{\mathbf{y}}^-_{j+1/2}(t)$ and at the right side by $\hat{\mathbf{y}}^+_{j+1/2}(t)$, respectively. Then, a third order reconstruction at the edge $x_{j+1/2}$ is of the form

$$
\text{CWENO} \;:\; \Big[\hat{\mathbf{y}}_{j-1}, \hat{\mathbf{y}}_j, \hat{\mathbf{y}}_{j+1}, \hat{\mathbf{y}}_{j+2}\Big] \;\mapsto\; \Big[\hat{\mathbf{y}}^-_{j+1/2}, \hat{\mathbf{y}}^+_{j+1/2}\Big]
$$

and the resulting semi-discretization reads

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{y}}_j(t) = -\frac{1}{\Delta x}\left[\hat{\mathbf{f}}\Big(\hat{\mathbf{y}}^-_{j+1/2}(t), \hat{\mathbf{y}}^+_{j+1/2}(t)\Big) - \hat{\mathbf{f}}\Big(\hat{\mathbf{y}}^-_{j-1/2}(t), \hat{\mathbf{y}}^+_{j-1/2}(t)\Big)\right].
$$

It is approximated with a strong stability preserving (SSP) Runge-Kutta method with three stages [29]. A third order SSP-CWENO scheme is applied with space discretization $\Delta x = 10^{-3}$, CFL-condition 0.99 and gravitational constant $g = 1$. All simulations are done with `Matlab`. The CWENO reconstruction is borrowed from the authors of [11] and compiled in `Matlab` as `C`-implementation.
We illustrate the analysis using a **dam break problem** [36]. The solution consists of a rarefaction wave, moving with negative speed, and a shock wave with positive speed. Both waves are connected by an intermediate state y_{m}. For given states $\bar{\mathbf{y}}_\ell = (\bar{\mathbf{h}}_\ell, \bar{\mathbf{q}}_\ell)^{\mathrm{T}}$ and $\bar{\mathbf{y}}_r = (\bar{\mathbf{h}}_r, \bar{\mathbf{q}}_r)^{\mathrm{T}}$ with $\bar{\mathbf{h}}_\ell \geq \bar{\mathbf{h}}_r > 0$ and $\bar{\mathbf{q}}_\ell = \bar{\mathbf{q}}_r = 0$, the dam break problem with initial states $y(0, x_\ell) = \bar{\mathbf{y}}_\ell$ and $y(0, x_r) = \bar{\mathbf{y}}_r$ for $x_\ell < 0 < x_r$ is solved by

$$y(t,x) = \begin{cases} \bar{\mathbf{y}}_\ell & \text{if } \ x < t\lambda^-(\bar{\mathbf{y}}_\ell), \\ y_{\mathrm{rf}}(t,x;\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r) & \text{if } \ t\lambda^-(\bar{\mathbf{y}}_\ell) \leq x < t\lambda^-\big(y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r)\big), \\ y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r) & \text{if } \ t\lambda^-\big(y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r)\big) \leq x < ts(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r), \\ \bar{\mathbf{y}}_r & \text{if } \ ts(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r) < x. \end{cases} \tag{21}$$

The expressions for the rarefaction wave y_{rf}, intermediate state y_{m} and shock speed s are found in [36]. We consider uniformly distributed left initial values $\bar{\mathbf{h}}_\ell(\xi) \sim U(3,4)$ and the deterministic right state $\bar{\mathbf{h}}_r = 1$.

6.1 Dimension Reduction by the Wiener-Haar Expansion

The Haar sequence [26, 40, 46] with level $J \in \mathbb{N}_0$ generates a gPC basis $\mathcal{S}_K$ with $K = 2^{J+1} - 1$ elements by

$$\mathcal{S}_K := \big\{1, \psi(\xi), \psi_{j,k}(\xi) \;\big|\; k = 0,\dots,2^j - 1,\ j = 1,\dots,J\big\} \quad \text{for}$$

$$\psi_{j,k}(\xi) := 2^{j/2}\psi\big(2^j\xi - k\big) \quad \text{and} \quad \psi(\xi) := \begin{cases} 1 & \text{if } \ 0 \leq \xi < 1/2, \\ -1 & \text{if } \ 1/2 \leq \xi < 1, \\ 0 & \text{else.} \end{cases}$$

Using a lexicographical order we identify the relation $\phi_1 = \psi$, $\phi_2 = \psi_{1,0}$ and $\phi_3 = \psi_{1,1}$ with the supports $\mathrm{supp}\{\psi\} = [0,1)$, $\mathrm{supp}\{\psi_{1,0}\} = [0, 1/2)$ and $\mathrm{supp}\{\psi_{1,1}\} = [1/2, 1)$.

