

# Qualitative Properties of Mathematical Model for Data Flow

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#### Abstract

In this paper, properties of a recently proposed mathematical model for data flow in largescale asynchronous computer systems are analyzed. In particular, the existence of special weak solutions based on propagating fronts is established. Qualitative properties of these solution are investigated, both theoretically and numerically.

**MSC** 35L65, 35L40

Keywords conservation laws, predictive performance

## 1 Introduction

The increasing number and diversity of processing units in modern supercomputers presents the significant challenge of understanding how data is distributed globally in these systems [4, 8]. With the help of new mathematical models, a more efficient utilization of available processing units might be possible. Recently, a mathematical model for the evolution of processed data in a

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large-scale asynchronous computer systems has been proposed [1]. In the current work, we analyze some qualitative properties of this model, with regard to front propagation, and present simulation results that highlight the theoretical findings.

In [1], a microscopic ordinary differential equation (ODE) model is constructed to describe how data evolves in a lattice of processors. It is assumed that each processor performs the same task, which is broken up into discrete stages. The rate at which a given processor moves data from stage to stage depends not only its intrinsic processing rate, but also on the availability of usable data in the processor and its neighbors. From this microscopic model, a macroscopic fluid model is derived in the limit of infinitely many processors and infinitely many stages. Here we briefly review the microscopic model and the macroscopic limit.

In a one-dimensional periodic lattice, the amount of data at time t that sits in stage  $k \in \{1, \ldots, k_{\max}\}$  of processor  $i \in \{1, \ldots, i_{\max}\}$  is given by  $q_{i,k}(t)$ . In the absence of any throttling, the flow of data between stages is given by a (processor dependent) rate  $a_i(t) \ge 0$ . However the actual flow may be reduced due to a lack of available data in a given processor or its nearest neighbors. Throttling due to a lack of usable data in the nearest neighbors is characterized by a parameter  $\eta := k_{\max}/i_{\max} > 0$  that, at the discrete level, is equal to the ratio of task stages to total processors. Thus as  $\eta$  increases the global effect of local slowdowns increases.

For each  $i \in \{1, \ldots, i_{\max}\}$  and  $k \in \{1, \ldots, k_{\max}\}$ , the evolution of  $q_{i,k}$  is modeled by the ordinary differential equation

$$\dot{q}_{i,k}(t) = f_{i,k-1}(t) - f_{i,k}(t), \quad q_{i,k}(0) = q_{i,k}^0, \quad f_{i,0}(t) = \bar{f}_i(t),$$
(1.1)

where  $q_{i,k}^0$  is the (known) initial amount of data present in each processor *i* at each stage *k* and  $\bar{f}_i(t)$  is the (known) inflow of unprocessed (or raw) data at processor *i* and time *t*. For  $k \in \{1, \ldots, k_{\max}\}$ ,  $f_{i,k}$  is the flow of data in processor *i* from stage *k* to k+1; its precise form depends of the parameters  $a_i$  and  $\eta$  and is given in [1, equation (4)].<sup>1</sup> For  $i \in \{1, \ldots, i_{\max}\}$  and  $k \in \{1, \ldots, k_{\max}\}$ ,  $Q_{i,k}(t)$  is the total amount of data that by time *t* has traversed the first k-1 stage in processor *i*:

$$Q_{i,k}(t) = \sum_{j=k}^{k_{\max}} q_{i,j}(t) + \int_0^t f_{i,k_{\max}}(s) \mathrm{d}s.$$
(1.2)

This quantity plays an important role in modeling processor throttling by neighbors.

A continuum approximation for (1.1) is derived in the limit of infinitely many processors  $(i_{\max} \to \infty)$  and stages  $(k_{\max} \to \infty)$ , subject to that constraint that  $\eta$  is a fixed, positive, finite, constant. For each  $i \in \{1, \ldots, i_{\max}\}$  and  $k \in \{0, 1, \ldots, k_{\max}\}$ , let

$$x_i = i\delta_x \quad \text{and} \quad z_k = k\delta_z,$$
 (1.3)

where  $\delta_z := 1/k_{\text{max}}$  is the fraction of work completed at each stage and  $\delta_x = 1/i_{\text{max}}$  is the fraction of work to be completed by each processor. Let the data density  $\rho$ , flux F, and processed data Pbe bounded, measurable functions on  $[0, \infty) \times [0, 1) \times [0, 1]$  such that

$$\rho(t, x_i, z_k) = \frac{q_{i,k}}{\delta_x \delta_z}, \qquad F(t, x_i, z_k) = \frac{f_{i,k}(t)}{\delta_x}, \quad \text{and} \quad P(t, x_i, z_k) = Q_{i,k}(t).$$
(1.4)

Further, let  $\rho_0: [0,1) \times [0,1] \to [0,\infty)$ ,  $\overline{F}: [0,\infty) \times [0,1) \to [0,\infty)$ , and  $\alpha: [0,\infty) \times [0,1) \times [0,1] \to [0,\infty)$  be (a.e) bounded measurable functions such that

$$\rho_0(x_i, z_k) = \frac{q_{i,k}^0}{\delta_x \delta_z}, \qquad \bar{F}(t, x_i) = \frac{\bar{f}_i(t)}{\delta_x}, \quad \text{and} \quad \alpha(t, x_i, z_k) = \frac{a_{i,k}(t)}{\delta_x}. \tag{1.5}$$

<sup>&</sup>lt;sup>1</sup>There is another parameter  $\beta \in [0, 1]$  in [1, Section2.1] which measures the effective strength of coupling between process. Here we assume  $\beta = 1$  and do not consider it any further.

Then for any smooth function  $\varphi$  on  $[0,\infty) \times [0,1) \times [0,1]$ , it follows from the ODE model (1.1) that

$$\delta_{z}\delta_{x}\sum_{i=1}^{i_{max}}\sum_{k=1}^{k_{max}}\varphi(t,x_{i},z_{k})\partial_{t}\rho(t,x_{i},z_{k}) - \delta_{x}\sum_{i=1}^{i_{max}}\sum_{k=0}^{k_{max-1}}\left(\varphi(t,x_{i},z_{k+1}) - \varphi(t,x_{i},z_{k})\right)F(t,x_{i},z_{k})$$
$$= \delta_{x}\sum_{i=1}^{i_{max}}\varphi(t,x_{i},0)\bar{F}(t,x_{i}) - \varphi(t,x_{i},1)F(t,x_{i},1) \quad (1.6)$$

