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MEAN-FIELD LIMIT OF A HYBRID SYSTEM FOR MULTI-LANE MULTI-CLASS TRAFFIC*

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Abstract. This article aims to study coupled mean-field equation and ODEs with discrete events motivated by vehicular traffic flow. Multi-lane traffic flow in presence of human-driven and autonomous vehicles is considered, with the autonomous vehicles possibly influenced by external policy makers. First a finite-dimensional hybrid system is developed based on the continuous Bando-Follow-the-Leader dynamics coupled with discrete events due to lane changing. The mean-field limit of the finite-dimensional hybrid system is rigorously derived by letting the number of human-driven vehicles go to infinity, and it consists of an infinite-dimensional hybrid system. The latter is described by coupled Vlasov-type PDE, ODEs and discrete events. In particular, the microscopic lane changing maneuver of the human-driven vehicles generates a source term to a Vlasov-type PDE.

Key words. Multi-lane traffic, autonomous vehicles, mean-field limit, hybrid systems, generalized Wasserstain distance

AMS subject classifications. 90B20 (Traffic problems), 34A38 (Hybrid systems), 35Q83 (Vlasov-like equations)

1. Introduction. Mathematical traffic models, depending on the scale at which they represent vehicular traffic, usually can be classified into different categories: microscopic, mesoscopic, macroscopic, and cellular. We refer to the survey papers [1, 4, 37], and reference therein, for general discussions about the models at various scales in the literature. In this paper, we focus on microscopic models and mesoscopic descriptions.

Microscopic models are discrete models of traffic flow that study the behavior of individual vehicles and predict their trajectories by means of ordinary differential equations (ODEs). One such model is the combined Bando [3] and Follow-the-Leader [41, 42] model that concerns both relaxation to an optimal velocity and interactions with the closest neighboring vehicle ahead. Mean-field equations, and in general models based on partial differential equations (PDEs), treat vehicular traffic as fluid flow, and aim to provide an aggregate and statistical viewpoint of traffic by capturing and predicting the main phenomenology of the microscopic dynamics. Within this context we would like to mention the most classical works [33, 39, 40] and recent developments, e.g. [9, 13, 24, 30, 38]. Thus, this scale of representation is useful and accurate in the limit of the dynamical system with infinitely many vehicles, and the link between the two descriptions can be formally and also rigorously established in generalized Wasserstain distance [21]. We point-out that this discussion is not restricted to traffic flow and of interest in many research areas, such as in biology [10, 12] or social [11] and economic dynamics [47].

In the present work, we aim to develop and study qualitative properties of models for traffic which are motivated by the idea of considering, simultaneously, two important aspects: lane changing maneuvers and heterogeneous composition of the flow. The former is one of the most common maneuvers and source of interaction among vehicles on motorways. Currently, multi-lane traffic is modeled either by two-dimensional models [23, 45], in which lane changing rules are not explicitly prescribed, or by treating lanes as discrete entities [25, 43]. The latter aspect, instead,

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is becoming more and more important with the increasingly interest in automated-driven vehicles and their effects within the vehicular traffic flow [26, 44].

We start by defining the microscopic dynamics. In the following we consider two classes of vehicles, one identified by human-driven vehicles and the other one by automated-driven vehicles. We use a Bando-Follow-the-Leader model for both classes, but the dynamics of autonomous vehicles differs from the dynamics of human-driven ones due to an additive control term which, in applications, may be provided by a remote controller [44]. The Bando-Follow-the-Leader model is reformulated by replacing the interaction with the closest vehicle ahead by a short-range interaction kernel which allows to write the system of ODEs in a convolution framework. Along with the continuous dynamics, we consider discrete dynamics generated by the lane changing rules, which are designed following [29]. The presence of both continuous and discrete dynamics leads us to a hybrid system, see [6, 19, 20, 35, 46]. Finally, we perform a mean-field limit for human-driven vehicles only, since autonomous vehicles are supposed to be a small percentage of the total flow on motorways. The trajectories of the hybrid system exhibit dicontinuities thus the limit procedure requires a generalization of the classical Arzel-Ascoli Theorem. This leads to a Vlasov-type PDEs with a source term [15, 22, 27], which is generated by the discrete lane changing rules. Such source term induces the measure solutions to change mass in time, thus the limit is obtained using the generalized Wasserstein distance [36]. Together with the continuous and discrete dynamics of the autonomous vehicles, we obtain a hybrid system with mean-field limit involved, for which we prove existence and uniqueness of solutions.

Our main result is thus a complete representation of multi-lane multi-class hybrid system at microscopic and mesoscopic scales together connected by a rigorous limiting procedures. This framework allows to study optimal control problems at multiple scales, in the same spirit as [5, 18]. We also notice that, even if our main example is vehicular traffic, the same framework may be adapted to model any hybrid system with multi-population at microscopic and mesoscopic scale, including social and crowd dynamics [17].

The paper is organized as follows. In Section 2, we briefly recall the basic models, notions, notations and preliminaries used in this article. Section 3 devotes to the definition of lane changing conditions and the study of well-posedness of the finite-dimensional hybrid system modeling multi-lane traffic at the microscopic level. In Section 4, we define a hybrid system involving mean-field limit of the finite-dimensional hybrid system involving human-driven vehicles and prove the existence and uniqueness of the trajectories of the mean-field hybrid system. Finally, Section 5 ends the paper with conclusions and outlook.

2. Notations, Definitions and Preliminaries.

In this section, we first recall some basic notations and definitions about traffic flow models and the generalized Wasserstein distance we use in this article. Then we list some well-known results about solutions to Carathéodory differential equations and to partial differential equations of Vlasov-type with source term. At last, we give a proof to a revised version of Arzel-Ascoli Theorem.

2.1. Traffic Flow Models.

In order to setup the mathematical formulation, in the following we consider a population of P cars on an open stretch road. To each vehicle, labeled by an index $i \in \{1, \dots, P\}$, we associate a vector of indices $\iota(i) = (i, i_L, i_F)$. Here $i_L \in \{1, \dots, P\}$ is the index of the vehicle in front of vehicle i (the leader) and $i_F \in \{1, \dots, P\}$ is the index of the vehicle flowing vehicle i (the follower). To fix notation, we assume that $i_L = 0$ if vehicle i is the first and $i_F = 0$ if vehicle i is the last.

The Follow-the-Leader (FtL) model, which was introduced in [41, 42], assumes that the acceleration of a vehicle is directly proportional to difference between the velocity of the vehicle in front and its own velocity, and is inversely proportional to their distance. Let (x_i, v_i) be the vector of position-velocity, with $v_i \geq 0$ and $h_i = x_{i_L} - x_i$ be the headway of the i-th vehicle. The main dynamics described by the FtL model is given by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \beta_i \frac{v_{i_L} - v_i}{(h_i)^2}, \quad i \in \{1, \dots, P\}, \end{cases} \tag{2.1}$$

where β_i is a positive parameter with appropriate dimension. If vehicle i is the first vehicle, then the dynamics of vehicle i is given by $\dot{x}_i = v_{\max}$, where $v_{\max}$ is a given maximum velocity, perhaps the speed limit. By system (2.1), one can see also a drawback of the FtL model: as long as the relative velocity $\Delta v_i = v_{i_L} - v_i$ is zero, the acceleration is zero. That is to say, even at high speeds, an extremely small headway is allowed.

The Bando model, proposed by Bando et al. in [3], fixed the aforementioned problems by associating each vehicle an optimal velocity function V which describes the desired velocity for the headway. A driver controls the acceleration or deceleration based on the difference between his/her own velocity and the optimal velocity. The optimal velocity is typically an increasing function of the headway, namely it tends to zero for a small headway and to the maximum value $v_{\max}$ for a large headway. The governing equation of the Bando model is as follows:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \alpha_i (V(h_i) - v_i), \quad i \in \{1, \dots, P\}, \end{cases} \tag{2.2}$$

where α_i is a positive parameter denoting the speed of response. The equilibrium point for this model is obtained when all vehicles travel at constant speed and have the same headway, see [31].

For the combined Bando-FtL model, which represents the basic model we consider in this work, the dynamics of the i-th vehicle is defined as follows: If $i_L \neq 0$, i.e., if vehicle i is not the first, then

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \alpha_i (V(h_i) - v_i) + \beta_i \frac{v_{i_L} - v_i}{(h_i)^2}, \quad i \in \{1, \dots, P\}, \end{cases} \tag{2.3}$$

where the headway is $h_i = x_{i_L} - x_i$. For simplicity, we take $\alpha_i = \alpha$, $\beta_i = \beta$ for all $i \in \{1, \dots, P\}$.

Now we will rewrite the Bando-FtL model, system (2.3), in convolutional form to justify the fact that drivers adjust their acceleration or deceleration according to the velocities of their front nearby vehicles, their own velocities and optimal velocities. For $T > 0$ fixed and $i = 1, \dots, P$, define a time dependent atomic probability measure on $\mathbb{R} \times \mathbb{R}^+$,

$$\mu_P(t) = \frac{1}{P} \sum_{i=1}^{P} \delta_{(x_i(t), v_i(t))} \tag{2.4}$$

supported on an absolutely continuous trajectories $t \in [0, T] \mapsto (x_i(t), v_i(t)) \in \mathbb{R} \times \mathbb{R}^+$. Let $\varepsilon_0 > 0$ be fixed. Define a convolution kernel $H_1 \colon \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ as $H_1(x, v) = \alpha h(x) \left(V(-x) - v\right) \chi_{[-\epsilon_0, 0]}(x)$, where $h \colon \mathbb{R} \mapsto \mathbb{R}$ is a suitable smooth and compactly supported function on $[-\epsilon_0, 0]$ and weights the strength of the interaction depending on the distance between two vehicles. Then, formally, the Bando-term in (2.3) can be rewritten as

$$H_1 *_1 \mu_P(x_i, v_i) = \frac{1}{P} \sum_{k=1}^{P} H_1(x_i - x_k, v_i) = \frac{\alpha}{P} \left(\sum_{k \in i_{\varepsilon_0}} h(x_i - x_k) \left(V(x_k - x_i) - v_i\right) \right),$$

where $*_1$ is the convolution with respect to the first variable, and $i_{\varepsilon_0} = \{k \colon 0 < x_k - x_i < \varepsilon_0\}$. Similarly, define a convolution kernel $H_2 \colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ as $H_2(x, v) = \beta h(x) \frac{-v}{x^2}$. Then, we formally rewrite the FtL-term of (2.3) as

$$H_2 * \mu_P(x_i, v_i) = \frac{1}{P} \sum_{k=1}^{P} H_2(x_i - x_k, v_i - v_k) = \frac{\beta}{P} \left(\sum_{k \in i_{\varepsilon_0}} h(x_i - x_k) \frac{v_k - v_i}{(x_i - x_k)^2} \right),$$

where $*$ is the (x, v)-convolution.

Formally, the Bando-FtL model (2.3) can be written using the convolutional kernels as follows

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H_1 *_1 \mu_P + H_2 * \mu_P)(x_i, v_i), \quad i = 1, \dots, P. \end{cases} \tag{2.5}$$

We observe again that the introduction of the range of interaction ϵ_0 allows each vehicle to interact with more than one vehicle ahead. Model (2.5) has thus a close link to bounded confidence models for opinion formation, flocking and swarming behaviors [32].

Next, we will focus also on descriptions based on PDEs. In particular, system (2.5) formally admits the following mean-field limit as $P \to \infty$:

$$\partial_t \mu + v\partial_x \mu + \partial_v((H_1 *_1 \mu + H_2 * \mu)\,\mu) = 0, \tag{2.6}$$

which gives a partial differential equation of Vlasov-type. Here μ represents the density distribution of the vehicles in position-velocity variables in a single lane. Equation (2.6) describes the evolution of the density distribution μ with respect to time in the mesoscopic level. This can be easily derived in a formal way following classical computations, e.g. see [7], by considering a test function $\varphi \in C_0^1(\mathbb{R}^2)$ and computing the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_P(t), \varphi\rangle$. Mean-field limits can be also rigorously derived [8].

2.2. The Generalized Wasserstein Distance. In this subsection, we recall the definitions and some properties related to the Wasserstein distance and the generalized Wasserstein distance. For a complete introduction to Wasserstein distance, see [48] and to generalized Wasserstein distance, see [36].