Figure 3 shows those basis elements, as they approximate the continuous uniform distribution of the left initial values (blue). While the first element ϕ_0 yields the mean (green), the remaining functions give the details (black). A zoom in the area of the shock reveals that it is described mostly by the mean ϕ_0 and the first detail function ϕ_1. In fact, we have $\hat{h}_2(t,x) = 0$ close to the right half of the shock, since the corresponding basis element ϕ_2 describes low initial heights, which result in slow shock speeds. Furthermore, the 1.0-confidence region and realisations, corresponding to the jumps in the approximated input distribution, are shown. The entropy according to Theorem 4.3 is plotted with respect the right y-axis in red. For initial states $\bar{\mathbf{y}} = (\bar{\mathbf{h}}, 0)^{\mathrm{T}}$, when the momentum is zero, we have the relation

$$\mathbb{E}\Big[\eta_0\Big(\mathcal{G}_K\big[\bar{\mathbf{y}}(\xi)\big]\Big)\Big] = \mathbb{E}\Big[\frac{g}{2}\mathcal{G}_K\big[\bar{\mathbf{h}}(\xi)\big]^2\Big] = \frac{g}{2}\big\|\hat{\mathbf{h}}\big\|_2^2 = \eta(\hat{\mathbf{y}}), \tag{22}$$

where η_0 is the pointwise entropy (14). This motivates the choice of the mean of pointwise entropies as a quantitative comparison. This choice is completely independent from our new results and only based on a Monte-Carlo simulation. Although the entropies of the intrusive formulation converge to the mean for the initial values, we do not claim that there is a convergence also for $t > 0$. Confidence regions and the mean of entropies are determined with 10^5 samples.

Apart from the shock, good agreement is observed for both the entropy and the presented statistics of interest. The main difference is that there is no longer a smooth expectation of the shock. This issue has been observed also for continuous input distributions [15, 44, 17].

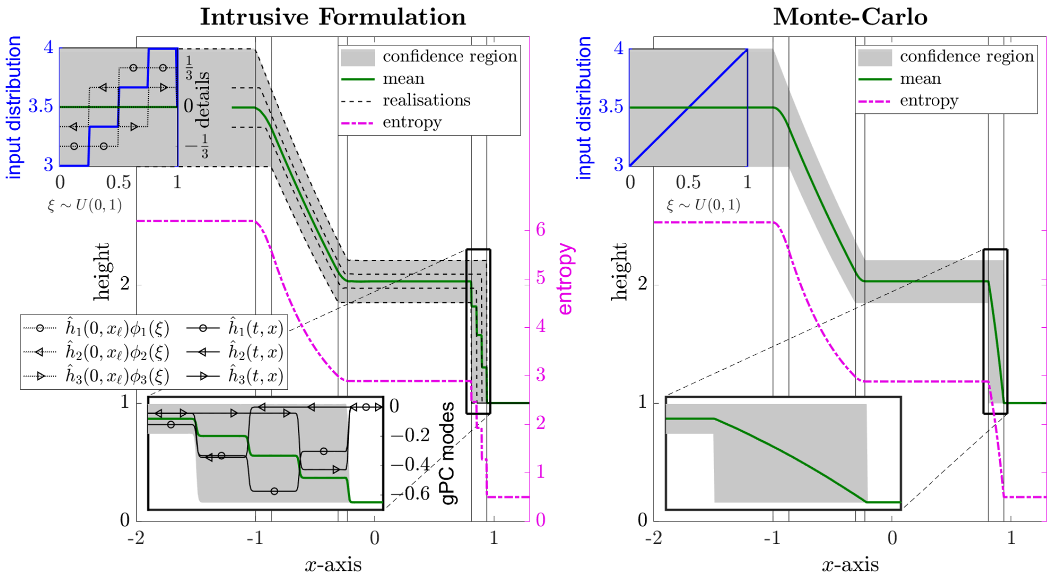

Fig. 3: Solution of the dam break problem for the intrusive formulation of Theorem 4.3 compared to a Monte-Carlo simulation in $t = 0.5$