In the limit  $\delta_z, \delta_x \to 0$  with  $\eta = \delta_x/\delta_z$  constant, (1.5) and (1.6) converge formally to a weak form of the partial differential equation (PDE)

$$\partial_t \rho(t, x, z) + \partial_z \Phi(\rho(t, x, z), \partial_x P(t, x, z)) = 0, \qquad (t, x, z) \in \mathbb{R}^+ \times \mathbb{T} \times (0, 1), \tag{1.7a}$$

$$\rho(0, x, z) = \rho_0(x, z), \quad (x, z) \in \mathbb{T} \times (0, 1),$$
(1.7b)

$$\Phi(\rho(t,x,0),\partial_x P(t,x,0)) = \overline{F}(t,x), \qquad (t,x) \in \mathbb{R}^+ \times (0,1), \tag{1.7c}$$

where  $\mathbb{R}^+ = (0, \infty)$ ,  $\mathbb{T}$  is the torus parameterized by  $x \in [0, 1)$ , and P and  $\Phi$  are given by

$$P(t, x, z) = \int_{z}^{1} \rho(t, x, y) dy + \int_{0}^{t} \Phi(\rho(s, x, 1), \partial_{x} P(s, x, 1)) ds,$$
(1.8a)

$$\Phi(\rho,\sigma) = \alpha \, w_1 \big( w_2(\rho,\sigma,-\sigma) \big), \tag{1.8b}$$

with

$$w_1(u) = \min\left\{\max\left\{\frac{u}{\rho_*}, 0\right\}, 1\right\}$$
(1.9a)

$$w_2(u, v_1, v_2) = \min\left\{u, \max\{u + \eta v_1, 0\}, \max\{u + \eta v_2, 0\}\right\}.$$
(1.9b)

The functions  $w_1$  and  $w_2$  (see Figure 1) model processor throttling and are arise during the formulation of the microscopic flux  $f_{i,k}$ . First  $w_2$  determines the amount of data available based on the local density and the density of left and right neighbors. Once  $w_2$  is specified,  $w_1$  computes the effective processing speed assuming a linear ramp, where the parameter  $\rho_* > 0$  is the minimum data density needed to operate at full capacity. As in the microscopic model, the parameter  $\eta$  controls degree of throttling due to the lack of available data in neighboring processors.

At first glance, (1.7) appears to include a recursive definition of the flux function  $\Phi$  due to equation (1.8a). However, after integrating (1.7a) over  $(s, z) \in [0, t] \times [0, 1]$  and applying (1.7c) and (1.7b), the result can substituted into (1.8a) to find the following formula for P:

$$P(t,x,z) = \int_0^t \bar{F}(s,x) ds + \int_0^1 \rho_0(x,z) dz - \int_0^z \rho(t,x,z) dz.$$
(1.10)

Alternatively, since  $\partial_z P(t, x, z) = -\rho(t, x, z)$ , (1.7a) can be rewritten as a closed Hamilton-Jacobi equation for P:

$$\partial_t P - \Phi(-\partial_z P, \partial_x P) = 0. \tag{1.11}$$

The reformulation of (1.7a) in terms of (1.11) has been the basis for the analysis and simulation performed in [1]. In the current work, we proceed by analyzing (1.7) directly in terms of  $\rho$ .

## 2 Qualitative properties of the mathematical model

In this section, we investigate qualitative properties of (1.7). We begin by simplifying the expression for  $\Phi$  in (1.8b) using the auxiliary function

$$W_2(s) = \min\{\overline{w}_2(s), \overline{w}_2(-s)\},\tag{2.1}$$





(b)  $w_2(u, v, -v)$  when  $\eta = 1$ ; see (1.9b)



(c) Function  $\overline{w}_2$  defined by equation (2.2).

Figure 1: Shape of the throttling functions used in the definition of the flux  $\Phi$  in (2.3).

where

$$\overline{w}_2(s) = \min\{1, \max\{1 + \eta s, 0\}\};$$
(2.2)

(see Figure 1). In terms of  $w_1$  and  $W_2$ ,  $\Phi$  takes the form

$$\Phi(\rho,\sigma) = \begin{cases} \alpha \ w_1\left(\rho W_2\left(\frac{\sigma}{\rho}\right)\right), & \rho \neq 0\\ 0, & \rho = 0. \end{cases}$$
(2.3)

Next for simplicity, we artificially extend the domain in z to  $[0, \infty)$  and assume initial data with compact support. This assumption allows us to ignore the contribution of the outflow in (1.8a). With this assumption, we modify the definition of P as follows:

**Assumption**: 
$$P(t, x, z) = \int_{z}^{\infty} \rho(t, x, y) dy \quad \forall t \ge 0, x \in \mathbb{T}, z \in [0, \infty).$$
 (2.4)

Hence, in the following we consider the model (1.7) on the extended domain

$$\mathbf{D} := \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^+ \tag{2.5}$$

and introduce the auxiliary variable

$$\sigma(t, x, z) := \partial_x \int_z^\infty \rho(t, x, y) \mathrm{d}y, \quad (t, x, z) \in \mathbf{D}.$$
 (2.6)

The resulting system for  $(\rho, \sigma) : \mathbf{D}^2 \to \mathbb{R}^2$  is then

$$\partial_t \rho + \partial_z \Phi(\rho, \sigma) = 0, \qquad (t, x, z) \in \mathbf{D},$$
 (2.7a)

$$\partial_x \rho + \partial_z \sigma = 0, \qquad (t, x, z) \in \mathbf{D},$$
 (2.7b)

$$\rho(0, x, z) = \rho_0(x, z), \qquad (x, z) \in \mathbb{T} \times \mathbb{R}^+, \tag{2.7c}$$

$$\rho(t, x, 0) = \rho_b(t, x), \ \sigma(t, x, 0) = \sigma_b(t, x), \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{T}.$$
(2.7d)

The relation of the quantities  $(\rho, \sigma)$  to the original model (1.7), modified by the previous Assumption, is as follows:

$$\sigma_b(t,x) = \partial_x \int_0^\infty \rho(t,x,y) dy, \qquad (2.8a)$$

$$\Phi(\rho_b(t,x),\sigma_b(t,x)) = \bar{F}(t,x), \qquad (2.8b)$$

$$\lim_{z \to \infty} \sigma(t, x, z) = 0 \tag{2.8c}$$

**Definition 1** (Weak solution). Given initial data  $\rho_0 \in L^{\infty}(\mathbb{T} \times \mathbb{R}^+)$  and boundary data  $(\rho_b, \sigma_b) \in L^{\infty}(\mathbb{R}^+ \times \mathbb{T})^2$ , we call  $(\rho, \sigma) \in [L^{\infty}(\mathbf{D})]^2$  a weak solution of (2.7) if for all smooth compactly supported functions  $\varphi : \mathbf{D} \to \mathbb{R}$  and  $\psi : \mathbf{D} \to \mathbb{R}$ ,

$$\int_{\mathbf{D}} \rho \,\partial_t \varphi + \Phi(\rho, \sigma) \,\partial_z \varphi \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{T} \times \mathbb{R}^+} \rho_0(x, z) \,\varphi(0, x, z) \,\mathrm{d}z \,\mathrm{d}x \\ + \int_{\mathbb{R}^+ \times \mathbb{T}} \Phi(\rho_b(t, x), \sigma_b(t, x)) \,\varphi(t, x, 0) \,\mathrm{d}x \,\mathrm{d}t = 0,$$
(2.9a)

$$\int_{\mathbf{D}} \rho \,\partial_x \psi + \sigma \,\partial_z \psi \,\mathrm{d}z \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^+ \times \mathbb{T}} \sigma_b(t, x) \,\psi(t, x, 0) \,\mathrm{d}x \mathrm{d}t = 0.$$
(2.9b)

We do not know whether a solution in the sense of Definition 1 exists for (2.7) with general initial and boundary data or whether such a solution is unique and depends continuously on the data.<sup>2</sup> However, for the particular initial and boundary data given in (2.18), we establish the existence of special solutions below.

