A COPY METHOD
TO CHANGE AN OBLIQUE VIEW INTO AN ORTHOGRAPHIC VIEW

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ABSTRACT: The use of a laser-copier to eliminate distortions of oblique pictorial drawings is possible. There are laser-copiers that can "copy" with different scaling factors for two perpendicular axes. When a picture is "copied" in one direction in the proportion 1:1 and in the perpendicular direction with a foreshortened proportion then the distortion is orthogonally affine. It will be shown that every simple drawable oblique view can be changed mechanically by special copying into an orthographic view which is much more complicated to construct but has a more realistic effect. Because the vividness of an orthographic view, looking perpendicularly on the image plane is much better than that of an oblique view, the effect of the illustration is improved by this change.

1 QUALITY OF AN OBLIQUE VIEW AND AN ORTHOGRAPHIC ONE

The more the projective beams and the visual rays (of the observer) correspond, the better is the realistic effect of a drawing. More precisely: looking onto a drawing, the bundle of the visual rays and the bundle of the projective beams should differ as little as possible. The degree of this correspondence varies with different ways of projection and is determined by the position of the observer.

Theoretically with PARALLEL PROJECTION this correspondence can be totally achieved, if the observer is looking out of the infinity and if the drawing is inclined to the axis of the view in the degree, that was the basis of the projection. If the distance between the illustration and the observer is infinite, the images on the retinas of both eyes are identical. Therefore it is possible – even though not out of infinity – to observe the results of parallel projection with both eyes open without any distortion of the proportions. (The results of central projection have – strictly speaking – to be observed with one eye closed.)

The realistic effect of a drawing will inevitably be reduced because the observer has to get close enough to the drawing, so that he can recognise the details. Then the projective beams and the visual rays are no longer exactly but only relatively corresponding. The capability of the human brain evens out satisfactorily these small deviations, if the main visual ray approximates the direction of the projection. (In this text the main visual ray is the extended axis of the optical system of the human eye.) But when the main visual ray and the direction of the projection deviate considerably, the result is serious distortion which can totally destroy the pictorial effect of the drawing.

The difficulty to get the eye in the right position depends on the position of the image plane to the projective beams:

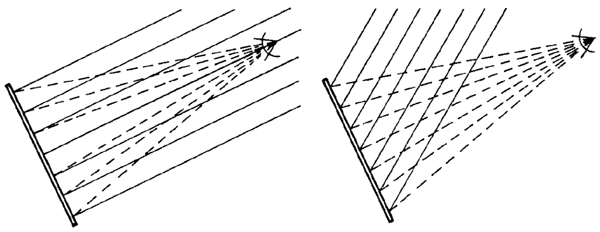

Fig. 1 Projective beams and visual rays

In an ORTHOGRAPHIC PROJECTION the direction of the projection in relation to the image plane is obvious. Without difficulties and without equipment the observer can position his eyes correctly. The projective beams and the visual rays almost correspond, if the eye is perpendicular above the image plane, which goes through the observed detail. For example, this condition is fulfilled, if the drawing is pinned for presentation on a wall and the observer is standing in front of it, or if the drawing is held in one's hands in such a way, that the visual rays are perpendicular to the image plane. The realistic effect of the drawing is weakend if the visual rays are oblique to the image plane; this happens e.g., if the drawing lies on a table and the observer is sitting or standing in front of it. But the human brain is able to interpret the drawing correctly in defiance of small distortions.

In OBLIQUE PROJECTION the bundle of the visual rays and the bundle of the projective beams are mostly not corresponding. As a rule the angle between the image plane and the projective rays is unknown and only laboriously to be found. Therefore the correspondence between the bundle of visual rays and the bundle of projective beams can nearly never be reached.
For architectural drawings the special case of oblique projection is favourable where the image plane is horizontal. These drawings are mostly constructed in such a way, that the angle between the image plane and projective beams is 45°. In these cases the observer can easily position his eyes correctly; this condition is met e.g., if the drawing lies on a table and the observer is sitting or standing in front of it. Hanging on a wall, these oblique views have a distorted effect to the standing observer.