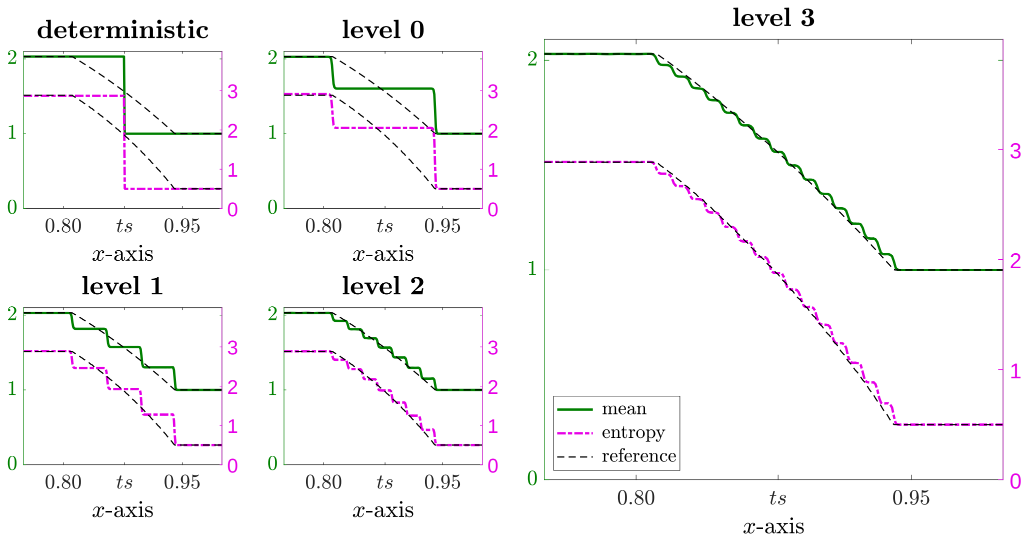

Fig. 4: Zoom on shock; semi-intrusive reference solution (dashed) with $K + 1 = 32$

Figure 4 consists of the amplifications of shocks and visualizes the regularity of truncated gPC expansions in more detail. For subcritical flows, which satisfy the assumption (16), the initial discontinuity splits into at most $K+1$ distinct waves that have positive speed. Here, $K+1$ waves move at slightly different speeds to the right. We choose as reference solution the level $J=4$ with $K+1=32$ basis elements. The gPC modes are determined in a semi-intrusive way as $\hat{y}_k^{\mathrm{ref}} := \mathbb{E}\big[y^{\mathrm{ref}}(t,x,\xi)\phi_k(\xi)\big]$, where $y^{\mathrm{ref}} = (h^{\mathrm{ref}}, q^{\mathrm{ref}})^{\mathrm{T}}$ denotes the reference solution (21). Expectations are computed by a Monte-Carlo method with 10^5 samples.

L^∞-error:	**rarefaction wave**				**shock**				**units**
level J	0	1	2	3	0	1	2	3	
$\hat{E}_K^{(\mathbb{E})}$	4.43	1.63	0.86	0.56	56.93	28.05	13.03	5.44	$[10^{-2}]$
$\hat{E}_K^{(\mathbb{V})}$	16.69	5.57	2.39	1.12	32.40	24.77	13.52	5.44	$[10^{-2}]$

L^1-error:	**rarefaction wave**				**shock**				**units**
level J	0	1	2	3	0	1	2	3	
$\hat{E}_K^{(\mathbb{E})}$	6.76	3.06	2.01	1.65	35.31	13.16	5.84	2.68	$[10^{-3}]$
$\hat{E}_K^{(\mathbb{V})}$	137.89	45.97	19.68	9.16	36.77	14.55	6.47	2.89	$[10^{-3}]$

	relative entropy for Cauchy problem				**units**
level J	0	1	2	3	
$\eta(\hat{y}^*\vert\hat{y})$	108.05	27.60	9.89	5.66	$[10^{-3}]$