The weak formulation (2.9) gives rise to a Rankine–Hugoniot jump condition [3]. Consider the surface

$$S := \{(t, x, z) \subset \mathbf{D} : z = \zeta(t, x)\}$$
(2.10)

for a differentiable function  $\zeta : \mathbb{R}_0^+ \times \mathbb{T} \to \mathbb{R}_0^+$  and assume that the functions  $\rho$  and  $\sigma$  satisfy (2.7) point-wise in the interior of **D**\S. Then standard arguments (see for example [5, Section 11.1.1]) can be used to establish that

$$\Phi(\rho_{\ell}, \sigma_{\ell}) - \Phi(\rho_r, \sigma_r) = \partial_t \zeta(t, x) \ (\rho_{\ell} - \rho_r), \qquad (2.11)$$

where

$$\rho_{\ell} = \rho(t, x, \zeta(t, x)^{-}), \qquad \sigma_{\ell} = \sigma(t, x, \zeta(t, x)^{-}), \qquad (2.12)$$

$$\rho_r = \rho(t, x, \zeta(t, x)^+), \qquad \sigma_r = \sigma(t, x, \zeta(t, x)^+).$$
(2.13)

## 2.1 Special solutions for front propagation

In this section, we investigate solutions to (2.7) that take the form of fronts; see Figure 2. Such solutions are important from the point of view of applications.

#### 2.1.1 Existence of Solutions

In the case of a single front, the initial and boundary conditions for  $\rho$  take the form

$$\rho_0(x,z) = rH(\zeta_0(x) - z) \quad \text{and} \quad \rho_b(t,x) = r,$$
(2.14)

where H denotes the Heaviside function and r > 0 is a positive constant. In addition, we enforce the consistency of  $\sigma_b$  as prescribed in (2.8a):

$$\sigma_b(t,x) = r\partial_x \zeta(t,x). \tag{2.15}$$

<sup>&</sup>lt;sup>2</sup>In [1], a continuous vanishing viscosity solution is established using known results from the mathematical literature on Hamilton-Jacobi equations. However, translating such a result to an  $L^1$ -theory for  $\rho$  would require P to be absolutely continuous. While possible, this additional regularity has not yet been determined.



Figure 2: Left: Contour plot of the density  $\rho(t, x, z)$  for a fixed time t. Processed data with constant density r > 0 is depicted in blue up to a stage of completion  $z = \zeta(t, x)$ . Zero data (grey) is prescribed for completion stages  $z > \zeta(t, x)$ , c.f. equation (2.18). Right: Similar plot of a density with constant values  $r_1$  and  $r_2$  and regions separated by functions  $\zeta_1(t, x)$  and  $\zeta_2(t, x)$ .

**Proposition 1.** Let  $\rho_*, \eta > 0$  be positive constants and let  $\alpha = \alpha(t, x) \in C^1(\mathbb{R}^+ \times \mathbb{T})$  be a strictly positive function. Given initial and boundary conditions of the form (2.14) and (2.15), where

$$0 < r < \rho_* \tag{2.16}$$

and  $\zeta_0 \in C^1(\mathbb{T})$  is non-negative, assume that there is a  $C^1$  solution  $\zeta : \mathbb{R}^+_0 \times \mathbb{T} \to \mathbb{R}^+$  that satisfies

$$\partial_t \zeta(t, x) = \frac{\alpha(t, x)}{\rho_*} W_2(\partial_x \zeta(t, x)), \quad \zeta(0, x) = \zeta_0(x).$$
(2.17)

Then there pair  $(\rho, \sigma)$  given by

$$\rho(t, x, z) = rH(\zeta(t, x) - z), \quad \sigma(t, x, z) = rH(\zeta(t, x) - z)\partial_x\zeta(t, x)$$
(2.18)

is a weak solution for (2.7) in the sense of Definition 1 with initial and boundary conditions given in (2.14) and (2.15)

Before proving this lemma, some remarks are in order.

1. The condition in (2.16) guarantees that  $\frac{r}{\rho_*}W_2(\frac{\sigma}{r}) < 1$ . In this case the flux function  $\Phi$  (c.f. (2.3)) is given by the simplified formula

$$\Phi(\rho,\sigma) = \begin{cases} \frac{\alpha\rho}{\rho_*} W_2\left(\frac{\sigma}{\rho}\right), & \rho \neq 0, \\ 0, & \rho = 0. \end{cases}$$
(2.19)

This formula and the Rankine–Hugoniot condition in (2.11) together form the key components of the existence proof.

- 2. While the condition  $r < \rho_*$  is sufficient to obtain (2.19), it is not necessary. Indeed  $W_2(\sigma/\rho) = 0$  for  $\sigma$  sufficiently large; in particular, (2.19) holds at any (x,t) such that  $\partial_x \zeta(t,x) > \eta^{-1}$ .
- 3. It is currently open as to whether equation (2.17) has a  $C^1$  solution. However, since  $\alpha, \rho_*$ , and  $W_2$  are non-negative, it is clear that any such solution will be non-decreasing. This fact is consistent with the notion that data is always processed toward completion.

4. A result similar to Proposition 1 can be obtained for initial data separated by a finite number of non-intersecting fronts. Consider for example the case of two fronts (see Figure 2-b):

$$\rho_0(x,z) = r_1 H \big(\zeta_{1,0}(x) - z\big) + r_2 \big[ H \big(\zeta_{2,0}(x) - z\big) - H \big(\zeta_{1,0}(x) - z\big) \big]$$
(2.20)

with positive constants  $r_1 \neq r_2$  and  $r_1 < \rho_*$  and  $r_2 < \rho^*$ . If  $0 < \zeta_{1,0}(x) < \zeta_{2,0}(x)$  for all  $x \in \mathbb{T}$ , then there is a weak solution  $(\rho, \sigma)$  in the sense of Definition 1 of the form

$$\rho(t, x, z) = (r_1 - r_2)H(\zeta_1(t, x) - z) + r_2H(\zeta_2(t, x) - z)$$
(2.21a)

$$\sigma(t, x, z) = (r_1 - r_2)\partial_x \zeta_1(t, x) H(\zeta_1(t, x) - z)$$
(2.21b)

$$+ r_2 \partial_x \zeta_2(t, x) H \big( \zeta_2(t, x) - z \big), \qquad (2.21c)$$

provided  $\zeta_1(0, x) = \zeta_{1,0}(x), \ \zeta_2(0, x) = \zeta_{2,0}(x)$ , and

$$\partial_t \zeta_1(t,x) = \frac{\alpha(t,x)}{\rho_*(r_2 - r_1)} \Big[ r_2 W_2 \Big( \partial_x \zeta_2(t,x) \Big) - r_1 W_2 \Big( \Big( \frac{r_1 - r_2}{r_1} \Big) \partial_x \zeta_1(t,x) + \frac{r_2}{r_1} \partial_x \zeta_2(t,x) \Big) \Big],$$
(2.22a)

$$\partial_t \zeta_2(t,x) = \frac{\alpha(t,x)}{\rho_*} W_2\Big(\partial_x \zeta_2(t,x)\Big). \tag{2.22b}$$

Proof of Lemma 1. With the  $C^1$  assumption on  $\zeta$ ,  $(\rho, \sigma)$  given by equation (2.18) clearly belongs to  $L^{\infty}(\mathbf{D})^2$ . By construction the given solution (2.18) fulfills boundary and initial condition pointwise and hence also in weak form. It therefore remains to verify that  $(\rho, \sigma)$  is a weak solution in the interior of  $\mathbf{D}$ .