Let $\mathcal{M}$ be the space of Borel measures with finite mass, $\mathcal{P}$ be the space of probability measures (the measures in $\mathcal{M}$ with unit mass) and $\mathcal{M}^p$ be the space of Borel measures with finite p-th moment on $\mathbb{R}^d$, where d is the dimension of the space. We also denote with $\mathcal{M}_0^{ac}$ the subspace of $\mathcal{M}$ of measures that are with bounded support and absolutely continuous with respect to the Lebesgue measure. Given a measure $\mu \in \mathcal{M}$, we denote with $|\mu| := \mu(\mathbb{R}^d)$ its mass. Given a Borel map $\gamma\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, the push-forward of μ by γ, $\gamma\#\mu$, is defined as for every Borel set $A \subset \mathbb{R}^d$, $\gamma\#\mu(A) := \mu(\gamma^{-1}(A))$. One can see that the mass of $\gamma\#\mu$ is identical to the mass of μ, i.e., $|\mu| = |\gamma\#\mu|$.

Given two probability measures $\mu,\ \nu \in \mathcal{P}$, a probability measure π on the product space $\mathbb{R}^d \times \mathbb{R}^d$ is said to be an admissible transference plan from μ to ν if the following properties hold:

$$\int_{y\in\mathbb{R}^d} \mathrm{d}\pi(x,y) = \mathrm{d}\mu(x), \quad \int_{x\in\mathbb{R}^d} \mathrm{d}\pi(x,y) = \mathrm{d}\nu(y). \tag{2.7}$$

We denote the set of admissible transference plans from μ to ν by $\Pi(\mu,\nu)$. Note that the set $\Pi(\mu,\nu)$ is always nonempty, since the tensor product $\mu\otimes\nu \in \Pi(\mu,\nu)$. To each admissible transference plan from μ to ν, π, one can define a cost as follows: $J[\pi] := \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\mathrm{d}\pi(x,y)$, where $|\cdot|$ represents the Euclidean norm. A minimizer of J in $\Pi(\mu,\nu)$ always exists. Furthermore, the space of probability measures with finite p-th moment, $\mathcal{P}\cap\mathcal{M}^p$, is a natural space in which J is finite. Thus for any two measures $\mu,\nu \in \mathcal{P}\cap\mathcal{M}^p$, one can define the following operator which is called Wasserstein distance $W_p(\mu,\nu) := \left(\min_{\pi\in\Pi(\mu,\nu)} J[\pi]\right)^{\frac{1}{p}}$. Note that if $\nu^{m,1} = \frac{1}{m}\sum_{k=1}^m \delta_{\xi_k^1}$ and $\nu^{m,2} = \frac{1}{m}\sum_{k=1}^m \delta_{\xi_k^2}$ are two atomic measures with $m \in \mathbb{Z}^+$, $\xi_k^1, \xi_k^2 \in \mathbb{R}^d$, then $W_1(\nu^{m,1},\nu^{m,2}) \leq \frac{1}{m}\sum_{k=1}^m |\xi_k^1 - \xi_k^2|$.

We additionally recall the following lemmas related to Wasserstein distance (see, e.g., Lemma 3.11, Lemma 3.13, Lemma 3.15, Lemma 4.7 in [8]).

LEMMA 2.1. *Let f_1 and $f_2\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ be two bounded Borel measurable functions. Then for every $\mu \in \mathcal{P}(\mathbb{R}^n)\cap\mathcal{M}^1(\mathbb{R}^n)$, one has*

$$W_1(f_1\#\mu, f_2\#\mu) \leq \|f_1 - f_2\|_{L^\infty(\operatorname{supp}\mu)}.$$

If in addition, f_1 is locally Lipschitz continuous, and $\mu,\nu \in \mathcal{P}(\mathbb{R}^n)\cap\mathcal{M}^1(\mathbb{R}^n)$ are both compactly supported on a ball B of $\mathbb{R}^n$, then

$$W_1(f_1\#\mu, f_1\#\nu) \leq L W_1(\mu,\nu),$$

where L is the Lipschitz constant of f_1 on B.

Now we state the following assumption on map $H\colon \mathbb{R}^{2d} \mapsto \mathbb{R}^d$:

$(H1)$ H is locally Lipschitz;

$(H2)$ H is of sub-linear growth, that is, there exists a constant $C > 0$ such that $|H(\xi)| \le C(1 + |\xi|)$, for all $\xi \in \mathbb{R}^{2d}$.

LEMMA 2.2. *Let H be a map satisfying condition (H1)-(H2). Let $R > 0$. Let $\mu, \nu\colon [0,T] \mapsto \mathcal{P}(\mathbb{R}^{2d}) \cap \mathcal{M}^1(\mathbb{R}^{2d})$ be continuous maps with respect to the first order Wasserstein distance W_1 both satisfying*

$$\operatorname{supp}\mu(t) \subset B(0,R) \text{ and } \operatorname{supp}\nu(t) \subset B(0,R),$$

for every $t \in [0,T]$. Then for every $\rho > 0$, there exists a constant $L_{\rho,R}$ such that

$$\|H * \mu(t) - H * \nu(t)\|_{L^\infty(B(0,\rho))} \le L_{\rho,R} W_1(\mu(t), \nu(t)).$$

Next, we recall the definition of the generalized Wasserstein distance on, $\mathcal{M}$, the space of Borel measures with finite mass on $\mathbb{R}^d$. For more detail, see [36].

DEFINITION 2.3. *Given $a, b \in (0,\infty)$ and $p \ge 1$, the generalized Wasserstein distance between two measures $\mu, \nu \in \mathcal{M}^p$ is*

$$W_p^{a,b}(\mu,\nu) \colon = \inf_{\substack{\tilde\mu,\tilde\nu \in \mathcal{M}^p \\ |\tilde\mu| = |\tilde\nu|}} \left(a\left(|\mu - \tilde\mu| + |\nu - \tilde\nu|\right) + b W_p(\tilde\mu, \tilde\nu)\right). \tag{2.8}$$

Remark 2.4. The standard Wasserstein distance is defined only for probability measures. Combing the standard Wasserstein distance and L^1 distance, the generalized Wasserstein distance can be applied to measures with different masses.

If μ_1 is absolutely continuous with repsect to $\mu \in \mathcal{M}$ and for every Borel set $A \subset \mathbb{R}^d$, $\mu_1(A) \le \mu(A)$, then we write $\mu_1 \le \mu$.

Remark 2.5. The infimum in equation (2.8) is always attained if one restrict the computation in equation (2.8) to $\tilde\mu \le \mu$, $\tilde\nu \le \nu$.

We recall some simple properties of the generalized Wasserstein distance, $W_p^{a,b}$. Compare the following proposition with Proposition 2 in [36].

PROPOSITION 2.6. *Let $\mu, \nu, \mu_1, \mu_2, \nu_1, \nu_2$ be measures in $\mathcal{M}^p$. The following properties of the generalized Wasserstein distance $W_1^{1,1}$ hold:*

$$W_1^{1,1}(k\mu, k\nu) \le k W_1^{1,1}(\mu,\nu) \text{ for } k \ge 0;$$
$$W_1^{1,1}(\mu_1 + \mu_2, \nu_1 + \nu_2) \le W_1^{1,1}(\mu_1,\nu_1) + W_1^{1,1}(\mu_2,\nu_2).$$

Similar to LEMMAS 2.1, 2.2, we have the following lemmas for the generalized Wasserstein distance.

LEMMA 2.7. *Let $f_1, f_2\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ be bounded Borel measureable functions. Then for every $\mu \in \mathcal{M}^1(\mathbb{R}^n)$, one has*

$$W_1^{1,1}(f_1\#\mu, f_2\#\mu) \le \|f_1 - f_2\|_{L^\infty(\operatorname{supp}\mu)}.$$

If in addition f_1 is locally Lipschitz continuous Borel measurable functions, Then for $\mu, \nu \in \mathcal{M}^1(\mathbb{R}^n)$ compactly supported on a ball B of $\mathbb{R}^n$,

$$W_1^{1,1}(f_1\#\mu, f_1\#\nu) \le \max\{L, 1\} W_1^{1,1}(\mu,\nu),$$

where L is the Lipschitz constant of f on B.

LEMMA 2.8. *Let H be a map satisfying condition (H1)-(H2). Let $R > 0$ be fixed. Let $\mu, \nu\colon [0,T] \mapsto \mathcal{M}^1(\mathbb{R}^{2d})$ be continuous maps with respect to the generalized Wasserstein distance $W_1^{1,1}$ both satisfying*

$$\operatorname{supp}\mu(t) \subset B(0,R) \text{ and } \operatorname{supp}\nu(t) \subset B(0,R),$$

for every $t \in [0,T]$. Then for every $\rho > 0$, there exists a constant $L_{\rho,R}$ such that

$$\|H * \mu(t) - H * \nu(t)\|_{L^\infty(B(0,\rho))} \leq L_{\rho,R} W_1^{1,1}(\mu(t), \nu(t)). \tag{2.9}$$

One can prove LEMMA 2.7 and LEMMA 2.8 by combining LEMMA 2.1, LEMMA 2.2, and the definition of generalized Wasserstein distance.

2.3. Carathéodory Differential Equations. In this section, we recall the following global existence and uniqueness result (see Theorem 6.2 in [17]) for Carathéodory differential equations. For further detailed discussions, see also [16].

THEOREM 2.9. *Let $T > 0$ and $n \geq 1$ be fixed. Consider a Carathéodory function $g\colon [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$. Assume that there exists a constant $C > 0$ such that for almost every $t \in [0,T]$ and every $y \in \mathbb{R}^n$, $|g(t,y)| \leq C(1 + |y|)$. Then given $y_0 \in \mathbb{R}^n$, there exists a solution $y(t)$ of $\dot{y}(t) = g(t, y(t))$ on the whole interval $[0,T]$ such that $y(0) = y_0$. Any such solution satisfies for every $t \in [0,T]$, $|y(t)| \leq (|y_0| + Ct)\, e^{Ct}$.*

If in addition, for every relatively compact open subset of $\mathbb{R}^n$, $|g(t,y_1) - g(t,y_2)| \leq L|y_1 - y_2|$ holds, then the solution is uniquely determined by the initial condition y_0 on the whole interval $[0,T]$.

2.4. Partial Differential Equations of Vlasov-type with Source Term. In this subsection, we consider partial differential equations of Vlasov-type.

Let H_1, H_2 be two maps satisfying condition (H1)-(H2). Let $T > 0$, $R > 0$ be fixed. Consider a continuous map $\mu\colon [0,T] \mapsto \mathcal{P}(\mathbb{R}^{2d}) \cap \mathcal{M}^1(\mathbb{R}^{2d})$ with respect to the first order Wasserstein distance, W_1, such that $\operatorname{supp}\mu(t) \subset B(0,R)$ for all $t \in [0,T]$, and a time dependent atomic measure $\nu(t)(y,w) = \frac{1}{M}\sum_{k=1}^{M} \delta_{(y_k(t), w_k(t))}$ supported on the absolutely continuous trajectories $t \mapsto (y_k(t), w_k(t))$, $k = 1, \ldots, M_j$. Then given an initial datum $P_0\colon = (x_0, v_0) \in \mathbb{R}^{2d}$, there exists a unique solution $P(t)\colon = (x(t), v(t))$ on the whole time interval $[0,T]$ to the following system of ODEs on $\mathbb{R}^{2d}$

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \left(H_1 *_1 (\mu + \nu) + H_2 * (\mu + \nu)\right)(x(t), v(t)). \end{cases}$$

Therefore, one can consider a family of flow maps

$$\mathcal{T}_t^{\mu,\nu}\colon P_0 \in \mathbb{R}^{2d} \mapsto P(t) \in \mathbb{R}^{2d}. \tag{2.10}$$

indexed by $t \in [0,T]$. Furthermore, the flow map $\mathcal{T}_t^{\mu,\nu}$ is Lipschitz continuous. In fact, let $\mu^q\colon [0,T] \mapsto \mathcal{P}(\mathbb{R}^{2d}) \cap \mathcal{M}^1(\mathbb{R}^{2d})$, $q = 1, 2$, be two continuous maps with respect to Wasserstein distance and be equi-compactly supported in $B(0,R)$. Let ν^1, ν^2 be two atomic measures supported on the respective absolutely continuous trajectories $t \mapsto (y_k^q(t), w_k^q(t))$, $q = 1, 2$ and $k = 1, \ldots, M$. Fix $r > 0$. Then there exist constants $\rho, L > 0$, such that whenever $|P_1| \leq r$,$|P_2| \leq r$,

$$\begin{aligned} &|\mathcal{T}_t^{\mu^1,\nu^1}(P_1) - \mathcal{T}_t^{\mu^2,\nu^2}(P_2)| \leq \\ \leq &e^{Lt}|P_1 - P_2| + \int_0^t e^{L(s-t)} \big\| \left(H_1 *_1 (\mu^1 + \nu^1) + H_2 * (\mu^1 + \nu^1)\right) \\ &- \left(H_1 *_1 (\mu^2 + \nu^2) + H_2 * (\mu^2 + \nu^2)\right) \big\|_{L^\infty(B(0,\rho))} \,\mathrm{d}s, \end{aligned} \tag{2.11}$$

for every $t \in [0,T]$. For more details, please see [17].