Tab. 1: Observed numerical errors for the dam break problem

Table 1 reports on numerical errors for the mean and the variance

$$
\begin{aligned}
E_K^{(\mathbb{E})}(t,x) &:= \Big|\mathbb{E}\big[h^{\mathrm{ref}}(t,x;\xi)\big] - \hat{h}_0(t,x)\Big|, \\
E_K^{(\mathbb{V})}(t,x) &:= \Big|\mathrm{Var}\big[h^{\mathrm{ref}}(t,x;\xi)\big] - \sum_{k=1}^{K+1} \hat{h}_k^2(t,x)\Big|.
\end{aligned}
$$

For each fixed point we obtain estimates $\hat{E}_K^{(\mathbb{E})}$ and $\hat{E}_K^{(\mathbb{V})}$ by a Monte-Carlo method with 10^5 samples. Table 1 is divided into the rarefaction wave for $x \in [-1.5, 0]$ and the shock for $x \in [0, 1.5]$. Then, for each level $J = 0, \ldots, 3$ with corresponding gPC order $K+1 = 2, 4, 8, 16$ the L^1-and L^∞-norms $\int |\cdot|\,\mathrm{d}x$ and $\sup_x |\cdot|$ are stated. Indeed, we observe a convergence for the mean and the variance. Furthermore, we show the relative entropy and use again the semi-intrusively computed gPC modes $\hat{y}_k^{\mathrm{ref}} \in \mathbb{R}^{64}$ as reference solution. We observe the expected decay also for this error measure. To verify the compatibility condition (3), we consider the L^2-error

$$
E_K^{(C)}(t,x) := \Big\|\mathrm{D}_{\hat{y}}\mu(\hat{y}) - \mathrm{D}_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\Big\|_2(t,x)
$$

and we expect it to be close to zero for smooth solutions. The compatibility condition is fulfilled up to numerical errors, which are two powers smaller than the spatial discretization.

6.2 Decay of Entropies

The entropy inequality (2) guarantees a decaying entropy, since spatial integration yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta\Big(\hat{y}(t,x)\Big)\,\mathrm{d}x \le 0 \quad \Longrightarrow \quad \int_{\mathbb{R}} \eta\Big(\hat{y}(t,x)\Big)\,\mathrm{d}x \le \int_{\mathbb{R}} \eta\Big(\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x. \tag{23}$$

Figure 5 aims to show this decay over time. We choose both the semi-intrusively computed gPC modes $\hat{y}_k^{\mathrm{ref}} \in \mathbb{R}^{64}$ and the mean of the pointwise entropies (22) as reference solution, which are computed for each time step with 10^5 samples. Indeed, all computed entropies are decreasing. For higher refinement level J the entropies are close to the reference solutions.

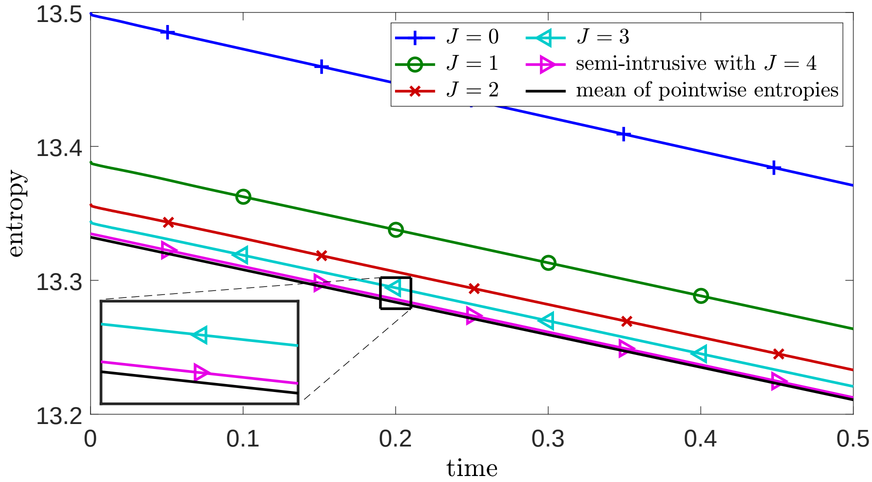

Fig. 5: Time evolution of entropies according to inequality (23)