We first verify that  $(\rho, \sigma)$  fulfills (2.9b). For  $\psi$  smooth and compactly supported in the interior of **D**, we need to show that:

$$\int_{\mathbf{D}} \left[ rH(\zeta(t,x)-z)\partial_x\psi(t,x,z) + rH(\zeta(t,x)-z)\partial_x\zeta(t,x)\partial_z\psi(t,x,z) \right] \mathrm{d}z\mathrm{d}x\mathrm{d}t = 0.$$
(2.23)

By definition of the Heaviside function,

$$\begin{split} \int_{\mathbf{D}} \left[ rH(\zeta(t,x)-z)\partial_x\psi(t,x,z) + rH(\zeta(t,x)-z)\partial_x\zeta(t,x)\partial_z\psi(t,x,z) \right] \mathrm{d}z\mathrm{d}x\mathrm{d}t \\ &= r\int_{\mathbb{R}^+_0\times\mathbb{T}} \int_0^{\zeta(t,x)} \left[ \partial_x\psi(t,x,z) + \partial_x\zeta(t,x)\partial_z\psi(t,x,z) \right] \mathrm{d}z\mathrm{d}x\mathrm{d}t \\ &= r\int_{\mathbb{R}^\times\mathbb{T}} \left[ \int_0^{\zeta(t,x)} \partial_x\psi(t,x,z)\mathrm{d}z + \partial_x\zeta(t,x)\psi(t,x,\zeta(t,x)) \right] \mathrm{d}x\mathrm{d}t \\ &= r\int_{\mathbb{R}^+\times\mathbb{T}} \frac{\partial}{\partial x} \left( \int_0^{\zeta(t,x)} \psi(t,x,z)\mathrm{d}z \right) \mathrm{d}x\mathrm{d}t = 0, \quad (2.24) \end{split}$$

where the integral in the last line vanishes due to periodicity of the domain with respect to x.

We next verify that  $(\rho, \sigma)$  fulfills (2.9a). Let  $S = \{(t, x, z) \subset \mathbf{D} : z = \zeta(t, x)\} \subset \mathbf{D}$  be the surface of the front described by  $\zeta$ , and let  $\mathbf{S}_+ := \{(t, x, z) \subset \mathbf{D} : z > \zeta(t, x)\}$  and  $\mathbf{S}_- := \{(t, x, z) \subset \mathbf{D} : z < \zeta(t, x)\}$ . For  $z < \zeta(t, x)$  the explicit form of  $(\rho, \sigma)$  implies the following relation

$$\frac{\sigma(t,x,z)}{\rho(t,x,z)} = \partial_x \zeta(t,x).$$
(2.25)

Hence according to (2.19),

$$\Phi(\rho(t,x,z),\sigma(t,x,z)) = \begin{cases} \frac{\alpha(t,x)r}{\rho_*} W_2(\partial_x \zeta(t,x)) & (t,x,z) \in \mathbf{S}_-\\ 0 & (t,x,z) \in \mathbf{S}_+. \end{cases}$$
(2.26)

Let  $\varphi$  be any smooth, compactly supported function in the interior of **D**. In  $\mathbf{S}_{\pm}$ ,  $\Phi(\rho, \sigma)$  and  $\rho$  are constant, and in  $\mathbf{S}_{+}$ , they both vanish. Hence,

$$\int_{\mathbf{D}} \rho \,\partial_t \varphi + \Phi(\rho, \sigma) \,\partial_z \varphi \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbf{S}_-} r \,\partial_t \varphi + \frac{\alpha r}{\rho_*} W_2(\partial_x \zeta) \,\partial_z \varphi \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}t$$
$$= \int_{\mathbb{T}} \int_{\{(t,z):z=\zeta(t,x)\}} \varphi \left( -r \partial_t \zeta + \frac{\alpha r}{\rho_*} W_2(\partial_x \zeta) \right) \frac{1}{\|(1,\partial_t \zeta)\|} \mathrm{d}A(t,z) \mathrm{d}x, \quad (2.27)$$

where dA(t, z) denotes the surface measure on the curve  $\{(t, z) : z = \zeta(t, x)\}$ . The final expression above vanishes the definition of  $\zeta$  given in (2.17). This finishes the proof.

#### 2.1.2 Properties of fronts and their associated weak solutions

It is the equation for the front  $\zeta$  in (2.17) that essentially determines the behavior of the special solutions (2.18). We assume again that  $0 < r < \rho_*$ . With the definition of  $W_2$  in (2.1), this equation takes the form

$$\partial_t \zeta(t,x) = \frac{\alpha(t,x)}{\rho_*} \times \begin{cases} 1 - \eta \, |\partial_x \zeta(t,x)|, & |\partial_x \zeta(t,x)| < \eta^{-1}, \\ 0, & |\partial_x \zeta(t,x)| \ge \eta^{-1}. \end{cases}$$
(2.28)

In general, we do not know if there exists a solution to (2.28), due to the discontinuous flux. However, we may consider particular solutions related to special initial conditions  $\zeta_0(x)$ . Those special solutions are used as test cases for numerical simulations in the next section.

• If for all  $(t, x) \in \mathbb{R}^+ \times \mathbb{T}$ ,  $|\partial_x \zeta(t, x)| < \eta^{-1}$  and  $\alpha(t, x) = \alpha$  is constant, then formally  $\partial_x \zeta(t, x)$  fulfills the conservation law

$$\partial_t(\partial_x\zeta)(t,x) + \frac{\alpha\eta}{\rho_*}\partial_x|(\partial_x\zeta)(t,x)| = 0, \quad (\partial_x\zeta)(0,x) = \partial_x\zeta_0(x) \tag{2.29}$$

with flux function f(u) = C|u| and  $C = \frac{\alpha \eta}{\rho_*}$ . It is known that there exists a weak (entropic) solution to (2.29) as long as  $\partial_x \zeta_0(x)$  is in  $L^{\infty}$ , has locally bounded variation, and is continuous from the left [2, Proposition 3.1]. Moreover, if  $\partial_x \zeta_0(x)$  is piecewise constant, this solution is given by piecewise constant states separate by a finite number of traveling discontinuities [2, Lemma 3.2]. We investigate such piece-wise solutions in Examples 2 and 3 in the numerical results of Section 3.