Given an initial condition $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d}) \cap \mathcal{M}^1(\mathbb{R}^{2d})$ of bounded support, we say that a measure $\mu(t)$ is a weak equi-compactly supported solution of the following Vlasov-type PDE with the initial datum μ_0,

$$\partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot [(H_1 *_1 (\mu + \nu) + H_2 * (\mu + \nu))\mu] = 0, \tag{2.12}$$

if (i) $\mu(0)=\mu_0$;
(ii) $\operatorname{supp}\mu(t)\subset B(0,R)$ for all $t\in[0,T]$, for some $R>0$;
(iii) for every $\varphi\in C_c^\infty(\mathbb{R}^{2d})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}}\varphi(x,v)\,\mathrm{d}\mu(t)(x,v)=\int_{\mathbb{R}^{2d}}\nabla\varphi(x,v)\cdot\tilde{\omega}_{H_1,H_2,\mu,\nu^j}(t,x,v)\,\mathrm{d}\mu(t)(x,v)$$

where $\tilde{\omega}_{H_1,H_2,\mu,\nu}(t,x,v)\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}^{2d}$ is defined as

$$\tilde{\omega}_{H_1,H_2,\mu,\nu}(t,x,v)\colon=(v,(H_1*_1(\mu+\nu)+H_2*(\mu+\nu))(x,v)). \tag{2.13}$$

Furthermore, following from Section 8.1 in [2], a measure $\mu(t)$ is a weak equi-compactly supported solution of equation (2.12) if and only if it satisfies condition (ii) and the measure-theoretical fixed point equation $\mu(t)=(\mathcal{T}_t^{\mu,\nu})\#\mu_0$ where the flow function $\mathcal{T}_t^{\mu,\nu}$ is defined in equation (2.10).

Now we consider solutions to the following Vlasov-type PDE with initial datum $\mu_0\in\mathcal{M}_0^{ac}(\mathbb{R}^{2d})\cap\mathcal{M}^1(\mathbb{R}^{2d})$ and source term S

$$\partial_t\mu+v\cdot\nabla_x\mu+\nabla_v\cdot[(H_1*_1(\mu+\nu)+H_2*(\mu+\nu))\mu]=S(\mu) \tag{2.14}$$

under the following hypotheses:

(S_1) $S(\mu)$ has uniformly bounded mass and support, that is, there exist Q,R, such that $S(\mu)(\mathbb{R}^{2d})\le Q$, and $\operatorname{supp}(S(\mu))\subset B(0,R)$;

(S_2) S is Lipschitz, that is, there exists L, such that, for any $\mu,\nu\in\mathcal{M}^1(\mathbb{R}^{2d})$, $W_1^{1,1}(S(\mu),S(\nu))\le LW_1^{1,1}(\mu,\nu)$.

A measure $\mu(t)$ is a weak solution of equation (2.14) with a given initial datum $\mu_0\in\mathcal{M}_0^{ac}(\mathbb{R}^{2d})\cap\mathcal{M}^1(\mathbb{R}^{2d})$, if $\mu(0)=\mu_0$ and if for every $\varphi\in C_c^\infty(\mathbb{R}^{2d})$, it holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}}\varphi(x,v)\,\mathrm{d}\mu(t)(x,v)=\\&=\int_{\mathbb{R}^{2d}}\varphi(x,v)\,\mathrm{d}S(\mu)(x,v)+\int_{\mathbb{R}^{2d}}\nabla\varphi(x,v)\cdot\tilde{\omega}_{H_1,H_2,\mu,\nu}(t,x,v)\,\mathrm{d}\mu(t)(x,v),\end{aligned}$$

where $\tilde{w}_{H_1,H_2,\mu,\nu}$ is as defined in (2.13).

Theorem 2.10. *Given an initial datum $\mu_0\in\mathcal{M}_0^{ac}(\mathbb{R}^{2d})\cap\mathcal{M}^1(\mathbb{R}^{2d})$, there exists a unique weak solution $\mu(t)$ to equation (2.14) under the hypotheses $(S_1),(S_2)$. Furthermore, $\mu(t)\in\mathcal{M}_0^{ac}(\mathbb{R}^{2d})\cap\mathcal{M}^1(\mathbb{R}^{2d})$.*

One can construct a weak solution $\mu(t)$ to equation (2.14) based on a Lagrangian scheme by sample-and-hold. Given a fixed $k\in\mathbb{N}^+$, define $\Delta t\colon=\frac{T}{2^k}$ and decompose the time interval $[0,T]$ in $[0,\Delta t],[\Delta t,2\Delta t],\dots,[(2^k-1)\Delta t,2^k\Delta t]$. We define
$\mu_k(0)\colon=\mu_0$;
$\mu_k((n+1)\Delta t)\colon=\mathcal{T}_{\Delta t}^{\mu_k(n\Delta t),\nu(n\Delta t)}\#\mu_k(n\Delta t)+\Delta tS(\mu_k(n\Delta t))$;
$\mu_k(t)\colon=\mathcal{T}_\tau^{\mu_k(n\Delta t),\nu(n\Delta t)}\#\mu_k(n\Delta t)+\tau S(\mu_k(n\Delta t))$,
where n is the maximum integer such that $t-n\Delta t\ge0$ and $\tau\colon=t-n\Delta t$. Then $\mu(t)=\lim_{k\to\infty}\mu_k(t)$ is the unique weak solution to equation (2.14). For more detail, please see [36].

2.5. A Revised Version of ArzelAscoli Theorem. In this subsection, we will provide a proof to a revised version of ArzelAscoli theorem.

Theorem 2.11. *Let K be a compact subset of $\mathbb{R}$ and let D be a complete and totally bounded metric space with metric d. Consider a sequence of functions $\{f_n\}_{n=1}^\infty$ in $C(K;D)$. If there exists $L>0$, such that the following is true: for any $\varepsilon>0$, there exists $N>0$, such that, whenever $n\ge N$,*

$$d(f_n(t),f_n(s))\le L|t-s|+\min\{\varepsilon,|t-s|\},\forall s,t\in K$$

then the sequence $\{f_n\}_{n=1}^\infty$ has a uniformly convergent sub-sequence.

Proof. First note that the subset $S = K \cap \mathbb{Q}$ of set $K \subset \mathbb{R}$ is countable and dense, that is, K is separable. We list the countably many elements of S as $\{t_1, t_2, t_3, \dots\}$.

We will find a sub-sequence of $\{f_n\}$ that converges point-wise on S by a standard diagonal argument.

Since D is complete and totally bounded, D is sequentially compact. Thus the sequence $\{f_n(t_1)\}_{n=1}^{\infty}$ in D has a convergent sub-sequence, which we will write using double subscripts, $\{f_{1,n}(t_1)\}_{n=1}^{\infty}$. Similarly, the sequence $\{f_{1,n}(t_2)\}_{n=1}^{\infty}$ also has a convergent sub-sequence $\{f_{2,n}(t_2)\}_{n=1}^{\infty}$. By proceeding in this way, we obtain a countable collection of sub-sequences of our original sequence $\{f_n\}_{n=1}^{\infty}$:

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3}, & \dots \\ f_{2,1} & f_{2,2} & f_{2,3}, & \dots \\ f_{3,1} & f_{3,2} & f_{3,3}, & \dots \\ \vdots & \vdots & \vdots & \dots \end{array}$$

where the sequence in the n-th row converges at the points $t_1, t_2, \dots, t_n$, and each row is a sub-sequence of its previous row. Let $\{g_n\}$ be the diagonal sequence produced in the previous step, i.e., $g_n = f_{n,n}$ for each $n \in \mathbb{N}$. Then the sequence $\{g_n\}$ is a sub-sequence of the original sequence $\{f_n\}$ that converges at each point of S.

Next, we will show that the sub-sequence $\{g_n\}$ of $\{f_n\}$ is uniformly convergent. Let $\varepsilon > 0$ be given and choose $\delta = \min\left\{\frac{\varepsilon}{6L}, \frac{\varepsilon}{6}\right\}$. Then there exists $N_1 > 0$, such that for every $n \geq N_1$, and for any $s, t \in K$ with $|s - t| < \delta$,

$$d(g_n(t), g_n(s)) \leq L|t-s| + \min\{\frac{\varepsilon}{6}, |t-s|\} \leq L\delta + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{3}.$$

Since K is compact, for any positive integer $M > \frac{1}{\delta}$, there exists a finite set $S_M \subset S$ such that $K \subset \bigcup_{s \in S_M} B_{\frac{1}{M}}(s)$. Since the sequence $\{g_n\}$ converges at each point of S_M, there exists $N_2 > 0$, such that whenever $n, m > N_2$,

$$d(g_n(s), g_m(s)) < \frac{\varepsilon}{3}, \quad \forall s \in S_M.$$

Let $t \in K$ be arbitrary but fixed. Then there exists some $s \in S_M$ such that $|s-t| < \delta$. In addition, let $N = \max\{N_1, N_2\}$. Then whenever $n, m > N$,

$$d(g_n(t), g_m(t)) \leq d(g_n(t), g_n(s)) + d(g_n(s), g_m(s)) + d(g_m(s), g_m(t)) < \epsilon.$$

Hence the sub-sequence $\{g_n\}$ of the original sequence $\{f_n\}$ is uniformly Cauchy. Since the metric space D is complete, $C(K; D)$ is complete with respect to the uniform metric. Thus the sub-sequence $\{g_n\}$ is uniformly convergent. ∎

3. The Finite-Dimensional Hybrid System. In this section, we specify the Bando-FtL model introduced in Section 2.1 to the case of lane changing maneuvers and multi-class vehicles.

In the case of multi-lane traffic, vehicles travel along multiple lanes with the possibility to change lane paying a cost related to such maneuver. We consider m lanes and assume that $j \in J = \{1, \dots, m\}$ is the index of lanes. Now, to each vehicle i, we associate a vector of indices $\iota(i) = (i, j, i_L, i_F)$, where $j \in J$ is the lane index of vehicle i, i_L and i_F are defined as in Section 2.1. Each individual vehicle has a continuous dynamic governed by system (2.5) before performing lane changing. Discrete dynamics of the vehicles will be generated due to lane changing. The presence of both continuous dynamics and discrete dynamics leads us to consider hybrid system, see [19, 35].

In particular, in the following we consider two classes of vehicles and split the population of P vehicles into M autonomous vehicles and N human-driven vehicles on an open stretch of road with m lanes. We let M_j and N_j be the number of autonomous vehicles and the number of human-driven vehicles on lane $j \in J = \{1, \dots, m\}$, respectively. Then $\sum_{j=1}^{m} M_j = M$ and $\sum_{j=1}^{m} N_j = N$.

First, we study the dynamics of the $M+N$ vehicles from the microscopic point of view. As in [17], we assume that we have a small amount M of autonomous vehicles that have a great influence on the population and a large amount N of human-driven vehicles which have a small influence on the population. Instead of controlling all vehicles, we just add controls on the M autonomous vehicles.