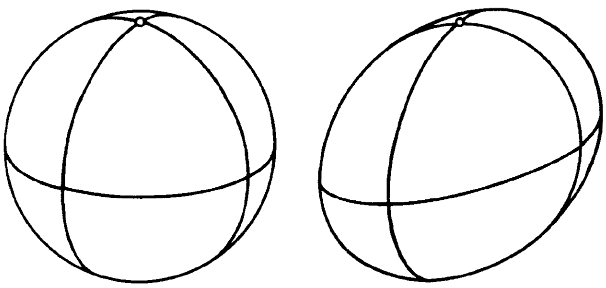

Fig. 2 Outlines of a sphere

The difference between the realistic effect of an orthographic view and an oblique view can be made visible in a simple way by the illustration of a sphere. In parallel projection the projective beams touch the sphere along a circle with the radius of the sphere. These beams form a straight circle cylinder.
–In orthographic projection this cylinder is perpendicularly cut by the image plane along a parallel circle. Therefore the outline of a sphere is a circle.
–In oblique projection this cylinder is cut by the image plane inclined along an ellipse. Therefore the outline of a sphere is an ellipse.

2 DERIVATION OF THE COPY METHHOD

It is well known that there is an AFFINITY between two (not parallel) planes, if one plane is projected onto the other plane by general parallel projection and the projective beams are not parallel to one of the planes. In these cases the following characteristics of the affinity can be used for construction:
–a point, which is not part of the axis of affinity, and its affine picture are on the same beam of affinity. All beams of affinity are parallel.
–a point on the axis of affinity and its picture are identical.
–the affine picture of a straight line is a straight line.
–a straight line and its affine picture intersect in a point of the axis of affinity or are both parallel to it.
–the affine pictures of parallel lines are parallel.
–the proportion of three points on a line and the proportion of their affine picture are identical. In particular the affine picture of the midpoint of a line segment is the midpoint of its affine picture.

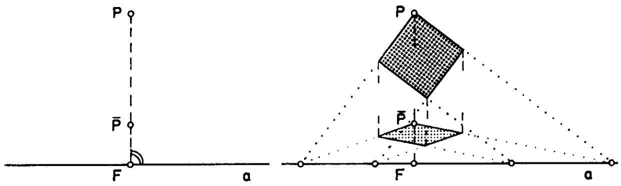

Fig. 3 Orthogonal affinity

The ORTHOGONAL AFFINITY (in a plane) is the special case of affinity where the rays of affinity are perpendicular to the axis a of the affinity. The degree of distortion v is the proportion of the length of the picture of a perpendicular straight line $\overline{F\bar{P}}$ and the true length of that line $\overline{FP}$

$$v = \frac{\overline{F\bar{P}}}{\overline{FP}}.$$

The phenomenon of affinity can easily be applied to the connection between two parallel views of an object which base on the same projective beams.

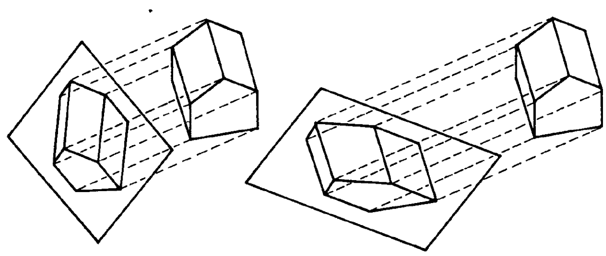

Fig. 4 Parallel projection

The beams of PARALLEL PROJECTION which “touch” an object form a prism (cylinder). A picture is formed by the intersection of this prism with the image plane:
–if the image plane is perpendicular to the projective beams (edges of the prism), an orthographic view is formed.
–if the image plane is inclined to the projective beams, an oblique view is formed.
An orthographic view and an oblique view are affine. The axis of affinity is the intersection line of both image planes. The beams of affinity are the projective beams. In general two parallel views are therefore affine, if they are based on the same projective beams.
Because of the more realistic effect of the

orthographic view the case in which one of the parallel views is an orthographic projection onto an inclined image plane is of particular interest. The following relation between an oblique view and an orthographic view is valid:

EVERY OBLIQUE VIEW OF AN OBJECT CAN BE TRANSFORMED USING ORTHOGONAL AFFINITY INTO AN ORTHOGRAPHIC VIEW.