Summary

We have introduced entropy and entropy flux pairs for stochastic Galerkin formulations of hyperbolic conservation laws. Two families have been introduced for Burgers' equation. One of these entropies has been used to ensure the uniqueness of a stochastic Galerkin square root. An important consequence is the bijective mapping between conserved and Roe variables, which has been used for a hyperbolic formulation of shallow water equations. For this system a generalization of the physical entropy to the stochastic Galerkin formulation has been presented. As important conclusions wellposedness of classical solutions as well as energy estimates and convergence of the polynomial chaos expansion have been stated. Numerical experiments have confirmed the theoretical findings.

7 Appendix

For the proof of Lemma 4.2 we recall $\mathrm{D}_{\hat\alpha}\mathcal{P}^{-1}(\hat\alpha) = -\mathcal{P}^{-1}(\hat\alpha)\mathrm{D}_{\hat\alpha}\mathcal{P}(\hat\alpha)\mathcal{P}^{-1}(\hat\alpha)$.

Proof of Lemma 4.2. Equation (9) follows from

$$\hat u^{\mathrm{T}}(\hat\omega)\mathcal{P}(\hat h) = \hat\beta^{\mathrm{T}}\mathcal{P}(\hat h)\mathcal{P}^{-1}(\hat\alpha) = \hat\alpha^{\mathrm{T}}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta)\mathcal{P}^{-1}(\hat\alpha) = (\hat\alpha * \hat\beta)^{\mathrm{T}},$$

where we have used symmetry and commutativity according to assumption. We calculate

$$\mathcal{P}(\hat\beta) = \left[\Big(\sum_{j=0}^{K}\hat\beta_j\langle\phi_i\phi_j,\phi_0\rangle\Big)\Big|\cdots\Big|\Big(\sum_{j=0}^{K}\hat\beta_j\langle\phi_i\phi_j,\phi_K\rangle\Big)\right]_{i=0,\dots,K} = \left[\mathcal{M}_0\hat\beta\Big|\cdots\Big|\mathcal{M}_K\hat\beta\right],$$
$$\mathcal{P}\big(\hat u(\hat\omega)\big)\hat\gamma = \mathcal{P}(\hat\gamma)\mathcal{P}^{-1}(\hat\alpha)\hat\beta = \mathcal{P}_1(\hat\omega)\hat\gamma,\ \mathcal{P}\big(\widehat{u^2}(\hat\omega)\big)\hat\gamma = \mathcal{P}(\hat\gamma)\mathcal{P}^{-2}(\hat\alpha)\mathcal{P}(\hat\beta)\hat\beta = \mathcal{P}_1^2(\hat\omega)\hat\gamma$$