Let (y, w) and (x, v) be the space-velocity variables of the autonomous vehicles and the human-driven vehicles, respectively. That is, the space-velocity of autonomous vehicle k and of human-driven vehicle i on j lane is (y_k^j, w_k^j) and (x_i^j, v_i^j), respectively. We consider the following atomic measures in $\mathcal{M}^+(\mathbb{R}\times\mathbb{R}^+)$ on each single lane

$$\mu_{N_j}(t) = \frac{1}{N_j}\sum_{i=1}^{N_j}\delta_{(x_i^j(t), v_i^j(t))}, \qquad \nu^j(t) = \frac{1}{M_j}\sum_{k=1}^{M_j}\delta_{(y_k^j(t), w_k^j(t))}. \tag{3.1}$$

From the microscopic point of view, the dynamics of vehicles on lane $j \in J$ without lane changing are

$$\begin{aligned} \dot{y}_k^j &= w_k^j; \\ \dot{w}_k^j &= \left(H_1 *_1 (\mu_{N_j} + \nu^j) + H_2 * (\mu_{N_j} + \nu^j)\right)(y_k^j, w_k^j) + u_k^j; \quad k = 1, \dots, M_j; \\ \dot{x}_i^j &= v_i^j; \\ \dot{v}_i^j &= \left(H_1 *_1 (\mu_{N_j} + \nu^j) + H_2 * (\mu_{N_j} + \nu^j)\right)(x_i^j, v_i^j); \quad i = 1, \dots, N_j, \end{aligned} \tag{3.2}$$

where $u_k^j\colon [0,T] \mapsto \mathbb{R}$ are measurable controls for $k = 1, \dots, M_j$ and $H_1\colon \mathbb{R}\times\mathbb{R}^+ \mapsto \mathbb{R}$, $H_2\colon \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ are locally Lipschitz convolution kernels with sub-linear growth. Particularly, there exists a constant $C > 0$ such that for all $(x_1, v_1) \in \mathbb{R}\times\mathbb{R}^+$ and $(x_2, v_2) \in \mathbb{R}\times\mathbb{R}$,

$$|H_1(x_1, v_1)| \le C(1 + |(x_1, v_1)|) \text{ and } |H_2(x_2, v_2)| \le C(1 + |(x_2, v_2)|). \tag{3.3}$$

Let $\Delta > 0$ be fixed. Vehicle n on $j \in J$ lane will perform lane changing to $j' = j+1$ or $j-1 \in J$ lane at time $t \in [0,T]$ if the following conditions occur:

$$\text{Safety: } \bar{a}_n^{j'}(t) \ge -\Delta \text{ and } \bar{a}_l^{j'}(t) \ge -\Delta; \qquad \text{Incentive: } \bar{a}_n^{j'}(t) \ge a_n^j(t) + \Delta.$$

In the case of $\bar{a}_n^{j+1}(t) = \bar{a}_n^{j-1}(t) \ge a_n^j(t)$ and $\bar{a}_n^{j'}(t), \bar{a}_l^{j'}(t) \ge -\Delta$, for $j' = j-1, j+1 \in J$, we assume that vehicle n changes from j lane to $j+1$ lane. Here l is the index of the first vehicle following vehicle n on the new lane if vehicle n changes lane at time t, $a_n^j(t)$ is the actual acceleration of vehicle n at time t on j lane, $\bar{a}_n^{j'}(t)$ and $\bar{a}_l^{j'}(t)$ are the expected accelerations of vehicle n on the new lane if vehicle n changes lane at time t, respectively. For instance, if vehicle n is an autonomous vehicle and if vehicle l is a human-driven vehicle, then at time $t \in [0,T]$,

$$\begin{aligned} a_n^j(t) &= \left(H_1 *_1 (\mu_{N_j}(t) + \nu^j(t)) + H_2 * (\mu_{N_j}(t) + \nu^j(t))\right)(y_n^j(t), w_n^j(t)) + u_n^j(t); \\ \bar{a}_n^{j'}(t) &= \left(H_1 *_1 (\mu_{N_{j'}}(t) + \nu^{j'}(t)) + H_2 * (\mu_{N_{j'}}(t) + \nu^{j'}(t))\right)(y_n^{j'}(t), w_n^{j'}(t)) + u_n^{j'}(t); \\ \bar{a}_l^{j'}(t) &= \left(H_1 *_1 (\mu_{N_{j'}}(t) + \tilde{\nu}^{j'}(t)) + H_2 * (\mu_{N_{j'}}(t) + \tilde{\nu}^{j'}(t))\right)(z_l^{j'}(t), \omega_l^{j'}(t)), \end{aligned}$$

where $(z_l^{j'}(t), \omega_l^{j'}(t))$ represents the position-velocity of vehicle l at time $t \in [0,T]$ on j' lane and

$$\tilde{\nu}^{j'} = \frac{1}{M_{j'}+1}\left(\sum_{\ell=1}^{M_{j'}}\delta_{(y_\ell^{j'}(t), w_\ell^{j'}(t))} + \delta_{\left(y_n^{j'}(t), w_n^{j'}(t)\right)}\right).$$

We assume that there are no two vehicles changing lane at the same time. It has been experimentally shown that lane changing are crucial for traffic safety, but lane changing are not frequent [28]. We assign each vehicle a timer over the whole time interval $[0,T]$. Let $N_\tau \in \mathbb{Z}^+$ be large and fixed and let $T_1 = \frac{T}{N_\tau}$. A vehicle would consider changing lane only when its timer reaches to T_1. Specifically, the timer τ_k^1 for autonomous vehicle k satisfies $\dot{\tau}_k^1 = 1, \tau_k^1(0) = \tau_{k,0}^1 \in [0, T_1)$ and the timer τ_i^2 for human-driven vehicle i satisfies $\dot{\tau}_i^2 = 1, \tau_i^2(0) = \tau_{i,0}^2 \in [0, T_1)$. In addition, the followings are true:

$$\begin{aligned} &\text{if } k_1 \neq k_2 \in \{1, \dots, M\}, \text{ then } \tau_{k_1,0}^1 \neq \tau_{k_2,0}^1; \\ &\text{if } i_1 \neq i_2 \in \{1, \dots, N\}, \text{ then } \tau_{i_1,0}^2 \neq \tau_{i_2,0}^2; \\ &\tau_{k,0}^1 \neq \tau_{i,0}^2 \text{ for any } k \in \{1, \dots, M\} \text{ and } i \in \{1, \dots, N\}. \end{aligned} \tag{3.4}$$

Besides, we reset the timer for each vehicle to be zero when it reaches to T_1. Here T_1 is called timer limit for all vehicles.

The presence of both continuous dynamics of vehicles governed by system (3.2) and discrete dynamics of vehicles caused by lane changing motivates us to consider the following finite-dimensional hybrid system.

For the definition of a hybrid system, we need to introduce the following notation. For each $\iota = \left(\{\iota_k^1\}_{k=1}^M, \{\iota_i^2\}_{i=1}^N\right) \in \mathbb{R}^{M+N}$,

$$\begin{aligned} A_\iota = \Bigg\{ &\left(y_k, w_k, \tau_k^1, x_i, v_i, \tau_i^2\right)_{\substack{k=1,\dots,M \\ i=1,\dots,N}} \in \left(\mathbb{R} \times \mathbb{R}^+ \times [0, T_1)\right)^2 : \\ &\exists k_1 \neq k_2 \in \{1, \dots, M\}, \text{ s.t. } \iota_{k_1}^1 = \iota_{k_2}^1 \wedge y_{k_1} = y_{k_2} \\ &\text{or } \exists i_1 \neq i_2 \in \{1, \dots, N\}, \text{ s.t. } \iota_{i_1}^2 = \iota_{i_2}^2 \wedge x_{i_1} = x_{i_2} \\ &\text{or } \exists k \in \{1, \dots, M\} \wedge i \in \{1, \dots, N\}, \text{ s.t. } \iota_k^1 = \iota_i^2 \wedge y_k = x_i \Big\}. \end{aligned}$$

DEFINITION 3.1. *A hybrid system is a* 6*-tuple* $\Sigma_1 = (\mathcal{L}, \mathcal{M}, U, \mathcal{U}, g, S)$ *where:*
H_1 $\mathcal{L} = \left\{\iota = (\iota_1^1, \iota_2^1, \dots, \iota_M^1, \iota_1^2, \iota_2^2, \dots, \iota_N^2), \iota_k^1, \iota_i^2 \in J, k = 1, \dots, M, i = 1, \dots, N\right\}$ *is the set of locations;*
H_2 $\mathcal{M} = \{\mathcal{M}_\iota\}_{\iota \in \mathcal{L}}$, *where* $\mathcal{M}_\iota = \left(\mathbb{R} \times \mathbb{R}^+ \times [0, T_1)\right)^M \times \left(\mathbb{R} \times \mathbb{R}^+ \times [0, T_1)\right)^N \setminus A_\iota$, *with* A_ι *is the set of states such that two cars are in same lane and position, see above.*
H_3 $U = \{U_\iota\}_{\iota \in \mathcal{L}}$, $U_\iota = I^M$, *where* $I \subset [0, U_{\max}]$ *is compact with* $U_{\max} > 0$*;*
H_4 $\mathcal{U} = \{\mathcal{U}_\iota\}_{\iota \in \mathcal{L}}$, $\mathcal{U}_\iota = \left\{u \colon [0, T] \mapsto U_\iota = I^M\right\}$*;*
H_5 $g = \{g_\iota\}_{\iota \in \mathcal{L}}$, $g_\iota \colon \mathcal{M}_\iota \times \mathcal{U}_\iota \mapsto (\mathbb{R}^3)^{M+N}$, $g_\iota = (w_k, a_k^1, 1, v_i, a_i^2, 1)$, *where*
$a_k^1 = \left(H_1 *_1 \left(\mu_{N_{\iota_k^1}} + \nu^{\iota_k^1}\right) + H_2 * \left(\mu_{N_{\iota_k^1}} + \nu^{\iota_k^1}\right)\right)(y_k, w_k) + u_k$, *and*
$a_i^2 = \left(H_1 *_1 \left(\mu_{N_{\iota_i^2}} + \nu^{\iota_i^2}\right) + H_2 * \left(\mu_{N_{\iota_i^2}} + \nu^{\iota_i^2}\right)\right)(x_i, v_i)$*;*
H_6 S *is a subset of* $LC(\Sigma_1)$, *where* $LC(\Sigma_1)$ *is the set of states for which a lane-changing can occur, that is,*

$$\begin{aligned} LC(\Sigma_1) = \Bigg\{ &\left(\iota, (y_k, w_k, \tau_k^1, x_i, v_i, \tau_i^2), \iota', (y_k', w_k', {\tau_k^1}', x_i', v_i', {\tau_i^2}')\right)_{\substack{k=1,\dots,M \\ i=1,\dots,N}} : \\ &\exists p_1 \in \{1, \dots, M\}, t_{p_1} \in [0, T], \ \textit{s.t. } \forall k \neq p_1, \\ &(y_k(t_{p_1}), w_k(t_{p_1}), \tau_k^1(t_{p_1}), \iota_k^1(t_{p_1})) = (y_k'(t_{p_1}), w_k'(t_{p_1}), {\tau_k^1}'(t_{p_1}), {\iota_k^1}'(t_{p_1})), \\ &\ \textit{and } (y_{p_1}(t_{p_1}), w_{p_1}(t_{p_1})) = (y_{p_1}'(t_{p_1}), w_{p_1}'(t_{p_1})), {\tau_{p_1}^1}'(t_{p_1}) = 0, \\ &{\iota_{p_1}^1}'(t_{p_1}) = (\iota_{p_1}^1(t_{p_1}) + 1)(1 - \delta_m(\iota_{p_1}^1(t_{p_1}))) \ \textit{or } (\iota_{p_1}^1(t_{p_1}) - 1)(1 - \delta_1(\iota_{p_1}^1(t_{p_1}))), \\ &(x_i(t_{p_1}), v_i(t_{p_1}), \tau_i^2(t_{p_1}), \iota_i^2(t_{p_1})) = (x_i'(t_{p_1}), v_i'(t_{p_1}), {\tau_i^2}'(t_{p_1}), {\iota_i^2}'(t_{p_1})), \\ &\ \textit{or } \exists p_2 \in \{1, \dots, N\}, t_{p_2} \in [0, T], \ \textit{s.t. } \forall i \neq p_2, \\ &(x_i(t_{p_2}), v_i(t_{p_2}), \tau_i^2(t_{p_2}), \iota_i^2(t_{p_2})) = (x_i'(t_{p_2}), v_i'(t_{p_2}), {\tau_i^2}'(t_{p_2}), {\iota_i^2}'(t_{p_2})), \\ &\ \textit{and } (x_{p_2}(t_{p_2}), v_{p_2}(t_{p_2})) = (x_{p_2}'(t_{p_2}), v_{p_2}'(t_{p_2})), {\tau_{p_2}^2}'(t_{p_2}) = 0, \\ &{\iota_{p_2}^2}'(t_{p_2}) = (\iota_{p_2}^2(t_{p_2}) + 1)(1 - \delta_m(\iota_{p_2}^2(t_{p_2}))) \ \textit{or } (\iota_{p_2}^2(t_{p_2}) - 1)(1 - \delta_1(\iota_{p_2}^2(t_{p_2}))), \\ &(y_k(t_{p_2}), w_k(t_{p_2}), \tau_k^1(t_{p_2}), \iota_k^1(t_{p_2})) = (y_k'(t_{p_2}), w_k'(t_{p_2}), {\tau_k^1}'(t_{p_2}), {\iota_k^1}'(t_{p_2})) \Bigg\}. \end{aligned}$$

Before actually defining a trajectory of hybrid system Σ_1, it is necessary to define its hybrid state first.

DEFINITION 3.2. *A hybrid state of the hybrid system* Σ_1 *is a* 7*-tuple* $(\iota, y, w, \tau^1, x, v, \tau^2)$, *where* $\iota \in \mathcal{L}$ *is the location,* $(y, w, \tau^1, x, v, \tau^2) \in \mathcal{M}_\iota$. *We denote by* $\mathcal{HS}_1$ *the* *set of all hybrid states of the hybrid system* Σ_1.

Now we will define a trajectory of hybrid system Σ_1.