Now the reasons for the validity of this claim will be given:

if the image plane π_n of the orthographic view is turned around the intersection line s of both image planes onto the image plane π_s of the oblique view, each point P_n of the plane π_n describes a circle. The plane ϵ of the circle is perpendicular to both image planes and their intersection line. After the fold-down P_n is in the position $\overline{P}_n$. Since the projective beam p through P_n and P_s is perpendicular to π_n, p is likewise in ϵ. The ray of affinity which connects P_s and $\overline{P}_n$ is the intersection line between ϵ and π_s and is therefore perpendicular to the axis s of affinity. Therefore the relation between both image planes is orthogonally affine.

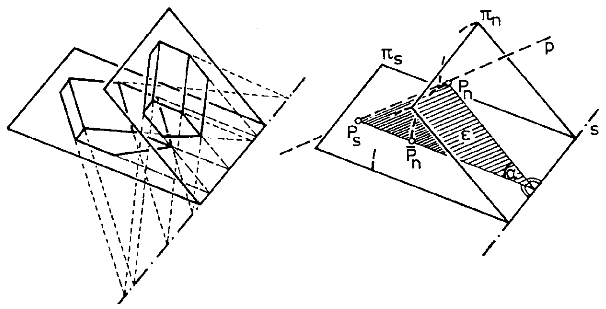

Fig. 5 Affinity betweeen two image planes

3 PRACTICAL APPLICATIONS

In order to change an oblique view with one “copy” into an orthographic view there are only the determinations of a perpendicular affinity to discover: the axis of the affinity and the degree of distortion. Unfortunately the determination of the necessary pieces needs in the case of an OBLIQUE PROJECTION ONTO AN INCLINED IMAGE PLANE lavish reflections and laborious constructions. This problem can be solved easily if the picture and the true length of a straight line perpendicular to the image plane can be found. These criteria are fulfilled by the often used OBLIQUE PROJECTION ONTO A HORIZONTAL IMAGE PLANE (bird's-eye-view) and the OBLIQUE PROJECTION ONTO A VERTICAL IMAGE PLANE (cavalier–perspective). In these cases the image plane of the oblique projection is parallel to a coordinate plane. The perpendicular angle between those two coordinate axes is invariant under the projection. In this case the third coordinate axis is logically perpendicular to the image plane and its projection shows the direction of the affinity. The axis of the affinity is perpendicular to that direction.

3.1 Orthographic view by transformation of an oblique view onto a horizontal image plane

In an oblique view onto a horizontal image plane the perpendicular angle between the x-axis and the y-axis of the coordinate system is invariant under the projection; in other words $\overline{x}$ and $\overline{y}$ are perpendicular. The z-axis is logically perpendicular to the image plane and the beams of affinity are parallel to its projection $\overline{z}$. The axis of the affinity is perpendicular to z.

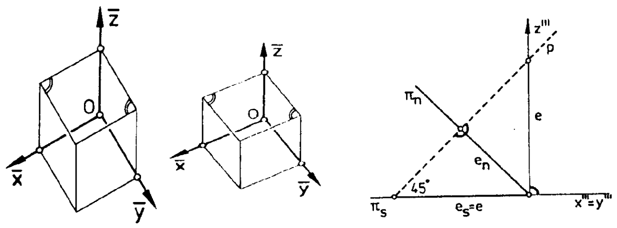

Fig. 6 Angle of the projective beams 45 °

The right illustration is a side view of the spatial situation where both image planes are projecting and therefore be seen as straight lines only. The angle between the projective beams p and the image plane π_s ((x,y)-plane) of the oblique view equals 45 ° if the drawing is constructed with a degree of distortion along the z-axis of 1:1 (therefore $e = e_s$). In the orthographic projection onto the image plane π_n the line segment e_n is shorter. For the transformation of an oblique view onto a horizontal image plane into an orthograhic view the degree v of distortion in direction of the rays of affinity is