to obtain the equations (10) and (11) as

$$\begin{aligned}
\mathrm{D}_{\hat\alpha}\left[\hat\beta^{\mathrm{T}}\mathcal{P}_1(\hat\omega)\hat\beta\right] &= -\left[\hat\beta^{\mathrm{T}}\mathcal{P}(\hat\beta)\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_k\mathcal{P}^{-1}(\hat\alpha)\hat\beta\right]_{k=0,\dots,K}\\
&= -\hat\beta^{\mathrm{T}}\mathcal{P}_1^2(\hat\omega),\\
\mathrm{D}_{\hat\beta}\left[\hat\beta^{\mathrm{T}}\mathcal{P}_1(\hat\omega)\hat\beta\right] &= \mathrm{D}_{\bar\beta}\left[\bar\beta^{\mathrm{T}}\mathcal{P}_1(\hat\omega)\bar\beta\right]\Big|_{\bar\beta=\hat\beta} + \mathrm{D}_{\bar\beta}\left[\hat\beta^{\mathrm{T}}\mathcal{P}(\bar\beta)\hat u(\hat\omega)\right]\Big|_{\bar\beta=\hat\beta}\\
&= 2\hat\beta^{\mathrm{T}}\mathcal{P}_1(\hat\omega) + \hat\beta^{\mathrm{T}}\mathcal{P}\big(\hat u(\hat\omega)\big)\\
&= 3\hat\beta^{\mathrm{T}}\mathcal{P}_1(\hat\omega),\\
\mathrm{D}_{\hat\alpha}\left[(\hat\alpha*\hat\alpha)^{\mathrm{T}}(\hat\alpha*\hat\beta)\right] &= \mathrm{D}_{\bar\alpha}\left[\bar\alpha^{\mathrm{T}}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta)\bar\alpha\right]\Big|_{\bar\alpha=\hat\alpha} + \mathrm{D}_{\bar\alpha}\left[\hat\alpha^{\mathrm{T}}\mathcal{P}(\bar\alpha)\mathcal{P}(\hat\beta)\hat\alpha\right]\Big|_{\bar\alpha=\hat\alpha}\\
&= 2\hat\alpha^{\mathrm{T}}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta) + \left[\hat\alpha^{\mathrm{T}}\mathcal{M}_0\mathcal{P}(\hat\beta)\hat\alpha\Big|\cdots\Big|\hat\alpha^{\mathrm{T}}\mathcal{M}_K\mathcal{P}(\hat\beta)\hat\alpha\right]\\
&= 3\hat\alpha^{\mathrm{T}}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta),\\
\mathrm{D}_{\hat\beta}\left[(\hat\alpha*\hat\alpha)^{\mathrm{T}}(\hat\alpha*\hat\beta)\right] &= \mathrm{D}_{\hat\beta}\left[\hat\alpha^{\mathrm{T}}\mathcal{P}^2(\hat\alpha)\hat\beta\right] = \hat\alpha^{\mathrm{T}}\mathcal{P}^2(\hat\alpha).
\end{aligned}$$

The matrices (12) and (13) follow from

$$\begin{aligned}
\mathrm{D}_{\hat\alpha}\left[\hat u(\hat\omega)\right] &= -\left[\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_0\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big|\cdots\Big|\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_K\mathcal{P}^{-1}(\hat\alpha)\hat\beta\right]\\
&= -\mathcal{P}_2(\hat\omega),\\
\mathrm{D}_{\hat\alpha}\left[\mathcal{P}_2(\hat\omega)\hat\beta\right] &= -2\mathrm{D}_{\bar\alpha}\left[\mathcal{P}_1(\hat\omega)\mathcal{P}^{-1}(\bar\alpha)\hat\beta\right]\Big|_{\bar\alpha=\hat\alpha}\\
&= -2\mathcal{P}_1(\hat\omega)\left[\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_0\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big|\cdots\Big|\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_K\mathcal{P}^{-1}(\hat\alpha)\hat\beta\right]\\
&= -2\mathcal{P}_1(\hat\omega)\mathcal{P}_2(\hat\omega),\\
\mathrm{D}_{\hat\beta}\left[\mathcal{P}_2(\hat\omega)\hat\beta\right] &= \mathrm{D}_{\hat\beta}\left[\mathcal{P}^{-2}(\hat\alpha)\,\hat\beta*\hat\beta\right] = 2\mathcal{P}_2(\hat\omega).
\end{aligned}$$

□

The proof of Lemma 5.1 is similar to [33, 12, 19, 20]. It is a slight adaptation which follows from exploiding the fact that systems endowed with entropies are symmetrizable, see e.g. [21, Th. 3.1]. Then, the matrix $\nabla^2_{\hat y}\eta(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)$ is symmetric.