DEFINITION 3.3. *Let* $(\iota_0, y_0, w_0, \tau_0^1, x_0, v_0, \tau_0^2) \in J^{M+N} \times \mathbb{R}^M \times (\mathbb{R}^+)^M \times [0, \delta_\tau)^M \times \mathbb{R}^N \times (\mathbb{R}^+)^N \times [0, \delta_\tau)^N$ *be given initial condition to the above hybrid system* Σ_1. *In addition, assume that the initial conditions* τ_0^1 *and* τ_0^2 *satisfy condition* (3.4). *A trajectory of the hybrid system* Σ_1 *with initial condition* $(\iota_0, y_0, w_0, \tau_0^1, x_0, v_0, \tau_0^2)$ *is a map* $\xi\colon [0,T] \mapsto \mathcal{HS}_1$, $\xi(t) = (\iota(t), y(t), w(t), \tau^1(t), x(t), v(t), \tau^2(t))$ *such that for* $k = 1, \ldots, M, i = 1, \ldots, N,$ *and* $n = 1, \ldots, N_\tau - 1$, *the following holds:*
1. $\tau_k^1(0) = \tau_{k,0}^1$, *and* $\tau_i^2(0) = \tau_{i,0}^2$;
2. $(y_k(0), w_k(0)) = (y_{k,0}, w_{k,0}) \in \mathbb{R} \times \mathbb{R}^+$ *and* $(x_i(0), v_i(0)) = (x_{i,0}, v_{i,0}) \in \mathbb{R} \times \mathbb{R}^+$;
3. $\iota_k^1(t) = \iota_{k,0}^1 \in J$ *on* $[0, \delta_\tau - \tau_{k,0}^1)$, $\iota_i^2(t) = \iota_{i,0}^2 \in J$ *on* $[0, \delta_\tau - \tau_{i,0}^2)$,
$\iota_k^1 = \iota_{k,n}^1$ *on* $(n\delta_\tau - \tau_{k,0}^1, (n+1)\delta_\tau - \tau_{k,0}^1)$, $\iota_i^2 = \iota_{i,n}^2$ *on* $(n\delta_\tau - \tau_{i,0}^2, (n+1)\delta_\tau - \tau_{i,0}^2)$;
$\iota_k^1 = \iota_{k,N_\tau}^1 \in J$ *on* $(N_\tau\delta_\tau - \tau_{k,0}^1, T]$, $\iota_i^2 = \iota_{i,N_\tau}^2 \in J$ *on* $(N_\tau\delta_\tau - \tau_{i,0}^2, T]$.
4. $\tau_k^1(n\delta_\tau - \tau_{k,0}^1) = 0$ *and* $\tau_i^2(n\delta_\tau - \tau_{i,0}^2) = 0$;
5. $\lim_{t \to (n\delta_\tau - \tau_{k,0}^1)^-} y_k(t) = y_k(n\delta_\tau - \tau_{k,0}^1)$, *and* $\lim_{t \to (n\delta_\tau - \tau_{i,0}^2)^-} x_i(t) = x_i(n\delta_\tau - \tau_{i,0}^2)$;
6. *For almost every* $t \in [0,T]$, *with* $u_k\colon [0,T] \mapsto I$ *a measurable control,*
$\frac{\mathrm{d}}{\mathrm{d}t}(y_k, w_k, \tau_k^1, x_i, v_i, \tau_i^2) = g_{\iota(t)}(y_k(t), w_k(t), \tau_k^1(t), x_i(t), v_i(t), \tau_i^2(t), u_k(t))$.

We shall derive the existence and uniqueness of the trajectory of hybrid system Σ_1 in the sense of DEFINITION 3.3. Let $\xi^j = (y^j, w^j)$ be the space-velocity of the autonomous vehicles in j lane. Recall that we denote by M_j and N_j the number of autonomous vehicles and the number of human-driven vehicles in j lane, respectively. Compare with Lemma 2.1 in [17], we have the following lemma.

LEMMA 3.4. *Given two locally Lipschitz convolution kernels with sub-linear growth* $H_1\colon \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ *and* $H_2\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, *and given* $\mu_n = \frac{1}{n}\sum_{l=1}^{n} \delta_{(x_l, v_l)}$ *an arbitrary atomic measure for* $(x_l, v_l) \in \mathbb{R} \times \mathbb{R}^+$, *with* $n \in \mathbb{Z}^+$, *we have, there exists a constant* $C > 0$ *such that* $|H_1 *_1 \mu_n(x,v)| \le C(1 + |(x,v)| + \frac{1}{n}\sum_{l=1}^{n} |(x_l, 0)|)$ *and* $|H_2 * \mu_n(x,v)| \le C(1 + |(x,v)| + \frac{1}{n}\sum_{l=1}^{n} |(x_l, v_l)|)$.

Proof. This is can be proved by the sub-linear growth of H_1 and H_2. ∎

As in [17], motivated by 1-Wasserstein distance, we endow space $\mathbb{R}^{2n}$ for any $n \in \mathbb{Z}^+$ with the following norm: for any $(x,v) \in \mathbb{R}^{2n}$, $\|(x,v)\| \colon = \frac{1}{n}\sum_{l=1}^{n} (|x_l| + |v_l|)$, and the metric induced by the above norm $\|\cdot\|$.

THEOREM 3.5. *Let* $H_1\colon \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$, $H_2\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *be locally Lipschitz convolution kernels with sub-linear growth. Then given an initial datum* $\xi_0 = (\iota_0, y_0, w_0, \tau_0^1, x_0, v_0, \tau_0^2)$, *there exists a unique trajectory* $\xi(t) = (\iota(t), y(t), w(t), \tau^1(t), x(t), v(t), \tau^2(t))$ *to the finite-dimensional hybrid system* Σ_1 *over the whole time interval* $[0,T]$. *Furthermore, both trajectories of the autonomous vehicles and the human-driven vehicles are Lipschitz continuous with respect to time over the time interval when there is no lane changing.*

Proof. Let $t_0 = \min_{\substack{k=1,\ldots,M \\ i=1,\ldots,N}} \{\delta_\tau - \tau_{k,0}^1, \delta_\tau - \tau_{i,0}^2\}$. Note that there is no vehicle changing lane over the time interval $[0, t_0)$ in any lane. In particular, for $t \in [0, t_0)$, the dynamics of autonomous vehicles on lane $j \in J$ satisfy

$$
\begin{aligned}
\dot{y}_k^j &= w_k^j; \\
\dot{w}_k^j &= \left(H_1 *_1 (\mu_{N_j} + \nu^j) + H_2 * (\mu_{N_j} + \nu^j)\right)(y_k^j, w_k^j) + u_k^j; \\
y_k^j(0) &= y_{k,0}^j; \\
w_k^j(0) &= w_{k,0}^j; \quad k = 1, \ldots, M_j;
\end{aligned}
\tag{3.5}
$$

For the sake of compact writing, we let $\xi^j(t) = (y^j(t), w^j(t)) \in \mathbb{R}^{2M_j}$ represent the trajectory of the autonomous vehicle on lane j over the time interval $[0, t_0)$ and re-write system (3.5) in the following form

$$\dot{\xi}^j(t) = g^j(t, \xi^j(t)), \tag{3.6}$$

where the right hand side is

$$g^j(t, \xi^j(t)) = (w^j(t), [\left(H_1 *_1 (\mu_{N_j} + \nu^j) + H_2 * (\mu_{N_j} + \nu^j)\right)(y^j_k, w^j_k) + u^j_k]_{k=1}^{M_j}).$$

Since H_1 and H_2 are locally Lipschitz with sub-linear growth, by LEMMA 3.4, we obtain

$$\|g^j(t, \xi^j(t))\| \le \bar{C}\left(1 + \|\xi^j(t)\|\right),$$

where $\bar{C} > 0$ is a constant depending on $C > 0$, $U_{\max} > 0$, but not depending on M or N. Thus the right hand side of equation (3.6) fulfills the sub-linear growth condition, by THEOREM 2.9, there exists a solution of system (3.5) on the interval $[0, t_0)$ such that $(y^j(0), w^j(0)) = (y^j_0, w^j_0)$. Moreover, for any $t \in [0, t_0)$,

$$\|\xi^j(t)\| \le (\|\xi^j_0\| + \bar{C}t_0)e^{\bar{C}t_0}.$$

In addition, the trajectory of the autonomous vehicles in j lane is Lipschitz continuous in time over the interval $[0, t_0)$. That is, for any $\tau_1, \tau_2 \in [0, t_0)$,

$$\|\xi^j(\tau_1) - \xi^j(\tau_2)\| \le \int_{\tau_1}^{\tau_2} \bar{C}(1 + \|\xi^j(s)\|)\,\mathrm{d}s \le \bar{C}(1 + (\|\xi^j_0\| + \bar{C}t_0)e^{\bar{C}t_0})|\tau_1 - \tau_2|.$$

Now for $n \ge 1$, let $t_n = \min\limits_{\substack{k=1,\dots,M\\ i=1,\dots,N}} \{\delta_\tau - \tau^1_k(t_{n-1}), \delta_\tau - \tau^2_i(t_{n-1})\}$. Then over the time interval $[t_{n-1}, t_n)$, $n \ge 1$, there is no vehicle changing lane. Similarly, one can show that the trajectory of the autonomous vehicles in j lane is unique and is Lipschitz continuous in time over the time interval $[t_{n-1}, t_n)$. Since the number of autonomous vehicle M is finite, one can repeat the above procedure for finitely many times to show that the trajectory of the autonomous vehicles on lane j is unique over the whole time interval $[0, T]$.

Furthermore, one can as well show that the trajectory of the human-driven vehicles on lane j is unique over the whole time interval $[0, T]$ and is Lipschitz continuous with respect to time over the time interval when there is no lane changing. ∎

4. The Mean-Field Limit to the Finite-Dimensional Hybrid System. In this section, we consider M autonomous vehicles and let the number of human-driven vehicles on each lane go to infinity on an open stretch of road with m lanes. We again just add controls on the M autonomous vehicles. It is possible to define a mean-flied limit of system (3.2) in the following sense: on lane $j \in J$, the population of vehicles can be represented by the vector of positions-velocities (y^j, w^j) of the autonomous vehicles coupled with the compactly supported non-negative measure $\mu^j \in \mathcal{M}^+(\mathbb{R} \times \mathbb{R}^+)$ of the human-driven vehicles in the position-velocity space. Then the mean-field limit will result in a coupled system of ODEs for (y^j, w^j) with control and a PDE for μ^j without control. Furthermore, the lane changing of the human-driven vehicles would lead to a source term to the PDE for μ^j. More specifically, the limit dynamics of vehicles on lane j when there is no autonomous vehicles changing lane is

$$\begin{aligned}
&\dot{y}^j_k = w^j_k;\\
&\dot{w}^j_k = \left(H_1 *_1 (\mu^j + \nu^j) + H_2 * (\mu^j + \nu^j)\right)(y^j_k, w^j_k) + u^j_k; \quad k = 1, \dots, M_j;\\
&\partial_t \mu^j + v^j \partial_x \mu^j + \partial_v\left(\left(H_1 *_1 (\mu^j + \nu^j) + H_2 * (\mu^j + \nu^j)\right)\mu^j\right) = S(\mu^{j-1}, \mu^j, \mu^{j+1}).
\end{aligned} \tag{4.1}$$

where $u^j_k \colon [0, T] \mapsto \mathbb{R}$ are measurable controls for $k = 1, \dots, M_j$, $H_1 \colon \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ and $H_2 \colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are locally Lipschitz convolution kernels with sub-linear growth satisfying equation (3.3), ν^j is as defined in (3.1) and the source term $S(\mu^{j-1}, \mu^j, \mu^{j-1})$ is defined as

$$\begin{aligned}
S(\mu^{j-1}, \mu^j, \mu^{j+1}) = &\left(S^{j-1,j}(\mu^{j-1}, \mu^j) - S^{j,j-1}(\mu^{j-1}, \mu^j)\right)(1 - \delta_{j,1})\\
&+ \left(S^{j+1,j}(\mu^j, \mu^{j+1}) - S^{j,j+1}(\mu^j, \mu^{j+1})\right)(1 - \delta_{j,m}),
\end{aligned} \tag{4.2}$$

with

$$S^{k,l}(\mu^k,\mu^l)=p([A^l-A^k-\Delta]_+)\mu^k, k,l\in\{j-1,j,j+1\} \quad \text{and} \quad k=l+1 \quad \text{or} \quad k=l-1. \tag{4.3}$$

Here $p\colon \mathbb{R}\mapsto[0,1]$ is increasing and is the probability of the large population of human-driven vehicles performing lane changing from k lane to l lane. In addition, if $a\leq 0$, then $p(a)=0$. We assume that the dimension of p to be $\frac{1}{t}$. This modeling choice is similar to [27, 22]. In addition, $A^l=H_1 *_1(\mu^l+\nu^l)+H_2*(\mu^l+\nu^l)$ is the average acceleration of vehicles on lane l. Equation (4.3) can be interpreted as the following: Let $\Delta>0$ be fixed. A large population of human-driven vehicles on lane k will perform lane changing to lane l with probability $p\in[0,1]$ if the following condition occur: $A^l>A^k+\Delta$.