• If  $\alpha$  is constant, and if  $0 \leq \partial_x \zeta_0(x) < \eta^{-1}$  locally in x, then for sufficiently small t,

$$\zeta(t,x) = \zeta_0 \left( x - \eta \frac{\alpha}{\rho_*} t \right) + \frac{\alpha}{\rho_*} t.$$
(2.30)

Similarly if  $-\eta^{-1} < \partial_x \zeta_0(x) \le 0$  locally, then for sufficiently small t,

$$\zeta(t,x) = \zeta_0 \left( x + \eta \, \frac{\alpha}{\rho_*} \, t \right) + \frac{\alpha}{\rho_*} \, t. \tag{2.31}$$

We use these formulas to make comparisons with numerical results in Examples 2-4 in Section 3.

• If locally  $|\partial_x \zeta_0(x)| \ge \eta^{-1}$ , then  $\zeta(t, x) = \zeta_0(x)$  for t sufficiently small. In this case the model predicts that all processors are stalled. However, the condition on  $\zeta_0$  cannot hold globally due to the periodicity assumption in x. We explore the behavior of the model for this scenario in Example 5.

## **3** Numerical simulations

In this section, we present numerical simulations that illustrate the behavior of special front-type solutions discussed in the previous section.

## 3.1 Discretization

Since the system in (2.7) is of mixed type, we approximate it by the following relaxed system of equations

$$\partial_t \rho + \partial_z \Phi(\rho, \sigma) = 0 \tag{3.1a}$$

$$\varepsilon \partial_t \sigma + \partial_x \rho + \partial_z \sigma = 0, \tag{3.1b}$$

where  $\epsilon > 0$  is a small parameter. Formally, (2.7a) and (2.7b) are obtained from (3.1) in the limit as  $\epsilon \to 0$ .

Before discretizing (3.1), we investigate hyperbolicity in the simplified case that  $\alpha(t, x) > 0$  is constant.<sup>3</sup> In this case, (3.1) takes the non-conservative form

$$\partial_t U + B^x(U)\partial_x U + B^z(U)\partial_z U = 0, \qquad (3.2)$$

where  $U = (\rho, \sigma)$  and

$$B^{x}(U) = \begin{pmatrix} 0 & 0 \\ 1 \\ \epsilon & 0 \end{pmatrix}, \quad B^{z}(U) = \begin{pmatrix} \partial_{\rho} \Phi & \partial_{\sigma} \Phi \\ 0 & \frac{1}{\epsilon} \end{pmatrix}.$$
(3.3)

Under the assumption (2.16) and provided that  $\rho > 0$ , the simplified form of  $\Phi$  in (2.19) implies that

$$\partial_{\rho}\Phi(\rho,\sigma) = \frac{\alpha}{\rho_*} (W_2(s) - sW_2'(s)), \quad \partial_{\sigma}\Phi(\rho,\sigma) = \frac{\alpha}{\rho_*} W_2'(s), \tag{3.4}$$

where  $s = \sigma / \rho$ .

The system (3.2) is hyperbolic if for any  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$  the matrix  $B(U, \xi) = \xi_1 B^x(U) + \xi_2 B^z(U)$  is diagonalizable in the field of real numbers [5]. Because the function  $W_2$  is not smooth,  $B(U,\xi)$  is not differentiable when s = 0 or  $s = \pm 1/\eta$ . However, away from these points,

$$B(U,\xi) = \begin{cases} \begin{pmatrix} 0 & 0\\ \frac{\xi_1}{\epsilon} & \frac{\xi_2}{\epsilon} \end{pmatrix}, & |s| > \frac{1}{\eta}, \\ \\ \left[ \begin{pmatrix} \xi_2 \frac{\alpha}{\rho_*} & -\xi_2 \operatorname{sgn}(s)\eta \frac{\alpha}{\rho_*}\\ \frac{\xi_1}{\epsilon} & \frac{\xi_2}{\epsilon} \end{pmatrix} \end{pmatrix}, & 0 < |s| < \frac{1}{\eta}, \end{cases}$$
(3.5)

and the eigenvalues of  $B(U,\xi)$  are

$$(\lambda_1, \lambda_2) = \begin{cases} \left(0, \frac{\xi_2}{\epsilon}\right), & |s| > \frac{1}{\eta}, \\ (\lambda_1^*, \lambda_2^*), & 0 < |s| < \frac{1}{\eta}, \end{cases}$$
(3.6)

where  $(\lambda_1^*, \lambda_2^*)$  are the roots of the characteristic polynomial

$$p(\lambda) = \lambda^2 - \xi_2 \left(\frac{\alpha}{\rho_*} + \frac{1}{\epsilon}\right) \lambda + \frac{\xi_2 \alpha}{\epsilon \rho_*} \left(\xi_2 + \operatorname{sgn}(s)\xi_1 \eta\right).$$
(3.7)

<sup>&</sup>lt;sup>3</sup>The hyperbolicity properties of (2.7a) will not change if  $\alpha$  is space or time–dependent, provided it is sufficiently smooth.



Figure 3: Domain of hyperbolicity of system (3.1). In both cases, the non-hyperbolic region (gray) is bounded by the  $\xi_1$ -axis and by  $\xi_2 = \pm C\xi_1$  with slope  $C = \frac{4\epsilon\alpha\rho_*\eta}{(\epsilon\alpha-\rho_*)^2} \to 0$  as  $\epsilon \to 0^+$ .

For  $|s| > \eta^{-1}$ , the system is strictly hyperbolic. For  $0 < |s| < \eta^{-1}$ , the polynomial p in (3.7) has discriminant

$$d_{\epsilon}(\xi_1,\xi_2) = \left(\frac{\xi_2}{\epsilon\rho_*}\right)^2 \left[ (\epsilon\alpha - \rho_*)^2 - 4\operatorname{sgn}(s)\epsilon\alpha\rho_*\eta\frac{\xi_1}{\xi_2} \right]$$
(3.8)

which, for  $\xi_2 \neq 0$ , is negative (so that  $\lambda_1^*$  and  $\lambda_2^*$  have non-zero imaginary component) if and only if

$$\operatorname{sgn}(s)\frac{\xi_1}{\xi_2} > \frac{(\epsilon\alpha - \rho_*)^2}{4\epsilon\alpha\rho_*\eta}.$$
(3.9)

For any  $\epsilon > 0$ , the non-hyperbolic region  $\{(\xi_1, \xi_2) : d_{\epsilon}(\xi_1, \xi_2) < 0\}$  is nontrivial; see Figure 3. However, for any fixed  $(\xi_1, \xi_2)$ , there exists an  $\epsilon^*(\xi_1, \xi_2)$  small enough that  $d_{\epsilon}(\xi_1, \xi_2) \ge 0$  for all  $\epsilon < \epsilon^*(\xi_1, \xi_2)$ . Hence as  $\epsilon to0^+$ , the non-hyperbolic region vanishes. The numerical method below is constructed by discretizing the relaxation system in (3.1) and then setting  $\epsilon = 0$ . This fact partially justifies the use of (3.1) even though it is not everywhere hyperbolic when  $\epsilon > 0$ .