Proof of Lemma 5.1. Due to the compatibility condition (3), and due to the symmetry of the matrix $\nabla^2_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})$ we obtain

$$\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y}) = \mathrm{D}_{\hat{y}}\eta(\hat{y})\hat{y}_t + \mathrm{D}_{\hat{y}}\mu(\hat{y})\hat{y}_x = \eta(\hat{y})_t + \mu(\hat{y})_x, \tag{24}$$

$$\begin{aligned}\hat{f}(\hat{y})_x^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y})(\hat{y}^* - \hat{y}) &= \hat{y}_x^{\mathrm{T}}\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y})(\hat{y}^* - \hat{y}) \\ &= \hat{y}_x^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^* - \hat{y}).\end{aligned} \tag{25}$$

For every non-negative C^1-function φ with compact support Rademacher's theorem yields that the approximation $\hat{y}$ and hence the auxiliary function $\tilde{\varphi} := \nabla_{\hat{y}}\eta(\hat{y})\varphi$ are differentiable almost everywhere. We obtain in the distributional sense

$$\tilde{\varphi}_t = \nabla^2_{\hat{y}}\eta(\hat{y})\hat{y}_t\varphi + \nabla_{\hat{y}}\eta(\hat{y})\varphi_t = \nabla^2_{\hat{y}}\eta(\hat{y})\big(\mathcal{R}(\hat{y}) - \hat{f}(\hat{y})_x\big)\varphi + \nabla_{\hat{y}}\eta(\hat{y})\varphi_t, \tag{26}$$

$$\tilde{\varphi}_x = \nabla^2_{\hat{y}}\eta(\hat{y})\hat{y}_x\varphi + \nabla_{\hat{y}}\eta(\hat{y})\varphi_x. \tag{27}$$

With equations (26) and (27) we conclude

$$\begin{aligned}
0 &= \tilde{\varphi}^{\mathrm{T}}\mathcal{R}(\hat{y}) + \tilde{\varphi}^{\mathrm{T}}\big(\hat{y}^* - \hat{y}\big)_t + \tilde{\varphi}^{\mathrm{T}}\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big)_x, \\
0 &= \int_0^T \int_{\mathbb{R}} \tilde{\varphi}_s^{\mathrm{T}}\big(\hat{y}^* - \hat{y}\big) + \tilde{\varphi}_x^{\mathrm{T}}\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big) - \tilde{\varphi}^{\mathrm{T}}\mathcal{R}(\hat{y})\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{R}} \tilde{\varphi}_0^{\mathrm{T}}\big(\hat{\mathcal{I}}^* - \hat{\mathcal{I}}\big)\,\mathrm{d}x \\
&= \int_0^T \int_{\mathbb{R}} \Big[\varphi\big(\mathcal{R}(\hat{y}) - \hat{f}(\hat{y})_x\big)^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y}) + \varphi_s\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big]\big(\hat{y}^* - \hat{y}\big) \\
&\quad + \Big[\varphi\hat{y}_x^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y}) + \varphi_x\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big]\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big) - \varphi\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{\mathbb{R}} \varphi_0\mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^* - \hat{\mathcal{I}}\big)\,\mathrm{d}x.
\end{aligned}$$

We rearrange these terms and use equation (25) to obtain

$$\begin{aligned}
&\int_0^{\mathrm{T}} \int_{\mathbb{R}} \mathrm{D}_{\hat{y}}\eta(\hat{y})\big(\hat{y}^* - \hat{y}\big)\varphi_s + \mathrm{D}_{\hat{y}}\eta(\hat{y})\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big)\varphi_x\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = -\int_0^T \int_{\mathbb{R}} \Big(\mathcal{R}(\hat{y}) - \hat{f}(\hat{y})_x\Big)^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y})(\hat{y}^* - \hat{y})\varphi - \hat{y}_x\nabla^2_{\hat{y}}\eta(\hat{y})\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big)\varphi \\
&\qquad + \mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}} \mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^* - \hat{\mathcal{I}}\big)\varphi_0\,\mathrm{d}x \\
&\quad = -\int_0^T \int_{\mathbb{R}} \mathcal{R}^{\mathrm{T}}(\hat{y})\nabla^2_{\hat{y}}\eta(\hat{y})(\hat{y}^* - \hat{y})\varphi \\
&\qquad - \hat{y}_x^{\mathrm{T}}\nabla^2_{\hat{y}}\eta(\hat{y})\Big[\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^* - \hat{y}) - \big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\big)\Big]\varphi\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \int_{\mathbb{R}} \mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^* - \hat{\mathcal{I}}\big)\varphi_0\,\mathrm{d}x.
\end{aligned} \tag{28}$$