Furthermore, system (4.1) implies that the acceleration of autonomous vehicle k on lane j is,

$$a_k^j=\left(H_1*_1(\mu^j+\nu^j)+H_2*(\mu^j+\nu^j)\right)(y_k^j,w_k^j)+u_k^j.$$

The k-th autonomous vehicle on lane j will perform lane changing to $j'=(j-1)(1-\delta_1(j))$ or $j'=(j+1)(1-\delta_m(j))$ lane if the following condition occur: $A^{j'}\geq a_k^j+\Delta$.

We again assign each autonomous vehicle a timer over the whole time interval $[0,T]$ such that there are no two autonomous vehicles changing lane at the same time. We define the timer τ_k^1 for autonomous vehicle k and the timer limit T_1 as before.

The continuous dynamics of vehicles governed by system (4.1) and the discrete lane changing dynamics of the autonomous vehicles lead us to consider the following hybrid system.

Definition 4.1. *A hybrid ODE-PDE system is a 6-tuple* $\Sigma_2=(\mathcal{L},\mathcal{M},U,\mathcal{U},g,S)$ *where*

H_1 $\mathcal{L}=\{\iota=(\iota_k)_{k=1,\dots,M},\iota_k\in J\}=J^M$ *is the set of locations;*

H_2 $\mathcal{M}=\{\mathcal{M}_\iota\}_{\iota\in\mathcal{L}}$, *where* $\mathcal{M}_\iota=(\mathbb{R}\times\mathbb{R}^+\times[0,\delta_\tau))^M\setminus A_\iota\times\left(\mathcal{M}^+(\mathbb{R}^2)\right)^m$,

$A_\iota=\Big\{\left(y_k,w_k,\tau_k^1\right)_{k=1,\dots,M}: y_k\in\mathbb{R},w_k\in\mathbb{R}^+,\tau_k^1\in[0,\delta_\tau)$ *and* $\exists k_1\neq k_2\in\{1,\dots,M\}$, *s.t.* $\iota_{k_1}=\iota_{k_2}$ *and* $y_{k_1}=y_{k_2}\}$;

H_3 $U=\{U_\iota\}_{\iota\in\mathcal{L}},U_\iota=I^M$, *where* $I\subset\mathbb{R}$ *is compact;*

H_4 $\mathcal{U}=\{\mathcal{U}_\iota\}_{\iota\in\mathcal{L}},\mathcal{U}_\iota=\left\{u\colon[0,T]\mapsto U_\iota=I^M\right\}$;

H_5 $g=\{g_\iota\}_{\iota\in\mathcal{L}}$,

$g_\iota\colon\mathcal{M}_\iota\times\mathcal{U}_\iota\mapsto(\mathbb{R}^3)^M, g_\iota((y_k,w_k,\tau_k^1,u_k,\mu^{\iota_k})_{k=1,\dots,M})=(v_k,a_k,1)_{k=1,\dots,M}$,
where $a_k=(H_1*_1(\mu^{\iota_k}+\nu^{\iota_k})+H_2*(\mu^{\iota_k}+\nu^{\iota_k}))(y_k,w_k)+u_k$;

H_6 S *is a subset of* $LC(\Sigma_2)$, *where*

$LC(\Sigma_2)=\left\{\left(\iota,(y_k,w_k,\tau_k^1,\mu^{\iota_k})_{k=1,\dots,M},\iota',\left(y_k',w_k',(\tau_k^1)',\mu^{\iota_k'}\right)_{k=1,\dots,M}\right):\right.$
$\exists p\in\{1,\dots,M\},t_p\in[0,T]$, *s.t.* $\forall k\neq p$,
$(y_k(t_p),w_k(t_p),\tau_k^1(t_p))=(y_k'(t_p),w_k'(t_p),(\tau_k^1)'(t_p)),\iota_k'(t_p)=\iota_k(t_p)$,
and $y_p(t_p)=y_p'(t_p),w_p(t_p)=w_p'(t_p),(\tau_p^1)'(t_p)=0,\iota_p'(t_p)=\iota_p(t_p)\pm1\big\}$.

Now we will define the hybrid state of the hybrid system Σ_2.

Definition 4.2. *A hybrid state of the hybrid system* Σ_2 *is a 5-tuple* (ι,y,w,τ^1,μ), *where* ι *is the location,* $(y,w,\tau^1,\mu)\in\mathcal{M}_\iota$. *We denote by* $\mathcal{HS}_2$ *the set of all hybrid states of the hybrid system* Σ_2.

Next we will give the definition of the trajectory of the hybrid system Σ_2.

Definition 4.3. *A trajectory of the hybrid system* Σ_2 *with initial condition* $(\iota_0,y_0,w_0,\tau_0^1,\mu_0)\in J^M\times\mathbb{R}^M\times(\mathbb{R}^+)^M\times[0,\delta_\tau)^M\times\left(\mathcal{M}^+(\mathbb{R}^2)\right)^m$ (*if* $k_1\neq k_2\in\{1,\dots,M\}$, *then* $\tau_{k_1,0}^1\neq\tau_{k_2,0}^1$) *is a map* $\xi\colon[0,T]\mapsto\mathcal{HS}_2$,
$\xi(t)=(\iota(t),y(t),w(t),\tau^1(t),\mu(t))$ *such that for* $k=1,\dots,M$ *and* $n=1,\dots,N_\tau-1$, *the following holds:*

1. $\tau_k^1(0)=\tau_{k,0}^1\in[0,\delta_\tau)$;
2. $(y_k(0),w_k(0))=(y_{k,0},w_{k,0})\in\mathbb{R}\times\mathbb{R}^+$;
3. *For* $t\in[0,\delta_\tau-\tau_{k,0}^1),\iota_k(t)=\iota_{k,0}\in J$,

$\iota_k(\cdot)$ *is constant in* $[n\delta_\tau - \tau_{k,0}, (n+1)\delta_\tau - \tau_{k,0})$, *and is equal to* $\iota_{k,n} \in J$;
4. $\tau_k^1(n\delta_\tau - \tau_{k,0}^1) = 0$;
5. $\lim_{t \to (n\delta_\tau - \tau_{k,0}^1)^-} y_k(t)$ *exists and is equal to* $y_k(n\delta_\tau - \tau_{k,0}^1)$;
6. *For every* $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$, *and for all* $t \in [0,T]$, $\mu^{\iota_k(t)}$ *satisfies* $\operatorname{supp}\mu^{\iota_k(t)} \subset B(0,R)$ *for some* $R>0$, *and for almost every* $t \in [0,T]$,
$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}\times\mathbb{R}^+} \varphi(x,v)\,\mathrm{d}\mu^{\iota_k(t)}(t)(x,v) = \int_{\mathbb{R}\times\mathbb{R}^+}\varphi(x,v)\,\mathrm{d}S(\mu^{\iota_k(t)-1},\mu^{\iota_k(t)},\mu^{\iota_k(t)+1})(t)(x,v)$
$+\int_{\mathbb{R}\times\mathbb{R}^+}\left(\nabla\varphi(x,v)\cdot\omega_{H_1,H_2,\mu^{\iota_k(t)},x^{\iota_k(t)},v^{\iota_k(t)}}(t,x,v)\right)\mathrm{d}\mu^{\iota_k(t)}(t)(x,v)$,
where $\omega_{H_1,H_2,\mu^{\iota_k(t)},x^{\iota_k(t)},v^{\iota_k(t)}}(t,x,v) :=$
$=\left(v, \left(H_1 *_1 (\mu^{\iota_k(t)}(t)+\nu^{\iota_k(t)}(t)) + H_2 * (\mu^{\iota_k(t)}(t)+\nu^{\iota_k(t)}(t))\right)(x,v)\right)$.
7. *For almost every* $t\in[0,T]$, *with* $u_k\colon[0,T]\mapsto I$ *a measurable control*
$\frac{\mathrm{d}}{\mathrm{d}t}(y_k(t),w_k(t),\tau_k^1(t)) = g_{\iota_k(t)}(y_k(t),w_k(t),\tau_k^1(t),u_k(t),\mu^{\iota_k(t)}(t))$.

Before actually proving the existence of trajectories of the hybrid system Σ_2 as in DEFINITION 4.3, it will be convenient to address the stability of the hybrid system Σ_2 with respect to the initial data first.

Let $t_0^1 = \min_{k=1,\dots,M}\left\{T_1 - \tau_{k,0}^1\right\}$. Then there is no autonomous vehicle changing lane over the time interval $[0,t_0^1)$ on any lane. As in THEOREM 3.5, it is enough to show the stability of the hybrid system Σ_2 with respect to the initial data over the time interval $[0,t_0^1)$. In particular, for $t\in[0,t_0^1)$, the dynamics of autonomous vehicles in $j\in J$ lane satisfy

$$
\begin{aligned}
\dot y_k^j &= w_k^j;\\
\dot w_k^j &= \left(H_1 *_1 (\mu^j+\nu^j) + H_2 * (\mu^j+\nu^j)\right)(y_k^j,w_k^j) + u_k^j;\\
y_k^j(0) &= y_{k,0}^j;\\
w_k^j(0) &= w_{k,0}^j;\quad k=1,\dots,M_j;
\end{aligned}
\tag{4.4}
$$

and the dynamics of human-driven vehicles in $j\in J$ lane satisfy

$$
\begin{aligned}
\partial_t\mu^j + v\partial_x\mu^j + \partial_v\left(\left(H_1 *_1 (\mu^j+\nu^j) + H_2*(\mu^j+\nu^j)\right)\mu^j\right) &= S(\mu^{j-1},\mu^j,\mu^{j+1}),\\
\mu^j(0) &= \mu_0^j,
\end{aligned}
\tag{4.5}
$$

where the atomic measure ν^j is defined as in equation (3.1) and the source term S is defined as in equation (4.2). Furthermore, we endow space $\mathcal{X}_n\colon \mathbb{R}^{2n}\times\mathcal{M}^+(\mathbb{R}\times\mathbb{R}^+)$ for any $n\in\mathbb{Z}^+$ with the following metric : for any $(y_1,w_1,\mu_1),(y_2,w_2,\mu_2)\in\mathcal{X}_n$,

$$
\|(y_1,w_1,\mu_1)-(y_2,w_2,\mu_2)\|_{\mathcal{X}_n} := \frac{1}{n}\sum_{k=1}^n\left(|y_{k,1}-y_{k,2}|+|w_{k,1}-w_{k,2}|\right) + W_1^{1,1}(\mu_1,\mu_2),
$$

where $W_1^{1,1}$ is the generalized Wasserstein distance in $\mathcal{M}^+(\mathbb{R}\times\mathbb{R}^+)$.