To derive a numerical scheme for (3.1), we first discretize in time with a first-order method. We are eventually interested in the  $\epsilon \to 0$  limit of the fully discretized scheme; hence the equation for  $\rho$  in (3.1a) is treated explicitly, but the equation for  $\sigma$  in (3.1b) is treated implicitly. Let  $\rho^n(x,z) \simeq \rho(n\Delta t, x, z)$  and  $\sigma^n(x,z) \simeq \sigma(n\Delta t, x, z)$  be the discrete approximations of  $\rho$  and  $\sigma$ , respectively, at time  $n\Delta t$ . We set

$$\rho^{n+1} = \rho^n - \Delta t \,\partial_z \Phi(\rho^n, \sigma^n), \tag{3.10a}$$

$$\sigma^{n+1} = \sigma^n - \frac{\Delta t}{\varepsilon} \left( \partial_x \rho^{n+1} + \partial_z \sigma^{n+1} \right).$$
(3.10b)

To discretize (3.10) in x and z, we introduce a bounded computational domain  $(x, z) \in [0, 1) \times [0, 1]$ , which is divided into  $N^x \times N^z$  uniform cells of size  $\Delta x \times \Delta z$ . The cell centers are denoted by  $(x_i, z_j)$ , for  $i \in \{1, \ldots, N^x\}$  and  $j \in \{1, \ldots, N^x\}$ ; and the cell edges are denoted by  $x_{i+\frac{1}{2}} = (i + \frac{1}{2})\Delta x$  and  $z_{j+\frac{1}{2}} = (j + \frac{1}{2})\Delta z$ , respectively, for  $i \in \{0, \ldots, N^x\}$  and  $j \in \{0, \ldots, N^z\}$ . The approximate cell averages of  $\rho^n$  and  $\sigma^n$  on the cell centered at  $(x_i, z_j)$  are denoted by

$$\overline{R}_{ij}^{n} \simeq \frac{1}{\Delta x \Delta z} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{z_{j-1/2}}^{z_{j+1/2}} \rho^{n}(x, z) \mathrm{d}x \mathrm{d}z$$
(3.11)

and

$$\overline{S}_{ij}^{n} \simeq \frac{1}{\Delta x \Delta z} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{z_{j-1/2}}^{z_{j+1/2}} \sigma^{n}(x, z) \mathrm{d}x \mathrm{d}z, \qquad (3.12)$$

respectively.

We use a Lax-Friedrichs approximation of  $\partial_z \Phi$  in (3.10a) and in (3.10b) a centered discretization for  $\partial_x \rho$  and a one-sided discretization of  $\partial_z \sigma$ . The resulting fully discrete scheme is

$$\overline{R}_{ij}^{n+1} = \overline{R}_{ij}^n - \frac{\Delta t}{\Delta z} \left( F_{i,j+1/2}^n - F_{i,j-1/2}^n \right) + \frac{1}{4} \left( \overline{R}_{i+1,j}^n - 2\overline{R}_{ij}^n + \overline{R}_{i-1,j}^n \right),$$
(3.13a)

$$\overline{S}_{ij}^{n+1} = \overline{S}_{ij}^n - \frac{\Delta t}{2\epsilon\Delta x} \left( \overline{R}_{i+1,j}^{n+1} - \overline{R}_{i-1,j}^{n+1} \right) - \frac{\Delta t}{\epsilon\Delta z} \left( \overline{S}_{i,j+1}^{n+1} - \overline{S}_{ij}^{n+1} \right),$$
(3.13b)

where

$$F_{i,j+\frac{1}{2}}^{n} = \frac{1}{2} \left( \Phi(\overline{R}_{i,j+1}^{n}, \overline{S}_{i,j+1}^{n}) + \Phi(\overline{R}_{ij}^{n}, \overline{S}_{ij}^{n}) - a(\overline{R}_{i,j+1}^{n} - \overline{R}_{ij}^{n}) \right).$$
(3.14)

The monotonicity of the numerical flux  $F^n_{i,j+\frac{1}{2}}$  is guaranteed provided that

$$\max_{(R,S)} \left\| \partial_{\rho} \Phi(R,S) \right\| =: a \le \frac{\Delta z}{\Delta t}.$$
(3.15)

This condition motivates an adaptive choice of  $\Delta t$ : At each time level  $t^n$  we set  $\Delta t = \frac{\Delta z}{a^n}$  for

$$a^{n} = \max_{(i,j):1 \le i \le N^{x}, 1 \le j \le N^{z}} \left\| \partial_{\rho} \Phi(\overline{R}^{n}_{ij}, \overline{S}^{n}_{ij}) \right\|.$$
(3.16)

The right-biased stencil in the discretization of  $\partial_z \sigma$  in equation (3.13b) deserves some discussion. Indeed, given that the boundary condition for  $\sigma$  is given at z = 0, it seems more natural to use a left-biased stencil. However, such an approach does not allow the boundary condition to be computed implicitly via the consistency relation in (2.8a). However, if  $\rho$  has compact support, then (2.8c) holds. As long as the support of the solution does not reach the boundary of the computational domain at z = 1, this condition can be enforced there, in an implicit fashion. This approach is consistent with the fact that  $\sigma$  at a given  $z_*$  is determined entirely by  $\rho$  at values of  $z > z_*$ . Moreover, numerical calculations confirm that enforcing the boundary condition in this way yields the most stable results.

The final scheme used in Section 3.2 is given by the formal limit of (3.13) in the limit  $\varepsilon \to 0$ :

$$\overline{R}_{ij}^{n+1} - \overline{R}_{ij}^n = -\frac{\Delta t}{\Delta z} \left[ F_{i,j+1/2}^n - F_{i,j-1/2}^n \right] + \frac{1}{4} \left( \overline{R}_{i+1,j}^n - 2\overline{R}_{ij}^n + \overline{R}_{i-1,j}^n \right),$$
(3.17a)

$$\overline{S}_{ij}^{n+1} = \frac{\Delta z}{2\Delta x} \left( \overline{R}_{i+1,j}^{n+1} - \overline{R}_{i-1,j}^{n+1} \right) + \overline{S}_{i,j+1}^{n+1}.$$
(3.17b)

This scheme above must be accompanied by boundary and initial conditions. The initial values  $\overline{R}_{ij}^0$  is obtained by integration of the initial data  $\rho_0$ :

$$\overline{R}_{ij}^{0} = \frac{1}{\Delta x \Delta z} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{z_{j-1/2}}^{z_{j+1/2}} \rho_0(x, y) \mathrm{d}x \mathrm{d}z.$$
(3.18)

The cell averages  $\overline{S}_{ij}^0$  are obtained by applying a midpoint rule to the consistency relation (2.8a):

$$\overline{S}_{ij}^{0} = \frac{1}{\Delta x} \sum_{\ell=j}^{N^{z}} \rho_{0}(x_{i+1/2}, z_{j}) - \rho_{0}(x_{i-1/2}, z_{j}).$$
(3.19)