$$v = \frac{1}{\sqrt{2}} \mathrel{\hat{=}} 71\%.$$

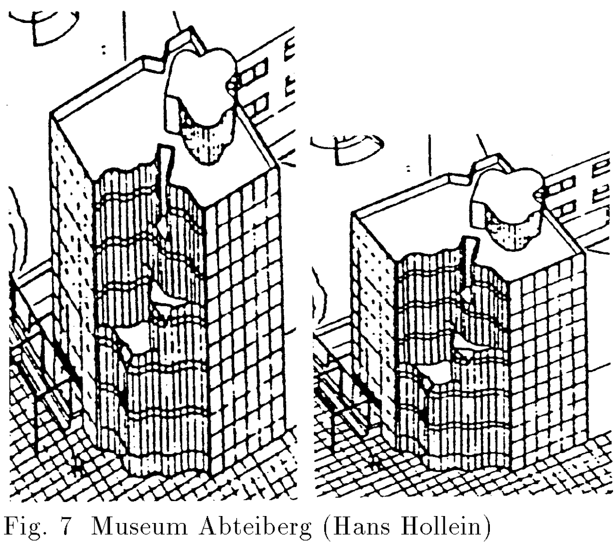

Fig. 7 Museum Abteiberg (Hans Hollein)

The above illustration (administrative building of the museum Abteiberg in Mönchengladbach, Hans Hollein) shows an oblique view and its distortion (71%) in the direction of the verticals into an orthographic view. The perpendicular angle between the horizontal axes of the coordinate system is distorted. (Stretching the oblique view with a degree of $w = \sqrt{2} \hat{=} 141\%$ perpendicular to the direction of the verticals has the same effect, but produces a bigger drawing.) The comparison with the aerial photograph of the object would show that the effect of the orthographic view is nearer to reality then the oblique view. But the improvement by the orthographic view is achieved only if the observer is looking perpendicular onto the drawing. This condition is mostly fulfilled if the drawing is pinned on a wall for presentation. Only if the drawings are lying on a table the bundle of projective beams of the oblique view might better approximate the visual rays of the observer. If an angle α of the projective beams of 45° is unfavourable for the object an oblique view onto a horizontal image plane can be constructed with a different degree of distortion ($\frac{e_s}{e}$) for the vertical line segments. For the transformation into an orthographic view the degree of distortion v of that oblique view can be ascertained with following equation:

$$v = \frac{e_n}{e_s} = \frac{e}{\sqrt{e^2+e_s{}^2}} = \sin\alpha.$$

The illustration on the left shows an oblique view based on a degree of distortion for the vertical line segments of 2:1. The illustration in the middle is an orthographic view as the result of its distortion:

$$v = \frac{1}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}} \hat{=} 45\%$$

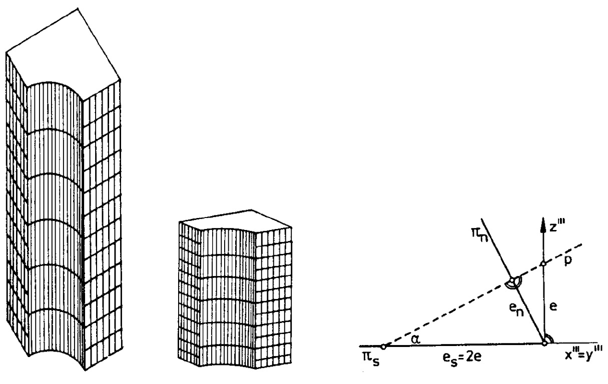

Fig. 8 Small angle of the projective beams

3.2 Distortion of ground-plan for the construction of an orthographic view

Architects sometimes want an inclined but nearly horizontal view of an object. Therefore the angle between the projective beams and the horizontal plane must be small. In this case it is not useful to draw an oblique view first. It is more convenient to distort the ground-plan and get the picture by drawing the verticals perpendicular to the axis of the affinity and using the true length of the lines on them.