LEMMA 4.4. *Let* $\mu^{j,q}$ *be two solutions to system* (4.5) *over the time interval* $[0,t_0^1)$ *with two different initial data* $\mu_0^{j,q}$, $q=1,2$. *Then there exists* $\bar C>0$ *such that,*

$$
\begin{aligned}
W_1^{1,1}(\mu^{j,1}(t),\mu^{j,2}(t)) \le \bar C\Big(&W_1^{1,1}(\mu_0^{j,1},\mu_0^{j,2}) + \\
&+\int_0^t \|(y^{j,1}(s),w^{j,1}(s),\mu^{j,1}(s)) - (y^{j,2}(s),w^{j,2}(s),\mu^{j,2}(s))\|_{\mathcal{X}_{M_j}}\,\mathrm{d}s\Big).
\end{aligned}
\tag{4.6}
$$

Proof. Let $\mu^{j,q}$ be two solutions to system (4.5) over the time interval $[0,t_0^1)$ with two different initial data $\mu_0^{j,q}$, $q=1,2$. Let $t\in[0,t_0^1)$ be fixed and let $\Delta t = \frac{t_0^1}{2^k}$ for a fixed $k\in\mathbb{N}^+$. Decompose the time interval $[0,t_0^1)$ into $[0,\Delta t]$, $[\Delta t, 2\Delta t], \dots, [(2^k-1)\Delta t, 2^k\Delta t)$. Let n be the maximum integer such that $t-n\Delta t\ge 0$, then $t\in[n\Delta t,(n+1)\Delta t)$. By section 2.4, we have, $\mu^{j,q}(t) = \lim_{k\to\infty}\mu_k^{j,q}(t)$, where $q = 1, 2$ and $\mu_k^{j,q}$ is defined as following:

$$
\begin{aligned}
&\mu_k^{j,q}(0) = \mu_0^{j,q},\\
&\mu_k^{j,q}((n+1)\Delta t) = \mathcal{T}_{\Delta t}^{\mu_k^{j,q}(n\Delta t),\nu^{j,q}(n\Delta t)} \# \mu_k^{j,q}(n\Delta t) + \Delta t S(\mu_k^{j,q}(n\Delta t)),\\
&\mu_k^{j,q}(t) = \mathcal{T}_{\tau}^{\mu_k^{j,q}(n\Delta t),\nu^{j,q}(n\Delta t)} \# \mu_k^{j,q}(n\Delta t) + \tau S(\mu_k^{j,q}(n\Delta t)),
\end{aligned}
$$

where $\tau = t - n\Delta t$ and $\nu^{j,q}(n\Delta t) = \frac{1}{M_j}\sum_{k=1}^{M_j} \delta_{\left(y_k^{j,q}(n\Delta t), w_k^{j,q}(n\Delta t)\right)}$, with $(y_k^{j,q}(n\Delta t), w_k^{j,q}(n\Delta t))$ being the vector of position-velocity of the k-th autonomous vehicle on lane j at time $n\Delta t$ when the initial data to system (4.5) is given by $\mu_0^{j,q}$. Note that

$$
\begin{aligned}
&W_1^{1,1}(\mu_k^{j,1}(t), \mu_k^{j,2}(t)) \le W_1^{1,1}\left(\tau S(\mu_k^{j,1}(n\Delta t)), \tau S(\mu_k^{j,2}(n\Delta t))\right)\\
&+ W_1^{1,1}\left(\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,1}(n\Delta t), \mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t)\right)\\
&+ W_1^{1,1}\left(\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t), \mathcal{T}_{\tau}^{\mu_k^{j,2}(n\Delta t),\nu^{j,2}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t)\right),
\end{aligned}
$$

where the last inequality is due to PROPOSITION 2.6. By the properties of the source term S, (S_2), and of the generalized Wasserstein distance $W_1^{1,1}$, PROPOSITION 2.6, there exists some constant L_S such that

$$
W_1^{1,1}\left(\tau S(\mu_k^{j,1}(n\Delta t)), \tau S(\mu_k^{j,2}(n\Delta t))\right) \le \tau L_S W_1^{1,1}(\mu_k^{j,1}(n\Delta t), \mu_k^{j,2}(n\Delta t)).
$$

Since the flow map $\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)}$ is Lipschitz, by LEMMA 2.7, there exists some constant L_1, such that,

$$
\begin{aligned}
&W_1^{1,1}\left(\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,1}(n\Delta t), \mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t)\right)\\
\le& L_1 W_1^{1,1}(\mu_k^{j,1}(n\Delta t), \mu_k^{j,2}(n\Delta t)).
\end{aligned}
$$

Since the flow maps $\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)}$ and $\mathcal{T}_{\tau}^{\mu_k^{j,2}(n\Delta t),\nu^{j,2}(n\Delta t)}$ are bounded and Borel measurable, by LEMMA 2.7, equation (2.11) and LEMMA 2.8, there exist $L_{\mathcal{T}}, \rho, L_* > 0$, such that

$$
\begin{aligned}
&W_1^{1,1}\left(\mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t), \mathcal{T}_{\tau}^{\mu_k^{j,2}(n\Delta t),\nu^{j,2}(n\Delta t)} \# \mu_k^{j,2}(n\Delta t)\right)\\
\le& \left\| \mathcal{T}_{\tau}^{\mu_k^{j,1}(n\Delta t),\nu^{j,1}(n\Delta t)} - \mathcal{T}_{\tau}^{\mu_k^{j,2}(n\Delta t),\nu^{j,2}(n\Delta t)} \right\|_{L^\infty(B(0,R))}\\
\le& L_* \int_{n\Delta t}^t e^{L_{\mathcal{T}}(s-t)} \left[\left(\frac{1}{M_j} \sum_{k=1}^{M_j} (|y_k^{j,1}(s) - y_k^{j,2}(s)| + |w_k^{j,1}(s) - w_k^{j,2}(s)|) \right) \right.\\
&\left. W_1^{1,1}(\mu_k^{j,1}(s), \mu_k^{j,2}(s)) \right] \mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&W_1^{1,1}(\mu_k^{j,1}(t), \mu_k^{j,2}(t)) \le (\tau L_S + L_1)\, W_1^{1,1}(\mu_k^{j,1}(n\Delta t), \mu_k^{j,2}(n\Delta t))\\
&+ L_* \int_{n\Delta t}^t e^{L_{\mathcal{T}}(s-t)} \left[\left(\frac{1}{M_j} \sum_{k=1}^{M_j} \left(|y_k^{j,1}(s) - y_k^{j,2}(s)| + |w_k^{j,1}(s) - w_k^{j,2}(s)| \right) \right) \right. \qquad (4.7)\\
&\left. + W_1^{1,1}(\mu_k^{j,1}(s), \mu_k^{j,2}(s)) \right] \mathrm{d}s.
\end{aligned}
$$

Similarly, there exists $L_2 > 0$, such that

$$
\begin{aligned}
&W_1^{1,1}(\mu_k^{j,1}(n\Delta t), \mu_k^{j,2}(n\Delta t)) \\
\le &(L_2 + \Delta t L_S) W_1^{1,1}(\mu_k^{j,1}((n-1)\Delta t), \mu_k^{j,2}((n-1)\Delta t)) \\
&+ L_* \int_{(n-1)\Delta t}^{n\Delta t} e^{L_\mathcal{T}(s-t)} \left[\left(\frac{1}{M_j} \sum_{k=1}^{M_j} \left(|y_k^{j,1}(s) - y_k^{j,2}(s)| + |w_k^{j,1}(s) - w_k^{j,2}(s)| \right) \right) \right. \\
&\left. + W_1^{1,1}(\mu_k^{j,1}(s) - \mu_k^{j,2}(s)) \right] \mathrm{d}s.
\end{aligned}
\tag{4.8}
$$

Combine with equations (4.7) and (4.8), and the definition of norm $\|\cdot\|_{\mathcal{X}_{M_j}}$, we obtain there exists C_0 such that

$$
\begin{aligned}
&W_1^{1,1}(\mu_k^{j,1}(t), \mu_k^{j,2}(t)) \le C_0 \left(W_1^{1,1}(\mu_0^{j,1}, \mu_0^{j,2}) \right. \\
&\left. + \int_0^t \|(y^{j,1}(s), w^{j,1}(s), \mu_k^{j,1}(s)) - (y^{j,2}(s), w^{j,2}(s), \mu_k^{j,2}(s))\|_{\mathcal{X}_{M_j}} \, \mathrm{d}s \right).
\end{aligned}
$$

Take $k \to \infty$ and consider the definition of $\mu^{j,p}$, $p = 1, 2$, we have, there exists $\bar{C}$ such that inequality (4.6) is true. ∎

THEOREM 4.5. *Let $(y^{j,i}, w^{j,i})$ be two solutions of system (4.4) relative to given respective initial data $(y_0^{j,i}, w_0^{j,i})$ and let $\mu^{j,i}$ be two solutions of system (4.5) relative to given respective initial data $\mu_0^{j,i}$, $i = 1, 2$ over the time interval $[0, t_0^1)$. Then there exists a constant $C > 0$ such that*

$$
\begin{aligned}
&\left\| \left(y^{j,1}(t), w^{j,1}(t), \mu^{j,1}(t) \right) - \left(y^{j,2}(t), w^{j,2}(t), \mu^{j,2}(t) \right) \right\|_{\mathcal{X}_{M_j}} \\
\le &C \left\| \left(y_0^{j,1}, w_0^{j,1}, \mu_0^{j,1} \right) - \left(y_0^{j,2}, w_0^{j,2}, \mu_0^{j,2} \right) \right\|_{\mathcal{X}_{M_j}}.
\end{aligned}
$$

Remark 4.6. THEOREM 4.5 implies that the trajectory of hybrid system Σ_2, if exists, is uniquely determined by the initial conditions.

Proof. By integration we have, for $t \in [0, t_0^1)$,

$$
y_k^{j,i}(t) = \int_0^t w_k^{j,i}(s)\,\mathrm{d}s + y_{k,0}^{j,i}, \quad i = 1, 2.
$$

Thus

$$
|y_k^{j,1}(t) - y_k^{j,2}(t)| \le |y_{k,0}^{j,1} - y_{k,0}^{j,2}| + \int_0^t |w_k^{j,1}(s) - w_k^{j,2}(s)|\,\mathrm{d}s.
\tag{4.9}
$$

In addition, by LEMMA 2.8, there exists a constant L_R, such that

$$
\begin{aligned}
&|w_k^{j,1}(t) - w_k^{j,2}(t)| \le |w_{k,0}^{j,1} - w_{k,0}^{j,2}| \\
&+ L_R \int_0^t \left(\frac{1}{M_j} \sum_{k=1}^{M_j} \left(|y_k^{j,1}(s) - y_k^{j,2}(s)| + |w_k^{j,1}(s) - w_k^{j,2}(s)| \right) \right. \\
&\left. + W_1^{1,1}(\mu^{j,1}(s), \mu^{j,2}(s)) \right) \mathrm{d}s.
\end{aligned}
\tag{4.10}
$$

Combine with equations (4.6) (4.9), (4.10), and the definition of the norm $\|\cdot\|_{\mathcal{X}_{M_j}}$, we have, there exists a constant C, s.t.,

$$
\|(y^{j,1}(t), w^{j,1}(t), \mu^{j,1}(t)) - (y^{j,2}(t), w^{j,2}(t), \mu^{j,2}(t))\|_{\mathcal{X}_{M_j}}
$$

$$\leq C\Big(\|(y_0^{j,1},w_0^{j,1},\mu_0^{j,1})-(y_0^{j,2},w_0^{j,2},\mu_0^{j,2}))\|_{\mathcal{X}_{M_j}}$$
$$+\int_0^t \|(y^{j,1}(s),w^{j,1}(s),\mu^{j,1}(s))-(y^{j,2}(s),w^{j,2}(s),\mu^{j,2}(s))\|_{\mathcal{X}_{M_j}}\,\mathrm{d}s\Big).$$

One can conclude the stability estimate by applying Gronwall's inequality. ∎

We shall now derive the existence of the trajectory of the hybrid system Σ_2. It is enough to show that the trajectories of the vehicles exist over the time interval $[0,t_0^1)$.

THEOREM 4.7. *On lane $j\in J$, let $(y_{k,0}^j,w_{k,0}^j)\in\mathbb{R}\times\mathbb{R}^+$, $k=1,\dots,M_j$, $\mu_0^j\in\mathcal{M}^+(\mathbb{R}\times\mathbb{R}^+)$ and $u_*\in L^1([0,T],\mathcal{U})$ be given. In addition, assume that μ_0^j is of bounded support in $B(0,R)$ for $R>0$. Then the trajectories of the vehicles exist on lane j over the time interval $[0,t_0^1)$.*

Proof. We will first construct a sequence of atomic measures to approximate the measure μ_0^j in generalized Wasserstein distance. For every $N\in\mathbb{N}^+$, consider the atomic measure

$$\mu_0^{j,N}=\sum_{i=1}^N\frac{\|\mu_0^j\|}{N}\delta_{\left(x_{i,0}^{j,N},v_{i,0}^{j,N}\right)},\tag{4.11}$$

such that $\lim_{N\to\infty}W_1^{1,1}(\mu_0^{j,N},\mu_0^j)=0$. Here we call $\frac{\|\mu_0^j\|}{N}$ the average mass of the human-driven vehicle on lane j.

In addition, fix a weakly convergent sequence $(u_N)_{N\in\mathbb{N}}$ in $L^1([0,T],\mathcal{U})$ of control functions such that $u_N\rightharpoonup u_*$ in $L^1([0,T],\mathcal{U})$. By THEOREM 3.5, for each initial datum $\xi_0^{j,N}=(y_0^j,w_0^j,x_0^{j,N},v_0^{j,N})$ depending on N, there exists a unique trajectory of the hybrid system Σ_1 with control u_N over the time interval $[0,t_0^1)$.