For  $\overline{S}_{ij}^n$  zero boundary conditions are prescribed at  $j = N^z$  consistent with the right-based stencil in the discretization of  $\partial_z \sigma$ :

$$\overline{S}_{i,N^z}^n = 0. aga{3.20}$$

For  $\overline{R}_{ij}^n$  boundary conditions at z = 0 are given by  $\rho_{bc}(t)$ . For the special solutions discussed in the previous section  $\rho_{bc}(t) = r$  and hence

$$\overline{R}_{i,1}^n = r. \tag{3.21}$$

Due to the central discretization of  $\partial_z \rho$ , boundary conditions for  $\overline{R}_{ij}^n$  at  $j = N^z$  need to be prescribed. Since in the simulation no data reaches this boundary we implement Neumann boundary conditions

$$\overline{R}_{i,N^z}^n = \overline{R}_{i,N^z-1}^n. \tag{3.22}$$

#### 3.2 Simulation results

**Example 1: validation with the microscopic model.** We consider the test problem introduced in [1] which is used to to validate the macroscopic model (2.7) and the numerical method (3.17) by comparing with a simulation of the microscopic model introduced in [1]. For this problem,  $\rho_* = 1.0$  and  $\eta = 1.0$ . The initial and boundary conditions are given by

$$\rho_0(x,z) = 1.5 \left( \sin(2\pi z) \right)^6 \chi_{[0,0.5]}(z), \quad \rho_b(t,x) = 0, \quad \sigma_b(t,x) = 0, \quad (3.23)$$

and the processor speed is

$$\alpha(x) = 1 - 0.4(\sin(\pi x))^2. \tag{3.24}$$

Both models are simulated up to a final time  $T_{\text{fin}} = 0.5$ .

The (x, z) domain for the macroscopic model is discretized using  $N^x = N^z = 800$  cells, and simulated with the algorithm in (3.17). In order to be consistent with the choice of  $\eta$ , the microscopic model uses  $i_{\text{max}} = 800$  processors and  $k_{\text{max}} = 800$  stages. The model is integrating in time using a forward Euler method.

In Figure 4 we show the initial density and the final density profiles for both models at time  $T_{\rm fin}$ . We observe qualitative agreement between the microscopic and the macroscopic solutions. In both cases, the data around x = 0.5 is processed more slowly than the data in the rest of the domain, due to a slower processing rate  $\alpha$  there.



In the remaining examples of this subsection, we explore the behavior of front-type solutions. Unless otherwise stated the following parameters are used in all simulations:

$$r = 0.5, \quad \rho_* = 0.8, \quad \alpha = 0.1, \quad T_{\text{fin}} = 2.0.$$
 (3.25)

The value of  $T_{\text{fin}}$  ensures that data does not reach z = 1; in particular the assumption in (2.4) holds. These parameter choices, along with the initial and boundary conditions ensure that

$$0 \le \rho(t, x, z) < \rho_*, \quad \text{for all } (t, x, z) \in \mathbf{D}, \tag{3.26}$$

in which case the flux  $\Phi$  is given by the simplified formula in (2.19).

**Example 2: front with constant profile.** We consider the case of an initial constant front, namely

$$\zeta_0(x) = \zeta_0 = 0.2, \quad \forall x \in \mathbb{T}, \tag{3.27}$$

which provides a trivial example for a piece-wise constant solution  $\partial_x \zeta$  to (2.29). Moreover, the evolution of this front is given explicitly in (2.30) and (2.31):

$$\zeta(t,x) = \zeta_0 + \frac{\alpha}{\rho_*}t. \tag{3.28}$$

In this example  $\eta$  is set to 0.5 although the solution in (3.28) is independent of this choice.

Numerical results are shown in Figure 5. Since the processing rate  $\alpha$  is constant the front moves with the same speed for each  $x \in [0, 1)$ , as expected. Moreover, the analytical and the numerical solutions provide the same position of the front at final time.

**Example 3:** V-shaped front with small  $\eta$ . We consider an initial condition characterized by a V-shape front under the condition that  $\eta |\partial_x \zeta| < 1$ ; cf. (2.28). This models the physical situation in which processors are not initially at the same stage of completion. Such a situation can be generated if a local group of processors is, at some previous time, slower than the rest [1]. Specifically, we let

$$\zeta_0(x) = (1 - 2z_0) |x - 0.5| + z_0 \tag{3.29}$$

where  $z_0 = \zeta_0(0.5) < 0.5$  is a fixed parameter.

Based on the theoretical findings in [2] for (2.29), we expect that the front at later times will be piecewise linear. Moreover, the formulas in (2.30) and (2.31) can be used to generate the local solution analytically. Indeed, under the assumption  $\eta |\partial_x \zeta| < 1$  a short calculation shows that the local, short time solution away from x = 0.5 is given by

$$\zeta_{\rm loc}(t,x) = \zeta_0(x) + \frac{\alpha}{\rho_*} \Big( 1 - \eta |\partial_x \zeta_0| \Big) t.$$
(3.30)

where  $|\partial_x \zeta_0| = (1 - 2z_0)$ . However at x = 0.5, Proposition 1 does not apply, since the front is not  $C^1$  there; hence (2.30) and (2.31) are not valid. We conjecture that at this point, the front moves forward at full speed  $\alpha/\rho_*$  and, as it catches up with neighboring points in the front, these points also move forward at full speed. This means that the global front solution for the front is given by

$$\zeta(t,x) = \max\left\{\zeta_{\text{loc}}(t,x), z_0 + \frac{\alpha}{\rho_*}t, \right\},\tag{3.31}$$

In Figure 6 we show numerical results for this test problem based on the algorithm in (3.17). We use a grid with  $N^x = N^z = 800$  computational cells and simulate the solution to a time horizon  $T_{\text{fin}} = 2$ . We set  $z_0 = 0.2$  (so that  $|\partial_x \zeta_0(x)| = 0.6$ ) and let  $\eta = 1/1.1$ . It is clear from these plots that the analytic solution (3.31) and the numerical solution provide the same position of the front.

**Example 4: Smooth front with small**  $\eta$ . We repeat the test problem above, except now the initial front is given by a smooth function:

$$\zeta_0(x) = 0.25\cos(2\pi x) + 0.3. \tag{3.32}$$

In order to fulfill the condition  $\eta |\partial_x \zeta| < 1$  we choose  $\eta^{-1} = \max_x \zeta_0(x) + 1 = \pi/2 + 1$ .

![](_page_14_Figure_0.jpeg)

Figure 5: Density profiles based on the a simulation of the constant front solution (3.28) with the scheme in (3.17). The simulations in panels (a)-(c) use  $N^x = N^z = 800$  computational cells, while in panel (d), numerical solutions are computed at different refinement levels. In the top row, the analytical front is marked by circles.