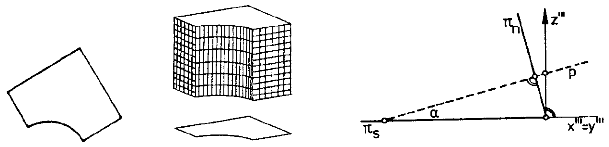

Fig. 9 Distortion of the ground-plan

The degree v of distortion in direction of the beams of affinity can be found with a trigonometrical function depending on the angle α between the projective beams and the image plane π_s: $v = \sin\alpha$. The lengthes of vertical line segments have to be shortend by $u = \cos\alpha$.

In order to avoid the necessity of the distortion of all elevations it is convenient to enlarge the distorted ground-plan by $\frac{1}{\cos\alpha}$. Therefore the ground-plan has to be distorted in direction of the beams of affinity by $\overline{v} = \frac{\sin\alpha}{\cos\alpha}$ and perpendicular to that it has to be stretched by $\overline{w} = \frac{1}{\cos\alpha}$. Now the true lenghtes of vertical line segments can be used for constructing the orthographic view based on the distorted ground-plan (axonometric ground-plan)

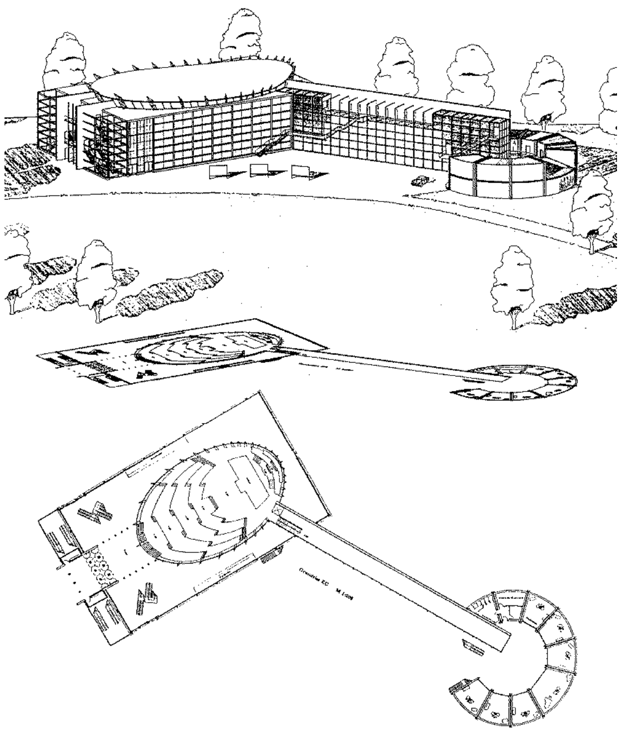

Fig. 10 Musical hall (Manfred Pütz)

3.3 Affine transformation of an elevation

In architecture one can often find complex figures in vertical planes (facades). For the construction of an orthographic view it is possible – similar as with the ground-plan – to use the orthogonal affinity which exists between the orthographic projection of a plane onto an image plane and its fold-down into the image plane.
The axis of affinity is perpendicular to the picture of the normal of the vertical plane. The position of the folded down facade-plane can be found by using the circle of Thales, if there are perpendicular lines in the facade. The degree of distortion in direction of the rays of affinity is found by the proportion of the distance of a point and its picture to the axis of affinity.
This copy-method can also be used very easily to get a side view of a vertical figure (ORTHOGRAPHIC VIEW ONTO A VERTICAL IMAGE PLANE); this problem has often to be solved in architecture. The axis of the orthogonal affinity is vertical; the degree of the distortion is given by the proportion between the length of a horizontal line segment on the image plane and its true length.