Denote the trajectories of the vehicles on lane j over the time interval $[0,t_0^1]$ with $\xi_N^j(t)=(y_N^j(t),w_N^j(t),\mu_N^j(t))\in\mathcal{X}_{M_j}$. Here we identify $(x_N^j(x),v_N^j(t))$, the vector of position-velocity of the human-driven vehicles with atomic measure $\mu_N^j(t)$ by means of equation (4.11). By THEOREM 3.5, the trajectories of the vehicles are Lipschitz continuous with respect to time over the time interval when there is no lane changing. Furthermore, note that the average mass of a human-driven vehicle $\frac{\|\mu_0^j\|}{N}\to0$ as $N\to\infty$. Thus there exists $L>0$, such that for any $\varepsilon>0$, there exists $\tilde N>0$, such that whenever $N\geq\tilde N$, $\|\xi_N^j(t)-\xi_N^j(s)\|_{\mathcal{X}_{M_j}}\leq L|t-s|+\min\{\varepsilon,|s-t|\}$. By THEOREM 2.11, there exists a sub-sequence, again denoted by $\xi_N^j(\cdot)=(y_N^j(\cdot),w_N^j(\cdot),\mu_N^j(\cdot))$ converging uniformly to a limit $\xi_*^j(\cdot)=(y_*^j(\cdot),w_*^j(\cdot),\mu_*^j(\cdot))$. We will first verify that $(y_*^j(\cdot),w_*^j(\cdot))$ is a solutions of system (4.4) for $\mu^j=\mu_*^j$ and $u^j=u_N^j$.

Note that $\xi_N^j\rightrightarrows\xi_*^j$ implies that

$$\begin{aligned}&(y_N^j(t),w_N^j(t))\rightrightarrows(y_*^j(t),w_*^j(t))\text{ in }[0,t_0^1);\\&(\dot y_N^j(t),\dot w_N^j(t))\rightharpoonup(\dot y_*^j(t),\dot w_*^j(t))\text{ in }L^1([0,t_0^1),\mathbb{R}\times\mathbb{R}^+);\\&\lim_{N\to\infty}W_1^{1,1}(\mu_N^j(t),\mu_*^j(t))=0.\end{aligned}$$

In particular, $\dot y_{k,*}^j(t)=w_{k,*}^j(t),$ for all $k=1,\dots,M_j$. Furthermore, let us denote now

$$\nu_N^j=\frac{1}{M_j}\sum_{k=1}^{M_j}\delta_{(y_{k,N}^j(t),w_{k,N}^j(t))}\text{ and }\nu_*^j=\frac{1}{M_j}\sum_{k=1}^{M_j}\delta_{(y_{k,*}^j(t),w_{k,*}^j(t))}.$$

By the uniform convergence of the trajectories and LEMMA 2.2, we have, as $N\to+\infty$, $W_1(\nu_N^j(t),\nu_*^j(t))\to0$. In addition, by the sublinear growth of H_1 and H_2, we have, as $N\to\infty$,

$$\begin{aligned}&(H_1*_1(\mu_N^j+\nu_N^j)+H_2*(\mu_N^j+\nu_N^j))(y_{k,N}^j(t),w_{k,N}^j(t))\\\rightrightarrows&(H_1*_1(\mu_*^j+\nu_*^j)+H_2*(\mu_*^j+\nu_*^j))(y_{k,*}^j(t),w_{k,*}^j(t)).\end{aligned}$$

By the weak convergence of u_N to u_* and of $\dot w_N^j$ to $\dot w_*^j$, for every $\tau \in [0, t_0^1]$, $\int_0^\tau \dot w_{k,*}^j(t)\,dt = \int_0^\tau \Big((H_1 *_1 (\mu_*^j + \nu_*^j) + H_2 * (\mu_*^j + \nu_*^j))(y_{k,*}^j(t), w_{k,*}^j(t)) + u_{k,*}^j(t) \Big)\,\mathrm{d}t$.

Now we will verify that μ_*^j is a solution to system (4.5) for $\nu^j = \nu_*^j$.

For any time $t \in [0, t_0^1]$, let N_1^j be the number of human-driven vehicles that still stay on lane j and let $(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t))$ be the location-velocity of the i-th human-driven vehicle that does not perform lane changing on lane j. Then we can track the position of those human-driven vehicles by an atomic measure

$\mu_{N,1}^j(t) = \sum_{i=1}^{N_1^j} \frac{\|\mu_0^j\|}{N} \delta_{\left(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t)\right)}$.

For all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, consider the following differentiation

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi, \mu_{N,1}^j(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N_1^j} \frac{\|\mu_0^j\|}{N} \varphi(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t)) \\
=&\frac{\|\mu_0^j\|}{N} \Bigg[\sum_{i=1}^{N_1^j} \partial_x \varphi(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t)) v_i^{j,N_1^j}(t) + \sum_{i=1}^{N_1^j} \partial_v \varphi(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t)) \\
&(H_1 *_1 (\mu_N^j + \nu_N^j) + H_2 * (\mu_N^j + \nu_N^j))(x_i^{j,N_1^j}(t), v_i^{j,N_1^j}(t)) \Bigg].
\end{aligned}
$$

Thus for all $s \in [0, t_0^1)$, we have

$$
\begin{aligned}
&\langle \varphi, \mu_{N,1}^j(s) - \mu_{N,1}^j(0)\rangle = \int_0^s \Bigg[\int_{\mathbb{R}\times\mathbb{R}^+} \partial_x \varphi(x,v) v \\
&+ \partial_v \varphi(x,v)(H_1 *_1 (\mu_N^j + \nu_N^j) + H_2 * (\mu_N^j + \nu_N^j))(x,v)\,\mathrm{d}\mu_{N,1}^j(t)(x,v) \Bigg]\,\mathrm{d}t.
\end{aligned}
$$

Furthermore,

$$
\lim_{N\to\infty} \langle \varphi, \mu_{N,1}^j(s) - \mu_{N,1}^j(0)\rangle = \langle \varphi, \mu_*^j - \mu_0^j\rangle. \tag{4.12}
$$

By dominate convergence theorem, we obtain the limit (possibly for a sub-sequence) that

$$
\begin{aligned}
&\lim_{N\to\infty} \int_0^s \int_{\mathbb{R}\times\mathbb{R}^+} (\nabla_x \varphi(x,v)\cdot v)\; d\mu_{N,1}^j(t)(x,v)\,\mathrm{d}t = \\
&= \int_0^s \int_{\mathbb{R}\times\mathbb{R}^+} (\nabla_x \varphi(x,v)\cdot v)\; d_{\mu_*^j}(t)(x,v)\,\mathrm{d}t,
\end{aligned} \tag{4.13}
$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$. Furthermore, by LEMMA 2.2 and LEMMA 2.8, we have, for every $\rho > 0$,

$$
\begin{aligned}
&\lim_{N\to\infty} \Big\| \Big(H_1 *_1 (\mu_N^j + \nu_N^j) + H_2 * (\mu_N^j + \nu_N^j) \Big) - \\
&- \big((H_1 *_1 (\mu_*^j + \nu_*^j) + H_2 * (\mu_*^j + \nu_*^j) \big) \Big\|_{L^\infty(B(0,\rho))} = 0.
\end{aligned}
$$

Now since $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ has compact support, we obtain

$$
\begin{aligned}
&\lim_{N\to\infty} \Big\| \partial_v \varphi \Big(\Big(H_1 *_1 (\mu_N^j + \nu_N^j) + H_2 * (\mu_N^j + \nu_N^j) \Big) \\
&- \Big((H_1 *_1 (\mu_*^j + \nu_*^j) + H_2 * (\mu_*^j + \nu_*^j) \Big) \Big) \Big\|_\infty = 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{k\to\infty} \int_0^s \int_{\mathbb{R}\times\mathbb{R}^+} \partial_v \varphi(x,v)(H_1 *_1 (\mu_N^j + \nu_N^j) + \\
&+ H_2 * (\mu_N^j + \nu_N^j))(x,v)\,\mathrm{d}\mu_{N,1}^j(t)(x,v)\,\mathrm{d}t \\
&= \int_0^s \int_{\mathbb{R}\times\mathbb{R}^+} \partial_v \varphi(x,v)(H_1 *_1 (\mu_N^j + \nu_N^j) + H_2 * (\mu_N^j + \nu_N^j))(x,v)\,\mathrm{d}\mu_{N,1}^j(t)(x,v)\,\mathrm{d}t.
\end{aligned} \tag{4.14}
$$

By the lane changing condition, we define

$$\begin{aligned}
\mu^j_{N,2}(t) = &\sum_{i=1}^{N_{j-1}} \frac{\|\mu_0^{j-1}\|}{N}\delta_{\left(x_i^{j-1,N_{j-1}}(t),v_i^{j-1,N_{j-1}}(t)\right)} p\left([A^j - A^{j-1} - \Delta]_+\right)\\
&- \sum_{i=1}^{N_j} \frac{\|\mu_0^{j}\|}{N}\delta_{\left(x_i^{j,N_j}(t),v_i^{j,N_j}(t)\right)} p\left([A^{j-1} - A^{j} - \Delta]_+\right)\\
&+ \sum_{i=1}^{N_{j+1}} \frac{\|\mu_0^{j+1}\|}{N}\delta_{\left(x_i^{j+1,N_{j+1}}(t),v_i^{j+1,N_{j+1}}(t)\right)} p\left([A^j - A^{j+1} - \Delta]_+\right)\\
&- \sum_{i=1}^{N_j} \frac{\|\mu_0^{j}\|}{N}\delta_{\left(x_i^{j,N_j}(t),v_i^{j,N_j}(t)\right)} p\left([A^{j+1} - A^{j} - \Delta]_+\right)
\end{aligned}$$

where

$$\begin{aligned}
A^{j-1} &= \Big(H_1 *_1 (\mu_N^{j-1}(t) + \nu_N^{j-1}(t)) + H_2 * (\mu_N^{j-1}(t) + \nu_N^{j-1}(t))\Big)(x,v),\\
A^{j} &= \Big(H_1 *_1 (\mu_N^{j}(t) + \nu_N^{j}(t)) + H_2 * (\mu_N^{j}(t) + \nu_N^{j}(t))\Big)(x,v),\\
A^{j+1} &= \Big(H_1 *_1 (\mu_N^{j+1}(t) + \nu_N^{j+1}(t)) + H_2 * (\mu_N^{j+1}(t) + \nu_N^{j+1}(t))\Big)(x,v).
\end{aligned}$$

Therefore, $\mu^j_N(t) = \mu^j_{N,1}(t) + \mu^j_{N,2}(t)$, and in addition,

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{i=1}^{N_{j-1}} \frac{\|\mu_0^{j-1}\|}{N}\delta_{\left(x_i^{j-1,N_{j-1}}(t),v_i^{j-1,N_{j-1}}(t)\right)} p\left([A^j - A^{j-1} - \Delta]_+\right)\\
=&\mu_*^{j-1} p\Big(\big[\big(H_1 *_1 (\mu^j_* + \nu^j_*) + H_2 * (\mu^j_* + \nu^j_*)\big) -\\
&- \big(H_1 *_1 (\mu^{j-1}_* + \nu^{j-1}_*) + H_2 * (\mu^{j-1}_* + \nu^{j-1}_*)\big) - \Delta\big]_+\Big) = S^{j-1,j}(\mu_*^{j-1},\mu_*^j).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\lim_{N\to\infty} \mu^j_{N,2}(t) = &\left(S^{j-1,j}(\mu_*^{j-1},\mu_*^j) - S^{j,j-1}(\mu_*^{j-1},\mu_*^j)\right)(1-\delta_{j,1})\\
&+ \left(S^{j+1,j}(\mu_*^{j},\mu_*^{j+1}) - S^{j,j+1}(\mu^{j},\mu^{j+1})(1-\delta_{j,m})\right) = S(\mu_*^{j-1},\mu_*^j,\mu_*^{j+1}).
\end{aligned} \tag{4.15}$$

The statement follows by combining equations (4.12), (4.13), (4.14), and (4.15). ∎

5. Conclusion. In this paper we have focused on a multi-lane multi-class description of vehicular traffic flow, where simultaneous presence of human-driven and autonomous vehicles has been considered.

The microscopic dynamics have been formulated by using a Bando-Follow-the-Leader type model, in which the interaction with the closest vehicle ahead is replaced by a space-dependent convolution kernel modeling interactions with the surrounding flow. Autonomous vehicles have been distinguished by control dynamics. Lane changing description has led to discrete events within the differential equations, and thus to a so-called hybrid system whose well-posedness has been studied.

Inspired by the empirical fact that the *penetration rate* of the autonomous vehicles is nowadays small, we have computed a mean-field limit for the dynamics of the human-driven vehicles only, leading to a coupled system of a PDE and ODEs with discrete events. The discrete lane changing descriptions for human-driven vehicles has been modeled by a source term of the corresponding Vlasov-type equation. Existence and uniqueness study of the trajectories of this system has been performed.

We point-out that the given application, based on traffic flow, inspiring this work is not restrictive, and many others may lead to the mathematical frameworks developed and studied here.

More precisely, we refer to all physical multi-agent systems that are intrinsically characterized by heterogeneity and instantaneous jumps in one of their states. For instance, these include also models for air traffic control [46], chemical process control [14] and manufacturing [34].

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