In Figure 7 we show numerical results for this test problem based on the algorithm in (3.17). We use a grid with  $N^x = N^z = 800$  computational cells and simulate the solution to a time horizon  $T_{\text{fin}} = 2$ . As in the previous example, (3.31) can be used to evaluate the location of the front analytically. The results in Figure 7 demonstrate that the numerical and analytical solutions agree even when  $\partial_x \zeta_0$  is not piece-wise constant.

**Example 5:** V-shaped front with large  $\eta$ . We consider again the initial condition (3.29) characterized by a V-shape front. The model parameters are the same as in Example 3, except that  $\eta$  is now large enough to ensure that  $\eta |\partial_x \zeta| > 1$ . In particular  $|\partial_x \zeta| = 0.6$ , but  $\eta = 10$ . In this case, we expect stalling for those processors  $x \neq 0.5$ ; see (2.28). Again at x = 0.5 the solution is not described by Lemma 1, since there the front is not  $C^1$ . We expect that the front at this point will proceed at the maximum speed  $\alpha/\rho_*$  and that the stalled points away front x = 0.5 will also begin to move at this rate after they have been overtaken. In other words, we conjecture that the

![](_page_15_Figure_0.jpeg)

Figure 6: Density profiles based on the simulation of the V-shaped initial front (3.29) with the scheme in (3.17) using  $N^x = N^z = 800$ . The analytic solution is computed using (3.31). In the top row, the analytical front is marked by circles.

global formula for the solution front is

$$\zeta(t,x) = \max\left\{\zeta_0(x), \, z_0 + \frac{\alpha}{\rho_*}t\right\}.$$
(3.33)

In Figure 8 we show numerical results for a grid with  $N^x = N^z = 800$  computational cells and a time horizon  $T_{\text{fin}} = 2$ . These results agree with the analytical solution in (3.33).

#### **3.3** Variations in the processor rate.

The ultimate purpose of the discrete model in (1.1) is to understand how variations in the processor rate affect computer performance, and whether these rates can be controlled to achieve a prescribed objective, such as faster throughput. For the continuum model in (2.7), such a control is specified via the function  $\alpha$ . In this section, we briefly explore the effects of adapting  $\alpha$  with the intent of a more extensive study in later work.

We assume that for all t,

$$\int_{\mathbb{T}} \alpha(t, x) \mathrm{d}x = \bar{\alpha} \tag{3.34}$$

in order to normalize the control action. Since  $\alpha$  is non-negative, so too is  $\bar{\alpha}$ ; we assume further that  $\bar{\alpha} > 0$ . Then several choices for  $\alpha$  can be envisioned. The simplest choice is to set  $\alpha$  equal to a constant. In this case, the normalization in (3.34) implies that  $\alpha = \bar{\alpha}$ . For a front-type solution, this simple choice will be compared with a policy that depends on initial data

$$\rho_0(x,z) = rH(\zeta_0(x) - z), \quad r < \rho_*, \quad \text{and} \quad \zeta_0(x) > 0,$$
(3.35)

![](_page_16_Figure_0.jpeg)

Figure 7: Density profiles based on the simulation of the V-shaped initial front in (3.29), using the scheme in (3.17) with  $N^x = N^z = 800$ . The analytic solution is computed using (3.31). In the top row, the analytical front is marked by circles.

Since the slope  $\partial_x \zeta(t, x)$  controls the throttling due to neighboring processors, the condition  $\partial_x \zeta(t, x) = 0$  is desirable.

We therefore propose a choice for  $\alpha$  that gives priority to processors that are trailing in the initial configuration. Specifically, we consider

$$\alpha(x) = C_{\alpha} \rho_* \left( \zeta_{\max} - \zeta_0(x) \right), \qquad (3.36)$$

where  $\zeta_{\max} > \max_{x \in \mathbb{T}} \zeta_0(x)$  is a fixed constant and  $C_{\alpha}$ , which depends on  $\zeta_0$ , is chosen so that (3.34) holds. This choice of  $\alpha$  will increase the speed of trailing processors

In order to compare the temporal and spatial evolution of  $\rho$  based on the policy (3.36) versus the constant  $\alpha = \bar{\alpha}$  policy, we compute some quantities of interest

$$\omega_1(t,z) := \int_0^t \int_{\mathbb{T}} \Phi(\rho(s,x,z), \sigma(s,x,z)) dx ds,$$
  

$$\omega_2(t,z) := \int_{\mathbb{T}} \Phi(\rho(t,x,z), \sigma(t,x,z) dx,$$
  

$$\omega_3(t,z) := \int_0^t \int_{\mathbb{T}} \rho(s,x,z) dx ds.$$
(3.37)

The quantity  $\omega_1$  measures the cumulative outflow at stage of completion z up to time t; the quantity  $\omega_2$  measures the current outflow at stage of completion z; and the quantity  $\omega_2$  measures the cumulative part density at stage of completion z. The first two quantities are indicators of the processed data whereas the last indicator shows the load of the processors, which may be related to the consumption of energy during computation. Larger values of  $\omega_1$  are preferable.

![](_page_17_Figure_0.jpeg)

Figure 8: Density profiles based on the simulation of the V-shaped initial front (3.29) with the scheme in (3.17) using  $N^x = N^z = 800$ . The analytical solution is computed using (3.33). In the top row, the analytical front is marked by circles.

![](_page_17_Figure_2.jpeg)

Figure 9: Density profiles for two different policies choices of  $\alpha$  with the scheme in (3.17) using  $N^x = N^z = 800$ .

For our numerical tests, we set  $T_{\text{fin}} = 1$  and  $\bar{\alpha} = 0.5$ . The initial front  $\zeta_0(x)$  is given as in (3.29) with  $z_0 = 0.1$ , in which case  $C_{\alpha} \approx 0.5/0.575$ . The quantities of interest are evaluated for z = 0.5 and z = 0.75. Figure 9 shows the initial density profile and the profiles at  $T_{\text{fin}} = 1$  for each of the two policies. For the policy based on (3.36), the control has been been active long enough to push the processors that were initially trailing ahead of the others. This means the control has been on for too long and should have been modified or turned off at an earlier time. While the results of the two policies appear similar, the quantities of interest  $w_i(t)$ ,  $i = 1, \ldots, 3$  displayed in Figure 10 demonstrate the differences. In particular, with respect to  $w_1$ , the policy in (3.36) outperforms the constant  $\alpha$  policy.

![](_page_18_Figure_0.jpeg)

Figure 10: Comparison of the quantities of interest  $w_i$ , i = 1, 2, 3 given in (3.37).

## 4 Summary

In this paper, we have analyzed a recently derived mathematical model for the evolution of processed data in large-scale, asynchronous computers [1]. After a suitable reformulation, the model is expressed as a system of partial differential equations of mixed type. It is possible to prove existence of (discontinuous) solutions to this system for a particular class of initial and boundary conditions. These solutions take the form of moving fronts. A numerical scheme for the reformulated system has also been designed based on a relaxation approximation. Numerical simulations based on this scheme demonstrate qualitative agreement with theoretical findings.

We have also briefly explored the effects of local processor speed on quantities of interest predicted by the model. In future work, we intend to investigate more extensively control mechanisms for optimizing important objectives related to performance of the large-scale computers.

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