The entropy inequality $\eta(\hat{y}^*)_t + \mu(\hat{y}^*)_x \leq 0$ and equation (24) imply in the distributional sense

$$\big(\eta(\hat{y}^*) - \eta(\hat{y})\big)_t + \big(\mu(\hat{y}^*) - \mu(\hat{y})\big)_x + \mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y}) \leq 0$$

which reads with equation (28) as

$$
\begin{aligned}
0 \leq & \int_0^T \int_{\mathbb{R}} \big(\eta(\hat{y}^*) - \eta(\hat{y})\big)\varphi_s + \big(\mu(\hat{y}^*) - \mu(\hat{y})\big)\varphi_x - \mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi \,\mathrm{d}x\,\mathrm{d}s \\
& + \int_{\mathbb{R}} \big(\eta(\hat{\mathcal{I}}^*) - \eta(\hat{\mathcal{I}})\big)\varphi_0 \,\mathrm{d}x \\
= & \int_0^T \int_{\mathbb{R}} \eta(\hat{y}^*|\hat{y})\varphi_s + \mu(\hat{y}^*|\hat{y})\varphi_x - \mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi \,\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^{\mathrm{T}} \int_{\mathbb{R}} \mathrm{D}_{\hat{y}}\eta(\hat{y})(\hat{y}^* - \hat{y})\varphi_s + \mathrm{D}_{\hat{y}}\eta(\hat{y})\Big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\Big)\varphi_x \,\mathrm{d}x\,\mathrm{d}s \\
& + \int_{\mathbb{R}} \big(\eta(\hat{\mathcal{I}}^*) - \eta(\hat{\mathcal{I}})\big)\varphi_0 \,\mathrm{d}x \\
= & \int_0^T \int_{\mathbb{R}} \eta(\hat{y}^*|\hat{y})\varphi_s + \mu(\hat{y}^*|\hat{y})\varphi_x \,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_{\mathbb{R}} -\mathcal{R}^{\mathrm{T}}(\hat{y})\nabla_{\hat{y}}^2\eta(\hat{y})(\hat{y}^* - \hat{y})\varphi \\
& + \hat{y}_x^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\Big[\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^* - \hat{y}) - \big(\hat{f}(\hat{y}^*) - \hat{f}(\hat{y})\big)\Big]\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{R}} \eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\varphi_0 \,\mathrm{d}x.
\end{aligned}
$$

In particular for the non-negative testfunction

$$
\varphi_\varepsilon(s,x;t) := \begin{cases} 1 & \text{if } s < t, \\ 1 - \frac{s-t}{\varepsilon} & \text{if } t < s < t+\varepsilon, \\ 0 & \text{if } t+\varepsilon < s \end{cases}
$$

we obtain for all Lebesgue points $t \in [0,T)$

$$
\begin{aligned}
0 \leq & \int_{\mathbb{R}} -\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \eta\Big(\hat{y}^*(s,x)\Big|\hat{y}(s,x)\Big)\,\mathrm{d}x + \int_{\mathbb{R}} \eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x \\
& - \int_0^T \int_{\mathbb{R}} \mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla_{\hat{y}}^2\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big)\varphi_\varepsilon(s,x;t) \\
& \qquad + \hat{y}_x^{\mathrm{T}}(s,x)\nabla_{\hat{y}}^2\eta\Big(\hat{y}(s,x)\Big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big) \\
& \qquad - \Big(\hat{f}\big(\hat{y}^*(s,x)\big) - \hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\varphi_\varepsilon(s,x;t)\,\mathrm{d}x\,\mathrm{d}s \\
\overset{\varepsilon\to 0}{\longrightarrow} & - \int_{\mathbb{R}} \eta\Big(\hat{y}^*(t,x)\Big|\hat{y}(t,x)\Big)\,\mathrm{d}x + \int_{\mathbb{R}} \eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x \\
& - \int_0^t \int_{\mathbb{R}} \mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla_{\hat{y}}^2\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big) \\
& \qquad + \hat{y}_x^{T}(s,x)\nabla_{\hat{y}}^2\eta\Big(\hat{y}(s,x)\Big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x) - \hat{y}(s,x)\Big) \\
& \qquad - \Big(\hat{f}\big(\hat{y}^*(s,x)\big) - \hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

□

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