3.4 Orthograhic view by transformation of an oblique view onto a vertical image plane

The explanations concerning the transformation of an oblique view onto a horizontal image plane can be transfered onto the oblique view onto a vertical image plane. Only the names of the axes have to be changed: the x-axis becomes the y-axis, the y-axis becomes the z-axis and the z-axis becomes the x-axis. Now the perpendicular angle between the y-axis and the z-axis is invariant under the projection; the direction for the distortion to get an orthographic view is the x-axis. The illustration shows an oblique view onto a vertical image plane (Part of the extension of the Parliament of the Netherlands) and its distortion (71% along x-axis) into an orthographic view.

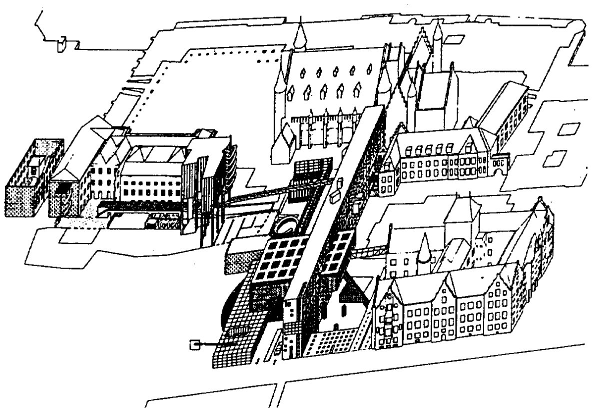

Fig. 11 Oblique view onto a vertical image plane (Extension of the Parliament of the Netherlands)

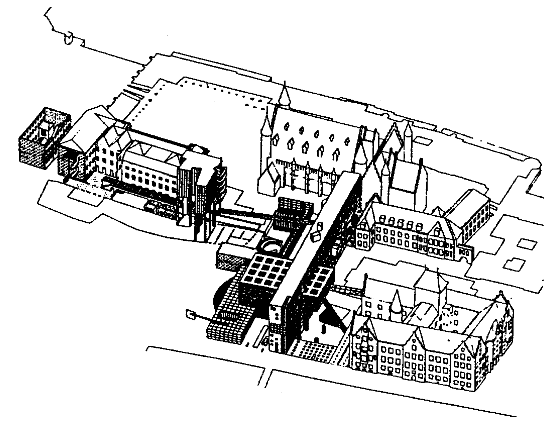

Fig. 12 Orthographic view

3.5 Oblique view by changing an orthograhic view

It is possible to reverse this process and transform every orthographic projection onto an inclined image plane into an oblique projection onto a VERTICAL image plane. The axis of affinity is the picture of the x-axis, the degree of distortion can be found using the circle of Thales to get the perpendicular angle between the y-axis and the z-axis.
For an oblique projection onto a HORIZONTAL image plane the axis of affinity is the picture of the z-axis; the degree of distortion can be found using the circle of Thales to get the perpendicular angle between the x-axis and the y-axis.

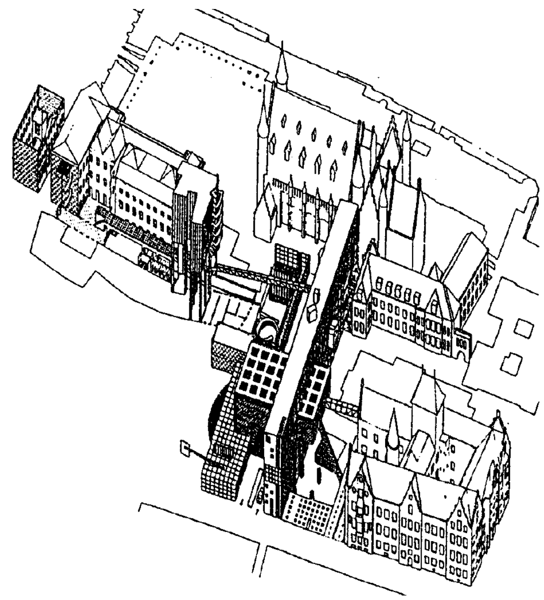

Fig. 13 Oblique view onto a horizontal image plane

4 REFERENCES

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